Stability of Plane Couette Flow
and Pipe Poiseuille Flow

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Abstract

This thesis concerns the stability of plane Couette flow and pipe Poiseuille flow in three space dimensions. The mathematical model for both flows is the incompressible Navier–Stokes equations. Both analytical and numerical techniques are used.

We present new results for the resolvent corresponding to both flows. For plane Couette flow, analytical bounds on the resolvent have previously been derived in parts of the unstable half-plane. In the remaining part, only bounds based on numerical computations in an infinite parameter domain are available. Due to the need for truncation of this infinite parameter domain, these results are mathematically insufficient.

We obtain a new analytical bound on the resolvent at $s = 0$ in all but a compact subset of the parameter domain. This is done by deriving approximate solutions of the Orr–Sommerfeld equation and bounding the errors made by the approximations. In the remaining compact set, we use standard numerical techniques to obtain a bound. Hence, this part of the proof is not rigorous in the mathematical sense.

In the thesis, we present a way of making also the numerical part of the proof rigorous. By using analytical techniques, we reduce the remaining compact subset of the parameter domain to a finite set of parameter values. In this set, we need to compute bounds on the solution of a boundary value problem. By using a validated numerical method, such bounds can be obtained. In the thesis, we investigate a validated numerical method for enclosing the solutions of boundary value problems.

For pipe Poiseuille flow, only numerical bounds on the resolvent have previously been derived. We present analytical bounds in parts of the unstable half-plane. Also, we derive a bound on the resolvent for certain perturbations. Especially, the bound is valid for the perturbation which numerical computations indicate to be the perturbation which exhibits largest transient growth. The bound is valid in the entire unstable half-plane.

We also investigate the stability of pipe Poiseuille flow by direct numerical simulations. Especially, we consider a disturbance which experiments have shown is efficient in triggering turbulence. The disturbance is in the form of blowing and suction in two small holes. Our results show the formation of hairpin vortices shortly after the disturbance. Initially, the hairpins form a localized packet of hairpins as they are advected downstream. After approximately 10 pipe diameters from the disturbance origin, the flow becomes severely disordered. Our results show good agreement with the experimental results.

In order to perform direct numerical simulations of disturbances which are highly localized in space, parallel computers must be used. Also, direct numerical simulations require the use of numerical methods of high order of accuracy. Many such methods have a global data dependency, making parallelization difficult. In this thesis, we also present the process of parallelizing a code for direct numerical simulations of pipe Poiseuille flow for a distributed memory computer.
Preface

This thesis contains five papers and an introduction.


The author of this thesis contributed to the ideas, performed the numerical computations and wrote the manuscript.

This paper is also part of the licentiate thesis [1].


The theoretical derivations were done in close cooperation between the authors, both of them contributing in an equal amount. The author of this thesis had the main responsibility for the computer implementations and wrote section 5 in the report. Malin Siklosi had the main responsibility for the literature studies and wrote sections 1-4 in the report.

This paper is also part of the licentiate thesis [1].


The author of this thesis contributed to the ideas, performed the mathematical derivations and wrote the manuscript.


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Chapter 1

Introduction

The main topic of this thesis is the stability of incompressible plane Couette flow and pipe Poiseuille flow. Plane Couette flow is the stationary flow between two infinite parallel plates, moving in opposite directions at a constant speed, and pipe Poiseuille flow is the stationary flow in an infinite circular pipe, driven by a constant pressure gradient in the axial direction. The mathematical model describing both flows is the Navier–Stokes equations. The reason for studying these flows is that they are simple examples of shear flows and the steady analytical solution is known in both cases. A better understanding of the stability of plane Couette flow and pipe Poiseuille flow could provide information useful for more complicated flows. Inducing or avoiding turbulence by active or passive control would have significant impact in various areas. For example, when mixing fuel and air in an engine, a high level of turbulence intensity is desired for an efficient mixing. The air flow around a moving car or airplane is turbulent in large regions which results in high skin friction. By reducing the areas where the flow is turbulent, lower fuel consumption could be achieved. Which mechanisms are important in the transition to turbulence is not understood. A further insight in this area is crucial for controlling turbulence.

Paper 1 concerns the stability of plane Couette flow by bounding the norm of the resolvent at the point \( s = 0 \) in the unstable half-plane. Previously derived bounds have been based on numerical computations in parts of an infinite parameter domain. We present new analytical results for the resolvent. These results imply a sharp bound on the resolvent in all but a compact subset of the infinite parameter domain. By reducing the domain where computations are needed to a compact set, it is possible to derive a mathematically rigorous bound by using a validated numerical method. This is the topic of paper 2, where we evaluate a method for proving existence and enclosures of solutions of boundary value problems by using numerical computations. Hence, paper 2 provides a way of making the bound on the resolvent in paper 1 rigorous.

Papers 3 – 5 concern the stability of pipe Poiseuille flow. In paper 3, resolvent bounds are derived in parts of the unstable half-plane. The remaining domain
grows with the Reynolds number. We also derive resolvent bounds for perturbations satisfying certain conditions. These bounds are valid in the entire unstable half-plane. No numerical computations are used in paper 3. In order to analyze the stability of pipe Poiseuille flow in detail, direct numerical simulations (DNS) can be used. Even in simple geometries like a pipe, such computations require the use of massive computing resources. The use of high order methods in codes for DNS can make parallelization difficult. This is the topic of paper 4, in which we describe the parallelization of a DNS code for pipe Poiseuille flow and show results of good parallel performance. In paper 5, results from the parallel DNS code are presented. Especially, we consider a spatially highly localized disturbance which in experiments have shown to be efficient in triggering turbulence.

The initial chapters in the thesis give a brief background to the topics of the five papers. In chapter 2, the Navier–Stokes equations are introduced, and some previous results are presented. The literature available on the Navier–Stokes equations is vast, and some references to books are given for further reading. Also, some classical cases of wall bounded shear flows are introduced in this chapter. In chapter 3, the stability of flows are discussed, with emphasis on the two shear flows considered in this thesis. Some previous results are presented, both of experimental and mathematical type. Special attention is given to the methods for analyzing stability considered in this thesis. In chapter 4, the use of numerical, approximate solutions for mathematical proofs is discussed. The idea of using computers for mathematically rigorous proofs is almost a contradiction. The inherent rounding errors in floating-point calculations and the necessity of finite dimensional models in a computer seem impossible to overcome. We give the basic ideas of how these obstacles can be conquered by using well known results from functional analysis and by using a different representation of real numbers when stored in a computer. The ideas are focused on the method implemented in paper 2. We also describe in detail how the method in paper 2 can be used to make the numerical part of the proof in paper 1 rigorous. Chapter 5 contains short summaries of the five papers in the thesis. The summaries are slightly more extensive than the corresponding abstracts and are included for the readers convenience.
Chapter 2

The Navier–Stokes Equations

A mathematical description of the flow of a viscous incompressible fluid was first derived in the early 19th century by Navier. Shortly after, others gave the equations a more firm mathematical foundation. The result was the widely known Navier–Stokes equations.

Given a domain \( \Omega \subset \mathbb{R}^n \), let \( u(t, x) = (u_1(t, x), \ldots, u_n(t, x)) \) be the velocity and \( p(t, x) \) the pressure at \((t, x) = (t, x_1, \ldots, x_n)\). The non-dimensionalized Navier–Stokes equations give the evolution of the flow as

\[
\begin{aligned}
  u_t + (u \cdot \nabla)u + \nabla p &= \frac{1}{R} \Delta u, \\
  \nabla \cdot u &= 0.
\end{aligned}
\]  

(2.1)

Here, \( R \) is the Reynolds number given by \( R = V L/\nu \), where \( V \) and \( L \) are typical velocity and length scales, respectively, and \( \nu \) is the kinematic viscosity of the fluid. The equations must also be supplemented with initial and boundary conditions.

It is well known that in two space dimensions, (2.1) has a unique solution for all times under some restrictions on the initial condition. In three space dimensions, there are local (in time) existence results which can be extended to global existence results if the initial condition is small enough in some suitable norm, see e.g. [44] p. 345. Since the analytical solution of (2.1) is only known in a few special cases, obtaining the solution usually involves the use of some numerical method. Existence and uniqueness results are then valuable. For further reading about the mathematical properties of the Navier–Stokes equations, we refer to [14, 44, 54].

2.1 Wall bounded shear flows

In shear flows, the fluid motion is dominated by sheets of fluid moving in different velocities parallel to each other. Although seemingly a substantial simplification, shear flows are present in more complicated flows when considering the flow sufficiently close to an object. In this section, we describe three classical wall bounded
shear flows which have been studied extensively; the main reason for their achieved popularity being that they all are analytical solutions of the Navier–Stokes equations.

All three flows concern the flow in simple geometries. Pipe Poiseuille flow, also known as Hagen–Poiseuille flow, is the (incompressible) flow in an infinite pipe of constant radius. The flow is driven by a constant non-zero pressure gradient in the axial direction. The other two classical wall bounded shear flows are plane Couette flow and plane Poiseuille flow. Both flows concern the flow between two infinite parallel plates. In plane Couette flow, the plates are moving in opposite directions at a constant speed and in plane Poiseuille flow, the plates are stationary and the flow is driven as in pipe Poiseuille flow, i.e. by a constant non-zero pressure gradient in the streamwise direction.

The stationary solutions of these flows are parallel flows, where the velocity only depends on the distance from the wall; for plane Couette flow, the solution is a velocity profile which varies linearly between the velocities of the two plates and in plane and pipe Poiseuille flow, the solution is a parabolic velocity profile in the direction of the negative pressure gradient, see Figure 2.1.

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![Figure 2.1: Stationary solution of plane Couette flow (left) and plane and pipe Poiseuille flow (right)](image)

The length scale used in the definition of the Reynolds number is in pipe flow the diameter of the pipe and in the channel flows half the distance between the plates. The velocity scale used is half the velocity difference between the plates in plane Couette flow and the maximum and mean velocity in plane and pipe Poiseuille flow, respectively. For the channel, the coordinate system is chosen such that $x_1$ is the streamwise direction, $x_2$ the direction normal to the plates and $x_3$ the spanwise direction. The plates are located at $x_2 = \pm 1$, i.e. the domain is $\Omega_c = \{x \in \mathbb{R}^3| -1 \leq x_2 \leq 1\}$. In the case of pipe Poiseuille flow, cylindrical coordinates, $(r, \phi, z)$, are used and the pipe radius is one, yielding the domain $\Omega_p = \{(r, \phi, z)|0 \leq r \leq 1, 0 \leq \phi \leq 2\pi, z \in \mathbb{R}\}$. The stationary solutions are now
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given by

\[
U = \begin{cases} 
  x_2 e_{x_1}, & \text{for plane Couette flow,} \\
  (1 - x_2^2) e_{x_1}, & \text{for plane Poiseuille flow,} \\
  (1 - r^2) e_z, & \text{for pipe Poiseuille flow.}
\end{cases}
\]

Plane Couette flow, plane Poiseuille flow and pipe Poiseuille flow have been extensively studied by applied mathematicians throughout the years, mainly because they are some of the simplest examples of flows available. However, despite their simplicity much is still unknown about the effects of perturbations on the stationary flows. A better understanding of the important mechanisms in these flows could have implications for other, more complicated, flows.
Chapter 3

Stability of shear flows

The field of hydrodynamic stability concerns the stability of various flows when subjected to disturbances. This is an important concept since a stationary unstable flow can not exist in reality. Also, a flow can be stable to some disturbances while unstable to others. A disturbance generating a perturbation which grows with time might lead to turbulence. Quantifying for which disturbances a flow is stable is of great importance in various applications. For an introduction to the field, we refer to the books [8, 9, 41].

Given a flow \( U, P \) which solves (2.1), the effect of a disturbance, \( u^0 \), can be investigated by considering the equations for the perturbed state. Let the deviation from the given flow be denoted by \( u, p \). Since both the given flow \( U, P \) and the perturbed state \( U + u, P + p \) satisfy (2.1), subtracting the equations yields

\[
\begin{align*}
\frac{\partial u}{\partial t} + (u \cdot \nabla)u + (U \cdot \nabla)u + (u \cdot \nabla)U + \nabla p &= \frac{1}{R}\Delta u, \\
\nabla \cdot u &= 0,
\end{align*}
\]

(3.1)

with initial condition \( u(x,0) = u^0 \).

In order to define stability, we need a norm to measure the size of the perturbation. The most commonly used norm is the \( L^2 \)-norm, since it corresponds to the kinetic energy of the perturbation. However, any norm can be used and in some cases other choices of norms might be more suitable.

The flow \( U \) is called stable to the disturbance \( u^0 \) if the norm of the resulting perturbation becomes arbitrarily small as time increases, i.e. if

\[
\lim_{t \to \infty} \|u(t)\| = 0.
\]

(3.2)

If the flow is stable to all disturbances, it is called globally stable. Usually, the flow is only stable to all disturbances which are small enough, i.e. to all disturbances satisfying \( \|u^0\| < \gamma \) for some \( \gamma > 0 \). This is known as conditional stability.

The type of stability a flow exhibits typically depends on the Reynolds number, \( R \). At low \( R \), the flow might be globally stable, while being conditionally stable at
higher $R$. Especially, some flows have a critical Reynolds number, $R_C$, such that for $R > R_C$, the flow is not conditionally stable. This means that there exists at least one infinitesimal disturbance such that the flow is not stable. Determining how the stability depends on the Reynolds number for different flows is of central interest in hydrodynamic stability.

Substantial insight in the stability properties of a flow can be obtained by experimental investigations. Numerous experiments have been performed over the years, both for plane Couette flow and pipe Poiseuille flow. Osborne Reynolds made extensive experimental investigations of pipe flow in the late 19th century; the main achievement of the experiments was the discovery that one non-dimensional number, subsequently named after him, characterized the stability of the flow. He also noted that for Reynolds numbers below $R \approx 2000$, pipe flow is globally stable. This value has in modern experiments been estimated to $R \approx 1800$ [30]. Although transition to turbulence may occur at higher Reynolds numbers, laminar flow can be maintained by avoiding disturbances. In highly controlled experiments, laminar pipe flow has been observed at $R \approx 10^5$ [31]. The highest Reynolds number for which plane Couette flow is globally stable has been determined in experiments to $R \approx 350$ [45].

From a mathematical point of view, the stability properties of a flow can be investigated in several different ways. The most straightforward way is to consider the eigenvalues of the linearized equations, i.e. of the equations (3.1) without the nonlinear term. If there exists an eigenvalue with positive real part, perturbations with a non-zero component in the direction of the corresponding eigenfunction will exhibit exponential growth. Determining the smallest Reynolds number which allows exponentially growing perturbations gives the critical Reynolds number, $R_C$. However, this does not imply that for subcritical Reynolds numbers, i.e. for $R < R_C$, the flow is stable to all perturbations, since the effect of the nonlinear term is ignored. Hence, the eigenvalues give no information about the possible conditional stability at lower Reynolds numbers.

An example of a flow with a critical Reynolds number is plane Poiseuille flow, which becomes linearly unstable at $R_C \approx 5772$ when the so called Tollmien–Schlichting wave becomes linearly unstable [28]. However, turbulence typically appears at much lower Reynolds numbers in reality. Also, the perturbation which requires the smallest amplitude for transition to turbulence at subcritical Reynolds numbers is not the Tollmien–Schlichting wave [35]. Hence, the spectrum gives poor information about the influence of different perturbations.

Even more misleading are the eigenvalues in the cases of plane Couette flow and pipe Poiseuille flow. Romanov proved in 1973 that plane Couette flow is linearly stable at all Reynolds numbers [38]. In experiments however, turbulence has been observed at Reynolds numbers as low as $R \approx 350$. Pipe Poiseuille flow is believed to be linearly stable at all Reynolds numbers, although formal proofs have only been derived for axisymmetric perturbations [11] as well as for certain non-axisymmetric perturbations [3]. In addition to these proofs, many numerical computations have been made indicating linear stability of pipe Poiseuille flow, see
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Despite this linear stability, turbulence may still be observed in pipe flow for Reynolds numbers above \( R \approx 1800 \) and is the typical state at high Reynolds numbers.

In the last two decades, the failure of eigenvalues to predict the stability of these flows has been explained by a phenomenon commonly referred to as transient growth, see e.g. [48] and the review article [39]. If all eigenvalues of the linearized equations have negative real part, linear theory predicts that all perturbations will eventually decay exponentially. However, linear effects may still cause considerable initial growth of a perturbation. This is due to non-orthogonality (in the considered scalar product) of the eigenfunctions of the linearized Navier–Stokes operator. Information about this transient growth is not captured by the eigenvalues but can be obtained by considering the resolvent or the \( \varepsilon \)-pseudospectrum. This is the topic of section 3.1.

Since both plane Couette flow and pipe Poiseuille flow are linearly stable, nonlinear effects are necessary for transition to turbulence. Computers are now powerful enough to simulate flows in simple geometries using direct numerical simulations (DNS). The possibility of high control of disturbances and detailed analysis of results makes DNS a powerful tool. Simulations can reveal which mechanisms are the most important during transition to turbulence. Such information is useful in control of turbulence. An improved ability to avoid or induce turbulence would have numerous applications; an efficient mixing of air and fuel in an engine is achieved with a high intensity of turbulence while airplanes would reduce fuel consumption if turbulence could be avoided. In section 3.2, direct numerical simulation is discussed further.

### 3.1 Stability by resolvent analysis

In order to analytically derive conditions for stability, the resolvent can be used. The resolvent is the solution operator of the Laplace transformed linearized problem. Assume that we have a bound on the norm of the resolvent in the entire unstable half-plane. Then it is possible to derive a bound on the solution of the forced linear problem. This bound is given in terms of the bound on the resolvent and the norm of the forcing. The linear bound is then extended to the nonlinear problem by treating the nonlinear term in the equation as part of the forcing. This is only possible if the forcing is sufficiently small. This condition gives a sufficient condition on the size of the perturbation under which nonlinear stability is guaranteed.

We first illustrate this method on a simple model problem, similar to the model problem treated in [13], before discussing results for plane Couette flow and pipe Poiseuille flow. For readers who are not interested in the details, the main steps in the proof of conditional stability of the model problem are the following: For the linear problem corresponding to the model problem (3.3), the resolvent bound (3.4) holds in the entire unstable half-plane Re\( (s) \geq 0 \). Using Parseval’s identity and scalar multiplication, this resolvent bound implies the bound (3.6) for the linear
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problem. If the forcing is sufficiently small, a bound for the nonlinear problem is obtained. By doing this also for the differentiated model problem, the bound (3.11) is obtained under the condition (3.12). The bound (3.11) implies stability for the nonlinear problem, i.e. we have proved conditional stability.

Model problem

Consider the following ordinary differential equation for \( v = (v_1, v_2)^T \),

\[
\begin{align*}
v_t &= L v + g(v) + f(t), \\
v(0) &= v^0,
\end{align*}
\]

where

\[
L = \begin{pmatrix} -R^{-1} & 0 \\ 1 & -2R^{-1} \end{pmatrix}, \quad g(v) = \begin{pmatrix} v_1v_2 \\ v_1^2 \end{pmatrix}.
\]

We are interested in how the stability of this system changes when the positive constant \( R \) grows.

Consider first the linear, unforced case \( g = f = 0 \) with initial condition \( v^0 = (v_1^0, v_2^0)^T \). Since, with \( R > 0 \), the eigenvalues of \( L \) are negative, we know that the solution decays exponentially for sufficiently large times. However, the short time behavior can be significantly different. The general solution of this problem is given by

\[
\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = v_1^0 \begin{pmatrix} 1 \\ R \end{pmatrix} e^{-t/R} + (v_2^0 - v_1^0 R) \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t/R}.
\]

We see that \( v_1 \) decays exponentially at all times. However, Taylor expanding \( v_2 \) at \( t = 0 \) shows that \( v_2 \) grows linearly for \( t \lesssim O(R) \). This is known as transient growth, and is due to the fact that the operator \( L \) is non-normal, i.e. the eigenvectors of \( L \) are non-orthogonal. In fact, the eigenvectors of \( L \) are \( (1, R)^T \) and \( (0, 1)^T \), i.e. they are increasingly non-orthogonal with increasing \( R \).

We now derive conditions for stability of the nonlinear problem. Since the resolvent method uses the Laplace transform, we consider (3.3) with homogeneous initial conditions. Note that (3.1) could be transformed to an equivalent homogeneous problem by e.g. introducing \( u = v + e^{-\delta t} u^0 \) for some \( \delta > 0 \). This would result in a forcing involving the initial perturbation, \( u^0 \), in the equations for \( v \).

Let \( \| \cdot \| \) and \( (\cdot, \cdot) \) denote the \( l^2 \)-norm and inner product of vectors and let \( \| \cdot \| \) denote the corresponding matrix norm. The linear problem corresponding to (3.3) is, after applying the Laplace transform,

\[
s\tilde{v} = L\tilde{v} + \tilde{f}(s).
\]

The solution operator \( \mathcal{R}(s) = (sI - L)^{-1} \) is known as the resolvent. With \( R > 0 \), the eigenvalues of \( L \) are negative. Hence, the resolvent is well defined in the entire
unstable half-plane, \( \text{Re}(s) \geq 0 \). For a normal operator, \( N \), the norm of the resolvent, \( R(s) = (sI - N)^{-1} \), is given by \( \|R(s)\| = \sup_{\lambda \in \sigma(N)} |s - \lambda|^{-1} \), where \( \sigma(N) \) is the spectrum of \( N \), see e.g. [12]. However, since \( L \) is non-normal, the norm of the resolvent is larger. Straightforward calculations give the sharp bound

\[
\|(sI - L)^{-1}\| \leq CR^2
\]

in the entire unstable half-plane.

We use this to bound the solution of the linear problem. By using Parseval’s identity, it follows that

\[
\int_0^T |v(t)|^2 dt \leq \int_0^\infty |v(t)|^2 dt \leq CR^2 \int_0^\infty |f(t)|^2 dt.
\]

For \( t \leq T \), the solution \( v(t) \) does not depend on \( f(t) \) for \( t > T \). Hence, we can set \( f(t) = 0 \) for \( t > T \) in the above expression, yielding

\[
\int_0^T |v(t)|^2 dt \leq CR^2 \int_0^T |f(t)|^2 dt.
\]

We also need a bound on \( |v(T)| \). Scalar multiplication of the linear equation corresponding to (3.3) with \( v \) gives

\[
\frac{1}{2} \frac{d}{dt} |v(t)|^2 = (v, v_t) = (v, Lv) + (v, f) \leq C_1 |v|^2 + \frac{1}{2} (|v|^2 + |f|^2),
\]

where \( C_1 \) is a bound on the range of \( L \). Integrating this from \( t = 0 \) to \( t = T \) and using (3.5) gives

\[
|v(T)|^2 \leq \tilde{C} R^2 \int_0^T |f(t)|^2 dt.
\]

Hence, we have the following bound for the linear problem,

\[
|v(T)|^2 + \int_0^T |v(t)|^2 dt \leq C_L R^4 \int_0^T |f(t)|^2 dt.
\]

Now, we will treat the nonlinear term as part of the forcing. For the nonlinear term, we have

\[
|g(v)|^2 \leq |v|^4.
\]

Assume that the solution of the nonlinear problem (3.3) satisfies

\[
|v(T)|^2 \leq 4R^4 K,
\]

\[
K = C_L \int_0^\infty |f(t)|^2 dt,
\]

for all times \( T \in [0, \infty) \). We prove this assumption by assuming that it is not true, thus deriving a contradiction. Since \( v(0) = 0 \), (3.8) must hold with strict
inequality for some initial time interval. Let \( T_0 > 0 \) be the smallest time such that there is equality in (3.8) and consider \( T \leq T_0 \). From the linear estimate (3.6) and the bounds (3.7) and (3.8), we have
\[
|v(T)|^2 + \int_0^T |v(t)|^2 dt \leq C_L R^4 \int_0^T |f(t) + g(v)|^2 dt
\]
\[
\leq 2C_L R^4 \int_0^T |f(t)|^2 + |v(t)|^4 dt
\]
\[
\leq 2R^4 K + 8C_L R^8 K \int_0^T |v(t)|^2 dt.
\]
We must now assume that the forcing is sufficiently small. Assume that
\[
1 - 8C_L R^8 K \geq \frac{1}{2}.
\]
(3.9)
Then for \( T \leq T_0 \), we have the following bound
\[
|v(T)|^2 + \frac{1}{2} \int_0^T |v(t)|^2 dt \leq 2R^4 K.
\]
(3.10)
Clearly, the assumption of equality in (3.8) at time \( T_0 \) can not be true and (3.10) must hold for all times \( T \in [0, \infty) \).

We also need a similar bound for \( |v_t| \). This is obtained by differentiating equation (3.3) with respect to \( t \). It is easily found that \( |q_t|^2 \leq 3(|v|^2 + |v_t|^2) \). By doing the same derivations as above for \( |v|^2 + |v_t|^2 \), it is found that
\[
|v(T)|^2 + |v_t(T)|^2 + \frac{1}{2} \int_0^T |v(t)|^2 + |v_t(t)|^2 dt \leq \tilde{C} R^4 \tilde{K},
\]
(3.11)
where
\[ \tilde{K} = C_L \int_0^\infty |f(t)|^2 + |f_t(t)|^2 dt. \]
If we assume \( f(t) \in H^1([0, \infty)) \), the right hand side of (3.11) is bounded. It follows that \( v(t) \in H^1([0, \infty)) \), which implies \( \lim_{t \to \infty} |v(t)| = 0 \). Thus, we have proved nonlinear stability under an assumption similar to (3.9) for \( \tilde{K} \) instead of \( K \), i.e. when
\[
\int_0^\infty |f(t)|^2 + |f_t(t)|^2 dt \leq \hat{C} R^{-8}.
\]
(3.12)

**Plane Couette flow and pipe Poiseuille flow**

It has been found that the eigenfunctions of the linearized operators of plane Couette flow and pipe Poiseuille flow are highly non-normal in the \( L^2 \)-inner product, see e.g. [36, 47]. This can cause significant transient growth, as explained in the
3.1. STABILITY BY RESOLVENT ANALYSIS

Previous section. Therefore, the resolvent or the closely related $\varepsilon$-pseudospectrum has been in focus the last twenty years. The $\varepsilon$-pseudospectrum of a linear operator, $L$, generalizes the concept of eigenvalues by defining $s$ to belong to the $\varepsilon$-pseudospectrum if $\|(sI - L)^{-1}\| \geq \varepsilon^{-1}$. Hence, the $\varepsilon$-pseudospectrum gives information of where the resolvent is large, as opposed to the spectrum which only give information of where the resolvent is infinite or non-existing. Computations of the $\varepsilon$-pseudospectrum for plane Couette flow and pipe Poiseuille flow can be found in e.g. [22, 47, 48].

For plane Couette flow, the resolvent, $R$, has been investigated by numerical and analytical techniques. In [13], the lower bound $\|R\| \geq CR^2$ was proved for the $L^2$-norm. Here and below, we use $C$ to denote any constant independent of the Reynolds number. Numerical computations in [13, 17] indicated this asymptotic dependence to hold in the entire unstable half-plane, i.e. $\|R\| \leq CR^2$. An analytical bound on the $L^2$-norm of the resolvent was derived in large parts of the unstable half-plane in [17], where also a new norm was introduced. Computations in [17] indicated $\|R\| \leq CR$ in the new norm. This is an optimal $R$-dependence since there is an eigenvalue with real part $-\text{Re}(\lambda) \sim R^{-1}$ [48]. For pipe Poiseuille flow, fewer analytical results about the resolvent have been derived. Numerical computations in [22] indicate the same dependence as for plane Couette flow, i.e. $\|R\| \leq CR^2$ for the $L^2$-norm.

As in the example in the previous section, a bound on the resolvent can be used to prove nonlinear stability. Using this technique, the upper bound $\beta \leq 5.25$ in the threshold amplitude dependence $R^{-\beta}$ was proved for wall bounded shear flows in [13], under the assumption of the resolvent bound $\|R\| \leq CR^2$. Although this upper bound on $\beta$ is not sharp, it serves as the only analytical proof of conditional stability of wall bounded shear flows. However, since the resolvent bounds available both for plane Couette flow and pipe Poiseuille flow are based on numerical computations in an infinite parameter domain, the proof is not fully rigorous.

This is the motivation of the first paper of this thesis [4]. We present a new sharp bound on the resolvent at the believed maximum $s = 0$. The bound is based on analytical estimates in all but a compact subset of the parameter domain. In this compact set, we use numerical computations to obtain a bound. Since the set is compact, the numerical bound can be made rigorous by using validated numerical methods. We explain in detail how this can be done in the next chapter. Using the same technique, we hope to bound the resolvent in the remaining part of the unstable half-plane in the future. Moreover, analytical bounds can provide more precise information about the resolvent than just the maximum in the unstable half-plane. Such information could be used to improve the upper bound on $\beta$, i.e. to sharpen the threshold amplitude for nonlinear stability.

The third paper of this thesis [3] concerns the resolvent of pipe Poiseuille flow. We derive analytical bounds on the resolvent in large parts of the unstable half-plane. Also, a bound valid in the entire unstable half-plane is derived for perturbations which satisfy certain relations involving the Reynolds number and the
wave numbers in the axial and azimuthal directions. Especially, this bound on the resolvent is valid for the perturbation which computations indicate to be the perturbation which exhibits largest transient growth [47].

3.2 Direct numerical simulations

In order to investigate the nonlinear behavior of a flow, direct numerical simulations (DNS) can be used. DNS means that the full nonlinear Navier–Stokes equations are solved such that all length scales are resolved. This requires large amounts of computer resources as well as numerical methods with high order of accuracy. DNS are therefore so far only possible at moderate Reynolds numbers and in simple geometries, and should therefore not be confused with an engineering tool for real-world problems.

DNS has, however, proven to be an excellent tool in research. For example, the threshold amplitude below which perturbations eventually decay has been examined by DNS; using different disturbances, the threshold was found to behave as $R^{-\beta}$, with $1 \leq \beta \leq 1.25$ for plane Couette flow and $1.6 \leq \beta \leq 1.75$ for plane Poiseuille flow [13, 19, 35]. The Reynolds number below which pipe Poiseuille flow is globally stable has been verified to $R \approx 1800$ by using DNS [52]. Also, DNS have yielded important understanding of the mechanisms behind transition to turbulence in boundary layers, see e.g. the review article [23].

Most DNS have so far been performed in planar geometries, since Cartesian coordinates can then be used. In pipe flow, high order methods can be used by considering the equations in cylindrical coordinates. However, this introduces smaller grid-cells near the center of the pipe, requiring a smaller time step. Also, additional difficulties arise from the polar “singularity” in the discretization. Some DNS codes have been developed for pipe Poiseuille flow, see e.g. [10, 18, 20, 21, 27, 29, 42, 49, 56]. However, these codes are typically of rather low order of accuracy and almost exclusively written for serial computers.

In the fourth paper of this thesis [2], we present the process of parallelizing a code for DNS of pipe Poiseuille flow for a distributed memory computer. The code is based on compact finite differences of high order of accuracy in the axial direction and Fourier and Chebyshev expansions in the azimuthal and radial directions, respectively [37]. These numerical techniques are computationally efficient but introduce a global data dependency. This makes parallelization difficult, since there is no way to divide the problem into smaller problems which are almost independent. We present our strategy of parallelization and show results on good efficiency of the parallel code.

The fifth paper of this thesis [5] concerns DNS of pipe Poiseuille flow. We use the parallel code developed in paper 4 in order to simulate a disturbance which is highly localized in space. The disturbance is a combination of suction and blowing through two small holes located close to each other and aligned such that they form a 45-degree angle with the pipe axis. The motivation for the simulations is
that experiments have shown that this disturbance is efficient in triggering turbulence. Our results show an initial formation of so-called hairpin vortices, which are known to play a central role in transition to turbulence in boundary layers [6]. The hairpins are initially advected downstream in an ordered and localized way. After approximately 10 pipe diameters, the perturbation changes from being localized to a globally disordered state. Our results show good agreement with the experiments.
Chapter 4

Computer-Assisted Proofs

The invention of the computer has had a tremendous impact on the field of applied mathematics. Problems that were practically impossible to solve 50 years ago are solved in fractions of a second today. However, these solutions are almost never true solutions. A numerical solution of a problem usually suffers from errors. One source of error is that the mathematical model might have infinite degrees of freedom, making finite dimensional approximations necessary. Deriving explicit bounds on the errors made by the approximations is usually difficult. Another source of error is the rounding error. Numbers like $\pi$, $\sqrt{2}$ cannot be stored exactly in a computer. Even for numbers that are stored exactly, floating-point arithmetic is not closed. This means that even if $x$ and $y$ can be stored exactly, there is no guarantee that e.g. $x + y$ can be stored exactly, making rounding necessary.

In this chapter, we give the basic ideas of how to prove existence and enclosures of solutions of elliptic boundary value problems. This is the topic of the second paper of this thesis [43]. We also explain why this topic is relevant for the first paper of this thesis.

4.1 Basic Ideas

In this section, we describe two methods for proving existence of solutions of elliptic boundary value problems. The first method was proposed by Nakao, and has been successfully used in various applications [25, 46, 50, 51, 53]. This is the method used in paper 2 of this thesis. The second method was proposed by Plum, and has also proved successful [7, 15, 32, 33, 34]. The methods are quite similar in some parts, and a combination of them has been used by Nagatou, Yamamoto and Nakao [26].

Both methods rely on an approximate, numerical solution, $u_h$, which can be derived by any numerical method. From the approximate solution, a suitable fixed-point equation, $w = T(w)$, for the error, $w = u - u_h$, is derived. The idea is to prove that $w = T(w)$ has a solution in a subset of a Banach space. The subset consists of
functions with norm smaller than an explicitly derived upper bound. This upper bound gives bounds on the magnitude of the error in the approximate solution, \( u_h \).

In order to prove the existence of a solution of the fixed-point equation, Nakao and Plum use the well known Schauder fixed-point theorem or Banach fixed-point theorem. The theorems state, see e.g. [55],

**Theorem 4.1.1** (Schauder fixed-point theorem). Let \( W \) be a non-empty, closed, bounded, convex subset of a Banach space \( X \). If \( T : W \to W \) is a compact operator, then there exists a \( w \in W \) such that \( w = T(w) \).

**Theorem 4.1.2** (Banach fixed-point theorem). Let \( W \) be a non-empty, closed subset of a complete metric space \( X \). If \( T : W \to W \) is a contraction on \( W \), then there exists a unique \( w \in W \) such that \( w = T(w) \).

Note that Theorem 4.1.2 ensures a unique solution in \( W \), which is not the case for Theorem 4.1.1.

Verifying that Theorem 4.1.1 or Theorem 4.1.2 can be applied to the derived fixed-point equation and finding a suitable subset are done in different ways in the approaches by Nakao and Plum.

In Nakao’s method, the fixed-point equation is divided into a finite dimensional part and an infinite dimensional part. The finite dimensional part is rewritten using the linearization, \( L_h \), of the finite dimensional projection of the given equation at the approximate solution, \( u_h \). This yields an equivalent fixed-point equation which is more likely to map the finite dimensional part of \( W \) into itself. Verifying the conditions of Theorem 4.1.1 or Theorem 4.1.2 for the finite dimensional part is done by explicitly inverting \( L_h \). The infinite dimensional part is treated by analytical methods, using e.g. a priori error bounds on the projection into the finite dimensional subspace.

Plum’s method uses the linearization, \( L \), of the infinite dimensional problem at the approximate solution, \( u_h \). Using a lower bound on the norm of \( L \) and a bound on the norm of the residual of \( u_h \), the conditions of Theorem 4.1.1 or Theorem 4.1.2 are verified by analytical and numerical techniques. The main difficulty is to derive the lower bound on the norm of \( L \). This is obtained from the eigenvalue of \( L \) or \( L^* L \) with smallest absolute value. Deriving an enclosure of this eigenvalue can be done by solving related finite-dimensional matrix eigenvalue problems which is suitable for computer implementation.

In both methods, the effect of the rounding errors in computations must be accounted for. This can be done by using interval arithmetic [24]. Interval arithmetic represents real numbers as closed intervals, where the upper and lower bounds of the intervals are floating-point numbers. Thus, all real numbers can be represented. By defining an arithmetic for the intervals, the effect of the rounding error can be rigorously accounted for in each arithmetic operation. This can be extended to all elementary functions used in computations, such that the functions take intervals as arguments and return intervals which encloses the range of the functions over the argument intervals.
4.2 Relation to Paper 1

In the first paper of this thesis, a bound on the resolvent for plane Couette flow is derived at the point \( s = 0 \). This is done by obtaining analytical bounds in all but a compact subset of an infinite parameter domain consisting of wave numbers in two space directions and the Reynolds number. In the remaining compact set, we use standard numerical computations for a finite set of these parameter values. However, although the subset of the parameter domain is bounded, it consists of infinitely many parameter values. Thus, this part of the proof is not rigorous. In this section, we describe how the method in paper 2 could be used to make also the numerical part of the proof rigorous.

We first summarize the procedure of making the proof in paper 1 rigorous. There are two separate problems in obtaining a rigorous bound. First, for a given choice of parameter values, how is a rigorous bound on the resolvent obtained by numerical computations? The solution to this is rather straightforward. In paper 1, the problem is reduced to solving a one-dimensional boundary value problem, computing quantities which depend on the solution, and showing that these quantities fulfill certain conditions. This can be done using the method described in paper 2. The second problem is that a rigorous resolvent bound is not only needed for one choice of parameter values but for infinitely many parameter values. This problem is solved by analytical means, resulting in Lemma 4.2.1. In short, the lemma states that if a rigorous resolvent bound is derived for one choice of parameter values, then this bound is valid in some neighborhood of the chosen parameter values. The size of this neighborhood is explicitly computable. Hence, rigorous resolvent bounds need to be derived only for a finite set of parameter values. In the rest of this section, we describe this strategy in further detail.

In paper 1, the numerical part of the proof concerns the boundary value problem

\[
\begin{align*}
\frac{d^2 w}{dx^2} - (iax + b^2) w(x) &= 0, \\
w(-1) &= 1, \\
w(1) &= 0,
\end{align*}
\]  

(4.1)

in the compact parameter domain \( \Sigma = \{a, b \in \mathbb{R} \mid a \in [1/16, 40^3], b^2 \in [0, a_1/3]\} \). For every combination of \( a \) and \( b \) in \( \Sigma \), we need to prove two things about the solution of (4.1). First, we need to prove that the \( L^2 \)-norm of the solution is bounded. Later in this section, we show that this holds for all parameter values in \( \Sigma \), see the remark after the proof of Lemma 4.2.1. Secondly, we need to prove that the matrix

\[
J = \begin{pmatrix}
  u'(-1) & -(u^*)'(1) \\
  u'(1) & -(u^*)'(-1)
\end{pmatrix}
\]  

(4.2)

is non-singular. Here, \((u^*)(x)\) denotes the complex conjugate of \( u(x) \). The matrix
elements are given by

\[ u'(-1) = \int_{-1}^{1} f_b(\sigma)w(\sigma)d\sigma, \] (4.3)

\[ u'(1) = \int_{-1}^{1} g_b(\sigma)w(\sigma)d\sigma, \] (4.4)

where

\[ f_b(\sigma) = \begin{cases} \frac{\sinh(b(\sigma-1))}{\sinh(2b)}, & b \neq 0, \\ \frac{\sigma-1}{2}, & b = 0, \end{cases} \] (4.5)

\[ g_b(\sigma) = \begin{cases} \frac{\sinh(b(\sigma+1))}{\sinh(2b)}, & b \neq 0, \\ \frac{\sigma+1}{2}, & b = 0. \end{cases} \] (4.6)

Note that the matrix (4.2) is non-singular if and only if \(|u'(-1)| \neq |u'(1)|\). Hence, for a given pair of parameters, \(a\) and \(b\), we need to enclose the solution \(w(x)\) of (4.1) and then derive rigorous enclosures of the absolute values of the integrals (4.3) and (4.4). This can be done with the method used in the second paper of this thesis. However, since we implemented the method in MATLAB, we were not able to obtain rigorous bounds when \(a\) is large. Using e.g. Fortran would hopefully be sufficient for covering the parameter domain we are interested in.

We are still left with the problem of having an infinite number of parameter values in \(\Sigma\). This can be handled with analytical techniques. By using information about how far from singular \(J\) is at a given point in \(\Sigma\), we are able to prove that \(J\) is non-singular in a neighborhood around this point. The result is summarized in the following lemma.

**Lemma 4.2.1.** If for \(a = A\) and \(b = B\), the solution \(W(x)\) of (4.1) is such that the matrix elements (4.3) and (4.4) satisfy

\[ |U'(-1)| - |U'(1)| \geq \alpha \] (4.7)

for some \(\alpha > 0\), then the matrix \(J\) given by (4.2) is non-singular for all parameter values \(a\) and \(b\) satisfying

\[ 8\beta(\|f_B\| + \|g_B\|) + (8\beta + 1)(\|f_b - f_B\| + \|g_b - g_B\|) < \frac{\alpha}{\|W\|}, \] (4.8)

where \(f\) and \(g\) are given by (4.5) and (4.6) and where

\[ \beta = |a - A| + |b^2 - B^2|. \]

Here, \(\| \cdot \|\) is the \(L^2\)-norm on \(\Omega = \{-1 \leq x \leq 1\}.

Before proving the lemma, note that it is not obvious that the quantity on the left hand side of (4.7) should be positive. However, numerical experiments indicate
this to always be the case. Of course, if the left hand side of (4.7) would be negative for some parameter combination, a similar lemma could be derived handling this case.

Proof. Consider some parameter values $a$ and $b$ satisfying (4.8) and denote the corresponding solution of (4.1) by $w(x)$. From (4.1), the difference $\tilde{w} = w - W$ satisfies

$$\tilde{w}'' - (i a x + b^2)\tilde{w} = (i(a - A)x + (b^2 - B^2))W, \quad \tilde{w}(\pm1) = 0.$$  

Taking the $L^2$-inner product of this equation with $\tilde{w}$, using integration by parts and taking the real part yields

$$\|\tilde{w}'\|^2 + b^2\|\tilde{w}\|^2 \leq (|a - A| + |b^2 - B^2|)\|W\||\tilde{w}| = \beta\|W\||\tilde{w}|.$$

Using a Poincaré inequality for $\tilde{w}$ and the relation $cd \leq c^2/(2\mu) + d^2\mu/2$, valid for all $c, d \in \mathbb{R}$, $\mu > 0$, we thus have the bound

$$\frac{1}{2}\|\tilde{w}'\|^2 + \left(\frac{1}{16} + b^2\right)\|\tilde{w}\|^2 \leq 4\beta^2\|W\|^2.$$  

(4.9)

Now, evaluating $|u'(1)|$ from (4.3), using $w = \tilde{w} + W$ and (4.9) gives

$$|u'(-1)| = \left|\int_{-1}^{1} (f_B(\sigma) + f_b(\sigma) - f_B(\sigma)) (\tilde{w}(\sigma) + W(\sigma))d\sigma\right|$$

$$\geq |U'(-1)| - \|f_B\||\tilde{w}| - \|f_b - f_B\|(\|\tilde{w}\| + \|W\|)$$

$$\geq |U'(-1)| - 8\beta\|f_B\||W| - (8\beta + 1)\|f_b - f_B\||W|.$$  

(4.10)

Similarly, using (4.4) yields

$$|u'(1)| \leq |U'(1)| + 8\beta\|g_B\||W| + (8\beta + 1)\|g_b - g_B\||W|.$$  

(4.11)

By (4.7), (4.8), (4.10), and (4.11), we have $|u'(-1)| - |u'(1)| > 0$ and thus $J$ is non-singular.

Remark. We stated earlier in this section that the $L^2$-norm of the solution of (4.1) is bounded for all $a$ and $b$ in $\Sigma$. Since (4.9) also holds when $W$ is the solution with $A$ and $B$ outside $\Sigma$, we can especially chose $A = B = 0$. Clearly, $\|W\|$ is then bounded, and it follows from (4.9) that $\|w\| = \|\tilde{w} + W\|$ is bounded in any bounded parameter domain.

Hence, Lemma 4.2.1 and the method used in paper 2 provides a possibility of deriving a rigorous bound on the resolvent in $\Sigma$, where the bound in paper 1 is not rigorous. One needs to find a finite set of points in $\Sigma$ such that $J$ is non-singular for these points and such that the neighborhoods, given by Lemma 4.2.1, cover $\Sigma$.

In order for $\Sigma$ to be covered, we must ensure that the measures of the neighborhoods do not become arbitrarily small even if $J$ is non-singular. This can only
happen if $\alpha$ in (4.7) becomes arbitrarily small somewhere in $\Sigma$. However, from (4.10) and (4.11), we know that the function $\gamma(a, b) \equiv |u'(1)| - |u'(-1)|$ is continuous with respect to $a$ and $b$. Since $\Sigma$ is a compact set, $\gamma(a, b)$ attains a minimum, $\alpha_{\text{min}}$, in $\Sigma$. Hence, if $J$ is non-singular in $\Sigma$, we can cover $\Sigma$ with a finite number of neighborhoods attained from using Lemma 4.2.1. Computations made in paper 1 indicate that $J$ is non-singular, and we believe this could be proved with the approach described in this section.

Finally, note that when computing the quantities in (4.8), all computations should be rigorous, using e.g. interval arithmetic. Since $\|f_B\|, \|g_B\|, \|f_b - f_B\|$ and $\|g_b - g_B\|$ can be derived explicitly, implementation using interval arithmetic is straightforward.
Chapter 5

Summary of Papers

5.1 Paper I: A Rigorous Resolvent Estimate for Plane Couette Flow

In this paper, we derive a rigorous bound on the resolvent for plane Couette flow at the point $s = 0$. We do this analytically by finding approximate solutions of the Orr–Sommerfeld equation while keeping track of the errors made by the approximations. This is not possible in the entire parameter domain. However, the remaining domain is bounded, and we use numerical computations to obtain a bound. Previously derived bounds at $s = 0$ have been based on computations in an infinite parameter domain, making rigorous results impossible. In a bounded domain, rigorous results can be derived by the use of numerical verification methods using interval arithmetic.

This paper is published online in Journal of Mathematical Fluid Mechanics and is entry [4] in the bibliography.

5.2 Paper II: On a Computer-Assisted Method for Proving Existence of Solutions of Boundary Value Problems

In paper 2, we investigate a method for proving existence of solutions of elliptic boundary value problems. The method was proposed by Nakao. We solve two problems using this method; a linear test problem and the one-dimensional viscous Burgers’ equation. For the first problem, the method works well. For Burgers’ equation however, the computational complexity becomes too large when the viscosity decreases. This is not surprising, since Burgers’ equation linearized at the correct solution rapidly becomes close to singular when the viscosity is decreased. We therefore reformulate the problem by replacing one of the boundary conditions with a global integral condition. This approach drastically reduces the computational complexity.

This paper is a technical report and is entry [43] in the bibliography.
5.3 Paper III: Resolvent Bounds for Pipe Poiseuille Flow

In paper 3, we derive an analytical bound on the resolvent of pipe Poiseuille flow in large parts of the unstable half-plane. This is done by scalar multiplying the linearized Navier–Stokes equations (in Cartesian coordinates) with the solution and using integration by parts. We also consider the linearized equations in cylindrical coordinates, Fourier transformed in axial and azimuthal directions. For certain combinations of the wave numbers and the Reynolds number, we derive an analytical bound on the resolvent of the Fourier transformed problem. In particular, this bound is valid for the perturbation which numerical computations indicate to be the perturbation that gives largest transient growth. Our bound has the same dependence on the Reynolds number as the computations give.

This paper is published in Journal of Fluid Mechanics and is entry [3] in the bibliography.

5.4 Paper IV: A Parallel Code for Direct Numerical Simulations of Pipe Poiseuille Flow

In this paper, we describe the process of parallelizing a serial code for direct numerical simulations of pipe Poiseuille flow for a distributed memory computer. The serial code, developed by Reuter and Rempfer, uses compact finite differences of at least eighth order of accuracy in the axial direction and Fourier and Chebyshev expansions in the azimuthal and radial directions, respectively. While these methods are attractive from a numerical point of view, they give a global data dependency which makes the parallelization procedure complex. In the resulting parallel code, the partitioning of the domain changes between partitioning in the axial direction and partitioning in the azimuthal direction as needed. We present results showing good speedup of the parallel code.

This paper is a technical report and is entry [2] in the bibliography.

5.5 Paper V: Direct Numerical Simulations of Localized Disturbances in Pipe Poiseuille Flow.

In this paper, we perform direct numerical simulations of pipe Poiseuille flow subjected to a disturbance which is highly localized in space. The disturbance is a combination of suction and blowing in two small holes, located such that they form a 45-degree angle with the pipe axis. We perform direct numerical simulations for the Reynolds number \( R = 5000 \). The results show a packet of hairpin vortices traveling downstream, each having a length of approximately one pipe radius. The perturbation remains highly localized in space while being advected downstream for approximately 10 pipe diameters. Beyond that distance from the disturbance origin the flow becomes severely disordered.
The stability of pipe Poiseuille flow is highly dependent on the specific disturbance used. The reason for studying this particular disturbance is that experiments by Mullin and Peixinho have shown that it is efficient in triggering turbulence, yielding a threshold dependence on the required amplitude as $\sim R^{-1.5}$ on the Reynolds number. The experiments also indicate an initial formation of hairpin vortices, with each hairpin having a length of approximately one pipe radius, independent of the Reynolds number in the range of $R = 2000$ to $3000$. Thus, our computations are in good agreement with the experiments.

This paper is submitted to Theoretical and Computational Fluid Dynamics and is entry [5] in the bibliography.
Bibliography


