Maximal Regularity of the Solutions for some Degenerate Differential Equations and their Applications

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Mathematics
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Abstract

This PhD thesis deals with the study of existence and uniqueness together with coercive estimates for solutions of certain differential equations.

The thesis consists of six papers (papers A, B, C, D, E and F), two appendices and an introduction, which put these papers and appendices into a more general frame and which also serves as an overview of this interesting field of mathematics.

In the text below the functions \( r = r(x) \), \( q = q(x) \), \( m = m(x) \) etc. are functions on \((-\infty, +\infty)\), which are different but well defined in each paper.

Paper A deals with the study of separation and approximation properties for the differential operator

\[
ly = -y'' + r(x)y' + s(x)\bar{y}'
\]

in the Hilbert space \( L_2 := L_2(\mathbb{R}) \), \( \mathbb{R} = (-\infty, +\infty) \), (here \( \bar{y} \) is the complex conjugate of \( y \)). A coercive estimate for the solution of the second order differential equation \( l y = f \) is obtained and its applications to spectral problems for the corresponding differential operator \( l \) is demonstrated. Some sufficient conditions for the existence of the solutions of a class of nonlinear second order differential equations on the real axis are obtained.

In paper B necessary and sufficient conditions for the compactness of the resolvent of the second order degenerate differential operator \( l \) in \( L_2 \) is obtained. We also discuss the two-sided estimates for the radius of fredholmness of this operator.

In paper C we consider the minimal closed differential operator

\[
Ly = -\rho(x)(\rho(x)y')' + r(x)y' + q(x)y
\]

in \( L_2(\mathbb{R}) \), where \( \rho = \rho(x) \), \( r = r(x) \) are continuously differentiable functions, and \( q = q(x) \) is a continuous function. In this paper we show that the operator \( L \) is continuously invertible when these coefficients satisfy some suitable conditions and obtain the following estimate for \( y \in D(L) \):

\[
\| -\rho(\rho y')' \|_2 + \| ry' \|_2 + \| qy \|_2 \leq c \| Ly \|_2,
\]
where $D(L)$ is the domain of $L$.

In papers D, E, and F various differential equations of the third order of the form

$$-m_1(x) (m_2(x) (m_3(x)y'))' + [q(x) + ir(x) + \lambda]y = f(x) \quad (0.1)$$

are studied in the space $L^p(\mathbb{R})$.

In paper D we investigate the case when $m_1 = m_3 = m$ and $m_2 = 1$. Moreover, in paper E the equation (0.1) is studied when $m_3 = 1$. Finally, in paper F the equation (0.1) is investigated under certain additional conditions on $m_j(x)$ ($j = 1, 2, 3$).

For these equations we establish sufficient conditions for the existence and uniqueness of the solution, and also prove an estimate of the form

$$\|m_1(x) (m_2(x) (m_3(x)y'))'\|_p^p + \|(q(x) + ir(x) + \lambda)y\|_p^p \leq c \|f(x)\|_p^p$$

for the solution $y$ of equation (0.1).
Preface

This PhD thesis consists of six papers (papers A, B, C, D, E and F), two appendices and an introduction, which puts these papers and appendices into a more general frame.


[F] R.D. Akhmetkaliyeva, On maximal regularity of singular third-order differential equations, *Luleå University of Technology, Department of Math-
* An abbreviated version of this paper is also published as:

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Introduction

This PhD thesis deals with the smoothness and approximation properties of solutions of differential equations defined in the Lebesgue space and having real and sometimes complex coefficients.

The main questions in the investigation of differential equations can be classified into the following three categories: existence, uniqueness and qualitative behavior of the solutions. The first two questions are responsible for the accordance of the equations as a mathematical model of the real process, and the third question is necessary to investigate in order to know more about the nature of the process. In the study of the qualitative behavior of solutions of linear and nonlinear differential equations we are interested in the following questions:

1) the smoothness of the solutions;
2) estimates of solutions in different weighted norms;
3) approximation properties of the solutions.

The problem of smoothness for solutions of elliptic equations and estimates of solutions in various norms are well studied in the case when the domain is bounded and the coefficients are reasonable "regular". In this case we have methods which are nowadays developed to the classical perfection and presented in detail in well-known monographs. A fairly complete bibliography of works in this field can be found e.g. in the books of O. A. Ladyzhenskaya and N. N. Ural’tseva [58], J.-L. Lions and E. Magenes [59].

Unfortunately, these methods are not applicable for differential equations given in an unbounded domain and with increasing (not integrable) coefficients. Studies of problems of this type were first made by W. N. Everitt and M. Giertz [39-44] as singular Sturm-Liouville problems. In particular, the formulation of the fundamental problem of separability for a differential operator belong to them. Moreover, in [39-44] the same authors basically elucidated the conditions on the potential function $q(x)$, providing the separability of the Sturm-Liouville operator

$$Ly(x) = -y''(x) + q(x)y(x), \quad x \in \mathbb{R}.$$
In these papers W. N. Everitt and M. Giertz called the indicated operator \textit{separable} in the space $L_2 = L_2(-\infty, +\infty)$, if from $y \in D(L)$, $Ly \in L_2$ it follows that $q(x)y$, $y'' \in L_2$.

Here and in the sequel $c (c_1, c_2, etc)$ denotes a positive constant which may be different in various places.

It is well-known that the separability of the operator $L$ is equivalent to the existence of the estimate

$$
\|y''\|_{L_2^1(R)} + \|qy\|_{L_2^1(R)} \leq c \left( \|Ly\|_{L_2^1(R)} + \|y\|_{L_2^1(R)} \right), \quad y \in D(L),
$$

(0.2)

where $D(L)$ is the domain of $L$. In [39-44] it was shown that if $\inf q(x) > -\infty$ and \(q^{-\frac{1}{2}}(x)'' q^{\frac{1}{4}}(x) \in L_1\), then the operator $L$ is separable. Moreover, an example of a separable operator $L$ with non-smooth potential $q$ was given. In the paper [45] an example of a non-separable operator $L$ with an infinitely differentiable but rapidly oscillating potential was given. Independently of each other F. V. Atkinson [13], K. H. Boimatov [23], [26], M. Otelbaev [88] and D. Z. Raimbekov [102] weakened the condition used by W. N. Everitt and M. Giertz. In particular, in [23], [88] and [102], the condition \(q^{-\frac{1}{2}}(x)'' q^{\frac{1}{4}}(x) \in L_1\) was replaced by weaker conditions (different by different authors), which are similar to the known conditions of Levitan - Titchmarsh, which commonly are used in investigations of the resolvent (concerning these conditions, see e.g. [25], [87] and [109]). In [88] the problem of separability was considered not only in the Hilbert space $L_2$, but also in non-Hilbert weighted spaces $L_{p,l}$ (where $l$ is a continuous weight function). Here $L_{p,l}$ is defined by the norm

$$
\|f\|_{p,l} := \left( \int_{-\infty}^{+\infty} |f(x)l(x)|^p dx \right)^{\frac{1}{p}} \quad (1 \leq p < +\infty).
$$

In particular, it was shown that the separability of the Sturm-Liouville operator holds for an extensive class of rapidly oscillating potentials (for example, $q(x) = e^{i|x|} \sin^2 e^{i|x|}$). Later on M. Otelbaev proposed a special method with local representation of the resolvent to solve the problem concerning the smoothness of solutions of some differential equations, which he called variational. Multivariate equations were considered in [24], where K. H. Boimatov essentially verbatim transferred results from [23] to a class of elliptic operators. The connection of separation with some concrete physical problems was noted in [83].

The existence and smoothness of solutions of nonlinear differential equations (with a singular potential) for unbounded domains equipped with the
Sturm-Liouville equation was considered by M. B. Muratbekov and M. Otelbaev [80]. Later on this problem was investigated in the works T. T. Amanova [9] and M. B. Muratbekov [78].

In [51] the authors investigated the separability of the nonlinear Sturm-Liouville operator

\[ Ly = -y'' + q(x, y)y \]

in the space \( L_1(-\infty, +\infty) \). Moreover, in [1], [17], [29], [74], [114] and [115] the differential expression

\[ Ly = -(P(x)y')' + Q(x)y, \quad x \in (-\infty, +\infty), \]

with operator coefficients was considered.

We have thus motivated the fact that in the case when the differential equation is given in an infinite domain and has unbounded coefficients, the problem of determining the estimates of separability of the type (0.2) for the corresponding differential operator is meaningful. The presence of estimates of separability allows us to accurately describe the class of functions, where the generalized solution of the singular boundary value problem for the differential equation belongs. At the same time the estimate of separability provides a precise description of the domain generated by the indicated singular boundary value problem for the differential operator. This domain is usually a weighted Sobolev space. Thus, if we have estimates of separability, then we can use the modern theory of function spaces to study qualitative properties of the solutions of singular differential equations. We recall that the famous scientist I. M. Gelfand considered that finding estimates of separability (maximal regularity) is one of the most central problems in the study of elliptic equations in the general theory of linear operators (see e.g. preface of the book [58, p.8]).

Separability of a wide class of linear elliptic differential operators was investigated in [11], [13], [23-44], [65-102] and [113-115], where, in particular, important smoothness and approximation properties of solutions of these equations and the spectral properties of the associated singular differential and integral operators were investigated. The methods of proofs in the indicated works are based on deep facts of the theory of embedding between function spaces, of spectral theory of operators, and also widely used advances in the theory of integral operators in function spaces and non-local apriori estimates of generalized solutions. These studies had an enormous influence on the development of the theory of singular differential equations, spectral theory of operators, the theory of weighted function spaces and integral operators in them.
Note that the results in all of these papers was concerned only with second order linear differential operators whose first order terms can be estimated in norms with the other terms involved. However, many practical problems lead us to study elliptic equations, whose properties depend strongly on the behavior of the components with intermediate derivatives of the solution involved and where we have no such norm estimate. Such equations are called in the literature degenerate differential equations. These include for example equations of Schrödinger type with intermediate members with unlimited potential from below, the Korteweg - de Vries type equation, where the coefficient of the junior term depends on the derivative of the unknown function, as well as the differential equation of oscillations in environments with resistance which is proportional to the velocity or acceleration (see [108]). Despite of this, the study of degenerate differential equations was carried out only in the symmetric case for the corresponding differential operators in [47], [54], [55] and [106], where, in particular, the problem of self-adjoint operators assessing their eigenvalues and determination of the structure of the spectrum was solved.

In stochastic analysis and in the theory of stochastic equations, the so-called generalized Ornstein-Uhlenbeck equation

$$Lu = -\Delta u + \nabla u \cdot b + cu = f(x)$$  \hspace{1cm} (0.3)

is widely used, where $x \in \mathbb{R}^n$, $b = b(x)$ is a continuously differentiable vector-valued function, $c = c(x)$ is a continuous function, $f(x) \in L_2(\mathbb{R}^n)$ and the point between the vectors denotes the scalar product in $\mathbb{R}^n$, see e.g. [12], [22], [34], [36], [46], [60], [61], [63], [66], [71], [98] and [100]. For example, in [98] the differential operator $L$ corresponding to (0.3) acts as the generator of the semigroup of the transition of the stochastic process, which determines the $n$-dimensional Brownian motion with a single covariant matrix.

A solution $u = u(x)$ from $L_2(\mathbb{R}^n)$ of the equation (0.3) is said to be regular if $u \in W_2^2(\mathbb{R}^n)$. Further, if the solution $u$ exists and satisfies the following estimate

$$\|\Delta u\|_2 + \|\nabla u \cdot b\|_2 + \|cu\|_2 \leq c (\|f\|_2 + \|u\|_2),$$

then we say that equation (0.3) is coercive solvable.

The function $b = b(x)$ in (0.3) is called displacement or drift. When $b = 0$, the equation (0.3) is the stationary Schrödinger equation, which has been systematically studied for a long time in connection with quantum-mechanical applications (see [53], [55], [81] and [103]). If the coefficient $b$ is different from zero and is not limited, then equation (0.3) is fundamentally different from the Schrödinger equation, since the expression $Lu$ is not always
obtained by adding to the Schrödinger operator an operator with respect to small perturbations.

The study of equation (0.3) is interesting not only from the theoretical point of view. It originally appeared in the fundamental paper [84] in connection with the description of the Brownian motion of particles. Studies of such scientists as M. Smoluchowski, A. Fokker, M. Plank, H.C. Burger, R. Furth, L. Zernike, S. Goudsmit, M.C. Wang and others were devoted to this issue in the early 20th century. An overview of their results can be found in [112]. Along with (0.3) we investigate a more general equation:

$$Au = -\nabla(Q\nabla u) + \nabla u \cdot b + cu = f(x),$$

(0.4)

where $Q = Q(x)$ is a real $n \times n$ matrix. The differential expression $Lu$ is a part of the Fokker-Planck and Cramer equations. In recent years the regularity of the solutions of equations (0.3) and (0.4) has been studied in [37], [38], [57], [68], [69] and [70]. Detailed information on other works can be found in the recently published monograph [22], which contains an extensive bibliography. The problems of the propagation of small oscillations in viscoelastic compressible media [105], [111], the dynamics of a stratified compressible fluid [48], the motion of particles in media with a resistance proportional to the velocity [108], as well as, biology [52] and financial mathematics [50] also lead to elliptic equations with displacement (0.3) and (0.4). The growth of the modulus of a vector (displacement) at infinity and its relation to the growth of other coefficients affects the solvability and regularity of equations (0.3) and (0.4). To compensate for the growth of $b$, the authors of [18], [49], [62], [67] and [99] regard both the solution and the right-hand side of these equations as elements of some weighted space whose weight is in some sense comparable with $b$. P. Cannarsa and V. Vespri [34] (see also [33]) introduced a weight $V$ with the same asymptotics as the real part of $c$ and assumed that $|b|/\sqrt{V}$ and $c/V$ are bounded. They established the existence and uniqueness of the solution of (0.4) in the weighted Sobolev space $H^1_V(\mathbb{R}^n)$.

A smaller amount of work is devoted to the nonweighted case. Moreover, A. Lunardi and V. Vespri [64] considered the case when $c$ is bounded, and the displacement $b$ has linear growth, and they proved the unique solvability of (0.3) with $f \in L_2(\mathbb{R}^n)$ in the space $H^1(\mathbb{R}^n)$. Using this result, G. Metafune [65] gave a characterization of the spectrum of the Ornstein-Uhlenbeck operator. Under certain conditions on bounded $Q$, $b$ and $c$, P.J. Rabier [101] proved the unique solvability of (0.4) in $H^2(\mathbb{R}^n)$. We also mention the paper of M. Sobajima [107], where in the case of linear and logarithmic growth, respectively, $b$ and $c$, the $m$-accretiveness and the $m$-sectoriality of the
operator $A$ in (0.4) were shown. Similar results were established in [108]. We
pronounce that in all these papers and in [70], [71] and [98] there remains to
investigate the case when the displacement $b$ in (0.4) have a stronger growth
and oscillation, so that it can not be controlled by diffusion $Q$ and potential $c$.

Moreover, the following result [77] of A.M. Molchanov for the Sturm-
Liouville operator is known: The resolvent $L^{-1}$ of the operator

$$L y = -y'' + q(x)y, \quad q \geq 1, \quad x \in \mathbb{R},$$

is compact in the space $L_2$ if and only if $\lim_{|x| \to \infty} \int_{x-d}^{x+d} q(t) dt = +\infty$ holds for each $d > 0$. The compactness of resolvents for a wide class of semibounded elliptic
operators for which there exists an extension in the sense of Friedrichs were
obtained in [21] and [94]. Such results for some non-self-adjoint operators are
proved by applying the known results obtained for semibounded operators
whose properties are close to self-adjoint operators.

In papers A, B and C of this PhD thesis (= [85], [3] and [86], respectively)
we study the more general case concerning a degenerate differential equation
having non-symmetric form. In the same papers we consider the question of
solvability (apparently for the first time) for a quasilinear degenerate differ-
ential equation. And we also investigate the spectral properties of the second
order degenerate differential operators with complex coefficients. The nature
of such operators is not close to semi-bounded operators. Hence, invertibility
of degenerate differential operators, the compactness of their inverse operator,
and other questions about the structure of the spectrum have not been
investigated so far. We considered an one-dimensional degenerate differential
operator of the form

$$l y := -y'' + r(x)y' + s(x)y', \quad x \in \mathbb{R},$$

with increasing "intermediate" coefficients. Here, the free term is equal to
zero and the members $r \frac{d}{dx}$ and $s \left( \frac{d}{dx} \right)$ do not depend on the operator $l_1 y \equiv -y''$, in other words, they are not infinitesimal perturbations of the operator
$l_1$ in some sense.

The problem about the structure of the spectrum for the degenerate operator $l$, as we saw above, has important practical applications. The fact
that the operator $l$ is not symmetric together with other problems raises an
issue about the invertibility of the operator $l$.

A number

$$\rho_A = \left[ \inf_{T \in \sigma_{\infty}(L_2)} \| A - T \|_{L_2 \to L_2} \right]^{-1},$$

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is called the radius of fredholmness of a bounded operator $A$ in $L_2$ \cite{92}. Here $\sigma_\infty(L_2)$ denotes the set of all compact operators in $L_2$.

In paper B we derived two-sided estimates of $\rho_{l-1}$ of the inverse operator of the above operator $l$.

Some criteria of compactness of the resolvent for semi-bounded operators are discussed in \cite{21}, \cite{77} and \cite{95}. Estimates of the radius of fredholmness of embedding operators of Sobolev spaces are established in \cite{92}.

The approaches developed in the above studies also allows us to study some classes of non semibounded differential operators, i.e., such energy spaces that are not enclosed in a Sobolev space. The non semibounded operators include all differential operators of odd order. Linear and nonlinear differential operators of odd order were investigated e.g. in \cite{7}, \cite{8}, \cite{10}, \cite{14-16}, \cite{19}, \cite{20}, \cite{79}, \cite{104} and \cite{110}. However, all of them except those of Zh. Zh. Aytkozha and M. B. Muratbekov \cite{15}, A. Birgebaev and M. Otelbaev \cite{20} and M. B. Muratbekov, M. M. Muratbekov and K. N. Ospanov \cite{79} were devoted to the case of a real potential and in \cite{15} and \cite{20} the case of a Hilbert space was considered. Odd order differential equations with singular complex coefficients in Banach space have not been studied systematically. Such equations constantly arise in the application of the projection methods, in particular in Fourier's method of separation of variables for solving partial differential equations.

In Papers D, E and F (=\cite{4}, \cite{5} and \cite{6}, respectively) we investigate some more general third order equations than those above. Usually, the previous mentioned authors only consider equations of the type

$$Ly = -y''' + q(x)y = f(x),$$

where $f = f(x) \in L_p(\mathbb{R})$, $\mathbb{R} = (-\infty, +\infty)$. However, we consider the more general case, when the coefficients are not constant in the leading term.

Before starting presentation of the results obtained in papers A, B, C, D, E and F we present a number of well-known necessary

Definitions, notations and auxiliary results

$\mathbb{R}^n$ is a $n$-dimensional real Euclidean space; in particular when $n = 2$ we obtain a two-dimensional Euclidean space of points $z = (x, y)$, where $-\infty < x < \infty$, $-\infty < y < \infty$.

$\Omega$ denotes an open domain in $\mathbb{R}^n$ and by $\overline{\Omega}$ we denote the closure of $\Omega$.

Let $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, where $\alpha_j \geq 0$ ($j = 1, 2, ...n$) are integers. We also use the notation $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$.

$C^l(\overline{\Omega})$, $l = 0, 1, 2, ...$, is the set of continuous functions with continuous
partial derivatives of order up to \( l \) inclusive in \( \bar{\Omega} \), which can be written as
\[
D^\alpha(u) := \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}, \quad \text{where } |\alpha| \leq l.
\]
\( C^\infty(\bar{\Omega}) \) denotes the space of infinitely differentiable functions in \( \bar{\Omega} \).

**Definition 0.1.** The set \( \{ x \in \Omega : u(x) \neq 0 \} \) is called the support of the function \( u = u(x) \) defined on the set \( \bar{\Omega} \) and it is denoted by \( \text{supp} u \).

\( C_0^\infty(\bar{\Omega}) \) denotes the set of infinitely differentiable and compactly supported functions in \( \bar{\Omega} \).

\( L_2 = L_2(\Omega) \) is the Hilbert space of Lebesgue measurable functions on \( \Omega \) with a finite norm
\[
\| u \|_{L_2(\Omega)} := \left[ \int_\Omega |u|^2 d\Omega \right]^{\frac{1}{2}}.
\]

\( W^k_2(\Omega) \) denotes the space of functions from \( L_2(\Omega) \) having all the generalized Sobolev derivatives up to order \( k \geq 1 \) also belonging to \( L_2(\Omega) \) with the norm
\[
\| u \|_{W^k_2(\Omega)} := \left[ \sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^2 d\Omega \right]^{\frac{1}{2}}.
\]

The domain of the operator \( A \) is denoted by \( D(A) \) and the range of \( A \) is denoted by \( R(A) \).

**Definition 0.2.** An operator \( A \) is called a bijection if, for any \( x_1 \) and \( x_2 \) belonging to \( D(A) \), such that \( Ax_1 = Ax_2 \), it follows that \( x_1 = x_2 \).

If \( A \) maps \( D(A) \) onto \( R(A) \) bijectively, then there exists an inverse mapping or inverse \( A^{-1} \) which maps \( R(A) \) onto \( D(A) \).

**Definition 0.3.** The operator \( A \) is said to be closed if, for every sequence \( \{ x_n \} \subset D(A) \), the fact that \( x_n \to x_0 \) and \( Ax_n \to y_0 \) implies that \( x_0 \in D(A) \) and \( y_0 = Ax_0 \).

If the operator \( A \) is not closed, then sometimes it can be extended to be closed. This operation is called the closure of the operator \( A \) and the operator is called closable.

A criterion to guarantee that an operator has a closed extension: an operator \( A \) has a closed extension if and only if the properties \( \{ x_n \} \subset D(A) \),
$x_n \to 0$ and $Ax_n \to y$ implies that $y = 0$.

**Definition 0.4.** An operator $A$ is said to be completely continuous if it maps every bounded set into a compact set or, for every bounded sequence $\{x_n\}$ of elements of $D(A)$, the sequence $\{Ax_n\}$ contains a convergent subsequence.

Let $X$ and $Y$ be normed spaces and let $A$ be a bounded operator from $X$ to $Y$. We define a functional $\varphi$ by

$$
\varphi(x) = (x, \varphi) = (Ax, f), \quad x \in X, \quad f \in Y^*, \quad (0.5)
$$

where $Y^*$ denotes the conjugate space of the space $Y$.

It is easy to see that $\varphi$ is linear and $D(\varphi) = X$. Hence, according to (0.5), for each $f \in Y^*$ there exists an element $\varphi \in X^*$, where $X^*$ is the conjugate space to $X$. Thus a linear continuous operator $\varphi = A^* f$ is given. This operator $A^*$ is called the adjoint of $A$.

**Definition 0.5.** An operator $A$ acting in the Hilbert space $L_2(\Omega)$ is said to be self-adjoint if it is symmetric, i.e., if the scalar product $\langle Au, v \rangle = \langle u, Av \rangle$ for any $u, v \in D(A)$ and from the identity

$$
\langle Au, v \rangle = \langle u, w \rangle,
$$

where $v$ and $w$ are fixed, $u$ is any element from $D(A)$, it follows that $v \in D(A)$ and $w = Av$.

Next we give the definition of Kolmogorov's $k$-widths and their properties.

Let $M$ be a centrally symmetric subset of $H$ ($H$ is a Hilbert space), i.e., $M = -M$.

The value

$$
d_k = \inf \sup \inf \|u - v\|, \quad k = 0, 1, 2, ...
$$

is called Kolmogorov's $k$-width of the set $M$, where $G_k$ is a subset with dimension $k$.

The $k$-widths $d_k$ ($k = 1, 2, ...$) have the following properties:

1) $d_0 \leq d_1 \leq d_2 \leq ...$

2) $d_k(M) \leq d_k(\tilde{M}), \quad \tilde{M} \subseteq M, \quad k = 1, 2, 3, ...

3) $d_k(nM) = nd_k(M), \quad n > 0, \quad nM = \{x' = nx, \ x \in M\}$. 
Let $L^l(\Omega, q)$ be the completion of $C^\infty_0(\Omega)$, defined by the norm
\[
\|(-\Delta)^{\frac{l}{2}}u\|_{L^p(\Omega)}^p + \int_{\Omega} q(t)|u(t)|^p dt,
\]
where $q(t)$ is a nonnegative function, $\Omega$ is an open (bounded or unbounded) set in $\mathbb{R}^n$, $l > 0$, $1 \leq p < \infty$. We pronounce that the space $L^l(\Omega, q)$ uniquely arises in many situations in the study of differential equations.

We also define the following function $q^*(x)$ introduced by M. Otelbaev (see [97]):
\[
q^*(x) = \inf_{Q_d(x) \subseteq \Omega} \left( d^{-1} : d^{-pl+n} \geq \int_{Q_d(x)} q(t) dt \right), \tag{0.6}
\]
where $Q_d(x)$ is a cube with sides equal to $d$ and with center $x \in \Omega$, $pl > n$.

**Definition 0.6.** Let $B_1$ and $B_2$ be Banach spaces. $B_1$ is said to be embedded in $B_2$ if $B_1$ is a subspace $B_2$ and there is a constant $c > 0$ such that
\[
\|x\|_{B_2} \leq c\|x\|_{B_1} \quad \text{for all} \quad x \in B_1.
\]
In this case we write $B_1 \hookrightarrow B_2$.

**Definition 0.7.** Let $B_1$ and $B_2$ be Banach spaces. Then a transformation $E$ mapping each element $x$ from $B_1$ to the same element in $B_2$ is called the embedding operator and denoted by $E : B_1 \rightarrow B_2$.

We also need the following important result of M. Otelbayev [93]:

**Theorem 0.1.** The embedding operator $E : L^l_p(\Omega, q) \hookrightarrow L_p$ is compact if and only if
\[
q^*(x) \rightarrow \infty \quad \text{when} \quad |x| \rightarrow \infty.
\]

Let $B_1$ and $B_2$ be Banach spaces and $B_1 \hookrightarrow B_2$.

**Definition 0.8.** The Kolmogorov $k$-width of the unit ball of the space $B_1$ in $B_2$ is called the $k$-width of the embedding $B_1 \hookrightarrow B_2$. 
We introduce a function $N(\lambda) = \sum_{d_k > \lambda} 1$ as the number of $k$-widths of the embeddings $B_1 \hookrightarrow B_2$ greater than $\lambda > 0$. $N(\lambda)$ is also called the distribution function of the $k$-widths $d_k$.

We also observe that the $k$-widths $d_k$ can be recovered from their distribution function using the formula

$$d_k = \inf\{\lambda > 0 : N(\lambda) \leq k\}, \text{ for any } k > 0.$$ 

Let $N(\lambda)$ be a distribution function of the $k$-widths $\{d_k\}$ related to the embedding $\circ L_p^l(\Omega, q) \hookrightarrow L_p^l$. Then the following theorem by M. Otelbayev (see [89], [94]) holds:

**Theorem 0.2** Let $pl > n$. Then the following estimates

$$c^{-1} \lambda^{-\frac{p}{l}} \mu \left( x \in \Omega : q^*(x) \leq \lambda^{-\frac{1}{l}} \right) \leq N(\lambda) \leq c \lambda^{-\frac{p}{l}} \mu \left( x \in \Omega : q^*(x) \leq \lambda^{-\frac{1}{l}} \right)$$

hold, where $\mu(\cdot)$ is the Lebesgue measure and $c$ depends only on $p$, $l$ and $n$.

It is easy to see that if $d = 1$ and the condition

$$\sup_{|x-y| \leq 1} \frac{q(x)}{q(y)} \leq c$$

holds, then $c_0^{-1} q^{pl-n}(x) \leq q^*(x) \leq c_0 q^{pl-n}(x)$, where $c_0 > 1$ and $q^*(x)$ is defined by (0.6). In this case Theorems 0.1 and 0.2 can be restated in terms of the function $q(x)$ in the following way:

**Theorem 0.3.** Let $pl > n$ and for a positive function $q(x)$ the condition (0.7) holds. Then the embedding operator $\circ L_p^l(\Omega, q) \hookrightarrow L_p^l$ is compact if and only if

$$q(x) \to \infty \text{ when } |x| \to \infty.$$ 

**Theorem 0.4.** Let $pl > n$ and the condition (0.7) holds. Then the following estimates

$$c^{-1} \lambda^{-\frac{p}{l}} \mu \left( x \in \Omega : q(x) \leq \lambda^{-\frac{1}{(pl-n)}} \right) \leq N(\lambda) \leq c \lambda^{-\frac{p}{l}} \mu \left( x \in \Omega : q(x) \leq \lambda^{-\frac{1}{(pl-n)}} \right)$$
hold, where \( \mu(\cdot) \) is the Lebesgue measure and \( c \) depends only on \( p \), \( l \) and \( n \).

**Theorem 0.5 (Schauder).** Let \( D \) be a nonempty closed bounded convex subset of a Banach space \( X \) and let the operator \( A : X \to X \) be compact and map \( D \) into itself. Then \( A \) has a fixed point in \( D \).

Now we are ready to briefly present the most important results of the papers A, B, C, D, E and F. In the sequel the functions \( r(x) \), \( s(x) \), \( m(x) \) etc. are functions on \((-\infty, +\infty)\), which are different but well defined in each paper.

In paper A we study a degenerate second order differential operator with complex coefficients. Let \( l \) be the closure in \( L_2 := L_2(\mathbb{R}) \), \( \mathbb{R} = (-\infty, +\infty) \) of the expression

\[
ly = -y'' + r(x)y' + s(x)\bar{y}'
\]

defined in the set \( C_0^\infty(\mathbb{R}) \) of all infinitely differentiable and compactly supported functions. Here \( r = r(x) \) and \( s = s(x) \) are complex-valued functions and \( \bar{y} \) is the complex conjugate to \( y \).

The operator \( l \) is said to be separable in \( L_2 \) if the following estimate holds:

\[
\|y''\|_2 + \|ry'\|_2 + \|s\bar{y}'\|_2 \leq c (\|ly\|_2 + \|y\|_2), \quad y \in D(l),
\]

where \( \|\cdot\|_2 \) is the \( L_2 \) norm. In this paper sufficient conditions for the invertibility and separability of the differential operator \( l \) are obtained. Moreover, spectral and approximate results for the inverse operator \( l^{-1} \) are achieved. Moreover, by using a separation theorem, which is obtained for the linear case, the solvability of the degenerate nonlinear second order differential equation

\[
-y'' + r(x,y)y' = F \quad (x \in \mathbb{R}) \quad (0.8)
\]

is proved.

**Definition 0.9.** A function \( y \in L_2 \) is called a solution of (0.8) if there is a sequence of twice continuously differentiable functions \( \{y_n\}_{n=1}^\infty \) such that \( \|\theta(y_n - y)\|_2 \to 0 \), \( \|\theta(Ly_n - f)\|_2 \to 0 \) as \( n \to \infty \) for any \( \theta \in C_0^\infty(\mathbb{R}) \).

We denote

\[
\alpha_{g,h}(t) = \|g\|_{L_2(0,t)} \|h^{-1}\|_{L_2(t,+\infty)} \quad (t > 0),
\]

\[
\beta_{g,h}(\tau) = \|g\|_{L_2(\tau,0)} \|h^{-1}\|_{L_2(-\infty,\tau)} \quad (\tau < 0),
\]
and
\[ \gamma_{g,h} = \max \left( \sup_{t>0} \alpha_{g,h}(t), \sup_{\tau<0} \beta_{g,h}(\tau) \right), \]
where \( g \) and \( h \) are given functions.

By \( C^{(1)}_{\text{loc}}(\mathbb{R}) \) we denote the set of functions \( f \) such that \( \psi f \in C^{(1)}(\mathbb{R}) \) for all \( \psi \in C_0^{\infty}(\mathbb{R}) \).

The main results of this paper are the following:

**Theorem 0.6.** Let the functions \( r \) and \( s \) satisfy the conditions
\[ r, s \in C^{(1)}_{\text{loc}}(\mathbb{R}), \quad \Re r - |s| \geq \delta > 0, \quad \gamma_{1,Re} r < \infty. \]
Then \( l \) is invertible and \( l^{-1} \) is defined in all \( L_2 \).

**Theorem 0.7.** Assume that the functions \( r \) and \( s \) satisfy the conditions
\[
\begin{align*}
\left\{ \begin{array}{l}
r, s \in C^{(1)}_{\text{loc}}(\mathbb{R}), \\
\Re r - \rho[|Im r| + |s|] \geq \delta > 0, \quad \gamma_{1,Re} r < \infty, \quad 1 < \rho < 2, \\
c^{-1} \leq \frac{\Re r(x)}{\Re r(\eta)} \leq c \text{ at } |x - \eta| \leq 1, \quad c > 1.
\end{array} \right.
\] 
(0.9)

Then, for \( y \in D(l) \) the estimate
\[ \|y''\|_2 + \|ry'\|_2 + \|s\gamma'\|_2 \leq c \|ly\|_2 \]
holds, i.e. the operator \( l \) is separable in \( L_2 \).

Two crucial Lemmas (Lemmas 2.1 and 2.2) to prove these theorems were not proved in Paper A. However, a detailed proofs of these Lemmas are included as Appendix [A1] of this PhD thesis. In particular, in the proof of one of these Lemmas was used an important result from the theory of Hardy type inequalities (see [56]).

**Theorem 0.8.** Assume that the functions \( r \) and \( s \) satisfy (0.9) and let
\[ \lim_{t \to +\infty} \alpha_{1,Re}(t) = 0, \quad \lim_{\tau \to -\infty} \beta_{1,Re}(\tau) = 0. \]
Then \( l^{-1} \) is completely continuous in \( L_2 \).

We assume that the conditions of Theorem 0.8 hold and consider the set
\[ M = \{ y \in L_2 : \|ly\|_2 \leq 1 \}. \]
Let
\[ d_k = \inf_{\Sigma_k \subset \{\Sigma_k\}} \sup_{y \in M} \inf_{w \in \Sigma_k} \|y - w\|_2 \quad (k = 0, 1, 2, \ldots) \]
be the Kolmogorov’s widths of the set \( M \) in \( L_2 \). Here \( \{\Sigma_k\} \) is a set of all subspaces \( \Sigma_k \) of \( L_2 \) whose dimensions are not greater than \( k \). By \( N_2(\lambda) \) we denote the number of widths \( d_k \), which are not smaller than a given positive number \( \lambda \). In particular, estimates of the width’s distribution function \( N_2(\lambda) \) are important in some approximation problems of solutions of the equation \( ly = f \). In paper A also the following Theorems were stated and proved:

**Theorem 0.9.** Assume that the conditions of Theorem 0.8 are fulfilled and let the function \( q = q(x) \) satisfy \( \gamma_{q, \Re r} < \infty \). Then the following estimates hold:
\[ c_1 \lambda^{-2} \mu \{x : |q(x)| \leq c_2^{-1} \lambda^{-1}\} \leq N_2(\lambda) \leq \frac{c_3}{\lambda^2} \mu \{x : |q(x)| \leq c_2 \lambda^{-1}\}, \]
where \( \mu \) is the Lebesgue measure.

**Theorem 0.10.** Let the function \( r \) be continuously differentiable with respect to both arguments and satisfy the following conditions
\[
\begin{align*}
& r \geq \delta_0 \sqrt{1 + x^2} \quad (\delta_0 > 0), \\
& \sup_{x, \eta \in \mathbb{R}, |x - y| \leq 1} \sup_{A > 0} \sup_{|C_1| \leq A, |C_2| \leq A} |C_1 - C_2| \leq A, \\
& \sup_{x, \eta \in \mathbb{R}, |x - y| \leq 1} \sup_{A > 0} \sup_{|C_1| \leq A, |C_2| \leq A} \frac{r(x, C_1)}{r(\eta, C_2)} < \infty.
\end{align*}
\]
Then there exists a solution \( y \) of (0.8), and
\[ \|y''\|_2 + \|r(\cdot, y)y'\|_2 < \infty. \]

In paper B, under conditions of separability, we obtained a necessary and sufficient condition for the compactness of the operator \( l^{-1} \) in \( L_2 \). Moreover, we derived two-sided estimates of \( \rho_{l^{-1}} \) of the operator \( l^{-1} \). The main result in paper B reads:

**Theorem 0.11.** Let the functions \( r = r(x) \) and \( s = s(x) \) be continuously differentiable and satisfy the conditions
\[
\begin{align*}
& |\Re r| - \rho |\Im r| + |s| \geq \delta > 0, \quad \gamma_{1, \Re r} < \infty, \quad \rho > 1, \\
& e^{-1} \leq \frac{\Re r(x)}{\Re r(\eta)} \leq c \quad x, \eta \in \mathbb{R}, \quad |x - \eta| \leq 1.
\end{align*}
\]
Then, for the Fredholm radius $\rho_{l-1}$ of the operator $l^{-1}$ the following estimates hold

$$c^{-1}_4 \leq \rho_{l-1} \leq c_4.$$ 

In paper C we consider the minimal closed differential operator

$$Ly = -\rho(x)(\rho(x)y')' + r(x)y' + q(x)y$$

in $L_2(\mathbb{R})$, where $\rho = \rho(x)$ and $r = r(x)$ are continuously differentiable functions, and $q = q(x)$ is a continuous function. We do not assume that $\rho$, $r$, $q$ are bounded in $\mathbb{R}$. In this paper we showed that the operator $L$ is continuously invertible when these coefficients satisfy some suitable conditions and obtained the following estimate for $y \in D(L)$

$$\| -\rho(\rho y')\|_2 + \| ry'\|_2 + \| qy\|_2 \leq c \| Ly\|_2,$$

where $D(L)$ is the domain of $L$, $\| \cdot \|_2$ is the norm in $L_2(\mathbb{R})$, and $c$ is independent of $y$.

**Theorem 0.12.** Let $\rho = \rho(x)$ be a bounded continuously differentiable function, and let $r = r(x)$ and $q = q(x)$ be continuous functions. Moreover, suppose that $\rho \geq 1$, $r$ and $q$ satisfy the conditions

$$r \geq \rho^2, \gamma_{1,\sqrt{r}} < \infty,$$

$$1 \leq \rho(x) \leq c(1 + x^2)^N \text{ for some } N > 0,$$

and $\gamma_{q,r} < \infty$. Then $L$ is continuously invertible and $L^{-1}$ is defined on the whole $L_2(\mathbb{R})$. Furthermore, there exists $c$ such that

$$\| -\rho(\rho y')\|_2 + \| ry'\|_2 + \| qy\|_2 \leq c \| Ly\|_2,$$

for any $y \in D(L)$.

Papers D, E and F are related in the sense that the most general results are presented and proved in paper F. However, some basic results, ideas and techniques to derive these general results are developed already in papers D and E (c.f. also Remark 0.1). For example, in paper D we consider the third order differential equation with unbounded coefficients:

$$(L + \lambda E)y := -m(x)(m(x)y')'' + [q(x) + ir(x) + \lambda]y = f(x), \quad (0.10)$$

where $f \in L_p$, $\lambda \geq 0$, and where $q(x)$, $r(x)$ are continuous functions and $m = m(x) \in C^{(2)}_{loc}(\mathbb{R})$. 
In this paper D we study questions of the existence and uniqueness of the solutions of (0.10) and conditions, which for a solution \( y \) of (0.10) the following estimate holds:

\[
\|m(x)(m(x)y')''\|^p_p + \|\left(q(x) + ir(x) + \lambda \right)y\|^p_p \leq c \|f(x)\|^p_p. \tag{0.11}
\]

We remark that when \( m(x) = 1 \) sufficient conditions for unique solvability of the equation (0.10) and the estimate of the form (0.11) for its solution in the spaces \( L_{p,l} \) were obtained by Zh. Zh. Aytkozha [14] and Zh. Zh. Aytkozha and M. B. Muratbekov [15].

In the case when \( m(x) = 1 \) and \( r(x) = 0 \) the existence and uniqueness questions for the solutions of (0.10) and also non-local estimates of the solutions and its derivatives have also been studied in [7], [8] and [104].

**Definition 0.10.** A function \( y(x) \in L_{p}(\mathbb{R}) \), is called a solution of (0.10), if there is a sequence \( \{y_n\}_{n=1}^{\infty} \) of continuously differentiable functions with compact support, such that \( \|y_n - y\|_p \to 0 \) and \( \|(L + \lambda E)y_n - f\|_p \to 0 \) as \( n \to \infty \).

By \( C^{(k)}(\mathbb{R}) \) \( (k = 1, 2, \ldots) \) we denote the set of all \( k \) times continuously differentiable functions \( \varphi(x) \) for which the value \( \sum_{j=0}^{k} \sup_{x \in \mathbb{R}} |\varphi^{(j)}(x)| \) is finite. Let \( W_\lambda(x) := \frac{|q(x) + \lambda + ir(x)|}{m^2(x)} \).

Our main results in paper D read:

**Theorem 0.13.** Assume that the functions \( q = q(x) \) and \( r = r(x) \) are continuous on \( R \), \( m = m(x) \in C^{(2)}_{loc}(\mathbb{R}) \) and that the following conditions hold:

\[
m(x) \geq 1, \quad \frac{q(x)}{m^2(x)} \geq 1, \quad r(x) \geq 1, \tag{0.12}
\]

\[
c^{-1} \leq \frac{m(x)}{m(\eta)} \frac{q(x)}{q(\eta)} \frac{r(x)}{r(\eta)} \leq c, \quad x, \eta \in \mathbb{R}, \quad |x - \eta| \leq 1, \quad \text{for some } c > 0, \tag{0.13}
\]

\[
|m^{(j)}(x)| \leq c_j m(x), \quad x \in \mathbb{R}, \quad \text{for some } c_j > 0, \quad j = 1, 2, \tag{0.14}
\]

and

\[
\sup_{|x - \eta| \leq 1} \frac{|W_\lambda(x) - W_\lambda(\eta)|}{|W_\lambda(x)|^\nu |x - \eta|^\mu} < +\infty, \quad 0 < \nu < \frac{\mu}{3} + 1, \quad \mu \in (0, 1], \quad \lambda \geq 0. \tag{0.15}
\]
Then there exists a number \( \lambda_0 \geq 0 \), such that the equation (0.10) has a solution \( y \) for all \( \lambda \geq \lambda_0 \).

**Theorem 0.14.** Let the functions \( q = q(x) \) and \( r = r(x) \) be continuous on \( \mathbb{R} \), \( m = m(x) \in C^{(3)}_{loc}(\mathbb{R}) \) and satisfy the conditions (0.12) - (0.15) and

\[
|m^{(3)}(x)| \leq c_3 m(x), \quad x \in \mathbb{R}.
\]

Then the solution of the equation (0.10) is unique and the estimate (0.11) holds.

In paper E we continue our investigations of this type by considering another similar case which required further developments of the methods. The obtained results were similar. So therefore we finish by describing the main results only in paper F. We only mention that a crucial Lemma (Lemma 4) in paper E was not proved there. However, in this PhD thesis a detailed proof of this Lemma is included as Appendix [E1].

In paper F we investigate the more general problem of existence and uniqueness of solutions of the third order differential equations

\[
-m_1(x) (m_2(x) (m_3(x)y')')' + (q(x) + ir(x) + \lambda) y = f(x),
\]

where \( x \in \mathbb{R} = (-\infty, +\infty) \), \( f \in L^p(\mathbb{R}) \), \( 1 < p \leq \infty \) and \( \lambda \geq 0 \). We assume that \( q(x) \), \( r(x) \), \( m_1(x) \) are continuous functions, and \( m_2(x) \in C^{(1)}_{loc}(\mathbb{R}) \), \( m_3(x) \in C^{(2)}_{loc}(\mathbb{R}) \). We also derive conditions so that for a solution \( y \) of (0.16) the following estimate holds:

\[
\left\| m_1(x) (m_2(x) (m_3(x)y')')' \right\|_p + \| (q(x) + ir(x) + \lambda) y \|_p \leq c \| f(x) \|_p. \tag{0.17}
\]

Our main results in Paper F are formulated in the following two Theorems:

**Theorem 0.15.** Assume that the functions \( q(x) \), \( r(x) \) and \( m_1(x) \) are continuous, \( m_2(x) \in C^{(1)}_{loc}(\mathbb{R}) \), \( m_3(x) \in C^{(2)}_{loc}(\mathbb{R}) \) and satisfy the following conditions:

\[
m_j(x) \geq 1 \quad (j = 1, 2, 3), \quad \frac{q(x)}{\prod_{k=1}^3 m_k^2(x)} \geq 1, \quad r(x) \geq 1, \quad (0.18)
\]

\[
c^{-1} \leq \frac{m_k(x)}{m_k(\eta)} \frac{q(x)}{q(\eta)} \frac{r(x)}{r(\eta)} \leq c, \quad (k = 1, 2, 3), \quad x, \eta \in \mathbb{R}, \quad |x - \eta| \leq 1, \quad (0.19)
\]
\[ |m'_2(x)| \leq cm_2(x), \quad |m'_3(x)| \leq cm_3(x) \quad (j = 1, 2), \quad x \in \mathbb{R}, \quad (0.20) \]

\[
\sup_{|x - \eta| \leq 1} \frac{|W_\lambda(x) - W_\lambda(\eta)|}{|W_\lambda(x)|^{\nu}|x - \eta|^\mu} < +\infty, \quad 0 < \nu < \frac{\mu}{3} + 1, \quad \mu \in (0, 1], \quad \lambda \geq 0. \quad (0.21)
\]

Then there exists a number \( \lambda_0 \geq 0 \), such that the equation (0.16) for all \( \lambda \geq \lambda_0 \) has a solution \( y \), where \( W_\lambda(x) : = \frac{|q(x) + \lambda + ir(x)|}{\prod_{k=1}^{n} m_k(x)} \).

**Theorem 0.16.** Let the functions \( q(x), r(x) \) be continuous, \( m_1(x) \in C_{loc}^{(3)}(\mathbb{R}), \quad m_2(x) \in C_{loc}^{(2)}(\mathbb{R}), \quad m_3(x) \in C_{loc}^{(2)}(\mathbb{R}) \) and satisfy the conditions (0.18)-(0.21) and

\[
\begin{cases}
|m_1^{(j)}(x)| \leq cm_1(x), & j = 1, 3, \\
|m_2^{(i)}(x)| \leq cm_2(x), & k = 2, 3, \\
|n_k^{(i)}(x)| \leq cm_3(x), & i = 1, 2, \quad x \in \mathbb{R}.
\end{cases}
\]

Then the solution \( y \) of the equation (0.16) is unique and the estimate (0.17) holds.

By a solution we mean a solution in the sense of Definition 0.10 but with (0.10) replaced (0.16).

**Remark 0.1.** As previously mentioned some basic results, ideas, and techniques, to prove these results were developed already in papers D and E. In particular, Theorem 0.15 coincides with Theorem 0.13 and Theorem 0.16 coincides with Theorem 0.14 for the case \( m_2(x) \equiv 1 \) and \( m_1(x) = m_3(x) \). Moreover, Theorem 0.15 coincides with Theorem 1 in paper C and Theorem 0.16 coincides with Theorem 2 for the case when \( m_3(x) = 1 \).
Bibliography


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Paper A
Separation and the existence theorem for second order nonlinear differential equation

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Abstract. Sufficient conditions for the invertibility and separability in $L_2(−\infty, +\infty)$ of the degenerate second order differential operator with complex-valued coefficients are obtained, and its applications to the spectral and approximate problems are demonstrated. Using a separability theorem, which is obtained for the linear case, the solvability of nonlinear second order differential equation is proved on the real axis.

Keywords: separability of the operator, complex-valued coefficients, completely continuous resolvent

Mathematics subject classifications: 34B40

1. Introduction and main results

A concept of the separability was introduced in the fundamental paper [1]. The Sturm-Liouville’s operator

$$Jy = −y'' + q(x)y, \quad x \in (a, +\infty),$$

is called separable [1] in $L_2(a, +\infty)$, if $y, −y'' + qy \in L_2(a, +\infty)$ imply $−y'', qy \in L_2(a, +\infty)$. From this it follows that the separability of $J$ is equivalent to the existence of the estimate

$$\|y''\|_{L_2(a,+\infty)} + \|qy\|_{L_2(a,+\infty)} \leq c \left( \|Jy\|_{L_2(a,+\infty)} + \|y\|_{L_2(a,+\infty)} \right), \quad y \in D(J), \quad (1.1)$$

where $D(J)$ is the domain of $J$. In [1] (see also [2, 3]) some criteria of the separability depended on a behavior $q$ and its derivatives has been obtained for $J$. Moreover, an example of non-separable operator $J$ with non-smooth potential $q$ was shown in this papers. Without differentiability condition on function $q$ the sufficient conditions for the separability of $J$ has been obtained in [4, 5]. In [6, 7] so-called Localization Principle of the proof for the separability of higher order binomial elliptic operators was developed in Hilbert space. In [8, 9] it was shown that local integrability and semiboundedness from below of $q$ are enough for separability of $J$ in $L_1(−\infty, +\infty)$. Valuation method of Green’s functions [1-3, 8, 9] (see also [10]), parametrix method [4, 5], as well as method of local estimates for the resolvents of some regular operators [6, 7] have been used in these works.

Sufficient conditions of the separability for the Sturm-Liouville’s operator

$$y'' + Q(x)y$$

have been obtained in [11-15], where $Q$ is an operator. A number of works were devoted to the separation problem for the general elliptic, hyperbolic and mixed-type operators.

An application of the separability estimate (1.1) in the spectral theory of $J$ has been shown in [15-18], and it allows us to prove an existence and a smoothness of solutions of nonlinear differential equations in unbounded domains [11, 17-20]. Brown [21] has shown that certain properties of positive solutions of disconjugate second

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order differential expressions imply the separation. The connection of separation with concrete physical problems has been noted in [22].

We denote \( L_2 := L_2(\mathbb{R}), \mathbb{R} = (-\infty, +\infty) \), the space of square integrable functions. Let \( l \) is a closure in \( L_2 \) of the expression \( ly = -y'' + r(x)y' + s(x)y' \) defined in the set \( C_0^\infty(\mathbb{R}) \) of all infinitely differentiable and compactly supported functions. Here \( r \) and \( s \) are complex valued functions, and \( \bar{y} \) is the complex conjugate to \( y \).

In this report we investigate some problems for the operator \( l \). Although the operator \( l \), similarly to the Sturm-Liouville operator \( J \), is a singular differential operator of second order, their properties are different. The theory of the Sturm-Liouville operator \( J \), in contrast to the operator \( l \), developing a long time, while the idea of research is often based on the positivity of the potential \( q(x) \) (see, eg, [1-20]). Because of the coefficients \( r \) and \( s \), are the methods developed for the Sturm-Liouville problems are often not applicable to the study of the operator \( l \). The spectral properties for self-adjoint singular differential operators of second order, without the free term, have been to a certain extent investigated; a review of literature can be found in [23, 24]. Note that the differential operator \( l \) is used, in particular, in the oscillatory processes in the medium with resistance depended on velocity [25, pp. 111-116].

The operator \( l \) is said to be separable in \( L_2 \) if the following estimate holds:

\[
\|y\|_2 + \|ry\|_2 + \|s\bar{y}\|_2 \leq c (\|y\|_2 + \|y\|_2), \quad y \in D(l),
\]

where \( \| \cdot \|_2 \) is the \( L_2 \)-norm. In the present communication the sufficient conditions for the invertibility and separability of the differential operator \( l \) are obtained. Moreover, spectral and approximate results for the inverse operator \( l^{-1} \) are achieved. Using a separation theorem, which is obtained for the linear case, the solvability of the degenerate nonlinear second order differential equation \(-y'' + r(x)y') = F (x \in \mathbb{R}) \) is proved.

Let’s consider the degenerate differential equation

\[
l y = -y'' + r(x)y' + s(x)y' = f. \tag{1.2}
\]

The function \( y \in L_2 \) is called a solution of (1.2) if there exists a sequence \( \{y_n\}_{n=1}^{+\infty} \) such that \( \|y_n - y\|_2 \to 0, \|ty_n - f\|_2 \to 0 \) as \( n \to +\infty \). If the operator \( l \) is separable, then the solution \( y \) of (1.2) belongs to the weighted Sobolev space \( W^2_2(\mathbb{R}, |r| + |s|) \) with the norm \( \|y\|_2 + \|(|r| + |s|)\bar{y}\|_2 \). So, the study of the qualitative behavior of solutions of (1.2) and spectral and approximate properties of \( l \) can be reduced to the investigation of embedding \( W^2_2(\mathbb{R}, |r| + |s|) \hookrightarrow L_2 \).

We denote

\[
\alpha_{g,h}(t) = \|g\|_{L_2(t,0)} \|1/h\|_{L_2(t, +\infty)} \quad (t > 0), \quad \beta_{g,h}(\tau) = \|g\|_{L_2(\tau,0)} \|1/h\|_{L_2(-\infty,\tau)} \quad (\tau < 0),
\]

\[
\gamma_{g,h} = \max \left( \sup_{t>0} \alpha_{g,h}(t), \sup_{\tau<0} \beta_{g,h}(\tau) \right),
\]

where \( g \) and \( h \) are given functions. By \( C^{(1)}_{loc}(\mathbb{R}) \) we denote the set of functions \( f \) such that \( \psi f \in C^{(1)}(\mathbb{R}) \) for all \( \psi \in C_0^\infty(\mathbb{R}) \).

**Theorem 1.** Let functions \( r \) and \( s \) satisfy the conditions

\[
r, s \in C^{(1)}_{loc}(\mathbb{R}), \quad \text{Re} \ r - |s| \geq \delta > 0, \quad \gamma_{1, \text{Re} \ r} < \infty. \tag{1.3}
\]

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Then \( l \) is invertible and \( l^{-1} \) is defined in all \( L_2 \).

**Theorem 2.** Assume that functions \( r \) and \( s \) satisfy the conditions

\[
\begin{cases}
  r, s \in C_{loc}^{(1)}(\mathbb{R}), & \Re r - \rho |\Im r| + |s| \geq \delta > 0, \quad \gamma_{1, \Re r} < \infty, \quad 1 < \rho < 2, \\
  c^{-1} \leq \frac{\Re r(x)}{\Re r(0)} \leq c \quad \text{at} \quad |x - \eta| \leq 1, \quad c > 1.
\end{cases}
\]

(1.4)

Then for \( y \in D(l) \) the estimate

\[
\|y\|_2 + \|ry\|_2 + \|sy\|_2 \leq c_1 \|ly\|_2
\]

holds, i.e. the operator \( l \) is separable in \( L_2 \).

We use the statement of Theorem 2 for proof of the following Theorems 3-5.

**Theorem 3.** Assume that functions \( r \) and \( s \) satisfy (1.4) and let \( \lim_{t \to +\infty} \alpha_{1, \Re r}(t) = 0 \), \( \lim_{r \to +\infty} \beta_{1, \Re r}(r) = 0 \). Then \( l^{-1} \) is completely continuous in \( L_2 \).

We assume that the conditions of Theorem 3 hold, and consider a set

\[
M = \{ y \in L_2 : \|ly\|_2 \leq 1 \}.
\]

Let

\[
d_k = \inf_{\Sigma_k \subseteq \Sigma} \sup_{w \in \Sigma} \inf_{y \in M} \|y - w\|_2 \quad (k = 0, 1, 2, ...)
\]

be the Kolmogorov’s widths of the set \( M \) in \( L_2 \). Here \( \{\Sigma_k\} \) is a set of all subspaces \( \Sigma_k \) of \( L_2 \) whose dimensions are not greater than \( k \). Through \( N_2(\lambda) \) denote the number of widths \( d_k \) which are not smaller than a given positive number \( \lambda \). Estimates of the width’s distribution function \( N_2(\lambda) \) are important in the approximation problems of solutions of the equation \( ly = f \). The following statement holds.

**Theorem 4.** Assume that the conditions of Theorem 3 be fulfilled, and let a function \( q \) satisfy \( \gamma_{q, \Re r} < \infty \). Then the following estimates hold:

\[
c_1 \lambda^{-2} \mu \left\{ x : |q(x)| \leq c_2 \lambda^{-1} \right\} \leq N_2(\lambda) \leq c_3 \lambda^{-2} \mu \left\{ x : |q(x)| \leq c_2 \lambda^{-1} \right\},
\]

where \( \mu \) is a Lebesgue measure.

**Example.** Assume that \( r = (1 + x^2)^\beta \ (\beta > 0) \) and let \( s = 0 \). Then the conditions of Theorem 2 are satisfied if \( \beta \geq 1/2 \). If \( \beta > 1/2 \), then the conditions of Theorem 4 are satisfied and the following estimates hold:

\[
c_4 \lambda^{-\frac{2\beta+1}{\beta}} \leq N_2(\lambda) \leq c_5 \lambda^{-\frac{2\beta+1}{\beta}}.
\]

Consider the following nonlinear equation

\[
Ly = -y'' + [r(x, y)]y' = f(x),
\]

(1.6)

where \( x \in \mathbb{R}, r \) is a real-valued function and \( f \in L_2 \).

A function \( y \in L_2 \) is called a solution of equation (1.6), if there exists a sequence of twice continuously differentiable functions \( \{y_n\}_{n=1}^\infty \) such that \( \|\theta(y_n - y)\|_2 \to 0 \), \( \|\theta(Ly_n - f)\|_2 \to 0 \) as \( n \to \infty \) for any \( \theta \in C_0^\infty(\mathbb{R}) \).
Theorem 5. Let the function $r$ be continuously differentiable with respect to both arguments and satisfy the following conditions

$$r \geq \delta_0 \sqrt{1 + x^2} \quad (\delta_0 > 0), \quad \sup_{x,y \in \mathbb{R}} \sup_{|x-y| \leq 1} \sup_{A>0} \sup_{|C_1| \leq A, |C_2| \leq A, C_1-C_2 \leq A} r(x,C_1) < \infty.$$  

Then there exists a solution $y$ of (1.6), and

$$\|y''\|_2 + \|r(\cdot, y)g'\|_2 < \infty.$$  

(1.7)

(1.8)

2. Auxiliary statements

The next statement is a corollary of the well known Muckenhoupt's inequality [26].

Lemma 2.1. Let functions $g$ and $h$ such that $\gamma_{g,h} < \infty$. Then for all $y \in C_0^\infty(\mathbb{R})$ the following inequality holds:

$$\int_{-\infty}^{\infty} |g(x)y(x)|^2 dx \leq C \int_{-\infty}^{\infty} |h(x)y'(x)|^2 dx.$$  

(2.1)

Moreover, if $C$ is a smallest constant for which estimate (2.1) holds, then $\gamma_{g,h} \leq C \leq 2\gamma_{g,h}$.

The following lemma is a particular case of Theorem 2.2 [23].

Lemma 2.2. Let the given function $h$ satisfy conditions

$$\lim_{x \to +\infty} \sqrt{x} \left( \int_{-\infty}^{x} h^{-2}(t) dt \right)^{\frac{1}{2}} = 0,$$

$$\lim_{x \to -\infty} \sqrt{|x|} \left( \int_{x}^{\infty} h^{-2}(t) dt \right)^{\frac{1}{2}} = 0.$$  

Then the set

$$F_K = \left\{ y : y \in C_0^\infty(\mathbb{R}), \quad \int_{-\infty}^{+\infty} |h(t)y'(t)|^2 dt \leq K \right\}, \quad K > 0,$$

is a relatively compact in $L_2(\mathbb{R})$.

Denote by $L$ a closure in $L_2$-norm of the differential expression

$$L_0 z = -z' + rz + sz$$  

(2.2)

defined on the set $C_0^\infty(\mathbb{R})$.

Lemma 2.3. Assume that functions $r$ and $s$ satisfy condition (1.3). Then the operator $L$ is boundedly invertible in $L_2$.  

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Proof. Let $L_{\lambda} = L + \lambda E$, where $\lambda \geq 0$, and $E$ be the identity map of $L_2$ to itself. Note that $L$ is separable if and only if $L_{\lambda} = L + \lambda E$ is separable for some $\lambda$. If $z$ is a continuously differentiate function with a compact support, then

$$
(L_{\lambda}z, z) = -\int_{\mathbb{R}} z'\bar{z}dx + \int_{\mathbb{R}} [(r + \lambda)|z|^2 + s\bar{z}^2]dx.
$$

(2.3)

But

$$
T := -\int_{\mathbb{R}} z'\bar{z}dx = \int_{\mathbb{R}} z\bar{z}'dx = -\bar{T}.
$$

Therefore $Re T = 0$ and from (2.3) it follows that

$$
Re (L_{\lambda}z, z) \geq c \int_{\mathbb{R}} [Re r + \lambda - |s||z|^2]dx.
$$

(2.4)

We estimate the left-hand side of inequality (2.4) by using the Holder’s inequality. Then by (1.3) we have $\|L_{\lambda}z\|_2 \geq \delta \|z\|_2$. This estimate implies that $L_{\lambda}$ is invertible. Let us proof that $L_{\lambda}^{-1}$ is defined in all $L_2$. Assume the contrary. Let $R(L_{\lambda}) \neq L_2$. Then there exists a non-zero element $z_0 \in L_2$ such that $z_0 \perp R(L_{\lambda})$. According to operator’s theory $z_0$ satisfies the equality

$$
L_{\lambda}^*z_0 := z_0' + (\bar{r} + \lambda)z_0 + sz_0 = 0,
$$

(2.5)

where $L_{\lambda}^*$ is an adjoint operator.

Let $\theta \in C_0^\infty(\mathbb{R})$ is a real function. Denote $\psi = \theta z_0$. From (2.5) it follows that $z_0 \in W^1_2,loc(\mathbb{R})$, then $\psi \in D(L_{\lambda}^*)$. Using (2.5), we get $L_{\lambda}^*\psi = \theta z_0$. Hence

$$
(L_{\lambda}^*\psi, \psi) = \int_{\mathbb{R}} \theta^2|z_0|^2dx.
$$

(2.6)

On the other hand using the expression $L_{\lambda}^*\psi$ we have

$$
Re (L_{\lambda}^*\psi, \psi) = \int_{\mathbb{R}} \theta^2[Re (\bar{r} + \lambda)|z_0|^2 + Re (sz_0^2)]dx \geq
$$

$$
\geq \int_{\mathbb{R}} \theta^2[Re \bar{r} + \lambda - |s||z_0|^2]dx.
$$

Hence by (2.6) the following estimate

$$
\delta \int_{\mathbb{R}} \theta^2|z_0|^2dx \leq \int_{\mathbb{R}} \theta^2|z_0|^2dx
$$

(2.7)

holds. Choose the function $\theta$ such that

$$
\theta(x) = \begin{cases} 1, & |x| \leq \xi \\ 0, & |x| \geq \xi + 1, \end{cases}
$$

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\[
0 \leq \theta \leq 1, \ |\theta^\prime| \leq C. \text{ Here } \xi > 0. \text{ Then it follows from (2.7)}
\]
\[
\delta \int_{-\xi-1}^{\xi+1} \theta^2 |z_0|^2 dx \leq C \left[ \int_{-\xi-1}^{-\xi} |z_0|^2 dx + \int_{\xi}^{\xi+1} |z_0|^2 dx \right].
\]

Since \(z_0 \in L_2\), passing to the limit as \(\xi \to +\infty\) in the last inequality, we have \(\|z_0\|_2 = 0\). Then \(z_0 = 0\). We obtain the contradiction, which gives that \(R(L_\lambda) = L_2\). The lemma is proved. \(\square\)

**Lemma 2.4.** Assume that functions \(r\) and \(s\) satisfy condition (1.4). Then \(L\) is separable in \(L_2\) and for \(z \in D(L)\) the following estimate holds:
\[
\|z\|^2_2 + \|rz\|^2_2 + \|s\|^2_2 \leq c \|Lz\|^2_2.
\] (2.8)

**Proof.** From inequality (2.4) it follows that
\[
\left\| \sqrt{\text{Re} \, r(\cdot) + \lambda z} \right\|^2_2 \leq c_1 \left\| \frac{1}{\sqrt{\text{Re} \, r(\cdot) + \lambda}} L_\lambda z \right\|^2_2.
\] (2.9)

It is easy to show that (2.9) holds for all \(z\) from \(D(L_\lambda)\).

Let \(\Delta_j = (j-1, j+1)\) \((j \in \mathbb{Z})\) and let \(\{\phi_j\}_{j=\infty}^{+\infty}\) be a sequence of functions from \(C_0^\infty(\Delta_j)\), such that
\[
0 \leq \phi_j \leq 1, \quad \sum_{j=-\infty}^{+\infty} \phi_j(x) = 1.
\]

We continue \(r(x)\) and \(s(x)\) from \(\Delta_j\) to \(\mathbb{R}\) so that its continuations \(r_j(x)\) and \(s_j(x)\) are bounded and periodic functions with period 2. Denote by \(L_{\lambda,j}\) the closure in \(L_2(\mathbb{R})\) of the differential operator \(-z^\prime + [r_j(x) + \lambda]z + s_j(x)z\) defined on \(C_0^\infty(\mathbb{R})\). Using the method which was applied for \(L_\lambda\) one can proof that \(L_{\lambda,j}\) are invertible and \(L_{\lambda,j}^{-1}\) are defined in all \(L_2\). In addition, the following inequality
\[
\left\| (\text{Re} \, r_j + \lambda)^{\frac{1}{2}} z \right\|^2_2 \leq c_2 \left\| (\text{Re} \, r_j + \lambda)^{-\frac{1}{2}} L_{\lambda,j}^{-1} z \right\|^2_2, \quad z \in D(L_{\lambda,j}),
\] (2.10)
holds. From estimate (2.10) by (1.4) it follows
\[
\|L_{\lambda,j} z\|^2_2 \geq c_3 \sup_{z \in \Delta_j} |\text{Re} \, r_j(x) + \lambda| \|z\|^2_2, \quad z \in D(L_{\lambda,j}).
\] (2.11)

Let us introduce the operators \(B_\lambda\) and \(M_\lambda\):
\[
B_\lambda f = \sum_{j=-\infty}^{+\infty} \phi_j(x) L_{\lambda,j}^{-1} \phi_j f, \quad M_\lambda f = \sum_{j=-\infty}^{+\infty} \phi_j(x) L_{\lambda,j}^{-1} \phi_j f.
\]

At any point \(x \in \mathbb{R}\) the sums of the right-hand side in these terms contain no more than two summands, therefore \(B_\lambda\) and \(M_\lambda\) is defined on all \(L_2\). It is easy to show that
\[
L_\lambda M_\lambda = E + B_\lambda.
\] (2.12)

Using (2.11) and properties of \(\phi_j\) \((j \in \mathbb{Z})\) we find that \(\lim_{\lambda \to +\infty} \|B_\lambda\| = 0\), hence there exists a number \(\lambda_0\) such that \(\|B_\lambda\| \leq 0.5\) for all \(\lambda \geq \lambda_0\). Then it follows from (2.12)
\[
L_\lambda^{-1} = M_\lambda(E + B_\lambda)^{-1}, \quad \lambda \geq \lambda_0.
\] (2.13)

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Using (2.13) and properties of $\varphi_j$ ($j \in \mathbb{Z}$) we have
\[ \|(\text{Re } r + \lambda)L_{\lambda,j}^{-1}f\|_2 \leq c_4 \sup_{j \in \mathbb{Z}} \|(\text{Re } r_j + \lambda)L_{\lambda,j}^{-1}\|_{L_2 \rightarrow L_2} \|f\|_2. \] (2.14)

Further, (1.4) and (2.11) imply that
\[ \sup_{j \in \mathbb{Z}} \|(\text{Re } r_j + \lambda)L_{\lambda,j}^{-1}F\|_{L_2(\mathbb{R})} \leq c_5 \sup_{x \in \mathbb{R}} \frac{[\text{Re } r(t) + \lambda]}{\inf_{t \in \mathbb{R}} [\text{Re } r(t) + \lambda]} \|F\|_{L_2(\mathbb{R})} \leq c_5 \sup_{x, t \in \mathbb{R}, |x - t| \leq s} \frac{\text{Re } r(x) + \lambda}{\text{Re } r(z) + \lambda} \|F\|_{L_2(\mathbb{R})} \leq c_6 \|F\|_{L_2(\mathbb{R})}. \]

From the last inequalities and (2.14) we obtain $\|(\text{Re } r + \lambda)z\|_2 \leq c_7 \|L_{\lambda,j}z\|_2$, $z \in D(L_{\lambda,j})$, therefore it follows from condition (1.4)
\[ \|z\|_2 + \|(r + \lambda)z\|_2 + \|s\|_2 \leq c_8 \|L_{\lambda,j}z\|_2. \]

When $\lambda = 0$ from this inequality we have estimate (2.8). The lemma is proved. \(\square\)

**Lemma 2.5.** Assume that functions $r$ and $s$ satisfy condition (1.3). Then for $y \in D(l)$ the estimate
\[ \|y\|_2 + \|y\|_2 \leq c \|y\|_2. \] (2.15)

**Proof.** Let $y \in C_0^{\infty}(\mathbb{R})$. Integrating by parts, we have
\[ (ly, y') = -\int_{\mathbb{R}} y''y'dx + \int_{\mathbb{R}} [r(x)|y'|^2 + s(x)(y')^2]dx. \] (2.16)

Since
\[ A := -\int_{\mathbb{R}} y''y'dx = \int_{\mathbb{R}} y'ydx = -\tilde{A}, \]
we see $\text{Re } A = 0$. Therefore, it follows from (2.16)
\[ \text{Re } (ly, y') \geq \int_{\mathbb{R}} [\text{Re } r - |s||y'|^2]dx \geq \delta \|y\|_2. \]

Hence, using the Holder’s inequality, the condition $\gamma_1 \text{Re } r < \infty$ in (1.3) and Lemma 2.1 we obtain (2.15) for any $y \in C_0^{\infty}(\mathbb{R})$. If $y$ is an arbitrary element of $D(l)$, then there exists a sequence $\{y_n\}_{n=1}^{\infty} \subset C_0^{\infty}(\mathbb{R})$ such that $\|y_n - y\|_2 \to 0$, $\|ly_n - ly\|_2 \to 0$ as $n \to \infty$. The estimate (2.15) holds for $y_n$. From (2.15) passing to the limit as $n \to \infty$ we obtain the same estimate for $y$. The lemma is proved. \(\square\)

A function $y \in L_2$ is called a solution of the equation
\[ ly \equiv -y'' + r(x)y' + s(x)y = f, \quad f \in L_2, \] (2.17)
if there exists a sequence $\{y_n\}_{n=1}^{\infty} \subset C_0^{\infty}(\mathbb{R})$ such that $\|y_n - y\|_2 \to 0$, $\|ly_n - f\|_2 \to 0$, $n \to \infty$.

**Lemma 2.6.** If functions $r$ and $s$ satisfy condition (1.3), then the equation (2.17) has a unique solution.
Proof. From (2.15) it follows that the solution \( y \) of (2.17) is unique and belongs to \( W^2_2(\mathbb{R}) \). Lemma 2.3 shows that \( L^{-1} \) is defined in all \( L_2 \). Then by the construction (2.17) is solvable. The proof is complete. \( \square \)

3. Proofs of Theorems 1-4

Proof of Theorem 1. From (1.3) and Lemma 2.6 we obtain that \( l \) is invertible and \( l^{-1} \) is defined in all \( L_2 \). \( \square \)

Proof of Theorem 2. From Lemma 2.4 it follows that \( L \) is separated in \( L_2 \) under condition (1.4). And consequently, by construction \( ly \equiv -y'' + r(x)y' + s(x)y' \) is separated in \( L_2 \) and the estimate (1.5) holds. The theorem is proved. \( \square \)

Proof of Theorem 3. The estimate (1.5) shows that \( l^{-1} \) maps \( L_2 \) into space \( \tilde{W}^2_2(\mathbb{R}) \) with the norm \( \|y''\|_{L_2} + \|r'y\|_{L_2} + \|s'y\|_{L_2} + \|y\|_{L_2} \). By condition of the theorem Lemma 2.2 implies that \( \tilde{W}^2_2(\mathbb{R}) \) is compactly embedded into \( L_2 \). The proof is complete. \( \square \)

Proof of Theorem 4. By Lemma 2.1 Theorem 2 implies that \( \|y''\|_{L_2} + \|r'y\|_{L_2} \leq c \|y\|_{L_2}, \ y \in D(l) \). Then Theorem 1 [27] gives the estimates in Theorem 4. \( \square \)

Proof of Theorem 5. Let \( \epsilon \) and \( A \) be positive numbers. We denote

\[
S_A = \left\{ z \in W^1_2(\mathbb{R}) : \|z\|_{W^1_2(\mathbb{R})} \leq A \right\}.
\]

Let \( \nu \) be an arbitrary element of \( S_A \). Consider the following linear “perturbed” equation

\[
l_{\alpha, \nu, \eta} y \equiv -y'' + [r(x, \nu(x)) + \epsilon(1 + \epsilon)^2]y' = f(x),
\]

(3.1)

Denote by \( l_{\alpha, \eta} \) the minimal closed operator in \( L_2 \) generated by expression \( l_{\alpha, \nu, \eta} y \). Since

\[
r_{\nu}(x) := r(x, \nu(x)) + \epsilon(1 + \epsilon)^2 \geq 1 + \epsilon(1 + \epsilon)^2,
\]

the function \( r_{\nu}(x) \) satisfies condition (1.3). Further, if \( |x - \eta| \leq 1 \) \( (x, z \in \mathbb{R}) \), then for \( \nu \in S_A \) we have

\[
|\nu(x) - \nu(\eta)| \leq |x - \eta| \|\nu'\|_{L^p} \leq |x - \eta| \|\nu\|_{W^1_2(\mathbb{R})} \leq A.
\]

(3.2)

It is easy to verify that

\[
\sup_{x, \eta \in \mathbb{R}, |x - \eta| \leq 1} \left( \frac{1 + 2^{\nu^2}}{(1 + \eta^2)^2} \right) \leq 9.
\]

Now we assume that \( \nu(x) = C_1, \ \nu(\eta) = C_2 \). Then by (1.7) and (3.2) we obtain

\[
\sup_{x, \eta \in \mathbb{R}, |x - \eta| \leq 1} \frac{r_{\nu}(x)}{r_{\nu}(\eta)} \leq \sup_{x, \eta \in \mathbb{R}, |x - \eta| \leq 1} A > 0 \ \sup_{C_1 \leq A, |C_1| \leq A, |C_1 - C_2| \leq A} \frac{r_{\nu}(\eta, C_2)}{r_{\nu}(\eta, C_2)} + 9\epsilon < \infty.
\]

Thus the coefficient \( r_{\nu}(x) \) in (3.1) satisfies the conditions of Theorem 2. Therefore, (3.1) has a unique solution \( y \) and for \( y \) the estimate

\[
\|y''\|_{L_2} + \left\| [r_{\nu}(\cdot, \nu(\cdot)) + \epsilon(1 + \epsilon)^2]y' \right\|_{L_2} \leq C_3 \|f\|_{L_2}
\]

(3.3)

holds (i.e. an operator \( l_{\alpha, \nu} \) is separated). By (1.7) and (2.1)

\[
\|y\|_{L_2} \leq C_0 \|ry'\|_{L_2}, \ |(1 + x^2)y|_{L_2} \leq C_4 \|(1 + x^2)y'\|_{L_2}.
\]

(3.4)
then the following relations hold:

$$Q \in L^2_{\nu, \epsilon},$$

w. Then by Kolmogorov-Frechct’s criterion the set $Q_A$ is compact in Sobolev’s space $W^2_2(\mathbb{R})$. Indeed, if $y \in Q_A$, $h \neq 0$ and $N > 0$, then the following relations hold:

$$\|y''(h) - y''(0)\|^2_{W^2_2(\mathbb{R})} = \int_{-\infty}^{+\infty} \left( |y''(t+h) - y''(t)|^2 + |y(t+h) - y(t)|^2 \right) dt =$$

$$\leq |h|^2 \int_{-\infty}^{+\infty} \left[ |y''(t)|^2 + |y'(t)|^2 \right] dt \leq C_6 \|f\|^2_{L^2_{\nu, \epsilon}} |h|^2,$n

$$\|y\|_{W^2_2(\mathbb{R})} = \int_{|y| \leq N} \left[ |y'(\eta)|^2 + |y(\eta)|^2 \right] d\eta \leq$$

$$\leq \int_{|y| \leq N} \left[ (1 + \eta^2) |y''(\eta)|^2 + (1 + \eta^2)^2 |y'(\eta)|^2 + (1 + \eta^2) |y(\eta)|^2 \right] d\eta \leq$$

$$\leq C_7 \|f\|^2_{L^2_{\nu, \epsilon}} (1 + N^2)^{-1}.$$

Expressions in the right-hand side of (3.6) and (3.7) tend to zero as $h \to 0$ and as $N \to +\infty$, respectively. Then by Kolmogorov-Frechct’s criterion the set $Q_A$ is compact in $W^2_2(\mathbb{R})$. Hence $P(\nu, \epsilon)$ is a compact operator.
Let us show that \( P(\nu, \epsilon) \) is continuous with respect to \( \nu \) in \( S_A \). Let \( \{\nu_n\} \subset S_A \) be a sequence such that \( \|\nu_n - \nu\|_{W^1_2(\mathbb{R})} \to 0 \) as \( n \to \infty \), and \( y_n \) and \( y \) such that \( L_{\nu_n, \epsilon} y = f, \ Y_{\nu_n, \epsilon} y_n = f \). Then it is enough to show that the sequence \( \{y_n\} \) converges to \( y \) in \( W^1_2(\mathbb{R}) \) - norm as \( n \to \infty \). We have
\[
P(\nu_n, \epsilon) - P(\nu, \epsilon) = y_n - y = L_{\nu_n, \epsilon}^{-1}[r(x, \nu_n(x)) - r(x, \nu(x))]y'_n.
\]
The functions \( \nu(x) \) and \( \nu_n(x) \) \((n = 1, 2, \ldots)\) are continuous. Then by conditions of the theorem the difference \( r(x, \nu_n(x)) - r(x, \nu(x)) \) is also continuous with respect to \( x \). Hence for each finite interval \([a, a]\), \( a > 0 \), we have
\[
\|y_n - y\|_{W^1_2(-\infty, \infty)} \leq C \max_{x \in [-\infty, \infty]} |r(x, \nu_n(x)) - r(x, \nu)| \cdot \|y'_n\|_{L_2(-\infty, \infty)} \to 0 \quad (3.8)
\]
as \( n \to \infty \). On the other hand, from Theorem 2 it follows that \( \{y_n\} \subset Q_A \), \( \|y_n\|_{W^1_2} \leq A \), \( \|y\|_{W^1_2} \leq A \). Since the set \( Q_A \) is compact in \( W^1_2(\mathbb{R}) \), \( \{y_n\} \) converges in the \( W^1_2(\mathbb{R}) \) - norm. Let \( z \) be the limit of \( \{y_n\} \). By properties of \( W^1_2(\mathbb{R}) \)
\[
\lim_{|x| \to \infty} y(x) = 0, \quad \lim_{|x| \to \infty} z(x) = 0. \quad (3.9)
\]
Since \( L_{\nu, \epsilon}^{-1} \) is the closed operator, from (3.8) and (3.9) we obtain \( y = z \). Then
\[
\|P(\nu_n, \epsilon) - P(\nu, \epsilon)\|_{W^1_2(\mathbb{R})} \to 0 \text{ as } n \to \infty.
\]
Summing up, we have that \( P(\nu, \epsilon) \) is the completely continuous operator in \( W^1_2(\mathbb{R}) \) and maps \( S_A \) to itself. Then by Schauder’s theorem \( P(\nu, \epsilon) \) has a fixed point \( y \) \( (P(y, \epsilon) = y) \) in \( S_A \). And consequently, \( y \) is a solution of the equation
\[
L_{\nu} y := -y'' + r(x, y, \epsilon + 1 + x^2) y' = f(x).
\]
By (3.3) for \( y \) the estimate
\[
\|y''\|_2 + \left| r(\cdot, y) + \epsilon(1 + x^2) \right| y'_2 \leq C_3 \|f\|_2
\]
holds.

Now, suppose that \( \{\epsilon_j\}_{j=1}^\infty \) is a sequence of positive numbers converged to zero. The fixed point \( y_j \in S_A \) of \( P(\nu, \epsilon_j) \) is a solution of the equation
\[
L_{\nu, \epsilon_j} y_j := -y''_j + r(x, y_j) + \epsilon_j(1 + x^2) y'_j = f(x).
\]
For \( y_j \) the estimate
\[
\|y''_j\|_2 + \left| r(\cdot, y_j) + \epsilon(1 + x^2) \right| y'_2 \leq C_3 \|f\|_2
\]
holds.

Suppose \( (a, b) \) is an arbitrary finite interval. From \( \{y_j\}_{j=1}^\infty \subset W^2_2(a, b) \) one can select a subsequence \( \{y_{j\ell}\}_{j=1}^\infty \) such that \( \|y_{j\ell} - y\|_{L_2(a, b)} \to 0 \) as \( j \to \infty \). A direct verification shows that \( y \) is a solution of (1.6). In (3.10) passing to the limit as \( j \to \infty \) we obtain (1.8). The theorem is proved. □
References


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Appendix to Paper A
APPENDIX TO PAPER A

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Abstract: In this appendix we include the proofs of some facts, which were omitted in Paper A, namely we present the proofs of the auxiliary Lemmas 2.1 and 2.2. Moreover, we give some examples of functions, which satisfy the conditions of Theorem 5.

We need the following Lemma to prove of Lemmas 2.1 and 2.2. It was given and proved in paper [3]. It was proved already earlier when $p = 2$ in [1].

**Lemma 1.** Assume that $1 \leq p \leq \infty$ and

$$M = \sup_{t>0} \left\{ \frac{1}{p} \int_0^t |g(x)|^p dx \right\} \left\{ \int_1^\infty |h(x)|^{-p'} dx \right\}^{1/p} < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1. \quad (1)$$

Then there exists a finite constant $C$ which the following inequality holds

$$\left\{ \int_0^\infty g(x) \int_x^\infty y(t) dt \right\} \frac{1}{p} \leq C \left\{ \int_0^\infty |h(x) y(x)|^p dx \right\}^{1/p}. \quad (2)$$

Moreover, if $C$ is a a smallest constant for which the estimate (2) holds, then

$$M \leq C \leq p^{\frac{1}{p'}} (p')^{\frac{1}{p'}} M.$$ 

**Remark 1.** Also a more general statement holds, namely that (1) holds if and only if (2) holds. This was an important step to start what is nowadays referred to as Hardy-type inequalities. The intensive development of this area is reflected in several papers and books. Here we just refer to the new book [2] by A. Kufner, L.-E. Persson and N. Samko, together with the references given there.

**Proof of Lemma 2.1.** Let $y \in C_0^{(1)}(R)$. We denote $R_- = (-\infty, 0)$, $R_+ = (0, +\infty)$. Then, since $y$ is a compactly supported function, supposing
that \( y(x) = \int_{-\infty}^{\tau} u(t) dt \) in Lemma 1 we have

\[
\int_0^{\infty} |q(x)y(x)|^2 dx \leq c^2 \int_0^{\infty} |r(x)y'(x)|^2 dx, \quad y \in C_0^{(1)}(R_+). \quad (3)
\]

If the constant \( c \) is the smallest constant that satisfies the estimate (3), then the estimates

\[
\sup_{t>0} \alpha_{q,r}(t) \leq c \leq 2 \sup_{t>0} \alpha_{q,r}(t)
\]

hold, where

\[
\alpha_{q,r}(t) = \left( \int_0^t |q(x)|^2 dx \right)^{\frac{1}{2}} \left( \int_0^t |r(x)|^{-2} dx \right)^{\frac{1}{2}}.
\]

Let \( y \in C_0^{(1)}(R_+) \). Then we make the change of variables \( x = -t \) and we obtain the equalities

\[
\int_0^{\infty} |q(x)y(x)|^2 dx = \int_{-\infty}^{0} |q(-t)y'(-t)|^2 dt
\]

and

\[
\int_0^{\infty} |r(x)y'(x)|^2 dx = \int_{-\infty}^{0} |r(-t)y'(-t)|^2 dt.
\]

Denoting by \( z(t) = y(-t) \), \( q_1(t) = q(-t) \) and \( r_1(t) = r(-t) \) \( (t < 0) \), and by using the inequality (3) we obtain the following estimate

\[
\int_{-\infty}^{0} |q_1(t)z(t)|^2 dt \leq c^2 \int_{-\infty}^{0} |r_1(t)z'(t)|^2 dt, \quad (4)
\]

where \( c \) is a constant in (3).

Further

\[
\beta_{q_1,r_1}(\tau) = \left( \int_0^{\tau} |q_1(x)|^2 dx \right)^{\frac{1}{2}} \left( \int_{-\infty}^{\tau} |r_1(x)|^{-2} dx \right)^{\frac{1}{2}} =
\]
\[
\left( \int_{0}^{-\tau} |q(t)|^2 dt \right)^{\frac{1}{2}} \left( \int_{-\tau}^{\infty} |r(t)|^{-2} dt \right)^{\frac{1}{2}} = \alpha_{q,r}(-\tau), \quad (\tau < 0). \tag{5}
\]

So, by (1), for any function \( z \in C^{(1)}(-\infty, 0) \) with compact support the estimate (4) holds, if
\[
M_1 = \sup_{\tau < 0} \beta_{q_1,r_1}(\tau) < \infty.
\]

Moreover, by (3), (5) and Lemma 1 we have that if \( c \) is the smallest constant for which the estimate (4) holds, then
\[
M_1 \leq c \leq 2M_1.
\]

By combining inequalities (3) and (4) we have that
\[
\int_{-\infty}^{+\infty} |q(x)y(x)|^2 dx = \int_{-\infty}^{0} |q(x)y(x)|^2 dx + \int_{0}^{+\infty} |q(x)y(x)|^2 dx \leq
\]
\[
\leq c_1^2 \int_{-\infty}^{0} |r(x)y'(x)|^2 dx + c^2 \int_{0}^{+\infty} |r(x)y'(x)|^2 dx, \tag{6}
\]
where \( y \in C^{(1)}_0(R) \). Obviously, (6) implies the following estimate:
\[
\left( \int_{-\infty}^{+\infty} |q(x)y(x)|^2 dx \right)^{\frac{1}{2}} \leq \max(c_1, c) \left( \int_{-\infty}^{+\infty} |r(x)y'(x)|^2 dx \right)^{\frac{1}{2}}.
\]

Furthermore, if \( c_1 \) and \( c \) are the smallest constants for which accordingly the estimates (2) and (3) hold, then
\[
\sup_{t > 0} \alpha_{q,r}(t) \leq c \leq 2 \sup_{t > 0} \alpha_{q,r}(t),
\]
\[
\sup_{\tau < 0} \beta_{q,r}(\tau) \leq c_1 \leq 2 \sup_{\tau < 0} \beta_{q,r}(\tau).
\]
Thus \( \max(c_1, c) \leq 2\gamma_{q,r} \) and \( \gamma_{q,r} \leq \max(c_1, c) \). The proof is complete.

**Proof of Lemma 2.2.** We write the function \( r(x) \) in the form \( r = r_* + r_+ \), here
\[
r_*(x) = \begin{cases} 
      r(x), & x \in (-\infty, 0] \\
      0, & x \in (0, +\infty),
   \end{cases}
\]
\[ r_+(x) = \begin{cases} 0, & x \in (-\infty, 0] \\ r(x), & x \in (0, +\infty) \end{cases} \]

Then the equality
\[ \|r\|_{L^2(\mathbb{R})} = \|r_+\|_{L^2(\mathbb{R}_+)} + \|r_-\|_{L^2(\mathbb{R}_-)} \]
holds. If we write the set
\[ F_K = \left\{ y : y \in C_0^{(1)}(\mathbb{R}), \int_{-\infty}^{+\infty} |r(t)y'(t)|^2 dt \leq K \right\}, \]
as a combination of sets
\[ F_{K_1}^+ = \left\{ y : y \in C_0^{(1)}(\mathbb{R}_+), \int_0^{+\infty} |r_+(t)y'(t)|^2 dt \leq K_1 \right\}, \quad K_1 > 0, \]
and
\[ F_{K_1}^- = \left\{ y : y \in C_0^{(1)}(\mathbb{R}_-), \int_{-\infty}^0 |r_-(t)y'(t)|^2 dt \leq K_2 \right\}, \quad K_2 > 0, \]
then it is enough to show the compactness of the sets \( F_{K_1}^+ \) and \( F_{K_1}^- \) to prove the compactness of \( F_K \).

If condition (2) is satisfied, then
\[ \sup_{x > 0} \frac{1}{x} \left( \int_{-\infty}^{+\infty} |r^{-2}(t)| dt \right)^{\frac{1}{2}} < \infty. \]

Therefore, according to Lemma 2.1, the set \( F_{K_1}^+ \) is bounded in the space \( L^2(\mathbb{R}_+) \). So, by Frechet-Kolmogorov Theorems it is enough to prove that there exists a number \( N_{\epsilon_1} > 0 \) for which the inequality
\[ \int_{N_1}^{\infty} |y|^2 dt \leq \epsilon_1, \quad y \in F_{K_1}^+, \quad N_1 \geq N_{\epsilon_1} \quad (7) \]
holds for any \( \epsilon_1 > 0 \). By Kac-Krein’s Theorem [1] the following estimates
\[ \int_{N}^{\infty} |y(t)|^2 dt = \int_{0}^{\infty} |y(t-N)|^2 dt \leq \]
\begin{align*}
&\leq c \sup_{x>0} x \left( \int_{x}^{\infty} r^{-2}(t-N) dt \right) \int_{0}^{\infty} |r(t-N)y'(t-N)|^2 dt \\
&\leq c \sup_{x>0} x \left( \int_{x+N}^{\infty} r^{-2}(t) dt \right) \int_{0}^{\infty} |r(t)y'(t)|^2 dt \\
&\leq c \sup_{x>N} x \int_{x}^{\infty} r^{-2}(t) dt
\end{align*}

hold. From these estimates and from the condition (2) we see that there exists a number \(N_1 > 0\) for which (7) holds for any \(\epsilon_1 > 0\). Similarly, the compactness of the set \(F_{\tilde{K}}\) can be shown by using the assumptions

\[
\lim_{x \to -\infty} \sqrt{|x|} \|r^{-1}\|_{L_2(-\infty, x)} = \lim_{x \to -\infty} \sqrt{|x|} \left( \int_{-\infty}^{x} r^{-2}(t) dt \right)^{\frac{1}{2}} = 0
\]

in Lemma 2.2. The proof is complete.

Finally, we present two examples of functions, which satisfy the conditions of Theorem 5.

**Example 1.** The function \(r_1(x, y) = 3 + 2x^4 + e^{5x} + y^2\) satisfies the conditions

\[r \geq \delta_0(1 + x^2) \quad (\delta_0 > 0)\]

and, for every number \(A > 0\),

\[\sup_{|x-y| \leq 1, |c_1| \leq A, |c_2| \leq A} \sup_{|c_1| \leq A, |c_2| \leq A} \frac{r(x, c_1)}{r(\eta, c_2)} < \infty\]

from Theorem 5. In fact,

\[r_1(x, y) \geq 3 + x^4 \geq 1 + 2(1 + x^4) \geq 2 + x^2 \geq 1 + x^2.\]

So (8) holds with \(r = r_1(x, y)\) and \(\delta_0 = 1\). Moreover if the numbers \(c_1, c_2\) and \(A\) satisfy conditions \(|c_1 - c_2| \leq A, |c_1| \leq A, \) and \(|c_2| \leq A\), then

\[\sup_{|x-\eta| \leq 1} \frac{r_1(x, c_1)}{r_1(\eta, c_2)} = \sup_{|x-\eta| \leq 1} \frac{3 + 2x^4 + e^{5x} + c_1^2}{3 + 2\eta^4 + e^{5\eta} + c_2^2} \leq\]

\[\leq \sup_{|x-\eta| \leq 1} \frac{2 + 2x^4}{3 + 2\eta^4} + \sup_{|x-\eta| \leq 1} \frac{e^{5x}}{e^{5\eta}} + \sup_{|x-\eta| \leq 1} \frac{3 + c_1^2}{3 + c_2^2} \leq\]

\[\leq \sup_{|x-\eta| \leq 1} \frac{2 + 2(|x-\eta| + |\eta|)^4}{3 + 2\eta^4} + \sup_{|x-\eta| \leq 1} e^{5|x-\eta|} + \]
\[ + \sup_{|x-\eta|\leq 1} \frac{3 + (|c_1 - c_2| + |c_2|)^2}{3 + c_2^2} \leq \]
\[ \leq \sup_{|\eta| \in \mathbb{R}} \frac{2 + 8(1 + \eta^4)}{3 + 2\eta^4} + e^5 + \sup_{|\eta| \in \mathbb{R}} \frac{3 + 2c_2^2 + A^2}{3 + c_2^2} \leq 4 + e^5 + \frac{1}{3}A^2, \]
so also (9) holds for each finite \( A \).

Example 2. The conditions (8), (9) are satisfied also by the function \( r_2(x, y) = 2 + x^2 + e^{y} \). In fact, obviously \( r_2(x, y) \geq 1 + x^2 \) so (8) holds for \( r = r_2(x, y) \) and \( \delta_0 = 1 \). Furthermore, if the number \( A > 0 \) is given and the inequalities \( |c_i| \leq A (i = 1, 2) \), \( |c_1 - c_2| \leq A \) are satisfied for the numbers \( c_1 \), \( c_2 \), then
\[ \sup_{|x-\eta|\leq 1} \frac{r_2(x, c_1)}{r_2(\eta, c_2)} = \sup_{|x-\eta|\leq 1} \frac{2 + x^2 + e^{c_1}}{2 + \eta^2 + e^{c_2}} \leq \sup_{|x-\eta|\leq 1} \frac{2 + x^2}{2 + \eta^2} + \sup_{|x-\eta|\leq 1} \frac{e^{c_1}}{2 + \eta^2 + e^{c_2}} \leq \]
\[ \leq \sup_{|x-\eta|\leq 1} \frac{2 + 2(1 + \eta^2)}{2 + \eta^2} + e^{|c_1 - c_2|} \leq 2 + e^A, \]
so (9) holds.

References

Compactness of the Resolvent of One Second Order Differential Operator

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Abstract. In this work a necessary and sufficient condition for the compactness of the resolvent of one second order degenerate differential operator in $L^2$ is obtained. We also discuss the two-sided estimates for the radius of fredholmness of this operator.

Keywords: degenerate operator, resolvent, compactness, radius of fredholmness.

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1. INTRODUCTION AND MAIN RESULTS

Let $L$ be a closure in $H = L^2(R)$ of the expression

$$L_y y = -y'' + r(x)y' + s(x)y$$

defined in the set $C_c^2(R)$ of twice continuously differentiable and compactly supported functions. Here $r$ and $s$ are complex-valued functions and $\bar{y}$ is the conjugate function to $y$. The operator $L$ does not have a coefficient by $y$ and $\bar{y}$. Such operators are called the degenerate operators.

The conditions of existence of the inverse operator $L^{-1}$ and its continuity were obtained in [1, 2]. Moreover in [1, 2] for $y \in D(L)$ was obtained the following estimate:

$$\|y\|_{L^2} + \|y'\|_{L^2} + \|\bar{y}\|_{L^2} \leq c\|\mathcal{L}y\|_{L^2}.$$  \hspace{1cm} (1)

Here the constant $c$ is not depend of $y$.

Definition. A number

$$\rho_\sigma = \left( \inf_{\mathcal{L}f \in L^2} \|I - \mathcal{T}\|_{L^2} \right)^{-\frac{1}{2}}$$

is called the radius of fredholmness of bounded operator $A$ in $L_2$ (see [3]). Here $\sigma(L_2)$ is the set of all compact linear operators in $L_2$.

In this paper, under the assumption that the inequality (1) holds, we will install a necessary and sufficient condition for the compactness of the operator $L^1$ in $L_2$. Also we will to discuss the two-sided estimates of the radius of fredholmness $\rho_\sigma$ of the operator $L^1$.

Some criterion of compactness of the resolvent for semi-bounded differential operators of Schrödinger type are discussed in [4, 5, 6]. Estimates of the radius of fredholmness of embedding operators of Sobolev spaces are established in [3].

We denote...
where \( g \) and \( h \) are given functions.

**Theorem 1.** Let the functions \( r \) and \( s \) be continuously differentiable and satisfy the conditions

\[
\Re \rho \geq \delta > 0, \quad \gamma_{r,s} < \alpha, \beta > 0, \quad \text{for } \delta \in \mathbb{R}.
\]

Then the resolvent \( L^{-1} \) of the operator \( L \) is compact in the space \( L^2 \) if and only if the following relations are fulfilled:

\[
\lim_{\nu \to 0} \alpha_{r,s} (\nu) = 0, \quad \lim_{\nu \to 0} \beta_{r,s} (\nu) = 0.
\]

**Theorem 2.** Let the functions \( r \) and \( s \) be continuously differentiable and satisfy the conditions (2) and (3). Then for radius of fredholmness \( \rho_{r,s} \) of the operator \( L^{-1} \) the following estimates holds:

\[
ce_2^{-1} \leq \rho_{r,s} \leq e_2,
\]

where

\[
\gamma_0 = \max \left( \sup_{t > 0} \alpha_{r,s} (t), \sup_{t < 0} \beta_{r,s} (t) \right).
\]

**2. AUXILIARY STATEMENTS AND PROOFS OF THE THEOREMS**

By \( \Omega \) we denote one of the semi-axes := \((0, +\infty)\) or := \((-\infty, 0)\). Let \( r_1 \neq 0 \) be a positive continuous function defined on \( \Omega \) and such that the following inequality holds:

\[
\|u\|_{L^2(\Omega)} \geq C \|u\|_{L^1(\Omega)},
\]

where \( u \) is arbitrary smooth function with compact support, and \( \| \cdot \|_{L^2(\Omega)} \) is the norm in \( L^2(\Omega) \). We denote by \( H_2(r_1, \Omega) \) the completion of the set \( C_0^2(\Omega) \) of twice differentiable and compactly supported functions defined on \( \Omega \) in the norm

\[
\|f\|_{H_2(r_1, \Omega)} := \|f\|_{L^2(\Omega)} + \|u\|_{L^1(\Omega)}.
\]

Under the above conditions on the function \( r_1 \) the space \( H_2(r_1, \Omega) \) is the norm space.
Let \( \eta \geq \delta > 0, \ x \in \Omega \). We continue \( \eta(x) \) to all \( R \) such that its continuation be even and by \( \eta^*(x) \) denote the function

\[
\eta^*(x) = \inf \left\{ d^{-1} : d^{-1} \geq \int_{\Gamma \setminus \frac{T}{2}} \eta^*(t)dt \right\}.
\]

**Lemma 1.** If the function \( \eta \geq \delta > 0 \) is continuous and satisfies the condition

\[
c^{-1} \leq \frac{\eta(x)}{\eta(t)} \leq c, \quad \text{where} \ x, t \in \Omega : |x - t| \leq \frac{d}{2}.
\]

then the inequalities

\[
\gamma_1 \eta(x) \leq \eta^*(x) \leq \gamma_2 \eta(x),
\]

hold. Here \( \gamma_1, \gamma_2 \) are positive constants independent of \( x \).

**Proof.** Denote \( d_{\eta} = \left[ \eta^*(x) \right]^{-1} \). Then by the continuity of \( \eta \), the following equality holds:

\[
d_{\eta}^{-1} = \eta^*(x) = \int_{\Gamma \setminus \frac{T}{2}} \eta^*(t)dt.
\]

From (7) and (5) we have the following two inequalities:

\[
\eta^*(x) \leq c \eta^*(x) \int_{\Gamma \setminus \frac{T}{2}} dt = c d_{\eta}^2(x) = c \left[ \eta^*(x) \right]^{-1} \eta^*(x),
\]

\[
\eta^*(x) \geq c^{-1} \eta^*(x) \int_{\Gamma \setminus \frac{T}{2}} dt = c^{-1} d_{\eta}^2(x) = c \left[ \eta^*(x) \right]^{-1} \eta^*(x).
\]

From these estimates it follows the inequalities (6), where \( \gamma_1 = \frac{1}{\sqrt{c}}, \gamma_2 = \sqrt{c} \). Lemma is proved.

We denote by \( E_\delta \) the embedding operator of the space \( H^r(\text{Re} r, R) \) to the space \( L^r(R) \), and by \( E_\delta \) the embedding operator of the space \( H^r(\text{Re} r, R) \) to the space \( L^r(R) \) and we write \( E_\delta : H^r(\text{Re} r, R) \to L^r(R) \), \( E_\delta : H^r(\text{Re} r, R) \to L^r(R) \).

**Lemma 2.** Let the function \( \text{Re} r(t) \geq \delta > 0 \) \( (t > 0) \) be continuous and satisfy the condition (3). Then the operator \( E_\delta : H^r(\text{Re} r, R) \to L^r(R) \) is compact if and only if the following equality holds:

\[
\lim_{t \to \infty} \alpha_{1, \text{Re} r}(t) = 0.
\]

**Proof.** Choosing \( d \leq 2 \) by Lemma 1 we have

\[
\gamma_1 \text{Re} r(x) \leq (\text{Re} r)^*(x) \leq \gamma_2 \text{Re} r(x)
\]
By Theorem 2 [3] the embedding operator \( E \) is compact if and only if the following equality holds:

\[
\lim_{r \to +\infty} \beta_{\text{max}}(r) = 0.
\]

From this and inequalities (9) follows (8). Lemma is proved.

**Lemma 3.** Let the function \( \text{Re} r(t) \geq \delta > 0 \) \((t < 0)\) be continuous and satisfy the condition (3). Then the embedding operator \( E : H^1(\text{Re} r, R^2) \to L_2(R^2) \) is compact if and only if the following equality holds:

\[
\lim_{r \to +\infty} \beta_{\text{max}}(r) = 0.
\]

**Proof.** We assume that \( u(-t) = V(t), \text{Re} r(-t) = r(t), t > 0 \), then

\[
\|H_{\text{L}^2}^{(r(t), R^2)}\| = \|H_{\text{L}^2}^{(r(t), R^2)}\| \quad \text{and} \quad \|H_{\text{L}^2}^{(r(t))}\| = \|H_{\text{L}^2}^{(r(t))}\|.
\]

Therefore, the embedding operators \( E : H^1(\text{Re} r, R^2) \to L_2(R^2) \) and \( E' : H^1(\tau, R^2) \to L_2(R^2) \) simultaneously compact. And by Lemma 2 \( E \) is a compact operator if and only if the following equality holds:

\[
\lim_{r \to +\infty} \sqrt{\frac{1}{r^2} \int \frac{dS}{|\tau|^2}} = \lim_{r \to +\infty} \sqrt{\frac{1}{r^2} \int |\text{Re} r(\tau)|^2} = \lim_{r \to +\infty} \beta_{\text{max}}(r) = 0.
\]

Lemma is proved.

We denote by \( H^1(\text{Re} r, R^2) \) \((R = (-\infty, +\infty))\) the completion of the set \( C^{(2)}_L(R) \) in following norm

\[
\|H_{\text{L}^2}^{(r(t), R^2)}\| = \|H_{\text{L}^2}^{(r(t), R^2)}\| \quad \text{and} \quad \|H_{\text{L}^2}^{(r(t))}\| = \|H_{\text{L}^2}^{(r(t))}\|.
\]

Let \( E \) be an embedding operator of space \( H^1(\text{Re} r, R^2) \) to the space \( L_2 \).

**Lemma 4.** Let the function \( \text{Re} r(x) \geq \delta > 0 \) \((x \in R)\) be continuous and satisfy the condition (3). Then \( E \) is compact if and only if the equalities (4) hold.

**Proof.** Let

\[
r_r(x) = \begin{cases} 
\text{Re} r(x), & x \in (-\infty, 0] \\
0, & x \in (0, +\infty)
\end{cases}
\]

and \( \tau_r(x) = \begin{cases} 
0, & x \in (-\infty, 0] \\
\text{Re} r(x), & x \in (0, +\infty)
\end{cases} \). Then \( \text{Re} r = r_r + r_\tau \) and

\[
\|\text{Re} r\|_{L^2} = \|H_{\text{L}^2}^{(r(t), R^2)}\| + \|H_{\text{L}^2}^{(r(t))}\|.
\]

The embedding operator \( E : H^1(\text{Re} r, R^2) \to L_2 \) is compact if and only if the operators \( E : H^1(\text{Re} r, R^2) \to L_2(R^2) \) and \( E' : H^1(\text{Re} r, R^2) \to L_2(R^2) \) are both compact. We will prove this.

Let \( E \) and \( E' \) be compact operators. Then from Lemma 2 and Lemma 3 follows the equalities (4). According to a well-known theorem of Frechet-Kolmogorov the embedding operator \( E \) is compact if and only if both of the following relations hold:

\[
\sup_{r \to +\infty} \|H_{\text{L}^2}^{(r(t), R^2)}\|_{L_2(R^2)} = 0, \quad N \to +\infty,
\]

(10)
and
\[
\sup_{K \subseteq \mathbb{R}, x, K} \left\| f_{(x, x)}^{(K, \infty, K)} \right\| 	o 0, \quad K \to -\infty.
\] (11)

By theorem Kats-Krein [7] we have
\[
\left\| f_{(x, x)}^{(K, \infty, K)} \right\| \leq c \sup_{j \in \mathbb{N}} \left\| f_{(x, x)}^{(j, \infty, j)} \right\| \left( \text{Re} \, r \right)^{-j} \left\| f_{(x, x)}^{(j, \infty, j)} \right\| \leq c_1 \sup_{j \in \mathbb{N}} \left\| f_{(x, x)}^{(j, \infty, j)} \right\|.
\]

From these inequalities by the first equality in (4) follows (10). Similarly, taking into account the second equality in (4) we have (11). Thus \( E \) is compact operator.

Now let \( E \) be a compact operator. Assume the contrary, let the operator \( E \) does not compact. Then, by definition of compactness there is a fundamental sequence
\[
\{ f_{(x)}^{(j, \infty, j)} \} \subseteq H_{(\text{Re} \, r, R)} \), \left\| f_{(x)}^{(j, \infty, j)} \right\| \leq 1,
\]
which does not converge in the space \( L_{2}(R) \). We choose the functions \( f_{(x)}^{(j, \infty, j)} \) such that
\[
f_{(x)}^{(j, \infty, j)} = 0, \quad x \in (0, \xi), \quad \xi > 0.
\]

We will continue \( f_{(x)}^{(j, \infty, j)} \) by zero on \( R \), and this continuation we denote by \( f_{(x)}^{(j, \infty, j)} \). Then the sequence \( \{ f_{(x)}^{(j, \infty, j)} \} \) is fundamental and converges in \( L_{2}(R) \), since \( \{ f_{(x)}^{(j, \infty, j)} \} \subseteq H_{(\text{Re} \, r, R)} \) and \( E \) is compact operator by assumption. Then, by our choice the sequence \( \{ f_{(x)}^{(j, \infty, j)} \} \) also converges in \( L_{2}(R) \). This is a contradiction. So the operator \( E \) is compact. If \( E \) be compact operator, then the embedding operator \( E \) also is compact. This fact proved similarly. Lemma is proved.

**Proof of Theorem 1.** If the functions \( r \) and \( s \) satisfy the conditions (2) and (3), then (see [1], Theorem 1 and Theorem 2) the inverse operator \( L^{-1} \) is a continuous operator mapping the space \( L_{2}(R) \) to \( H_{2}(\text{Re} \, r,R) \). Hence by Lemma 4 the operator \( L^{-1} \) is compact if and only if the condition (4) holds. The theorem is proved.

**Proof of Theorem 2.** In [3] (Theorem 3) we obtained the two-sided estimates for the radius of fredholmness of above embedding operator \( E_{r} \). The two-sided estimates for the radius of fredholmness of embedding operator \( E_{r} \) will be proved similarly. From this using Lemma 1 in [8] we will have the following inequalities:
\[
0 \leq c_{2} \leq c_{1} \leq \rho_{2} \rho_{3} \leq c_{2}.
\]

Theorem is proved.

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**REFERENCES**

Paper C
SOME INEQUALITIES FOR SECOND ORDER DIFFERENTIAL OPERATORS WITH UNBOUNDED DRIFT

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Abstract. We study coercive estimates for some second-order degenerate and damped differential operators with unbounded coefficients. We also establish the conditions for invertibility of these operators.

1 Introduction

For the Sturm-Liouville operator $l_0y = -y'' + q(x)y$ ($x \in \mathbb{R}$), coercive estimates and other properties associated with Sobolev spaces are well known (see [1, 3, 4, 15]). Properties of the operator $ly = -y'' + ry' + qy$ with the intermediate coefficient $r$ subordinated to the potential $q$ in some sense, are studied in [5, 9].

In this work, we consider the minimal closed differential operator $Ly = -\rho(x)(\rho(x)y')' + ry' + q(x)y$ in $L^2(\mathbb{R})$, where $\rho, r$ are continuously differentiable functions, and $q$ is a continuous function. We do not assume that $\rho, r, q$ are bounded in $\mathbb{R}$. The aim of this work is to show that the operator $L$ is continuously invertible when these coefficients satisfy some conditions and to obtain the following estimate for $y \in D(L)$

$$
\|\rho(\rho y')'\|_2 + \|ry'\|_2 + \|qy\|_2 \leq C \|Ly\|_2,
$$

(1.1)

where $D(L)$ is the domain of $L$, $\| \cdot \|_2$ is the norm in $L^2(\mathbb{R})$, and $C$ independent of $y$.

Estimate (1.1) already implies that the domain of $L$ coincides with the subspace generated by the norm $\|\rho(\rho y')'\|_2 + \|ry'\|_2 + \|qy\|_2$. This fact enables us to use the methods of the embedding theory of weighted Sobolev spaces for studying many important properties (for example, regularity, spectral or approximation properties) of $L$ (see [8, 12, 13, 16]).

The operator $L$ has numerous applications in mathematical physics and stochastic processes. For example, in the theory of Brownian motion the Ornstein-Uhlenbek operator is used (see [10]), which is an operator of type $L$, and the Fokker-Plank and Kramer differential operators are generalizations of the Ornstein-Uhlenbek operator. The Ornstein-Uhlenbek operator was studied in works of M. Smoluchowski, A. Fokker,
M. Plank, H.C. Burger, R. Furth, L. Zernike, S. Goudsmitt, M.C. Wang (see [20] and the references therein). On the other hand, the operator $L$ is used to describe the problem of the propagation of small oscillations in a viscoelastic compressible medium [17, 19]. Also, the operator $L$ is used in the study of the vibrational motion in mediums with resistance, where the resistance depends on the velocity [18].

Recently in works of J. Pruss, R. Shnaubelt, A. Rhandi, G. Da Prato, V. Vespri, P. Clement, G. Metafune, D. Pallara, M. Hieber, L. Lorenzi and others the following Ornstein-Uhlenbek-type operator $A_0 u = -\text{div}(a \nabla u) + F \cdot \nabla u - Vu$ was investigated with various properties (see [2] and references therein). In this works are imposed the additional conditions which are sufficient to control the drift term $F \cdot \nabla u$ by $-\text{div}(a \nabla u)$ and $Vu$.

The results of the present paper show that if the intermediate coefficient $r$ is quickly growing, then the one dimensional operator $L$ is invertible and has regular properties. Estimate (1.1) is useful for evolutionary partial differential equations associated with the operator $L$ (see [7]).

The paper is organized as follows. In Section 2 we prove several auxiliary statements and the invertibility of the operator

$$ly = -\rho(\rho y)' + ry'$$

for a certain class of $\rho$ and $r$. In Section 3 we prove inequality (1.1) under some additional conditions. We present some examples in Section 4.

Inequality (1.1) for operator $l$ in the case $\rho = 1$ was obtained in [11]. The coercive estimate of $L$ in $L^2(\mathbb{R})$ was proved in [14].

We denote by $C(\mathbb{R})$ the class of the continuous functions, and by $C^{(s)}(\mathbb{R})$ ($s = 1, 2, ...$) the class of all $s$ times continuously differentiable functions and by $C^{(s)}_0(\mathbb{R})$ ($s = 1, 2, ...$) the subset of all compactly supported functions in $C^{(s)}(\mathbb{R})$.

2 Auxiliary statements and existence of the resolvent for a degenerate operator

Denote by $l$ the closure in $L_2(\mathbb{R})$ of the differential expression

$$l_0 y = -\rho(\rho y)' + ry'$$

on $C^{(2)}_0(\mathbb{R})$, where $\rho \in C^{(1)}(\mathbb{R})$, $r \in C(\mathbb{R})$. The operator $l$ is a degenerate operator, since it does not have the lower-order term. The domain $D(l)$ is contained in the space $L_2(\mathbb{R})$ only in the case when the functions $\rho$ and $r$ satisfy some additional conditions.

In this section, we give some sufficient conditions for bounded invertibility of the operator $l$. We denote

$$\alpha_{g,h}(t) = \|g\|_{L_2(0,t)} \left\| h^{-1} \right\|_{L_2(t, +\infty)} (t > 0),$$

$$\beta_{g,h}(\tau) = \|g\|_{L_2(\tau,0)} \left\| h^{-1} \right\|_{L_2(-\infty, \tau)} (\tau < 0),$$
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\[ \gamma_{g,h} = \max \left( \sup_{t>0} \alpha_{g,h}(t), \sup_{t<0} \beta_{g,h}(t) \right) \]

where \( g \) and \( h \) are given functions.

**Lemma 2.1.** [11]. Let \( g \) and \( h \) be continuous functions on \( \mathbb{R} \) and \( \gamma_{g,h} < \infty \). Then for any \( y \in C^{(1)}_0(\mathbb{R}) \) the following inequality holds:

\[ \int_{-\infty}^{\infty} |g(x)y(x)|^2 \, dx \leq c_1 \int_{-\infty}^{\infty} |h(x)y'(x)|^2 \, dx. \]

Moreover, the least such constant \( c_1 \) satisfies \( \gamma_{g,h} \leq c_1 \leq 2\gamma_{g,h} \).

**Lemma 2.2.** Let \( \rho \in C^{(1)}(\mathbb{R}) \) and \( r \in C(\mathbb{R}) \) satisfy the following conditions

\[ r \geq 1, \gamma_{1,\sqrt{r}} < \infty. \]  

Then for \( y \in D(l) \) the following estimate holds:

\[ \|\sqrt{r}y'\|_2 + \|y\|_2 \leq (1 + \sqrt{2\gamma_{1,\sqrt{r}}}) \left\| \frac{1}{\sqrt{r}}ly \right\|_2. \]

**Proof.** Let \( y \in C^{(2)}_0(\mathbb{R}) \). Integrating by parts, we have

\[ (ly, y') = \int_{\mathbb{R}} r(x)(y')^2 dx. \]

By Hölder’s inequality,

\[ |(Ly, y')| \leq \left\| \frac{1}{\sqrt{r}}L_0y \right\|_2 \left\| \sqrt{r}y' \right\|_2. \]

Since \( r \geq 1 \), from (2.3) and (2.4) it follows that

\[ \left\| \sqrt{r}y' \right\|_2 \leq \left\| \frac{1}{\sqrt{r}}L_0y \right\|_2. \]

On the other hand, using Lemma 2.1, we get

\[ \|y\|_2 \leq 2\gamma_{1,\sqrt{r}} \left\| \sqrt{r}y' \right\|_2. \]

Then

\[ \left\| \sqrt{r}y' \right\|_2 + \|y\|_2 \leq (1 + 2\gamma_{1,\sqrt{r}}) \left\| \sqrt{r}y' \right\|_2. \]

So, using (2.5) we obtain that (2.2) holds for any \( y \in C^{(2)}_0(\mathbb{R}) \).

Let \( y \in D(l) \). Then there exists a sequence \( \{y_n\}_{n=1}^\infty \subset C^{(2)}_0(\mathbb{R}) \) such that \( \|y_n - y\|_2 \to 0, \|y_n - ly\|_2 \to 0 \) as \( n \to \infty \). Since (2.2) holds for all \( y_n \) \( (n \in \mathbb{N}) \), then passing to limit as \( n \to \infty \) we obtain the desired estimate for \( y \in D(l) \). \[ \square \]
Theorem 2.1. Let \( r \in C(\mathbb{R}) \), \( \rho \in C^{(1)}(\mathbb{R}) \) be such that
\[
\rho \geq \rho^2. \gamma_1, \sqrt{\gamma} < \infty
\] (2.6)
and for some \( N > 0 \) the following inequality holds
\[
1 \leq \rho(x) \leq c_2 \left(1 + x^2\right)^N. \tag{2.7}
\]
Then the operator \( l \) is invertible and the inverse operator \( l^{-1} \) is defined on the whole \( L_2(\mathbb{R}) \).

Proof. Inequality (2.2) implies that the inverse \( l^{-1} \) exists. It suffices to show that \( R(l) = L_2(\mathbb{R}) \). Assume that \( R(l) \neq L_2(\mathbb{R}) \). Then there exists a non-zero element \( v \in L_2(\mathbb{R}) \) such that \( v \perp R(l) \). It follows that
\[
l^*v \equiv (\rho(\rho v)' + (rv)') = 0,
\]
where \( l^* \) is the adjoint operator of \( l \). Put \( \rho v = z \), then
\[
\left( \rho z' + \frac{r}{\rho} z \right)' = 0,
\]
or
\[
\left( z \exp \left[ \int_a^x \frac{r(t)}{\rho^2(t)} \, dt \right] \right)' = \frac{c}{\rho} \exp \left( \int_a^x \frac{r(t)}{\rho^2(t)} \, dt \right),
\]
where \( c \) is a constant.

If \( c \neq 0 \), then we can assume that \( c = -1 \). Inequalities (2.6), (2.7) imply that
\[
\left( z(x) \exp \left[ \int_a^x \frac{r(t)}{\rho^2(t)} \, dt \right] \right)' \leq c_1 < 0, \quad x \in (a, +\infty).
\]
Hence (2.6) and (2.7) imply that \( v \notin L_2(\mathbb{R}) \).

If \( c = 0 \), then we have
\[
v = \frac{c_2}{\rho(x)} \exp \left. \left[ - \int_a^x \frac{r(t)}{\rho^2(t)} \, dt \right] \right|.
\]
By (2.7), there exists \( x_0 < a \) such that \( |v(x)| \geq \delta > 0 \) for any \( x \leq x_0 \). So \( v \notin L_2(\mathbb{R}) \).

Hence, we obtained a contradiction. Thus \( R(l) = L_2(\mathbb{R}) \). \( \square \)

Definition 1. \( l \) is called separable in \( L_2(\mathbb{R}) \), if there exists \( c > 0 \) such that
\[
\|\rho(\rho y')'\|_2 + \|ry'\|_2 \leq c_3 \|y\|_2 \tag{2.8}
\]
for all \( y \in D(l) \).

Put \( \rho y' = z \). Then
\[
l y = -\rho z' + \frac{r}{\rho} z.
\]
Let $\lambda \geq 0$, and $\rho$ be a bounded function. We define $K_\lambda : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ as follows:

$$K_\lambda z = -z' + \left( \frac{r}{\rho^2} + \lambda \right) z, \quad z \in D(K_\lambda),$$

where $D(K_\lambda)$ is the domain of $K_\lambda$. Note that $K_\lambda$ is separable in $L_2(\mathbb{R})$, if for some $c_4 > 0$,

$$\|z''\|_2 + \left\| \left( \frac{r}{\rho^2} + \lambda \right) z \right\|_2 \leq c_4 \|K_\lambda z\|_2$$

for all $z \in D(K_\lambda)$.

**Lemma 2.3.** Let $\rho \in C^{(1)}(\mathbb{R})$, $1 \leq \rho \leq s$, $r \in C(\mathbb{R})$ satisfy (2.2). Then $l$ is separable in $L_2(\mathbb{R})$ if and only if

$$K_\lambda z = -z' + \left( \frac{r}{\rho^2} + \lambda \right) z$$

is separable in $L_2(\mathbb{R})$ for some $\lambda \geq 0$.

**Proof.** Assume that $l$ is separable in $L_2(\mathbb{R})$. Put $\rho y' = z$. Then

$$\| - \rho z' \|_2 + \left\| \frac{r}{\rho} z \right\|_2 \leq c_5 \|\rho^{-1} K_0 z\|_2.$$

Hence,

$$\| - z' \|_2 + \left\| \frac{r}{\rho} z \right\|_2 \leq c_5 \|K_0 z\|_2. \quad (2.9)$$

It is easy to check that for any $z \in D(K_\lambda)$ the following estimate holds:

$$\left\| \sqrt{\frac{r}{\rho^2} + \lambda} z \right\|_2 \leq \left\| \frac{1}{\sqrt{\frac{r}{\rho^2} + \lambda}} K_\lambda z \right\|_2. \quad (2.10)$$

Therefore,

$$\left( \frac{1}{s^2} + \lambda \right) \|z\|_2 \leq \|K_\lambda z\|_2, \quad z \in D(K_\lambda). \quad (2.11)$$

By (2.9) and (2.11), we have that

$$\| - z' \|_2 + \left\| \left( \frac{r}{\rho^2} + \lambda \right) z \right\|_2 \leq c_5 \|K_0 z\|_2 + \lambda \|z\|_2 \leq (c_5 + 2) \|K_\lambda z\|_2. \quad (2.12)$$

So, $K_\lambda$ is separable in $L_2(\mathbb{R})$.

Let $K_\lambda$ be separable in $L_2(\mathbb{R})$, i.e.

$$\| - z' \|_2 + \left\| \left( \frac{r}{\rho^2} + \lambda \right) z \right\|_2 \leq c_6 \|K_\lambda z\|_2, \quad z \in D(K_\lambda).$$

By (2.11), we obtain that

$$\|K_\lambda z\|_2 \leq \|K_0 z\|_2 + \frac{\lambda}{\lambda + 1/s^2} \|K_\lambda z\|_2.$$
hence
\[ \| K_\lambda z \|_2 \leq (s^2 \lambda + 1) \| K_0 z \|_2. \]

So, it follows that
\[
\| -\rho z' \|_2 + \| \frac{r}{\rho} z \|_2 \leq s \left[\| -z' \|_2 + \| \frac{r}{\rho} z \|_2 \right] \leq c_6 \| K_\lambda z \|_2 + \lambda \| z \|_2
\]
\[
\leq (c_6 + 1) \| K_\lambda z \|_2 \leq 2c_6 (s^2 \lambda + 1) \| K_0 z \|_2.
\]

Taking \( z/\rho = y' \), we get that
\[
\| -\rho \frac{r}{\rho} (\rho y')' \|_2 + \| ry' \|_2 \leq c_7 \| ly \|_2.
\]

**Lemma 2.4.** Let \( \rho \in C^{(1)}(\mathbb{R}) \), \( 1 \leq \rho \leq s \) and \( r \in C(\mathbb{R}) \). Suppose that
\[
\sup_{|x-\eta| \leq 2} \left| r(x) - r(\eta) \right| < \infty \tag{2.13}
\]

and condition (2.2) hold. Then \( l \) is separable in \( L_2(\mathbb{R}) \).

**Proof.** By Lemma 2.3, it is enough to prove that \( K_\lambda \) is separable in \( L_2(\mathbb{R}) \) for some \( \lambda \geq 0 \).

Theorem 2.1 implies that \( K_\lambda \) is continuously invertible on \( L_2(\mathbb{R}) \) for all \( \lambda \geq 0 \). Next, we show a useful representation of \( K_\lambda^{-1} \). Let \( \Delta_j = (j - 1, j + 1) \) (\( j \in \mathbb{Z} \)), and \( \{\varphi_j\}_{j=-\infty}^{+\infty} \) be a sequence in \( C_0^\infty(\Delta_j) \) such that
\[
0 \leq \varphi_j \leq 1, \left| \varphi_j'(x) \right| \leq m \ (j \in \mathbb{Z}), \sum_{j=-\infty}^{+\infty} \varphi_j^2(x) = 1.
\]

We extend the restriction of \( r(x)\rho^{-2}(x) \) to the interval \( \Delta_j \) to \( \mathbb{R} \) as a piecewise continuous function \( \psi_j(x) \) with period 2. Let \( K_{\lambda,j} \) be the closure in \( L_2(\Delta_j) \) of the differential operator \(-z' + (\psi_j(x) + \lambda) z \) on \( C_0^{(1)}(\Delta_j) \). Similarly to (2.10), we obtain that
\[
\left\| \sqrt{\psi_j + \lambda} \right\|_{2,\Delta_j} \leq \left\| \frac{1}{\sqrt{\psi_j + \lambda}} K_{\lambda,j} z \right\|_{2,\Delta_j}, \ z \in C_0^{(1)}(\Delta_j), j \in \mathbb{Z}.
\]

Hence,
\[
\left( \frac{1}{s^2} + \lambda \right) \| z \|_{2,\Delta_j} \leq \| K_{\lambda,j} z \|_{2,\Delta_j}, \ z \in D(K_{\lambda,j}), j \in \mathbb{Z}. \tag{2.14}
\]

So, \( K_{\lambda,j}^{-1} \) exists. On the other hand, by Theorem 2.1, \( K_{\lambda,j}^{-1} \) is defined on the whole \( L_2(\Delta_j) \).

Define \( B_\lambda \) and \( M_\lambda \) as follows:
\[
B_\lambda f = \sum_{j=-\infty}^{+\infty} \varphi_j'(x) K_{\lambda,j}^{-1} \varphi_j f, \quad M_\lambda f = \sum_{j=-\infty}^{+\infty} \varphi_j(x) K_{\lambda,j}^{-1} \varphi_j f, \ f \in L_2(\mathbb{R}).
\]
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Since \( \text{supp} \varphi_j \subset \Delta_{j-1} \cup \Delta_j \cup \Delta_{j+1} \) (\( j \in \mathbb{Z} \)), at each point \( x \in \mathbb{R} \) the sums of the right-hand side of \( B_\lambda \) and \( M_\lambda \) contain no more than two summands, so \( B_\lambda \) and \( M_\lambda \) are well-defined on the whole \( L_2(\mathbb{R}) \). Moreover, it is clear that

\[
K_\lambda M_\lambda = E - B_\lambda. \tag{2.15}
\]

Notice that in \((j,j+1) \) (\( j \in \mathbb{Z} \)) only the functions \( \varphi_j \) and \( \varphi_{j+1} \) are not equal to zero. So, we have that

\[
\|B_\lambda f\|_2^2 = \left\| \sum_{j=-\infty}^{+\infty} \varphi_j(x)K_{\lambda,j}^{-1}\varphi_j f \right\|_2^2 = \int_{-\infty}^{+\infty} \left| \sum_{j=-\infty}^{+\infty} \varphi_j(x)K_{\lambda,j}^{-1}\varphi_j f \right|^2 \, dx
\]

\[
= \sum_{i=-\infty}^{+\infty} \int_{\Delta_i}^i \left( \sum_{j=-\infty}^{+\infty} |\varphi_j(x)| \left| \left[ K_{\lambda,j}^{-1}(\varphi_j f) \right](x) \right| \right)^2 \, dx
\]

\[
\leq 2 \sum_{i=-\infty}^{+\infty} \left( \int_{\Delta_i}^i |\varphi_j|^2 |K_{\lambda,j}^{-1}(\varphi_i f)|^2 \, dx + \int_{\Delta_{i+1}}^i |\varphi_{i+1}|^2 |K_{\lambda,i+1}^{-1}(\varphi_{i+1} f)|^2 \, dx \right)
\]

\[
= 4 \sum_{i=-\infty}^{+\infty} \int_{\Delta_i}^i |\varphi_j(x)|^2 \left| K_{\lambda,j}^{-1}(\varphi_i f)(x) \right|^2 \, dx.
\]

Furthermore

\[
\|B_\lambda f\|_2^2 \leq 4m^2 \sum_{j=-\infty}^{+\infty} \left( \|K_{\lambda,j}^{-1}\|_{L_2(\Delta_j)\rightarrow L_2(\Delta_j)} \|\varphi_j f\|_{L_2(\Delta_j)}^2 \right)
\]

\[
\leq 8m^2 \sup_{j \in \mathbb{Z}} \|K_{\lambda,j}^{-1}\|_{L_2(\Delta_j)\rightarrow L_2(\Delta_j)}^2 \int_{\mathbb{R}} \left( \sum_{j} \varphi_j^2 \right) |f|^2 \, dx
\]

\[
= 8m^2 \sup_{j \in \mathbb{Z}} \|K_{\lambda,j}^{-1}\|_{L_2(\Delta_j)\rightarrow L_2(\Delta_j)}^2 \|f\|_2^2.
\]

By inequality (2.14),

\[
\|K_{\lambda,j}^{-1}\|_{L_2(\Delta_j)\rightarrow L_2(\Delta_j)} \leq \frac{s^2}{1 + s^2\lambda}.
\]

Thus \( \|B_\lambda f\|_2 \leq \frac{2\sqrt{2}m^2}{1 + s^2\lambda} \|f\|_2, \) \( f \in L_2(\mathbb{R}) \). Let \( \lambda_0 = \left(4\sqrt{2} \right)m s^2 - 1 \) \( s^{-2} \). Then

\[
\|B_\lambda\|_{L_2(\mathbb{R})\rightarrow L_2(\mathbb{R})} \leq \frac{1}{2}
\]

holds for any \( \lambda \geq \lambda_0 \). So, \( E + B_\lambda \) (\( \lambda \geq \lambda_0 \)) is invertible. By (2.15), we get

\[
K_\lambda^{-1} = M_\lambda (E - B_\lambda)^{-1}, \lambda \geq \lambda_0. \tag{2.16}
\]
Now, we can prove (2.8). Let \( m_1 = \sup_{|x| \leq 2} \frac{r(x)}{\rho(x)} \). By (2.16) and the properties of \( \varphi_j (j \in \mathbb{Z}) \), we obtain that
\[
\left\| \left( \frac{r}{\rho^2} + \lambda \right) K_{\lambda} f \right\|_2 \leq 4\sqrt{2} (m_1 s^2 + 1) \|f\|_2.
\]
Then, for \( \lambda \geq (4\sqrt{2} ms^2 - 1)s^{-2} \), we have that
\[
\|z\|_2 + \left\| \left( \frac{r}{\rho^2} + \lambda \right) z \right\|_2 \leq (1 + 4\sqrt{2} + 4\sqrt{2} m_1 s^2) \|K_{\lambda} z\|_2.
\] (2.17)

Put \( z = \rho y' \). By (2.17), we get that
\[
\|\rho (\rho y')\|_2 + \|\rho y'\|_2 \leq 8\sqrt{2} ms (1 + 4\sqrt{2} + 4\sqrt{2} m_1 s^2) \|y\|_2, y \in D(\ell),
\] (2.18)
hence \( \ell \) is separable. \( \square \)

3 Separability of the damped differential operator

Denote by \( L \) the closure in \( L^2(\mathbb{R}) \) of the differential expression
\[
\tilde{L} y = -\rho (\rho y')' + \rho y' + qy
\]
on \( C_0^{(2)}(\mathbb{R}) \), where \( \rho \) is a continuously differentiable function, \( r \) and \( q \) are continuous functions.

**Theorem 3.1.** Let \( \rho \) be a bounded continuously differentiable function, \( r \) and \( q \) be continuous functions. Suppose that \( \rho \geq 1 \), \( r \) and \( q \) satisfy conditions (2.2), (2.17) and \( \gamma_{q,r} < \infty \). Then \( L \) is continuously invertible, and \( L^{-1} \) is defined on the whole \( L^2(\mathbb{R}) \). Furthermore, there exists \( c_8 \) such that
\[
\|\rho (\rho y')\|_2 + \|\rho y'\|_2 + \|qy\|_2 \leq c_8 \|L y\|_2,
\] (3.1)
for any \( y \in D(L) \).

**Proof.** We consider the equation
\[
L y = f.
\] (3.2)
A function \( y \in L^2(\mathbb{R}) \) is called a solution to (3.2), if there is a sequence \( \{y_n\}_{n=1}^{\infty} \subset C_0^{(2)}(\mathbb{R}) \) such that \( \|y_n - y\|_2 \to 0 \), \( \|L y_n - f\|_2 \to 0 \) \((n \to +\infty)\). It is clear that \( L \) is continuously invertible if and only if there exists a unique solution \( y \) to (3.2) for each \( f \in L^2(\mathbb{R}) \). Putting \( x = at \) \((a > 0)\), we rewrite (3.2) in the following form:
\[
-\tilde{\rho}(t) (\tilde{\rho}(t) \tilde{y}^{\prime}_t)' + 1/\alpha \tilde{r}(t) \tilde{y}^{\prime}_t + 1/\alpha^2 \tilde{q}(t) \tilde{y} = \tilde{f},
\] (3.3)
where
\[
\tilde{y}(t) = y(at), \tilde{\rho}(t) = \rho(at), \tilde{r}(t) = r(at), \tilde{q}(t) = q(at), \tilde{f}(t) = f(at)/a^2.
\]
Let
\[ \hat{l}_0\tilde{y} = -\dot{\rho}(t)\dot{\hat{y}} + \ddot{\hat{y}}/a \tilde{y}, \]
then from (3.3) we obtain
\[ \|\hat{l}_0\tilde{y} + \dot{\hat{y}}(t)/a^2\tilde{y} = \tilde{f}(t) \].
(3.4)
Note that \( \dot{\hat{y}}/a \) satisfies the conditions of Lemma 2.3, so the operator \( \hat{l}_0 \) is continuously invertible. By (2.18),
\[ \| -\dot{\rho}(t)(\dot{\hat{y}}(t))/\tilde{y} \|_2 + \| \ddot{\hat{y}}/a \tilde{y} \|_2 \leq T \| \hat{l}_0\tilde{y} \|_2, \forall \tilde{y} \in D(\hat{l}_0), \]
(3.5)
where \( T = 8\sqrt{2}m(1 + 4\sqrt{2} + 4\sqrt{2}m_1s^2) \).

It is clear that \( 2\gamma_{0,\hat{a}} = 1/a \gamma_{0,\tilde{a}} \). By Lemma 2.1 and (3.5),
\[ \left\| \frac{1}{a^2}\tilde{q}\tilde{y} \right\|_2 \leq 2\gamma_{0,\hat{a}} \| \ddot{\hat{y}} \|_2 \leq 2\gamma_{0,\hat{a}}a^{-2} \| \ddot{\hat{y}}' \|_2 \leq \frac{2T\gamma_{0,\hat{a}}}{a^2} \| \hat{l}_0\tilde{y} \|_2. \]
Choose \( a = 2\sqrt{T\gamma_{0,\hat{a}}} \), then
\[ \left\| \frac{1}{a^2}\tilde{q}\tilde{y} \right\|_2 \leq \frac{1}{2} \| \hat{l}_0\tilde{y} \|_2. \]
(3.6)
By Theorem 1.16 in Chapter IV of \cite{6}, \( \hat{l}_0 + \frac{1}{a^2}\tilde{q}(t)E \) is invertible and \( R\left( \hat{l}_0 + \frac{1}{a^2}\tilde{q}(t)E \right) = L_2(\mathbb{R}) \). Let \( \tilde{y} \) be a solution to (3.4). Then, by (3.5) and (3.6), we get that
\[ \| -\dot{\rho}(t)(\dot{\hat{y}}(t))' \|_2 + \| \ddot{\hat{y}}/a \tilde{y} \|_2 + \| \ddot{\hat{y}}' \|_2 \leq T \left(1 + \frac{2\gamma_{0,\hat{a}}}{a^2}\right) \| \hat{l}_0\tilde{y} \|_2. \]
(3.7)
On the other hand,
\[ \| \hat{l}_0\tilde{y} \|_2 \leq \left\| \left( \hat{l}_0 + \frac{1}{a^2}\tilde{q}(t)E \right) \tilde{y} \right\|_2 + \left\| \frac{1}{a^2}\tilde{q}\tilde{y} \right\|_2. \]
(3.8)
Using (3.4) and (3.6), we obtain that
\[ \left\| \frac{1}{a^2}\tilde{q}\tilde{y} \right\|_2 \leq \left\| \left( \hat{l}_0 + \frac{1}{a^2}\tilde{q}(t)E \right) \tilde{y} \right\|_2, \]
and
\[ \| \hat{l}_0\tilde{y} \|_2 \leq \left\| \left( \hat{l}_0 + \frac{1}{a^2}\tilde{q}(t)E \right) \tilde{y} \right\|_2 + \left\| \frac{1}{a^2}\tilde{q}\tilde{y} \right\|_2 \leq 2 \left\| \left( \hat{l}_0 + \frac{1}{a^2}\tilde{q}(t)E \right) \tilde{y} \right\|_2. \]
(3.9)
So, (3.7) and (3.9) imply that the inequality
\[ \| -\dot{\rho}(\dot{\hat{y}}(t))' \|_2 + \| \ddot{\hat{y}}/a \tilde{y} \|_2 + \| \ddot{\hat{y}}' \|_2 \leq \left[ T \left(1 + \frac{2\gamma_{0,\hat{a}}}{a^2}\right) \right] \| \tilde{f} \|_2 \]
holds for any solution \( \tilde{y} \) to (3.4). Let \( t = x/a \). Rewriting the above formula, we obtain (3.1).
4 Examples

1. Let $L_0 y = (1 + x^2) \left( (1 + x^2) y' \right)' + (5 + x^4) y'$. Then all conditions of Theorem 2.1 are satisfied. Hence, $L_0$ is invertible, and $L_0^{-1}$ is continuous.

2. We consider

$$ Ly = -y'' + (1 + x^2)^\omega y' + |x|^\sigma y, $$

where $\omega > 0$, $\sigma \geq 0$. If $\omega \geq \sigma/2 + 3/4$, then the conditions of Theorem 3.1 are satisfied. So $L$ has a bounded inverse $L^{-1}$, and there exists $c_9 > 0$ such that

$$ \|y''\|_2 + \left\| (1 + x^2)^\omega y' \right\|_2 + \| |x|^\sigma y\|_2 \leq c_9 \|Ly\|_2 $$

for all $y \in D(L)$.

3. By Theorem 3.1, $\tilde{L} y = -y'' + \exp(1 + x^2)y' + \exp|x|y$ is continuously invertible on $L_2(\mathbb{R})$. Moreover, for all $y \in D(\tilde{L})$,

$$ \|y''\|_2 + \left\| \exp(1 + x^2) y' \right\|_2 + \| \exp|x|y\|_2 \leq c_{10} \left\| \tilde{L}y \right\|_2, $$

where $c_{10}$ is independent of $y$. 
References


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Paper D
Some new results concerning a class of third-order differential equations

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We consider the following third-order differential equation with unbounded coefficients:

\[ -m(x) \left( m(x)y' \right)' + \left[ q(x) + ir(x) + \lambda \right] y = f(x). \]

Some new existence and uniqueness results are proved, and precise estimates of the norms of the solutions are given. The obtained results may be regarded as a unification and extension of all other results of this type.

Keywords: differential equations; existence; uniqueness; estimates of solutions

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1. Introduction and statement of the main results

Linear and nonlinear equations of odd order are sometimes called nonclassical equations of mathematical physics. The study of boundary value problems and qualitative properties of solutions of such equations begun fairly late and is reflected e.g. in the works of Kozhanov et al. [1] and some others. As important representatives of such equations, we mention the Korteweg-de Vries equation and its modifications arising in the theory of distribution of long waves of small finite amplitudes, as well as the composite type equations arising in problems of hydrodynamics.

Questions of smoothness of solutions of differential equations are of great interest due to their importance for applications (e.g. for many problems of gas dynamics, hydrodynamics, hydromechanics, etc.). The case with bounded domains and smooth scalar coefficients are well understood and sufficiently well described in the known literature. In the case with unbounded domains, the problem of separability was started in 1970 by Everitt and Giertz [2,3]. They proved some interesting separation results for the Sturm–Liouville differential operator

\[ L y(x) = -y''(x) + q(x)y(x), \quad x \in \mathbb{R}, \]

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in the space $L_2(R)$. They studied the following question: what are the conditions on $q(x)$ such that if $y(x) \in L_2(R)$, $R = (-\infty, +\infty)$, and $Ly(x) \in L_2(R)$ imply both of $y''(x)$ and $q(x)y(x) \in L_2(R)$. More fundamental results of separation of differential operators were obtained later by the same authors.\cite{4,5} A number of results concerning the property referred to as the separation of differential operators was investigated, e.g. by Boimatov\cite{6}, Otelbaev\cite{7}, Zettle\cite{8}. Some separation criteria and inequalities associated with linear second-order differential operators were studied by Brown et al.\cite{9,10}.

Recently, in\cite{11}, Ospanov and Akhmetkaliyeva have derived sufficient conditions for the invertibility and separability in $L_2(-\infty; +\infty)$ of the degenerate second-order differential operator

$$Ly = -y'' + r(x)y' + s(x)y'$$

and also presented some results concerning separability of not semibounded differential operators (see also\cite{12}).

Moreover, in\cite{13}, Zayed et al. derived some new results on the separation of linear and nonlinear Schrödinger-type operators with operator potentials in Banach spaces. Furthermore, in\cite{14}, the same authors studied the separation of the elliptic differential operator

$$Ly(x) = -\sum_{i,j=1}^{n} \left[ D_i \left( P_{ij}(x)D_j y(x) \right) - P_{ij}(x)b_i(x)b_j(x)y(x) \right] + V(x)y(x),$$

with the operator potential $V(x) \in C^1(R^n, L(H_1))$, in the weighted Hilbert space $L_{2,q}(R^n, H_1)$, where $P_{ij}(x)$ and $b_i(x)$ are real-valued continuous functions (here $D_i := \frac{\partial}{\partial x_i}$).

It is also important to mention that sufficient conditions for the separability of the nonlinear operator

$$Ly = -y'' + q(x, y(x))y + \lambda y, \quad \lambda > 0, \quad x \in R,$$

have been established by Otelbaev and Birgebaev\cite{15}.

In the Hilbert case, the differential operator corresponding to the linear equation of odd order is not semibounded; thus, to estimate the intermediate derivatives of the solution, not to mention the highest derivative, is a nontrivial problem.

We also note that Aliev\cite{16} and Amanova\cite{17} proved the separation of the differential operator $L$ in $L_p(R)\ (1 \leq p < \infty)$, generated by the expression $-y'' + q(x)y$, under some natural assumptions. In particular, they proved that for all sufficiently large $\lambda > 0$, the operator $L + \lambda E$ has a bounded inverse operator in $L_p(R)$.

In\cite{18}, Sapenov and Shuster studied the problem if the solutions of the differential equation

$$-y^{(2n+1)}(t) + q(t)y(t) = f(t) \in L_p(R), \quad t \in R, \quad n \in Z_+,$$

belongs to a weighted Lebesgue space under certain conditions imposed on $q(t)$.

Moreover, very recently in\cite{19}, Ospanov et al. proved some new results concerning the smoothness and approximative properties of generalized solutions of the nonlinear equation

$$(-1)^k y^{2k+1} + [q(x, y) + \lambda + ir(x, y)]y = f, \quad k \in Z_+,$$

on $R$. Here, $\lambda \geq 0$ is a constant, and $q = q(x, y)$ and $r = r(x, y)$ are given functions, which may increase near infinity.
Let $1 < p < +\infty$. By $L_p \equiv L_p(R)$, $R = (-\infty, +\infty)$, we as usual denote the space of functions with finite norm

$$\|\psi\|_p := \left(\int_R |\psi(x)|^p dx\right)^{1/p}.$$ 

Let $\lambda \geq 0$. Assume that $L_\lambda$ is the closure in $L_p$ of the differential expression $-m(x) (m(x)y')'' + [q(x) + ir(x) + \lambda]y$, defined on the set $C^\infty_c(R)$ of infinitely differentiable and compactly supported functions.

We consider the third-order differential equation

$$L_\lambda y := -m(x) (m(x)y')'' + [q(x) + ir(x) + \lambda]y = f(x), \quad (1)$$

where $f \in L_p$, $\lambda \geq 0$, and where $m$, $q$, and $r$ are coefficients to be defined later on.

In this paper, we study questions of the existence and uniqueness of the solutions of (1) and conditions, which for a solution $y$ of (1) the following estimate holds:

$$\|m(x)(m(x)y')''\|_p + \|[q(x) + ir(x) + \lambda]y\|_p \leq c \|f(x)\|_p. \quad (2)$$

We remark that in the applications of well-known projection methods (e.g. Fourier or Laplace transformations) to multi-dimensional differential equations and with coefficients depending on a single variable, we usually obtain ordinary differential equations with complex coefficients. Therefore, the term $ir(x)$ in the Equation (1) is important for the study of coercive solvability of multi-dimensional differential equations with odd-order derivative of unknown function and unlimited coefficients.

We also note that to study Equation (1) with $ir(x)$ is nontrivial, since the differential operator corresponding to this equation loses the property of symmetry and it is not subject to the area of a well-developed theory of self-adjoint operators.

The introduction of the term $m(x)$ is an important difference of the Equation (1) considered by us from the models studied in [15,19–23]. This function may belong to a quite wide class, it has a growth about infinity and the aim of this paper is to get conditions for the unique solvability and coercive estimates, discribed clearly in terms of the functions $m$, $q$, and $r$.

Our results open the possibility to investigate linear partial differential equations of odd order with variable coefficients of the highest derivatives of the unknown function.

The results are obtained for a positive $m$, which can grow indefinitely as a polynomial or as an exponential function of $x$ (see the condition (5)).

We note that with the introduction of the term $m(x)$ of the Equation (1), there are new difficulties, in particular, Equation (1) may degenerate depending on the growth rate and you can not apply to standard theory of integral equations.

In the case of $m(x) = 1$, sufficient conditions for unique solvability of the Equation (1) and the estimate of the form (2) for its solution when $r(x) \equiv 0$ can be found in [15,19,20], and when $r(x) \geq 1$ in [21–23].

In [20–22], the question of the solvability of the following third-order differential equation with real coefficients at the free term

$$y''' + [q(x) + \lambda] = f(x),$$
defined on the whole real axis and the question of separability of the corresponding differential operator \( Ly \equiv y''' + [q(x) + \lambda] \) in the Hilbert space \( L_2(-\infty, +\infty) \) was investigated. These works together with [16,17] laid the foundation for the study of singular equations of odd order. Moreover, in [15], the question on the solvability of the quasilinear equation

\[
y''' + [q(x, y) + ir(x, y) + \lambda] = f(x), \quad x \in R = (-\infty, +\infty),
\]

\((**\))

where

\[
q(x, v) \to +\infty, \quad r(x, v) \to +\infty \quad \text{at} \quad x^2 + v^2 \to +\infty \quad (**)
\]

was investigated. In [23], the unique solvability of the linear equation

\[
y''' + [q(x) + ir(x) + \lambda] = f(x)
\]

and separability of the corresponding operator in the case when \( q \) and \( r \) are not required to satisfy the condition \((*)\) were investigated.

Finally, we mention that in [15,20–23], the Equation (1) of the third order was investigated for the case \( m = 1 \).

Hence, the results in this paper can be regarded as a unification and extension of all of the results in [15,20–23] and [19] (in the case \( k = 1 \)).

There are mainly two methods to study the solvability and smoothness properties of solutions of differential equations defined on an infinite domain. The first of them is the variational method based on uniform a priori estimates of the solutions. It is mainly used when the differential operator corresponding to the equation is semi-bounded. Such equations are of even-order. The second method developed in [24] is usually called the method of Titchmarsh. It is an effective construction of such an integral operator, which is close to the resolvent of differential equations and it can replace this resolution for large values of the present parameter \( \lambda \) (see [15,20–23]). The Titchmarsh method is used by us as well as by the authors of the papers [15,19–23]. One novelty of this work, in comparison of the papers [15,19–23], is the method for selecting the kernel of the indicated integral operator and to establish the necessary properties of it, which allows us to use the Titchmarsh method with appropriate modifications associated with the introduction of new functions \( m \) and \( r \).

**Definition 1** A function \( y(x) \in L_p(R) \) is called a solution of (1) if there is a sequence \( \{y_n\}_{n=1}^\infty \) of three times continuously differentiable functions with compact support, such that \( \|y_n - y\|_p \to 0 \) and \( \|L\lambda y_n - f\|_p \to 0 \) as \( n \to \infty \).

By \( C^{(k)}(R) \) \((k = 1, 2, \ldots)\), we denote the set of all \( k \) times continuously differentiable functions \( \varphi(x) \) for which the value \( \sum_{j=0}^k \sup_{x \in R} |\varphi^{(j)}(x)| \) is finite. Let \( W_2(x) := \frac{|g(x) + \lambda + ir(x)|}{m^2(x)} \).

Our main results in this paper read:
Theorem 1 Assume that the functions $q = q(x)$ and $r = r(x)$ are continuous on $R$, $m = m(x) \in C^{(2)}_{loc}(R)$ and that the following conditions hold:

$$
m(x) \geq 1, \quad \frac{q(x)}{m^4(x)} \geq 1, \quad r(x) \geq 1, \quad (3)$$

$$\frac{c^{-1}}{m(x)} \leq \frac{q(x)}{m(\eta)} \cdot \frac{r(x)}{r(\eta)}, \quad x, \eta \in R, \quad |x - \eta| \leq 1, \quad \text{for some } c > 0 \quad (4)$$

$$|m^{(j)}(x)| \leq c_j m(x), \quad x \in R, \quad \text{for some } c_j > 0, \quad j = 1, 2. \quad (5)$$

$$\sup_{|x - \eta| \leq 1} \frac{|W_\lambda(x) - W_\lambda(\eta)|}{|W_\lambda(x)|^\nu |x - \eta|^\mu} < +\infty, \quad 0 < \nu < \frac{\mu}{3} + 1, \quad \mu \in (0, 1], \quad \lambda \geq 0. \quad (6)$$

Then there exists a number $\lambda_0 \geq 0$, such that the Equation (1) has a solution $y$ for all $\lambda \geq \lambda_0$.

Theorem 2 Let the functions $q = q(x)$ and $r = r(x)$ be continuous on $R$, $m = m(x) \in C^{(3)}_{loc}(R)$ and satisfy the conditions (3)–(6) and

$$|m^{(3)}(x)| \leq c_3 m(x), \quad x \in R. \quad (7)$$

Then the solution of the Equation (1) is unique and the estimate (2) holds.

We will now give some nontrivial examples where the conditions of Theorems 1 and 2 hold.

Example 1 Let in the Equation (1)

$$m \equiv m_1 = \sqrt{5 + x^2}, \quad q \equiv q_1 = 5 + 3x^4, \quad r \equiv r_1 = 10 + 6x^4.$$  

First, we note that the conditions (3), (5), and (7) are easily verified. Let us now verify condition (4). In fact,

$$\sup_{|x - \eta| \leq 1} \frac{m(x)}{m(\eta)} = \sup_{|x - \eta| \leq 1} \sqrt{\frac{5 + x^2}{5 + \eta^2}} \leq \sup_{x \in R} \sqrt{\frac{5 + (\eta + 1)^2}{5 + \eta^2}} \leq \sqrt{2},$$

Similarly, we can show that

$$\sup_{|x - \eta| \leq 1} \frac{q(x)}{q(\eta)} < \infty, \quad \sup_{|x - \eta| \leq 1} \frac{r(x)}{r(\eta)} < \infty.$$

Furthermore,

$$W_\lambda(x) = \sqrt{5} \frac{5 + 3x^4}{5 + x^2},$$

so that

$$|W_\lambda(x) - W_\lambda(\eta)| \leq \sqrt{5} \frac{(5 + x^2)|x^4 - \eta^4| + (5 + 3x^2)|x^2 - \eta^2|}{(5 + x^2)(5 + \eta^2)}.$$

Taking into account the inequality $|x - \eta| \leq 1$, we have that

$$\sup_{|x - \eta| \leq 1} \frac{|W_\lambda(x) - W_\lambda(\eta)|}{|W_\lambda(x)||x - \eta|} < \infty.$$
which means that also (6) is satisfied. Therefore, by Theorems 1 and 2, we find that the equation
\[- \sqrt{5 + x^2} \left( \sqrt{5 + x^2} y' \right)'' + [5 + 3x^4 + i(10 + 6x^4) + \lambda]y = f, \quad x \in \mathbb{R},\]
has a solution \( \hat{y} \) for any function \( f \) from \( L_p(R) \) (\( 1 < p < +\infty \)), and the following inequality holds:
\[ \left\| \sqrt{5 + x^2} \left( \sqrt{5 + x^2} \hat{y}' \right)'' \right\|_p + \left\| [5 + 3x^4 + i(10 + 6x^4) + \lambda] \hat{y} \right\|_p \leq c \| f \|_p. \]

**Example 2** Let in the Equation (1)
\[ m = m_2 = \sqrt{1 + x^2}, \quad q = r = q_2 = 20 + \exp(3x) + (1 + x^2)^2. \]
We will check the conditions of Theorems 1 and 2. Conditions (3), (5), and (7) are easily verified. Furthermore,
\[ \sup_{|x-\eta| \leq 1} \frac{q_2(x)}{q_2(\eta)} \leq \sup_{|x-\eta| \leq 1} \frac{\exp(3x)}{\exp(3\eta)} + \sup_{|x-\eta| \leq 1} \frac{(1 + x^2)^2}{(1 + \eta^2)^2} \leq 20 + \exp 3 + 4. \]
The other inequalities in (4) can be checked in the same way as in Example 1. In addition, it is easy to check that \( q_2(x) \leq cq_2(\eta) \), which implies the validity of the inequality (6). Thus, all the conditions of Theorems 1 and 2 are fulfilled. Hence, we conclude that the equation
\[- \sqrt{1 + x^2} \left( \sqrt{1 + x^2} y' \right)'' + [20 + \exp(3x) + (1 + x^2)^2 + \lambda + i(20 + \exp(3x) + (1 + x^2)^2)]y = g, \quad x \in \mathbb{R},\]
has a solution \( \hat{y} \) for any function \( g \) from \( L_p(R) \) (\( 1 < p < +\infty \)), the following inequality holds:
\[ \left\| \sqrt{1 + x^2} \left( \sqrt{1 + x^2} \hat{y}' \right)'' \right\|_p + \left\| [20 + \exp(3x) + (1 + x^2)^2 + \lambda + i(20 + \exp(3x) + (1 + x^2)^2)] \hat{y} \right\|_p \leq c \| g \|_p. \]

In order to make the proofs of Theorems 1 and 2 more clear, some auxiliary Lemmas and other preliminaries are given in Section 2. Some identities and inequalities in these Lemmas are of independent interest. The detailed proofs can be found in Section 3.

### 2. Preliminaries

Let \( \xi_s = \xi_s(x), \ s = 1, 2, 3, \) be the roots of equation \( m^2(x)\xi^3 - r(x) + i(q(x) + \lambda) = 0, \) where \( m(x), r(x), \) and \( q(x) \) satisfy the properties of Theorem 1. From the conditions of Theorem 1, it follows that \( 0 < \arg \xi_1 < \pi, \ \pi < \arg \xi_j < 2\pi, \ j = 2, 3. \) We introduce the function
\[ M_0(x, \eta, \lambda) = \begin{cases} \frac{1}{3m^2(x)} \frac{\xi^2_1}{\xi_s}, & -\infty < \eta < x, \\ \frac{1}{3m^2(x)} \sum_{j=2}^{3} \frac{\xi^2_j}{\xi_s}, & x < \eta < +\infty. \end{cases} \]
By a direct computation, we get the following equalities:

\[
\begin{align*}
\frac{\partial^j M_0(x, \eta, \lambda)}{\partial \eta^j} \bigg|_{x=\eta-0} &= \frac{\partial^j M_0(x, \eta, \lambda)}{\partial \eta^j} \bigg|_{x=\eta+0}, \quad j = 0, 1, \\
\frac{\partial^2 M_0(x, \eta, \lambda)}{\partial \eta^2} \bigg|_{x=\eta-0} - \frac{\partial^2 M_0(x, \eta, \lambda)}{\partial \eta^2} \bigg|_{x=\eta+0} &= -\frac{1}{m^2(x)},
\end{align*}
\]

and

\[
-m(x) \left( m(x) \frac{\partial M_0(x, \eta, \lambda)}{\partial \eta} \right)^{\prime \prime} \eta \eta + [q(x) + ir(x) + \lambda] M_0(x, \eta, \lambda) = 0.
\]

Let the function \( d(\eta) \in C_0^\infty(-1, 1) \) be such that

\[
d(\eta) = \begin{cases} 1, & |\eta| \leq \frac{1}{2}, \\ 0, & |\eta| \geq 1. \end{cases}
\]

We denote

\[
M_1(x, \eta, \lambda) = \left[ (q(\eta) + ir(\eta)) - \frac{m^2(\eta)}{m^2(x)} (q(x) + ir(x)) \right] M_0(x, \eta, \lambda) d(\eta - x),
\]

\[
M_2(x, \eta, \lambda) = -\left[ 2m'(\eta)m(\eta)d(\eta - x) + 3m^2(\eta)d'(\eta - x) \right] \frac{\partial^2 M_0(x, \eta, \lambda)}{\partial \eta^2} \bigg|_{x=\eta-0} \times \frac{\partial M_0(x, \eta, \lambda)}{\partial \eta}
\]

\[
-\left[ m''(\eta)m(\eta)d'(\eta - x) + 4m'(\eta)m(\eta)d''(\eta - x) + 3m^2(\eta)d'''(\eta - x) \right]
\]

\[
\times M_0(x, \eta, \lambda),
\]

and

\[
M_3(x, \eta, \lambda) = M_0(x, \eta, \lambda) d(\eta - x).
\]

We also introduce the following integral operators:

\[
( M_j(\lambda)f)(\eta) = \int_R M_j(x, \eta, \lambda)f(x)dx \quad (j = 1, 2, 3).
\]

The following statement is known ([25, p.902]):

**Lemma 1** Let \( 1 < p < +\infty, \ k(x, \eta) \) be a continuous function and

\[
(Kv)(\eta) := \int_R k(x, \eta)v(x)dx.
\]

Then,

\[
\|K\|_{L_p \rightarrow L_p} \leq \sup_{\eta \in R} \left[ |k(x, \eta)| + |k(\eta, x)| \right] dx.
\]

Our next auxiliary result reads:
Lemma 2 Let all the conditions of Theorem 1 be satisfied. Then the operators $M_j(\lambda)$, $j = 1, 2, 3$, are continuous in the space $L_p$, and the following estimates hold ($\lambda \geq 0$):

$$
\|M_1(\lambda)\|_{L_p \rightarrow L_p} \leq \frac{c_1}{b^{\mu+3-3\nu}_x(\eta)}, \quad \mu \in (0, 1], \quad 0 < \nu < \frac{\mu}{3} + 1,
$$

(12)

$$
\|M_2(\lambda)\|_{L_p \rightarrow L_p} \leq \frac{c_2}{b^{2}_x(\eta)},
$$

(13)

and

$$
\|M_3(\lambda)\|_{L_p \rightarrow L_p} \leq \frac{c_3}{m^2(\eta)b^3_x(\eta)}.
$$

(14)

Here $b^x_\lambda(x) = \sqrt[3]{\frac{|r(x) - i(q(x) + \lambda)|}{m^2(x)}}$.

Remark 1 The statement in Lemma 2 remains true if the condition $r(x) \geq 1$ in (3) is replaced by the condition $r(x) \leq -1$, but we do not need this fact in our further investigations.

Also the following identity of independent interest is crucial for the proof of Theorem 1:

Lemma 3 Let the conditions of Theorem 1 be satisfied. Then the following equality holds:

$$
L_\lambda [M_3(\lambda)f](\eta) = f(\eta) + [M_1(\lambda)f](\eta) + [M_2(\lambda)f](\eta).
$$

(15)

Let the functions $m(x)$, $q(x)$, $r(x)$ satisfy the conditions of Theorem 2, and let $p'$ denote the conjugate number of $p$, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. We denote by $(L_\lambda)'$ an operator acting in the space $L_{p'}(R)$ such that

$$(L_\lambda y, z) = (y, (L_\lambda)'z), \quad y \in D(L_\lambda), \quad z \in D((L_\lambda)') .
$$

Obviously, it yields that

$$(L_\lambda)'z \equiv (m(x)(m(x)z)' + q(x) + \lambda - ir(x))z .
$$

We consider the differential equation

$$(L_\lambda)'z \equiv (m(x)(m(x)z)' + q(x) + \lambda - ir(x))z = g(x),
$$

(16)

where the function $m(x) \geq 1$ is continuous together with its derivatives up to third order, $q(x)$ and $r(x)$ are continuous real-valued functions, $\lambda \geq 0$ and $g(x) \in L_{p'}(R)$.

The next Lemma is crucial for the proof of Theorem 2.

Lemma 4 Let the continuous functions $q(x)$, $r(x)$ and the function $m(x) \in C^{(3)}_{loc}(R)$ satisfy the conditions (3)–(7). Then there exists a number $\lambda_1 \geq 0$, such that the Equation (16), for all $\lambda \geq \lambda_1$, has a solution.

Before we can prove Lemma 4, we introduce some notations and give some further auxiliary statements.
Let, $\zeta_j = \zeta_j(x), j = 1, 2, 3$, be the roots of the equation $m^2(\lambda)\zeta^3_j - r(x) - ir(x) + \lambda = 0$. From the assumptions of Lemma 4, it follows that $0 < arg\zeta_j < \pi, j = 1, 2,$ and $\pi < arg\zeta_3 < 2\pi$. We introduce the function

$$N_0(x, \eta, \lambda) = \begin{cases} -\frac{1}{3m^2(x)} \sum_{j=1}^{2} \frac{e^{i(x-\eta)\zeta_j}}{\zeta_j^2}, & -\infty < \eta < x, \\ \frac{1}{3m^2(x)} \zeta_3^2, & x < \eta < +\infty. \end{cases} \quad (17)$$

By direct computations, we easily obtain the following:

$$\frac{\partial^j N_0(x, \eta, \lambda)}{\partial \eta^j} \bigg|_{x=\eta=0} = \frac{\partial^j N_0(x, \eta, \lambda)}{\partial \eta^j} \bigg|_{x=\eta=+0}, \quad j = 0, 1, \quad (18)$$

$$\frac{\partial^2 N_0(x, \eta, \lambda)}{\partial \eta^2} \bigg|_{x=\eta=0} = -\frac{\partial^2 N_0(x, \eta, \lambda)}{\partial \eta^2} \bigg|_{x=\eta=+0} = -\frac{1}{m^2(x)}, \quad (19)$$

$$\left( m(x) (m(x)N_0(x, \eta, \lambda))''_{\eta} \right)' + [q(x) - ir(x) + \lambda]N_0(x, \eta, \lambda) = 0. \quad (20)$$

Moreover, we denote

$$N_1(x, \eta, \lambda) = \left[ (q(\eta) - ir(\eta)) - \frac{m^2(\eta)}{m^2(x)} (q(x) - ir(x)) \right] N_0(x, \eta, \lambda) d(\eta - x),$$

$$N_2(x, \eta, \lambda) = \left[ 4m'(\eta) m(\eta) d(\eta - x) + 3m^2(\eta) d'(\eta - x) \right] \frac{\partial^2 N_0(x, \eta, \lambda)}{\partial \eta^2} + \left[ 3m''(\eta) m(\eta) d(\eta - x) + 2(m'(\eta))^2 d(\eta - x) + 8m(\eta) m'(\eta) d'(\eta - x) + 3m^2(\eta) d_{\eta}(\eta - x) \right] \frac{\partial N_0(x, \eta, \lambda)}{\partial \eta} + \left[ m''(\eta) m'(\eta) d(\eta - x) + m'''(\eta) m(\eta) d(\eta - x) + 3m''(\eta) m(\eta) d'_0(\eta - x) + 2(m'(\eta))^2 d'_{\eta}(\eta - x) + 4m'(\eta) m(\eta) d''_{\eta}(\eta - x) + m^2(\eta) d'''_{\eta}(\eta - x) \right] \right] \times N_0(x, \eta, \lambda),$$

and

$$N_3(x, \eta, \lambda) = N_0(x, \eta, \lambda) d(\eta - x).$$

We also introduce the following integral operators

$$\left( N_j(\lambda f) (\eta) \right) := \int_R N_j(x, \eta, \lambda) f(x) dx, \quad j = 1, 2, 3.$$

**Lemma 5** Let all the conditions of Lemma 4 be satisfied. Then the operators $N_j(\lambda)$ are continuous in the space $L_{p'}$, and the following estimates hold ($\lambda \geq 0$):

$$\|N_1(\lambda)\|_{L_{p'} \rightarrow L_{p'}} \leq \frac{c_1}{b_1(\lambda)}, \quad \beta \in (0, 1], \quad 0 < \alpha < \frac{\beta}{2} + 1, \quad (21)$$

$$\|N_2(\lambda)\|_{L_{p'} \rightarrow L_{p'}} \leq \frac{c_2}{b_2(\lambda)}, \quad (22)$$
According to our choice $M_j$ for the function $f$ we obtain the following estimates: (3)–(6) of Theorem 1 and (25)–(26) for the functions $M_j$. Proof of Lemma 2 Under the assumptions of Theorem 1 for the functions $M_j$, there exists a constant $\sigma > 0$ such that $Im \xi_j \geq \sigma$ and $Im \xi_j \leq -\sigma$ ($j = 2, 3$). Then, for the function $M_0$ defined by (8), we have the following estimates:

$$\|M_0(x, \eta, \lambda)\| \leq \left\{ \begin{array}{ll} \frac{1}{3m^2(x)} \frac{1}{b_2^2(x)} e^{-\sigma(x-\eta)b_2(x)}, & -\infty < \eta < x, \\
\frac{1}{2} \frac{3m^2(x)}{b_2^2(x)} e^{\sigma(x-\eta)b_2(x)}, & x < \eta < +\infty, \end{array} \right. \quad (25)$$

and

$$\left| \frac{\partial^j M_0(x, \eta, \lambda)}{\partial \eta^j} \right| \leq \left\{ \begin{array}{ll} \frac{1}{3m^2(x)} \frac{1}{b_2^{-j}(x)} e^{-\sigma(x-\eta)b_2(x)}, & -\infty < \eta < x, \\
\frac{1}{2} \frac{3m^2(x)}{b_2^{-j}(x)} e^{\sigma(x-\eta)b_2(x)}, & x < \eta < +\infty, \end{array} \right. \quad (26)$$

According to our choice $M_j(x, \eta, \lambda) = 0$ for $|x - \eta| > 1$. Taking into account the conditions (3)–(6) of Theorem 1 and (25)–(26) for the functions $M_j(x, \eta, \lambda)$, $j = 0, 1, 2$, at $|x - \eta| \leq 1$, we obtain the following estimates:

$$|M_1(x, \eta, \lambda)| \leq \left\{ \begin{array}{ll} \tilde{c} \frac{m^2(\eta)}{|x - \eta|^3} b_2^{3/2}(x) e^{-\sigma(x-\eta)b_2(x)} \frac{m^2(x)}{b_2^2(x)}, & -\infty < \eta < x, \\
\tilde{c} \left( \frac{m^4(\eta)}{|x - \eta|^3} b_2^{3/2}(x) e^{-\sigma(x-\eta)b_2(x)} \frac{m^2(x)}{b_2^2(x)}, & x < \eta < +\infty. \right. \end{array} \right. \quad (27)$$

$$|M_2(x, \eta, \lambda)| \leq \left\{ \begin{array}{ll} \frac{m^2(\eta)}{m^2(x)} \sum_{k=0}^{\tilde{c}} \frac{e^{-\sigma(x-\eta)b_2(x)}}{b_2^2(x)}, & -\infty < \eta < x, \\
\frac{m^2(\eta)}{m^2(x)} \sum_{k=0}^{\tilde{c}} \frac{e^{\sigma(x-\eta)b_2(x)}}{b_2^2(x)}, & x < \eta < +\infty. \end{array} \right. \quad (28)$$

and

$$|M_3(x, \eta, \lambda)| \leq \left\{ \begin{array}{ll} \frac{1}{3m^2(x)} \frac{b_2^2(x)}{b_2^2(x)} e^{-\sigma(x-\eta)b_2(x)}, & -\infty < \eta < x, \\
\frac{1}{2} \frac{3m^2(x)}{b_2^2(x)} e^{\sigma(x-\eta)b_2(x)}, & x < \eta < +\infty. \end{array} \right. \quad (29)$$
We shall estimate the norms $\| M_j(\lambda) \|_{L^p \to L^p}$ of the operators $M_j(\lambda), \ j = 1, 2, 3$, by using Lemma 1 and the inequalities (27)–(27). We have that

$$\| M_1(\lambda) \|_{L^p \to L^p} \leq \sup_{\eta \in \mathbb{R}} \int_R \left( |M_1(x, \eta, \lambda)| + |M_1(\eta, x, \lambda)| \right) dx$$

$$\leq c_1 \sup_{\eta \in \mathbb{R}} \int_R \left( \frac{m^2(\eta) \exp[-\sigma(x - \eta)b_2(x)]}{b_2^2(x)} + \frac{m^2(\eta) \exp[-\sigma(x - \eta)b_2(\eta)]}{b_2^2(\eta)} \right)$$

$$\times \left| \frac{q(\eta) + \lambda + ir(\eta)}{m^2(\eta)} - \frac{q(x) + \lambda + ir(x)}{m^2(x)} \right| dx.$$ 

By using (3), (4), (6), we obtain that

$$\| M_1(\lambda) \|_{L^p \to L^p} \leq c_2 \sup_{\eta \in \mathbb{R}} \int_R \left( \exp[-\sigma(x - \eta)cb_1(\eta)] + \frac{\exp[-\sigma(x - \eta)b_2(\eta)]}{b_2^2(\eta)} \right)$$

$$\times \left| \frac{q(\eta) + \lambda + ir(\eta)}{m^2(\eta)} \right|^{\nu} \left| \eta - x \right|^{\mu} dx.$$ 

Hence, by making the change of variable $\eta - x = \frac{1}{\sigma b_1(\eta)} z$, we find that

$$\| M_1(\lambda) \|_{L^p \to L^p} \leq c_3 \frac{|q(\eta) + \lambda + ir(\eta)|^{\nu}}{(cb_1(\eta))^{\mu+3}} + c_4 \frac{|q(\eta) + \lambda + ir(\eta)|^{\nu}}{(b_2(\eta))^{\mu+3}} = c_5 \frac{|q(\eta) + \lambda + ir(\eta)|}{m^2(\eta)} \eta \geq 1 + \lambda.$$ 

Moreover, according to the condition (3), we have that

$$\frac{|q(\eta) + \lambda + ir(\eta)|}{m^2(\eta)} \geq \sqrt{1 + \lambda}.$$ 

Therefore, from the previous inequality we obtain (12). Furthermore, in view of the conditions (3)–(5) of Theorem 1, we can deduce that

$$\| M_2(\lambda) \|_{L^p \to L^p} \leq \sup_{\eta \in \mathbb{R}} \int_R \left( |M_2(x, \eta, \lambda)| + |M_2(\eta, x, \lambda)| \right) dx$$
\[
\leq c_2 \sup_{\eta \in \mathbb{R}} \int_{\eta-1}^{\eta+1} \left( \frac{m^2(\eta)}{m^2(x)} e^{-\sigma(x-\eta)b_2(x)} + \frac{m^2(\eta)}{m^2(x)} e^{-\sigma(x-\eta)b_2(x)} \right) dx
\]

Hence, by calculating the integrals, we obtain that

\[
\|M_2(\lambda)\|_{L_p \to L_p} \leq \tilde{c}_3 \sup_{\eta \in \mathbb{R}} \left( \frac{1}{\sigma b_3(\eta)} \left( 1 - e^{-\sigma b_3(\eta)} \right) \right.
\]

Finally, we shall prove the inequality (14). According to the estimate (29), it yields that

\[
\|M_3(\lambda)\|_{L_p \to L_p} \leq \tilde{c}_3 \sup_{\eta \in \mathbb{R}} \left( \frac{1}{\sigma b_3(\eta)} \left( 1 - e^{-\sigma b_3(\eta)} \right) \right.
\]

Thus, by taking the condition (4) into account, we find that

\[
\|M_3(\lambda)\|_{L_p \to L_p} \leq \tilde{c}_3 \sup_{\eta \in \mathbb{R}} \left( \frac{2}{3m^2(\eta)} \frac{\exp[-\sigma(x-\eta)b_2(x)]}{b_3^2(\eta)} + \frac{2}{3m^2(\eta)} \frac{\exp[-\sigma(x-\eta)b_2(\eta)]}{b_3^2(\eta)} \right) dx
\]

The proof is complete.
Proof of Lemma 3  Obviously, it yields that

\[ L_\lambda [M_3(\lambda) f](\eta) \]

\[ = -m(\eta) \left( m(\eta) \left( \int_{-\infty}^{\eta} M_0(x, \eta, \lambda) d(\eta - x) f(x) dx + \int_{\eta}^{+\infty} M_0(x, \eta, \lambda) d(\eta - x) f(x) dx \right) \right) ' ' \eta \eta \]

\[ + (q(\eta) + i r(\eta) + \lambda) \left( \int_{-\infty}^{\eta} M_0(x, \eta, \lambda) d(\eta - x) f(x) dx + \int_{\eta}^{+\infty} M_0(x, \eta, \lambda) d(\eta - x) f(x) dx \right)(\eta). \]

Moreover,

\[ \frac{d}{d\eta} (M_3(\lambda) f)(\eta) \]

\[ = \left[ \int_{-\infty}^{\eta} M_0(x, \eta, \lambda) d(\eta - x) f(x) dx + \int_{\eta}^{+\infty} M_0(x, \eta, \lambda) d(\eta - x) f(x) dx \right]' \eta \]

\[ = M_0(x, \eta, \lambda) d(\eta - x) f(x)|_{x=\eta=0} - M_0(x, \eta, \lambda) d(\eta - x) f(x)|_{x=\eta=0} \]

\[ + \int_{-\infty}^{\eta} M_0'(x, \eta, \lambda) d(\eta - x) f(x) dx + \int_{\eta}^{+\infty} M_0(x, \eta, \lambda) d'(\eta - x) f(x) dx \]

\[ + \int_{-\infty}^{\eta} M_0'(x, \eta, \lambda) d(\eta - x) f(x) dx + \int_{\eta}^{+\infty} M_0(x, \eta, \lambda) d'(\eta - x) f(x) dx. \]

According to (9), we have that

\[ M_0(x, \eta, \lambda) d(\eta - x) f(x)|_{x=\eta=0} - M_0(x, \eta, \lambda) d(\eta - x) f(x)|_{x=\eta=0} = 0. \]

By using this fact we find that

\[ \frac{d}{d\eta} \left( m(\eta) M_3'(\lambda) f \right)(\eta) = m'(\eta) \int_{R} M_0'(x, \eta, \lambda) d(\eta - x) f(x) dx \]

\[ + m'(\eta) \int_{R} M_0(x, \eta, \lambda) d'(\eta - x) f(x) dx + m(\eta) \int_{R} M_0''(x, \eta, \lambda) d(\eta - x) f(x) dx \]

\[ + 2m(\eta) \int_{R} M_0'(x, \eta, \lambda) d'(\eta - x) f(x) dx + m(\eta) \int_{R} M_0(x, \eta, \lambda) d''(\eta - x) f(x) dx. \]
Theorem 1. Hence, we leave out the details. Lemma 4 is a simple consequence of Lemmas 3 and 2.

Proof of Theorem 1

By using Lemma 4, we conclude that the operator \( L(\lambda) \) has a bounded inverse \( G^{-1}(\lambda) \) in \( L_p \). Hence, by letting \( h = [E + M_1(\lambda) + M_2(\lambda)] f \), in view of the relation (15) in Lemma 3, we obtain that \( L_\lambda[M_3(\lambda)G^{-1}(\lambda)h](\eta) = h \). Thus, for all \( \lambda \), \( \lambda \geq \lambda_0 \), it yields that the Equation (1) for any right-hand side \( f \) has the solution. The proof is complete.

We are now ready to present the

Proof of Theorem 1

By using the estimates (12) and (13) in Lemma 2, we conclude that there exists a number \( \lambda_0 > 0 \) such that the inequality \( \| M_1(\lambda) \|_{L_p} + \| M_2(\lambda) \|_{L_p} \leq \frac{1}{2} \) holds if \( \lambda \geq \lambda_0 \). Therefore, the operator \( G(\lambda) := E + M_1(\lambda) + M_2(\lambda) \) has a bounded inverse \( G^{-1}(\lambda) \) in \( L_p \). Hence, by letting \( h = [E + M_1(\lambda) + M_2(\lambda)] f \), in view of the relation (15) in Lemma 3, we obtain that \( L_\lambda[M_3(\lambda)G^{-1}(\lambda)h](\eta) = h \). Thus, for all \( \lambda \), \( \lambda \geq \lambda_0 \), it yields that the Equation (1) for any right-hand side \( f \) has the solution. The proof is complete.

The proofs of Lemmas 5, 6 can be carried out similarly as the proofs of Lemmas 2, 3 and Theorem 1. Hence, we leave out the details. Lemma 4 is a simple consequence of Lemmas 5 and 6.

We are now ready to prove Theorem 2.

Proof of Theorem 2

By using Lemma 4, we conclude that the operator \( (L_\lambda)' \), acting in the space \( L_{p'}(R) \) at \( \lambda \geq \lambda _{1} \), has a right inverse, which is defined on \( L_{p'}(R) \). Thus, \( \ker ((L_\lambda)')^* = [0] \), where \( (L_\lambda)'^* \) is the adjoint operator of \( (L_\lambda)' \). From here, since \( ((L_\lambda)')^* \) is an extension of the operator \( L_\lambda \), we have that \( \ker L_\lambda = [0] \), \( \lambda \geq \tilde{\lambda} = \max(\lambda_0, \lambda_1) \). Thus, the operator \( L_\lambda \) is a boundedly invertible operator in the space \( L_{p'}(R) \) and, in fact, we have that

\[
(L_\lambda)^{-1} = M_3(\lambda)G^{-1}(\lambda), \quad \lambda \geq \tilde{\lambda} = \max(\lambda_0, \lambda_1).
\]

(30)
Let $y$ be a solution of the Equation (1), where $\lambda \geq \tilde{\lambda} = \max(\lambda_0, \lambda_1)$. We shall prove the estimate (2). By using (30), Lemma 1 and the conditions (3)–(6) we have that
\[
\left\| (q + \lambda + ir)(L_\lambda)^{-1} \right\|_{L_p \rightarrow L_p} = \left\| (q + \lambda + ir)M_3(\lambda)G^{-1}(\lambda) \right\|_{L_p \rightarrow L_p}
\leq c \sup_{\eta \in \mathbb{R}} \int_{\eta - 1}^{\eta + 1} b_3(\eta) b_\lambda^{-2}(x) \exp[-\sigma |x - \eta| b_\lambda(x)] dx
\leq c_1 \sup_{\eta \in \mathbb{R}} b_\lambda(\eta) \int_{\eta - 1}^{\eta + 1} \exp[-\sigma |x - \eta| b_\lambda(x)] dx < \infty.
\]
From this and (1), we conclude that
\[
\left\| m(x) \left( m(x)y' \right)' \right\|_p \leq c \left( \| f \|_p + \| y \|_p \right).\]
Finally, by combining the last two estimates we obtain (2). The proof is complete. □

References

Paper E
On Solvability of Third-Order Singular Differential Equation

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Abstract. In this paper some new existence and uniqueness results are proved and maximal regularity estimates of the solutions of third-order differential equation with unbounded coefficients are given.

Keywords: Differential equation · Differential operator · Non-semibounded operator · Finite function · Separability · Coercive estimate · Closure · Bounded invertibility · Inverse operator · Unbounded domain · Adjoint operator

1. Introduction and Main Results

In this paper we consider questions of the existence and uniqueness of solutions of equation

\[(L + \lambda E)y \equiv -p_1(x)(p_2(x)y')' + (q(x) + ir(x) + \lambda)y = f(x), \tag{1}\]

where \(\lambda \geq 0\) is a constant, and \(q\) and \(r\) are given functions, \(f \in L_p\). We study conditions, which providing for solution \(y\) of (1) the following estimate:

\[
\left\|p_1(x)(p_2(x)y')'\right\|_p + \|q(x) + ir(x) + \lambda\|_p \leq c \|f(x)\|_p. \tag{2}\]

The separation of differential expressions was early studied by W.N. Everitt and M. Giertz [4], and they proved some fundamental results. Later on a number of results concerning the property referred to as separation of differential expressions have been obtained e.g. by K.Kh. Boimatov [3], M. Otelbaev [9], A. Zettl [10] and A.S. Mohamed [5]. Some very recent results in this direction was presented and proved in [6] and [1]. In this paper we give the solvability results for (1) with unbounded coefficients \(p_1\) and \(p_2\). With respect to other operators the separation results have been obtained in [7, 8, 2].

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Let \( 1 < p < +\infty \). By \( L_p \equiv L_p(\mathbb{R}), \mathbb{R} = (-\infty, +\infty) \), we denote the space of functions with finite norm
\[
\| \varphi \|_p = \left( \int_{\mathbb{R}} |\varphi(x)|^p dx \right)^{\frac{1}{p}}.
\]

**Definition 1.** A function \( y(x) \in L_p(\mathbb{R}) \) is called a solution of (1), if there is a sequence of three times continuously differentiable functions with compact support \( \{y_n\}_{n=1}^\infty \) such that \( \|y_n - y\|_p \to 0 \) and \( \|(L + \lambda E)y_n - f\|_p \to 0 \) as \( n \to \infty \).

By \( C^{(k)}_p(\mathbb{R}) \) \( (k = 1, 2, \ldots) \) we denote the set of all \( k \) times continuously differentiable functions \( \varphi(x) \) that \( \sum_{j=0}^k \sup_{x \in \mathbb{R}} |\varphi^{(j)}(x)| \) is finite. Let \( W_2(x) = \frac{\|q(x) + \lambda + ir(x)\|}{p_1(x)p_2(x)} \).

Our main results in this paper are following Theorems 1 and 2.

**Theorem 1.** Assume that the functions \( p_1(x), q(x) \) and \( r(x) \) are continuous, \( p_2 \in C^{(1)}_{loc}(\mathbb{R}) \) and satisfy the conditions
\[
p_1(x) \geq 1, \quad p_2(x) \geq 1, \quad \frac{q(x)}{p_1(x)p_2(x)} \geq 1, \quad r(x) \geq 1, \quad (3)
\]
\[
c_0^{-1} \leq \frac{p_j(x)}{p_j(\eta)} \frac{q(x)}{q(\eta)} \frac{r(x)}{r(\eta)} \leq c_0, \quad j = 1, 2, \quad x, \eta \in \mathbb{R}, \quad |x - \eta| \leq 1, \quad (4)
\]
\[
|p_2'(x)| \leq c_1 p_2(x), \quad x \in \mathbb{R}, \quad (5)
\]
\[
\sup_{x, \eta \in \mathbb{R}, |x - \eta| \leq 1} \frac{|W_2(x) - W_2(\eta)|}{|W_2(x)||x - \eta|^\beta} < +\infty, \quad 0 < \alpha < \frac{\beta}{3} + 1, \quad \beta \in (0, 1], \quad \lambda \geq 0. \quad (6)
\]

Then there exists a number \( \lambda_0 \geq 0 \), such that the equation (1) for all \( \lambda \geq \lambda_0 \) has a solution \( y \).

In (4), (5) and elsewhere, \( c_n \) \((n = 0, 1)\) denotes a fixed constant which, in general, may be different in the various places it is used.

**Theorem 2.** Let the functions \( q(x), r(x) \) be continuous, \( p_1 \in C^{(3)}_{loc}(\mathbb{R}), p_2 \in C^{(2)}_{loc}(\mathbb{R}) \) and satisfy conditions (3), (4), (6) and
\[
|p_1^{(j)}(x)| \leq c_1 p_1(x) \quad (j = 1, 3), \quad |p_2^{(k)}(x)| \leq c p_2(x) \quad (k = 1, 2), \quad x \in \mathbb{R}. \quad (7)
\]

Then the solution of the equation (1) is unique and the estimate (2) holds.

**2 Auxiliary statements**

Below we suppose that conditions of Theorem 1 are fulfilled.
Let \( \xi_s = \xi_s(x) \) \( (s = 0, 1, 2) \) be the roots of the equation

\[
p_1(x)p_2(x)\xi^3 - r(x) + i(q(x) + \lambda) = 0.
\]

From conditions of the Theorem 1 it follows that \( 0 < \arg \xi_0 < \pi \) and \( \pi < \arg \xi_j < 2\pi \), \( j = 1, 2 \).

We introduce the following kernels:

\[
M_0(x, \eta, \lambda) = \begin{cases} 
- \frac{1}{\xi_0(x)p_2(x)} \frac{\xi_0(x)}{\xi_0(x)} \xi_0^3 \int_{-\infty}^{x} \frac{e^{i(x - \eta)\xi_0}}{\xi_0^3} d\eta, & -\infty < \eta < x \\
\frac{1}{\xi_0(x)p_2(x)} \sum_{i=1}^{2} \frac{\xi_0(x)}{\xi_0(x)} \xi_0^3 \int_{-\infty}^{x} \frac{e^{i(x - \eta)\xi_0}}{\xi_0^3} d\eta, & x < \eta < +\infty,
\end{cases}
\]

(8)

\[
M_1(x, \eta, \lambda) = p_1(\eta)p_2(\eta) \left[ \frac{q(\eta) - ir(\eta) + \lambda}{p_1(\eta)p_2(\eta)} - \frac{q(\eta) - ir(\eta) + \lambda}{p_1(\eta)p_2(\eta)} \right] M_0(x, \eta, \lambda) \omega(\eta - x),
\]

\[
M_2(x, \eta, \lambda) = - \left[ p_1(\eta)p_2(\eta) \omega(\eta - x) + 3p_1(\eta)p_2(\eta) \omega'(\eta - x) \right] M_0^{\prime\prime}(x, \eta, \lambda) - \\
\left[ 2p_1(\eta)p_2(\eta) \omega''(\eta - x) + 3p_1(\eta)p_2(\eta) \omega'''(\eta - x) \right] M_0'(x, \eta, \lambda) - \\
\left[ p_1(\eta)p_2(\eta) \omega''\eta(\eta - x) + p_1(\eta)p_2(\eta) \omega'''\eta\eta(\eta - x) \right] M_0(x, \eta, \lambda),
\]

and

\[
M_3(x, \eta, \lambda) = M_0(x, \eta, \lambda) \omega(\eta - x),
\]

where the function \( \omega(\eta) \in C^\infty_\eta(-1, 1) \) is such that

\[
\omega(\eta) = \begin{cases} 
1, & |\eta| \leq 1/2 \\
0, & |\eta| \geq 1.
\end{cases}
\]

It is easily to get the following equalities:

\[
\frac{\partial^j M_0(x, \eta, \lambda)}{\partial \eta^j} \bigg|_{x = 0} = \frac{\partial^j M_0(x, \eta, \lambda)}{\partial \eta^j} \bigg|_{x = 0}, \quad j = 0, 1,
\]

(9)

\[
\frac{\partial^2 M_0(x, \eta, \lambda)}{\partial \eta^2} \bigg|_{x = 0} = \frac{\partial^2 M_0(x, \eta, \lambda)}{\partial \eta^2} \bigg|_{x = 0} = - \frac{1}{p_1(x)p_2(x)},
\]

(10)

\[
-p_1(x) \left( p_2(x) \frac{\partial^2 M_0(x, \eta, \lambda)}{\partial \eta^2} \right)' + [q(x) + ir(x) + \lambda] M_0(x, \eta, \lambda) = 0.
\]

(11)

We define the operators \( M_j(\lambda), \quad (j = 1, 3) \) by means of the following equalities:
\((M_j(\lambda)f)(\eta) = \int_R M_j(x, \eta, \lambda)f(x)dx\quad (j = 1, 3)\).

The following statement is well-known (see [6]).

**Lemma 1.** Let \(1 < p < +\infty\) and \(k(x, \eta)\) be continuous function and

\[(Kv)(\eta) = \int_R k(x, \eta)v(x)dx.\]

Then

\[\|K\|_{L^p \to L^p} \leq \sup_{\eta, R} \left(\int_R |k(x, \eta)| + |k(\eta, x)| \right) dx.\]

**Lemma 2.** Let all of the conditions of Theorem 1 be satisfied. Then the operators \(M_j(\lambda), j = 1, 3,\) are continuous in the space \(L_p\) and following estimates hold (\(\lambda \geq 0\)):

\[
\|M_1(\lambda)\|_{L^p \to L^p} \leq \frac{c}{b_\lambda^{\beta+3-3\alpha}(\eta)}, \quad \beta \in (0, 1), \quad 0 < \alpha < \frac{\beta}{3} + 1, \tag{12}
\]

\[
\|M_2(\lambda)\|_{L^p \to L^p} \leq \frac{c}{b_\lambda^\alpha(\eta)}, \tag{13}
\]

\[
\|M_3(\lambda)\|_{L^p \to L^p} \leq \frac{c}{p_1(\eta)p_2(\eta)b_\lambda^\alpha(\eta)}, \tag{14}
\]

where \(b_\lambda(x) = \sqrt{W_\lambda(x)}\).

**Proof.** Under the assumptions of Theorem 1 for the functions \(q(x), r(x)\) and \(p_j(x)\) \((j = 1, 2)\) there exists a constant \(\sigma > 0\) such that \(Im \xi_1 \geq \sigma\) and \(Im \xi_1 \leq -\sigma (l = 1, 2)\). Then from (8) we can derive that

\[
|M_0(x, \eta, \lambda)| \leq \begin{cases} 
\frac{1}{3p_1(x)p_2(x)}e^{\sigma(x-\eta)b_\lambda(x)}b_\lambda^\alpha(\xi) & \text{if } -\infty < \eta < x,
\frac{2}{3p_1(x)p_2(x)}e^{\sigma(x-\eta)b_\lambda(x)}b_\lambda^\alpha(\xi) & \text{if } x < \eta < +\infty
\end{cases} \tag{15}
\]

and

\[
\left|\frac{\partial^jM_0(x, \eta, \lambda)}{\partial \eta^j}\right| \leq \begin{cases} 
\frac{1}{3p_1(x)p_2(x)}e^{\sigma(x-\eta)b_\lambda(x)}b_\lambda^{2\alpha-\alpha}(\xi) & \text{if } -\infty < \eta < x,
\frac{2}{3p_1(x)p_2(x)}e^{\sigma(x-\eta)b_\lambda(x)}b_\lambda^{2\alpha-\alpha}(\xi) & \text{if } x < \eta < +\infty
\end{cases} \tag{16}
\]

where \(j=1,2\). According to our choice, \(M_j(x, \eta, \lambda) = 0\) at \(|x - \eta| > 1\). Taking into account conditions (3), (4), (5) and (6) of Theorem 1 and (15), (16) for functions \(M_j(x, \eta, \lambda)\) \((j = 0, 1, 2)\) at \(|x - \eta| \leq 1\), we obtain the following estimates:
\[ |M_1(x, \eta, \lambda)| \leq \begin{cases} c p_1(\eta) p_2(\eta) \right| x - \eta \right|^{\beta} b^3 \left| x - \eta \right|^{\alpha - 2} \left( \eta \right) p_1(\eta) p_2(\eta) \right| x - \eta \right|^{\alpha - 2} \left( \eta \right) p_1(\eta) p_2(\eta) \right| x < \eta < +\infty, \end{cases} \]

\[ |M_2(x, \eta, \lambda)| \leq \begin{cases} \frac{p_1(\eta) p_2(\eta)}{p_1(\lambda) p_2(\lambda)} \sum_{k=0}^{\infty} \frac{e^{\alpha - \eta \beta} b_k(\lambda)}{b_k(\lambda)} \right| x - \eta \right|^{2} b^2 \left( \eta \right) p_1(\lambda) p_2(\lambda) \right| x < \eta < +\infty, \end{cases} \]

and

\[ |M_3(x, \eta, \lambda)| \leq \begin{cases} \frac{1}{3 \pi_1(\lambda) p_2(\lambda)} \right| x - \eta \right|^{\alpha - 2} b^2 \left( \eta \right) p_1(\lambda) p_2(\lambda) \right| x < \eta < +\infty, \end{cases} \]

We will estimate the norms \( \|M_j(\lambda)\|_{L_p^r - L_p^\rho} \) \( (j = 1, 2, 3) \) of the operators \( M_j(\lambda) \) using Lemma 1 and inequalities (17), (18), (19) and the conditions (3), (4) and (6).

Then making the change of variable \( \eta = \frac{x}{\sigma \beta \lambda(\eta)} \), we obtain

\[ \|M_j(\lambda)\|_{L_p^r - L_p^\rho} \leq \frac{c}{(b_\lambda(\eta))^{\beta + \lambda}} + \frac{c}{(b_\lambda(\eta))^{\beta + \lambda}} = \frac{c}{(b_\lambda(\eta))^{\beta + \lambda}}. \]

(3) implies \( \frac{|q(\eta) + \lambda + \sigma r(\eta)|}{p_1(\eta) p_2(\eta)} \geq \sqrt{1 + \lambda} \). Therefore, from the previous inequality we obtain (12). Inequalities (13) and (14) are proved similarly. The lemma is proved.

Using the definitions of \( M_j(\lambda) \) \( (j = 1, 2, 3) \) and equalities (9), (10) and (11), we prove the following Lemma.

**Lemma 3.** Let the conditions of Theorem 1 be satisfied. Then the following equality holds:

\[ (L + \lambda E) [M_3(\lambda) f](\eta) = f(\eta) + [M_1(\lambda) f](\eta) + [M_2(\lambda) f](\eta). \]

### 3. Proofs of the main results

**Proof of Theorem 1.** By estimates (12) and (13), there exists a number \( \lambda_0 > 0 \), such that \( \|M_1(\lambda)\|_{L_p^r - L_p^\rho} + \|M_2(\lambda)\|_{L_p^r - L_p^\rho} \leq 1/2 \) for any \( \lambda \geq \lambda_0 \). Then the operator \( G(\lambda) = E + M_1(\lambda) + M_2(\lambda) \) has a bounded inverse \( G^{-1}(\lambda) \) in \( L_p^r \). Let \( h = [E + M_1(\lambda) + M_2(\lambda)] f \). By (20), we obtain \( (L + \lambda E) [M_3(\lambda) G^{-1}(\lambda) h](\eta) = h. \)
So, for all \( \lambda \geq \lambda_0 \) the function \( y = M_3(\lambda)G^{-1}(\lambda)f \) is a solution to equation (1). The proof is complete.

Let the functions \( p_i, \ (i = 1, 2) \), \( q, r \) satisfy the conditions of Theorem 2, and a number \( \sigma' \) such that \( \frac{1}{p} + \frac{1}{p'} = 1 \). We denote by \((L + \lambda E)'\) an operator acting in the space \( L_{p'}(R) \) and such that \((L + \lambda E)y, z) = (y, (L + \lambda E)'z), \ y \in D(L + \lambda E), \ z \in D((L + \lambda E)').\) It is clear, that

\[
(L + \lambda E)'z \equiv (p_2(x) \ (p_1(x)z)'\)'' + (q(x) + \lambda - i\sigma(x))z.
\]

We consider the following differential equation:

\[
(L + \lambda E)'z \equiv (p_2(x) \ (p_1(x)z)'\)'' + (q(x) + \lambda - i\sigma(x))z = g(x), \quad (21)
\]

where \( p_j(x) \geq 1 \ j = 1, 2 \) are continuous together with derivatives up to third and second order respectively, and \( q(x) \) and \( r(x) \) are continuous real-valued functions, \( \lambda \geq 0, \ g(x) \in L_{p'}(R). \)

The following lemma is proved similarly to Theorem 1.

**Lemma 4.** Let the continuous functions \( q(x) \), \( r(x) \) and the functions \( p_1 \in C^{(2)}_{loc}(R), \)
\( p_2 \in C^{(2)}_{loc}(R) \) satisfy the conditions (3), (4), (6) and (7). Then there exists a number \( \lambda_1 \geq 0, \) such that for all \( \lambda \geq \lambda_1 \) the equation (21) has solution.

**Proof of Theorem 2.** Lemma 4 implies that the operator \((L + \lambda E)\)' at \( \lambda \geq \lambda_1 \) has a right inverse, which defined on whole \( L_{p'}(R) \). So \( \ker((L + \lambda E)')^* = \{0\} \), where \((L + \lambda E)'^*\) is an adjoint operator to \((L + \lambda E)'\). It is clear that \((L + \lambda E)', \) is an extension of the operator \( L + \lambda E, \) hence we have \( \ker(L + \lambda E) = \{0\}, \forall \lambda \geq \hat{\lambda} = \max(\lambda_0, \lambda_1) \). Thus, the operator \( L + \lambda E \) is a boundedly invertible in the space \( L_{p'}(R) \) and by proof of the Theorem 1,

\[
(L + \lambda E)^{-1} = M_3(\lambda)G^{-1}(\lambda), \quad \lambda \geq \hat{\lambda} = \max(\lambda_0, \lambda_1) \quad (22)
\]

Let \( y \) be a solution of equation (1), where \( \lambda \geq \hat{\lambda} = \max(\lambda_0, \lambda_1) \). We shall prove the estimate (2). By (22), Lemma 1 and the conditions (3), (4), (5) and (6), we have

\[
\|(q + \lambda + i\sigma)(L + \lambda E)^{-1}\|_{L_p \to L_p} = \|(q + \lambda + i\sigma)M_3(\lambda)G^{-1}(\lambda)\|_{L_p \to L_p} \leq
\]

\[
\sup_{\eta \in R} \int_{\eta - 1}^{\eta + 1} b_3(\eta) b_2^{-2}(x) \exp[-\sigma|x - \eta|b_3(x)] dx \leq
\]

\[
\sup_{\eta \in R} \int_{\eta - 1}^{\eta + 1} \exp[-\sigma|x - \eta|b_3(x)] dx < \infty.
\]

By (1), we get \( \|p_1(x) (p_2(x)y)'\|_p \leq c \left( \|f\|_p + \|y\|_p \right). \) Combining the last two estimates we obtain (2). The theorem is proved.
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References

Appendix to Paper E
APPENDIX TO PAPER E

by

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Abstract: In this appendix we include the proof of Lemma 4, which was omitted in Paper E.

Before we can prove Lemma 4, we introduce some notations and give some further auxiliary statements.

We consider an integral, which is defined in the following form:

\[ \int_{-\infty}^{+\infty} \frac{e^{i(x-\eta)\xi}}{p_1(x)p_2(x)} \left( \xi^3 - \frac{r(x)+i(q(x)+\lambda)}{p_1(x)p_2(x)} \right) d\xi. \]

By the residue Theorem,

\[ \tilde{M}_0(x, \eta, \lambda) = \sum_{k=1}^{3} \text{res}_{\xi_k} g(\xi) \]

where \( \text{res}_{\xi_k} g(\xi) \) stands for the set of residues of the involved function \( g(\xi) \).

Here the numbers \( \xi_k = \xi_k(x) \) (\( k = 1, 2, 3 \)) are the roots of the equation

\[ p_1(x)p_2(x)\xi^3 - r(x) - i(q(x) + \lambda) = 0. \]

From the assumptions in Lemma 4, i.e. \( r(x) \geq 1, q(x) \geq 1 \) and \( \lambda \geq 0 \), we get that

\[ \arg \left( \frac{r(x)+i(q(x)+\lambda)}{p_1(x)p_2(x)} \right) \in \left[ 0, \frac{\pi}{2} \right). \]

Then \( 0 < \arg \xi_j < \pi \) (\( j = 1, 2 \)) and \( \pi < \arg \xi_3 < 2\pi \). So we know that

\[ \text{Im} \xi_1 > 0, \text{Im} \xi_2 > 0 \text{ and } \text{Im} \xi_3 < 0. \]

We denote \( \phi = \arg \xi_k \) and it is clear that \( \phi_2 = \phi_1 + \frac{2\pi}{3}, \phi_3 = \phi_2 + \frac{2\pi}{3} \). Therefore, when finding the value of the residues at the points \( \xi_1 \) and \( \xi_2 \) we assume that \( \eta - x > 0 \), and at the point \( \xi_3 \) we suppose that \( \eta - x < 0 \), so

\[ \text{res}_{\xi_k} \frac{e^{i(x-\eta)\xi}}{\xi^3 - \frac{r+(q+\lambda)i}{p_1(x)p_2(x)}} = \frac{e^{i(x-\eta)\phi}}{3^{\frac{1}{2}} \left| \frac{r+(q+\lambda)i}{p_1(x)p_2(x)} \right|^3 (\cos \phi_k + i \sin \phi_k)} \quad (k = 1, 2, 3). \]
Thus, the function $\widetilde{M}_0(x, \eta, \lambda)$ can be written as follows

$$
\widetilde{M}_0(x, \eta, \lambda) = \begin{cases} 
-\frac{1}{3p_1(x)p_2(x)} \sum_{j=1}^{2} \frac{e^{i(x-\eta)\xi_j}}{\xi_j^2}, & -\infty < \eta < x, \\
1 \frac{e^{i(x-\eta)\xi_3}}{3p_1(x)p_2(x)}\xi_3^2, & x < \eta < +\infty.
\end{cases}
$$

(1)

We obtain the following equations by direct calculations:

$$
\left. \frac{\partial^j \tilde{M}_0(x, \eta, \lambda)}{\partial \eta^j} \right|_{x=0} = \left. \frac{\partial^j \tilde{M}_0(x, \eta, \lambda)}{\partial \eta^j} \right|_{x=+0}, \text{ } j = 0, 1, \tag{2}
$$

$$
\left. \frac{\partial^2 \tilde{M}_0(x, \eta, \lambda)}{\partial \eta^2} \right|_{x=0} - \left. \frac{\partial^2 \tilde{M}_0(x, \eta, \lambda)}{\partial \eta^2} \right|_{x=+0} = -\frac{1}{p_1(x)p_2(x)}, \tag{3}
$$

and

$$
\left( p_2(x) \left( \frac{\partial}{\partial \eta} \tilde{M}_0(x, \eta, \lambda) \right) \right)'' + \left( q(x) + ir(x) + \lambda \right) \tilde{M}_0(x, \eta, \lambda) = 0. \tag{4}
$$

Next, we introduce the following notations

$$
\widetilde{M}_1(x, \eta, \lambda) = \left[ (q(\eta) + ir(\eta) + \lambda) - \frac{p_1(\eta)p_2(\eta)}{p_1(x)p_2(x)} (q(x) + ir(x) + \lambda) \right] \tilde{M}_0(x, \eta, \lambda) \omega(\eta - x),
$$

$$
\widetilde{M}_2(x, \eta, \lambda) = \left[ (3p_1'(\eta)p_2(\eta) + 2p_1(\eta)p_2'(\eta)) \omega(\eta - x) + 3p_2(\eta)p_1(\eta)\omega'(\eta - x) \right] \tilde{M}''_{0\eta}(x, \eta, \lambda) - \left[ (4p_1'(\eta)p_2'(\eta) + 3p_1''(\eta)p_2(\eta) + p_1(\eta)p_2''(\eta)) \omega(\eta - x) + (6p_1'(\eta)p_2(\eta) + 4p_1(\eta)p_2'(\eta))\omega'(\eta - x) + 3p_1(\eta)p_2(\eta)\omega''_{\eta\eta}(\eta - x) \right] \tilde{M}_{0\eta\eta}(x, \eta, \lambda) + \left[ (p_1'(\eta)p_2''(\eta) + 2p_1''(\eta)p_2'(\eta) + p_1'''(\eta)p_2(\eta)) \omega(\eta - x) + (4p_1'(\eta)p_2'(\eta) + 3p_1''(\eta)p_2(\eta) + p_1(\eta)p_2''(\eta))\omega'(\eta - x) + (3p_1'(\eta)p_2(\eta) + 2p_1(\eta)p_2'(\eta))\omega''_{\eta\eta}(\eta - x) + p_1(\eta)p_2(\eta)\omega'''_{\eta\eta\eta}(\eta - x) \right] \tilde{M}_{0\eta\eta\eta}(x, \eta, \lambda),
$$

and

$$
\tilde{M}_3(x, \eta, \lambda) = \tilde{M}_0(x, \eta, \lambda) \omega(\eta - x),
$$

2
where

\[
\omega(\eta) = \begin{cases} 
1, & |\eta| \leq \frac{1}{2} \\
0, & |\eta| \geq 1,
\end{cases}
\]

and \( \omega(\eta) \in C_0^\infty(-1, 1) \). We also define the integral operators:

\[
\left( \widetilde{M}_j(\lambda)f \right)(\eta) := \int \widetilde{M}_j(x, \eta, \lambda)f(x)dx, \quad (j = 1, 2, 3).
\]

The proof of Lemma 4 is based on two auxiliary Lemmas A1 and A2 of independent interest.

**Lemma A1.** Let all the assumptions in Lemma 4 be fulfilled. Then the operators \( \widetilde{M}_j(\lambda) \) are continuous in the space \( L_{p'} \) and the following estimates

\[
\left\| \widetilde{M}_1(\lambda) \right\|_{L_p \to L_{p'}} \leq \frac{c_1}{b_\lambda^\alpha(\eta)}, \quad \mu \in (0, 1], \quad 0 < \nu < \frac{\mu}{3} + 1,
\]

\[
\left\| \widetilde{M}_2(\lambda) \right\|_{L_p \to L_{p'}} \leq \frac{c_2}{b_\lambda(\eta)},
\]

hold.

**Proof of Lemma A1.** By the assumptions of Lemma 4 for the functions \( q(x), r(x) \) and \( p_s(x) \) \((s = 1, 2)\) there exists a constant \( \delta > 0 \) such that \( \text{Im} \xi_j \geq \delta \) \((j = 1, 2)\) and \( \text{Im} \xi_3 \leq -\delta \). Therefore, from (1) it follows that

\[
\left| \widetilde{M}_0(x, \eta, \lambda) \right| \leq \begin{cases} 
\frac{2}{3p_1(x)p_2(x) b_\lambda(x)} e^{-\delta(x-\eta)b_\lambda(x)}, & -\infty < \eta < x, \\
1 & x < \eta < +\infty,
\end{cases}
\]

and

\[
\left| \frac{\partial^i \widetilde{M}_0}{\partial \eta^j} \right| \leq \begin{cases} 
\frac{2}{3p_1(x)p_2(x) b_\lambda^{2-j}(x)} e^{-\delta(x-\eta)b_\lambda(x)}, & \eta \in (-\infty, x), \\
1 & \eta \in (x, +\infty), \quad j = 1, 2.
\end{cases}
\]

We note that if \( |x-\eta| > 1 \), then \( \widetilde{M}_j(x, \eta, \lambda) = 0 \). Moreover when \( |x-\eta| \leq 1 \), by using the assumptions in of Lemma 4 and inequalities (8) and (9) we have that
\[ |\widetilde{M}_1(x, \eta, \lambda)| \leq \begin{cases} 
 c p_1(x) p_2(x) |x - \eta|^{\nu \beta_{\lambda}(x) - 2} \frac{e^{-\sigma(x-\eta)b_{\lambda}(x)}}{p_1(x)p_2(x)}, & \eta < x, \\
 c p_1(x) p_2(x) |x - \eta|^{\nu \beta_{\lambda}(x) - 2} \frac{e^{\sigma(x-\eta)b_{\lambda}(x)}}{p_1(x)p_2(x)}, & x < \eta, 
\end{cases} \tag{10} \]

and
\[ |\widetilde{M}_2(x, \eta, \lambda)| \leq \begin{cases} 
 \frac{p_1(\eta) p_2(\eta)}{p_1(x)p_2(x)} \sum_{k=0}^{2} c_k \frac{e^{-\sigma(x-\eta)b_{\lambda}(x)}}{b_{\lambda}^k(x)}, & -\infty < \eta < x, \\
 \frac{p_1(\eta) p_2(\eta)}{p_1(x)p_2(x)} \sum_{k=0}^{2} c_k \frac{e^{\sigma(x-\eta)b_{\lambda}(x)}}{b_{\lambda}^k(x)}, & x < \eta < +\infty. 
\end{cases} \tag{11} \]

Now we will estimate norms of the operators \(\widetilde{M}_j(\lambda)\) \((j = 1, 2, 3)\) defined by (5) by using inequalities (10)-(11) and Lemma 1 in paper E:
\[ \left\| \widetilde{M}_1(\lambda) \right\|_{L^p_* - L^p_*} \leq \sup_{\eta \in R} \int_{R} \left[ |\widetilde{M}_1(x, \eta, \lambda)| + |\widetilde{M}_1(\eta, x, \lambda)| \right] dx \leq \]
\[ \leq c \sup_{\eta \in R} \int_{\eta-1}^{\eta+1} \left( \frac{p_1(\eta) p_2(\eta)}{p_1(x)p_2(x)} \frac{e^{-\delta(x-\eta)b_{\lambda}(\eta)}}{b_{\lambda}^2(\eta)} + \frac{p_1(\eta) p_2(\eta)}{p_1(x)p_2(x)} \frac{e^{-\sigma(x-\eta)b_{\lambda}(\eta)}}{b_{\lambda}^2(\eta)} \right) \times \]
\[ \times \left| \frac{q(\eta) + \lambda + ir(\eta)}{p_1(\eta)p_2(\eta)} - \frac{q(x) + \lambda + ir(x)}{p_1(x)p_2(x)} \right| dx. \]

Moreover, if we use the following conditions in Theorem 1 of paper E
\[ p_1(x) \geq 1, \quad p_2(x) \geq 1, \quad \frac{q(x)}{p_1(x)p_2(x)} \geq 1, \quad r(x) \geq 1, \tag{12} \]
\[ c_0^{-1} \leq \frac{p_j(x)}{p_j(\eta)}, \quad \frac{q(x)}{q(\eta)}, \quad \frac{r(x)}{r(\eta)} \leq c_0, \quad j = 1, 2, \quad x, \eta \in R, \quad |x - \eta| \leq 1, \tag{13} \]
and
\[ \sup_{x, \eta \in R; |x - \eta| \leq 1} \left| \frac{W_{\lambda}(x) - W_{\lambda}(\eta)}{W_{\lambda}(x)^\alpha |x - \eta|^{\beta}} \right| < +\infty, \quad 0 < \alpha < \frac{\beta}{3} + 1, \quad \beta \in (0, 1], \tag{14} \]

then we find that
\[ \left\| \widetilde{M}_1(\lambda) \right\|_{L^p_* - L^p_*} \leq \]
According to condition (12) we get

\[ H \leq c \sup_{\eta \in R} \int_{\eta-1}^{\eta+1} \left( e^{-\delta(x-\eta)c_{b_\lambda}(\eta)} + e^{-\delta(x-\eta)b_{\lambda}(\eta)} \right) \left| \frac{q(\eta) + \lambda + ir(\eta)}{p_1(\eta)p_2(\eta)} \right|^\nu |\eta - x|^\mu \, dx. \]

In this inequality, we make the change of variables \( \eta - x = \frac{1}{\sigma b_\lambda(\eta)} z \) and get the following estimate:

\[ \left\| \tilde{M}_1(\lambda) \right\|_{L^\nu_p - L^\mu_p} \leq \left( \frac{c}{(cb_{\lambda}(\eta))^{\mu+\frac{3}{2}}} \right) \left( \frac{q(\eta) + \lambda + i r(\eta)}{p_1(\eta)p_2(\eta)} \right)^\nu \left( \frac{\sigma^\nu}{p_1(\eta)p_2(\eta)} \right)^{\frac{\mu}{3}}. \]

According to condition (12) we get

\[ \left| q(\eta) + \lambda + i r(\eta) \right| \geq \sqrt{1 + \lambda}, \]

so from the previous inequality we obtain (6).

Now let us prove the estimate (7). Taking into account the assumptions of Lemma 4 we have that

\[ \left\| \tilde{M}_2(\lambda) \right\|_{L^\nu_p - L^\mu_p} \leq c \sup_{\eta \in R} \int_{\eta-1}^{\eta+1} \left[ \left| \tilde{M}_2(x, \eta, \lambda) \right| + \left| \tilde{M}_2(x, \eta, \lambda) \right| \right] dx \leq \]

\[ \leq c \sup_{\eta \in R} \int_{\eta-1}^{\eta+1} \left( \frac{p_1(\eta)p_2(\eta)}{p_1(\eta)p_2(\eta)} e^{-\sigma(x-\eta)b_\lambda(x)} + \frac{p_1(\eta)p_2(\eta)}{p_1(\eta)p_2(\eta)} e^{-\sigma(x-\eta)b_\lambda(x)} \right) + \]

\[ + \frac{p_1(\eta)p_2(\eta)}{p_1(\eta)p_2(\eta)} e^{-\sigma(x-\eta)b_\lambda(x)} + \frac{p_1(\eta)p_2(\eta)}{p_1(\eta)p_2(\eta)} e^{-\sigma(x-\eta)b_\lambda(x)} + \]

\[ + \frac{p_1(\eta)p_2(\eta)}{p_1(\eta)p_2(\eta)} e^{-\sigma(x-\eta)b_\lambda(x)} + \frac{p_1(\eta)p_2(\eta)}{p_1(\eta)p_2(\eta)} e^{-\sigma(x-\eta)b_\lambda(x)} \right) dx. \]

Hence, calculating the integrals and using conditions (12), (13), (14) and

\[ |p_1^{(j)}(x)| \leq c_1 p_1(x) \quad (j = 1, 3), \quad |p_2^{(k)}(x)| \leq c_2 p_2(x) \quad (k = 1, 2), \quad x \in R, \]

we obtain that

\[ \left\| \tilde{M}_2(\lambda) \right\|_{L^\nu_p - L^\mu_p} \leq c \sup_{\eta \in R} \int_{\eta-1}^{\eta+1} \left[ e^{-\sigma(x-\eta)b_\lambda(x)} + e^{-\sigma(x-\eta)b_\lambda(x)} + e^{-\sigma(x-\eta)b_\lambda(x)} \right] dx = \]
\[= c \sup_{\eta \in R} \left[ \frac{(1 - e^{-\sigma b_\lambda(\eta)})}{\sigma b_\lambda(\eta)} + \frac{(1 - e^{-\sigma b_\lambda(\eta)})}{\sigma b_\lambda^2(\eta)} + \frac{(1 - e^{-\sigma b_\lambda(\eta)})}{\sigma b_\lambda^3(\eta)} \right] \leq \frac{c}{b_\lambda(\eta)},\]

so (7) is proved. The proof is complete.

**Lemma A2.** Let the assumptions of Lemma 4 be fulfilled. Then the equality

\[(L + \lambda E)' \left[ \tilde{M}_3(\lambda)f \right](\eta) = f(\eta) + \left[ \tilde{M}_1(\lambda)f \right](\eta) + \left[ \tilde{M}_2(\lambda)f \right](\eta) \quad (15)\]

holds.

**Proof of Lemma A2.** By the definition of the operator \((L + \lambda E)'\) it yields that

\[(L + \lambda E)' \left[ \tilde{M}_3(\lambda)f \right](\eta) = \left( p_2(\eta) \begin{array}{cc} p_1(\eta) \left( \int_{-\infty}^{\eta} \tilde{M}_0(x, \eta, \lambda) \omega(\eta - x) f(x) dx + \int_{\eta}^{+\infty} \tilde{M}_0(x, \eta, \lambda) \omega(\eta - x) f(x) dx \right) \right) \right) \right) + (q(\eta) + ir(\eta) + \lambda) \times \]

\[\times \left( \int_{-\infty}^{\eta} \tilde{M}_0(x, \eta, \lambda) \omega(\eta - x) f(x) dx + \int_{\eta}^{+\infty} \tilde{M}_0(x, \eta, \lambda) \omega(\eta - x) f(x) dx \right) (\eta).\]

We note that

\[\frac{d}{d\eta} \left( p_1(\eta) \tilde{M}_3(\lambda)g \right)(\eta) = p'_1(\eta) \int_R \tilde{M}_0 \omega(\eta - x) g(x) dx + \]

\[+ p_1(\eta) \int_R \tilde{M}_0' \omega(\eta - x) g(x) dx + p_1(\eta) \int_R \tilde{M}_0 \omega'(\eta - x) g(x) dx.\]

If we define

\[A := \frac{d}{d\eta} \left( p_1(\eta) \tilde{M}_3(\lambda)g \right),\]

then

\[\frac{d^2}{d\eta^2} \left( p_2(\eta)A \right)(\eta) = p'_2(\eta)A + 2p'_2(\eta)A' + p_2(\eta)A''.\]
where

\[
A' = p_1(\eta) \int_\mathbb{R} \tilde{M}_{0\eta} \omega(\eta - x)g(x)dx + 2p_1'(\eta) \int_\mathbb{R} \tilde{M}_{0\eta} \omega(\eta - x)g(x)dx +
\]

\[
+ 2p_1(\eta) \int_\mathbb{R} \tilde{M}_{0\eta} \omega'(\eta - x)g(x)dx + p_1''(\eta) \int_\mathbb{R} \tilde{M}_{0\eta} \omega(\eta - x)g(x)dx +
\]

\[
+ 2p_1'(\eta) \int_\mathbb{R} \tilde{M}_{0\eta} \omega(\eta - x)g(x)dx + p_1(\eta) \int_\mathbb{R} \tilde{M}_{0\eta} \omega(\eta - x)g(x)dx
\]

and

\[
A'' = 3p_1'(\eta) \int_\mathbb{R} \tilde{M}_{0\eta} \omega(\eta - x)g(x)dx + 3p_1(\eta) \int_\mathbb{R} \tilde{M}_{0\eta} \omega'(\eta - x)g(x)dx +
\]

\[
+ 3p_1''(\eta) \int_\mathbb{R} \tilde{M}_{0\eta} \omega(\eta - x)g(x)dx + 6p_1'(\eta) \int_\mathbb{R} \tilde{M}_{0\eta} \omega'(\eta - x)g(x)dx +
\]

\[
+ 3p_1(\eta) \int_\mathbb{R} \tilde{M}_{0\eta} \omega''(\eta - x)g(x)dx + p_1''(\eta) \int_\mathbb{R} \tilde{M}_{0\eta} \omega(\eta - x)g(x)dx +
\]

\[
+ 3p_1'(\eta) \int_\mathbb{R} \tilde{M}_{0\eta} \omega(\eta - x)g(x)dx + 3p_1'(\eta) \int_\mathbb{R} \tilde{M}_{0\eta} \omega'(\eta - x)g(x)dx +
\]

\[
+ p_1(\eta) \int_\mathbb{R} \tilde{M}_{0\eta} \omega'(\eta - x)g(x)dx.
\]

Hence we obtain equality (15) by using the relations (2), (3) and (4) and together with the definitions of \( \tilde{M}_j(x, \eta, \lambda) \) \((j = 1, 2, 3)\). The proof is complete.

**Proof of Lemma 4.** According to estimates (6) and (7) in Lemma A1 there exists a number \( \lambda_1 > 0 \) such that the inequality

\[
||\tilde{M}_1(\lambda)||_{L_{\rho'}-L_{\rho'}} + ||\tilde{M}_2(\lambda)||_{L_{\rho'}-L_{\rho'}}
\]

holds for any \( \lambda \geq \lambda_1 \). Then there is a finite inverse operator \( \Phi^{-1}(\lambda) \) of the operator \( \Phi(\lambda) = E + \tilde{M}_1(\lambda) + \tilde{M}_2(\lambda) \) in the space \( L_{\rho'} \). Thus, by denoting \( h = [E + \tilde{M}_1(\lambda) + \tilde{M}_2(\lambda)] \), from equality (15) in Lemma A2 we obtain the equality

\[
(L + \lambda E)'[\tilde{M}_3(\lambda)\Phi^{-1}(\lambda)h](\eta) = h.
\]

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Therefore the function $y = \tilde{M}_3(\lambda)\Phi^{-1}(\lambda)g$ is a solution of the equation

$$(L + \lambda E)z \equiv (p_2(x)(p_1(x)z)'') + (q(x) + \lambda - ir(x))z = g(x)$$

for any $\lambda \geq \lambda_0$. The proof is complete.

**Remark.** By using the same arguments as in the proofs of (6) and (7) in Lemma A1 we can also prove the following estimate:

$$\|\tilde{M}_3(\lambda)\|_{L^p \to L^p} \leq \frac{c_3}{p_1(\eta)p_2(\eta)b_3^3(\eta)}.$$
Paper F
On Maximal Regularity of Singular Third-Order Differential Equations

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Remark: An abbreviated version of this paper is also published as follows:


Abstract: The paper is devoted to the study of the correct and coercive solvability in the space $L_p(-\infty, +\infty)$ ($1 < p < \infty$) of non-self-adjoint third-order equations with variable leading coefficients on the whole real axis. The lowest coefficient of the equation is complex-valued, it is not bounded, and, generally speaking, it is not differentiable. Sufficient conditions for the unique solvability of this equation for any of its right-hand sides are found, and estimates for the maximal regularity of the solution are obtained.

AMS subject classification (MSC 2010): 35J70

Keywords and Phrases: Differential equation, singular third-order differential equation, non-semibounded operator, maximal regularity, separability, coercive estimate, closure, bounded invertibility, inverse operator, unbounded domain, adjoint operator, extension of the operator.

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1 Introduction and main results

In this paper we investigate the following equation

\[(l + \lambda E)y := -m_1(x)(m_2(x)y')' + (q(x) + ir(x) + \lambda) y = f(x), \quad (1)\]

where \(x \in R = (-\infty, +\infty), f \in L_p(R), 1 < p \leq \infty\) and \(\lambda \geq 0\). We assume that \(q(x), r(x), m_1(x)\) are continuous functions, and \(m_2 \in C^{(1)}_{loc}(R), m_3 \in C^{(2)}_{loc}(R)\).

The equation (1) is well studied when \(m_k(x) = 1 (k = 1, 2, 3)\) in [1-5]. In the case when \(m_2(x) = 1\) and \(m_1(x) = m_3(x)\) the sufficient conditions for the unique solvability of the equation (1) and the coercive estimate for its solution were established in [6]. For the case \(m_1(x) = m_3(x)\) see also [7]. In [6] and [7] reviews of studies conducted in this direction were given.

**Definition.** A function \(y(x) \in L_p(R)\) is called a solution of (1), if there is a sequence of three times continuously differentiable functions with compact support \(\{y_n\}_{n=1}^{\infty}\) such that \(\|y_n - y\|_p \to 0\) and \(\|(l + \lambda E)y_n - f\|_p \to 0\) as \(n \to \infty\).

Our main results in this paper are formulated in the following two theorems:

**Theorem 1.** Assume that the functions \(q(x), r(x)\) and \(m_1(x)\) are continuous, \(m_2 \in C^{(1)}_{loc}(R), m_3 \in C^{(2)}_{loc}(R)\) and satisfy the following conditions:

\[
m_j(x) \geq 1 \quad (j = 1, 2, 3), \quad \frac{q(x)}{\prod_{k=1}^{3} m_k^2(x)} \geq 1, \quad |r(x)| \geq 1, \quad (2)
\]

\[
c^{-1} \leq \frac{m_k(x)}{m_k(\eta)} \frac{q(x)}{q(\eta)} \frac{r(x)}{r(\eta)} \leq c, \quad (k = 1, 2, 3), \quad x, \eta \in R, \quad |x - \eta| \leq 1, \quad (3)
\]

\[
|m_2'(x)| \leq cm_2(x), \quad |m_3'(x)| \leq cm_3(x) \quad (j = 1, 2), \quad x \in R, \quad (4)
\]

\[
\sup_{|x-\eta| \leq 1} \frac{|W_{\lambda}(x) - W_{\lambda}(\eta)|}{|W_{\lambda}(x)|^\nu |x-\eta|^\mu} < +\infty, \quad 0 < \nu < \frac{\mu}{3} + 1, \quad \mu \in (0, 1], \quad \lambda \geq 0, \quad (5)
\]

where \(W_{\lambda}(x) := \frac{|q(x) + \lambda + ir(x)|}{\prod_{k=1}^{3} m_k(x)}\). Then there exists a number \(\lambda_0 \geq 0\), such that the equation (1) has a solution \(y\) for all \(\lambda \geq \lambda_0\).
Theorem 2. Let the functions \( q(x), r(x) \) be continuous, \( m_1 \in C^{(3)}_{\text{loc}}(R) \), \( m_2 \in C^{(2)}_{\text{loc}}(R) \), \( m_3 \in C^{(2)}_{\text{loc}}(R) \) and satisfy the conditions (2)-(5) and

\[
\begin{align*}
|m_{1,j}(x)| &\leq cm_1(x), & j = 1, 3, \\
|m_{k,i}(x)| &\leq cm_k(x), & k = 2, 3, & i = 1, 2, & x \in R.
\end{align*}
\]  

Then the solution \( y \) of the equation (1) is unique and the following estimate

\[
\left\| m_1(x) (m_2(x) (m_3(x)y')') \right\|_p^p + \| (q(x) + ir(x) + \lambda)y \|_p^p \leq c \| f(x) \|_p^p
\]  

holds.

In Section 2 some auxiliary statements are stated and some necessary notations introduced. In particular, Lemma 2 is crucial and of independent interest. The detailed proofs of these main theorems and auxiliary statements are given in Section 3.

2 Auxiliary statements

We denote by \( \xi_s = \xi_s(x) \) \( (s = 0, 1, 2) \) the roots of the following algebraic equation of third order

\[
\prod_{k=1}^{3} m_k(x) \xi^3 - r(x) + i(q(x) + \lambda) = 0.
\]  

According to our limitations (2) on the function \( m_j(x) \) \( (j = 1, 2, 3) \), \( r(x) \) and \( q(x) \) we have that \( \frac{\pi}{2} < \arg \xi_0 < \frac{2\pi}{3}, \pi < \arg \xi_j < 2\pi, \ j = 1, 2. \) We consider the function \( M_0 = M_0(x, \eta, \lambda) \):

\[
M_0(x, \eta, \lambda) = \begin{cases}
\frac{1}{3 \prod_{k=1}^{3} m_k(x)} \frac{e^{i(x-\eta)\xi_0}}{\xi_0^2}, & -\infty < \eta < x, \\
\frac{1}{3 \prod_{k=1}^{3} m_k(x)} \sum_{j=1}^{2} \frac{e^{i(x-\eta)\xi_j}}{\xi_j^2}, & x < \eta < +\infty.
\end{cases}
\]
It is easy to verify that this function satisfies the following relations (10) - (12):

\[ \frac{\partial^j M_0(x, \eta, \lambda)}{\partial \eta^j} \bigg|_{x=\eta-0} = \frac{\partial^j M_0(x, \eta, \lambda)}{\partial \eta^j} \bigg|_{x=\eta+0}, \quad j = 0, 1, \]  

(10)

\[ \frac{\partial^2 M_0(x, \eta, \lambda)}{\partial \eta^2} \bigg|_{x=\eta-0} - \frac{\partial^2 M_0(x, \eta, \lambda)}{\partial \eta^2} \bigg|_{x=\eta+0} = -\frac{1}{\prod_{k=1}^3 m_k(x)}, \]  

(11)

and

\[
-m_1(x) \left( m_2(x) \left( m_3(x) \frac{\partial M_0(x, \eta, \lambda)}{\partial \eta} \right)_0 \right)_0 + m_1(x) \left( \frac{\partial^2 M_0(x, \eta, \lambda)}{\partial \eta^2} \right) = 0. \]  

(12)

Using \( M_0(x, \eta, \lambda) \) we define the following functions \( M_j(x, \eta, \lambda), \quad (j = 1, 2, 3) \):

\[ M_1(x, \eta, \lambda) = \]  

\[ \left[ (q(\eta) + ir(\eta) + \lambda) - \frac{3}{\prod_{k=1}^3 m_k(x)} (q(x) + ir(x) + \lambda) \right] M_0(x, \eta, \lambda) \theta(\eta - x), \]  

\[ M_2(x, \eta, \lambda) = - \left[ 2m_1(\eta)m_2(\eta)m_3'(\eta)\theta(\eta - x) + m_1(\eta)m_2'(\eta)m_3(\eta)\theta(\eta - x) + \right. \]  

\[ + 3m_1(\eta)m_2(\eta)m_3(\eta)\theta'(\eta - x) \right] \frac{\partial^2 M_0(x, \eta, \lambda)}{\partial \eta^2} - \]  

\[ - \left[ m_1(\eta)m_2'(\eta)m_3(\eta)\theta'(\eta - x) + m_1(\eta)m_2(\eta)m_3''(\eta)\theta(\eta - x) + \right. \]  

\[ + 4m_1(\eta)m_2(\eta)m_3'(\eta)\theta''(\eta - x) + 2m_1(\eta)m_2'(\eta)m_3(\eta)\theta'(\eta - x) + \]  

\[ + 3m_1(\eta)m_2(\eta)m_3(\eta)\theta''(\eta - x) \right] \frac{\partial M_0(x, \eta, \lambda)}{\partial \eta} - \]  

\[ - \left[ m_1(\eta)m_2'(\eta)m_3(\eta)\theta''(\eta - x) + m_1(\eta)m_2(\eta)m_3''(\eta)\theta'(\eta - x) + \right. \]  

\[ + 2m_1(\eta)m_2(\eta)m_3'(\eta)\theta'''(\eta - x) + \]  

\[ + m_1(\eta)m_2'(\eta)m_3(\eta)\theta'''(\eta - x) + m_1(\eta)m_2(\eta)m_3(\eta)\theta'''(\eta - x) \right] ] M_0(x, \eta, \lambda), \]
and
\[ M_3(x, \eta, \lambda) = M_0(x, \eta, \lambda) \theta(\eta - x), \]
where
\[ \theta(\eta) = \begin{cases} 1, & |\eta| \leq \frac{1}{2} \\ 0, & |\eta| \geq 1, \end{cases} \]
and \( \theta(\eta) \in C_0^\infty(-1, 1) \). We also define the integral operators:
\[ (M_j(\lambda)f)(\eta) = \int_{\mathbb{R}} M_j(x, \eta, \lambda)f(x)dx, \quad (j = 1, 2, 3). \]

The following lemma is known, which is a consequence of a more general assertion from [8] (p. 902).

Lemma 1. Let \( K \) be an integral operator such that
\[ (K \nu)(\eta) = \int_{\mathbb{R}} k(x, \eta)\nu(x)dx, \]
where \( k(x, \eta) \) is a continuous function. Then, for \( p > 1 \),
\[ \|K\|_{L^p \rightarrow L^p} \leq \sup_{\eta \in \mathbb{R}} \int_{\mathbb{R}} \left| |k(x, \eta)| + |k(\eta, x)| \right| dx. \]

Remark 1. We notice that the right-hand side of the last inequality does not depend on \( p \).

We introduce the function
\[ b_\lambda(x) = \frac{|r(x) - i(q(x) + \lambda)|}{\prod_{k=1}^{3} m_k(x)}, \]
which exists when the conditions of Theorem 1 are satisfied.

The next Lemma is crucial and of independent interest.
Lemma 2. Let the conditions of Theorem 1 be satisfied and \( \lambda \geq 0 \). Then the operators \( M_j(\lambda) \) \((j = 1, 2, 3)\) and bounded operators in the space \( L_p(R) \) for all \( 1 < p < +\infty \), and their norms satisfy the following estimates:

\[
\|M_1(\lambda)\|_{L_p \to L_p} \leq \frac{c_1}{b_\lambda^{\frac{4}{3} - 3\nu}(\eta)}, \quad \mu \in (0, 1], \quad 0 < \nu < \frac{\mu}{3} + 1,
\]

(13)

\[
\|M_2(\lambda)\|_{L_p \to L_p} \leq \frac{c_2}{b_\lambda(\eta)},
\]

(14)

and

\[
\|M_3(\lambda)\|_{L_p \to L_p} \leq \frac{c_3}{\prod_{k=1}^{3} m_k(x)b_\lambda^2(\eta)}.
\]

(15)

We denote by \( L + \lambda E \) \((\lambda \geq 0)\) the closure in \( L_p \) of the differential expression

\[
(l + \lambda E)y \equiv -m_1(x) (m_2(x) (m_3(x)y')')' + [q(x) + ir(x) + \lambda]y, \quad \lambda \geq 0,
\]

defined on the set \( C_0^\infty(R) \) of infinitely differentiable and compactly supported functions.

Lemma 3. Let the conditions of Theorem 1 be satisfied. Then the following equality holds:

\[
(L + \lambda E) [M_3(\lambda)f](\eta) = f(\eta) + [M_1(\lambda)f](\eta) + [M_2(\lambda)f](\eta).
\]

(16)

Let the functions \( m_k(x) \) \(k = 1, 2, 3, q(x)\) and \( r(x)\) satisfy the conditions of Theorem 2, and \( p'\) as usual be defined by \( \frac{1}{p} + \frac{1}{p'} = 1\). We denote by \( (L + \lambda E)' \) an operator acting in the space \( L_{p'}(R) \) and such that

\[
((L + \lambda E)y, z) = (y, (L + \lambda E)'z), \quad y \in D(L + \lambda E), \quad z \in D((L + \lambda E)')
\]

. Obviously

\[
(L + \lambda E)'z := \left( \mu_1(x) (\mu_2(x) (\mu_1(x)z)'), \quad (q(x) + \lambda - ir(x)) z.
\]

We will consider the differential equation

\[
(L + \lambda E)'z :=
\]

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\[ := \left( m_3(x) (m_2(x) (m_1(x) z)')' + (q(x) + \lambda - i r(x)) z \right) = g(x), \quad (17) \]

where the function \( m_1(x) \geq 1 \) is continuous together with the derivatives up to third order and \( m_2(x) \geq 1 \) is continuous together with derivatives up to second order, \( m_3(x) \geq 1 \) is continuously differentiable, and \( q(x), r(x) \) are continuous real-valued functions, \( \lambda \geq 0 \) and \( g(x) \in L_T'(R) \).

For the proof of Theorem 2 we also need the following result closely related to Theorem 1.

**Lemma 4.** Let the functions \( q(x), r(x) \) be continuous and the functions \( m_1 \in C^{(3)}_{loc}(R), m_2 \in C^{(2)}_{loc}(R), m_3 \in C^{(1)}_{loc}(R) \) satisfy the conditions (2), (3), (5) and (6). Then there exists a number \( \lambda_1 \geq 0 \), such that the equation (17) has a solution for all \( \lambda \geq \lambda_1 \).

### 3 Proofs

**Proof of Lemma 2.** First we consider the algebraic equation (8). By the assumptions of Theorem 1 with respect to the functions \( q(x), r(x), m_k(x) \) \( (k = 1, 2, 3) \), there exists a constant \( \sigma > 0 \) such that the roots of equation (8) satisfy the following inequalities

\[ \text{Im } \xi_0 \geq \sigma, \quad \text{Im } \xi_j \leq -\sigma \quad j = 1, 2. \]

Hence, from (9) we have that

\[ |M_0(x, \eta, \lambda)| \leq \begin{cases} 
\frac{1}{3 \prod_{k=1}^{3} m_k(x)} e^{-\sigma(x-\eta)b_{\lambda}(x)}, & -\infty < \eta < x, \\
\frac{2}{3 \prod_{k=1}^{3} m_k(x)} e^{\sigma(x-\eta)b_{\lambda}(x)} & x < \eta < +\infty.
\end{cases} \]
By making a straightforward calculation, we find the derivatives of $M_0(x, \eta, \lambda)$ up to the second order can be estimated as follows:

$$\left| \frac{\partial^j M_0}{\partial \eta^j} \right| \leq \begin{cases} 
 \frac{1}{3 \prod_{k=1}^{3} m_k(x)} \frac{e^{-(x-\eta)b_\lambda(x)}}{b^{2-j}_\lambda(x)}, & \eta \in (-\infty, x), \\
 \frac{2}{3 \prod_{k=1}^{3} m_k(x)} \frac{e^{\sigma(x-\eta)b_\lambda(x)}}{b^{2-j}_\lambda(x)}, & \eta \in (x, +\infty), \ j = 1, 2.
\end{cases}$$

(18)

According to our definition $M_j(x, \eta, \lambda) = 0$ for $|x - \eta| > 1$. When $|x - \eta| \leq 1$, according to (18) and taking into account conditions (2) - (5) for the functions $M_j(x, \eta, \lambda) = 0$ ($j = 0, 1, 2$), we have that

$$|M_1| \leq \begin{cases} 
 c \prod_{k=1}^{3} m_k(x) |x - \eta|^\mu b_\lambda^{\lambda-2}(x) \frac{e^{-(x-\eta)b_\lambda(x)}}{3 \prod_{k=1}^{3} m_k(x)}, & \eta < x, \\
 c \prod_{k=1}^{3} m_k(x) \mu_2(\eta) |x - \eta|^\mu b_\lambda^{\lambda-2}(x) \frac{e^{\sigma(x-\eta)b_\lambda(x)}}{3 \prod_{k=1}^{3} m_k(x)}, & x < \eta.
\end{cases}$$

(19)

$$|M_2| \leq \begin{cases} 
 \prod_{k=1}^{3} m_k(\eta) \sum_{k=0}^{2} c_k \frac{e^{-(x-\eta)b_\lambda(x)}}{b_\lambda^k(x)}, & -\infty < \eta < x, \\
 \prod_{k=1}^{3} m_k(\eta) \sum_{k=0}^{2} c_k \frac{e^{\sigma(x-\eta)b_\lambda(x)}}{b_\lambda^k(x)}, & x < \eta < +\infty.
\end{cases}$$

(20)
and

\[
|M_3| \leq \begin{cases} 
\frac{1}{3} \prod_{k=1}^3 m_k(x) \frac{e^{-\sigma(x-\eta)b_\lambda(x)}}{b_\lambda^2(x)}, & -\infty < \eta < x, \\
\frac{2}{3} \prod_{k=1}^3 m_k(x) \frac{e^{\sigma(x-\eta)b_\lambda(x)}}{b_\lambda^2(x)}, & x < \eta < +\infty.
\end{cases}
\tag{21}
\]

Now we apply the estimates (19), (20), and (21), and also Lemma 1 for estimating the norms of the operators \(M_j(\lambda)\) \((j = 1, 2, 3)\). In this way we find that the following inequalities hold:

\[
\|M_1(\lambda)\|_{L_p \to L_p} \leq \sup_{\eta \in \mathbb{R}} \int \frac{\mu}{\nu} \left| q(\eta) + \lambda + ir(\eta) \right| \left| \frac{\prod_{k=1}^3 m_k(\eta)}{\prod_{k=1}^3 m_k(x)} - \frac{\prod_{k=1}^3 m_k(\eta)}{\prod_{k=1}^3 m_k(x)} \right| dx.
\]

Further, if we use conditions (2), (3), (5), then we find that

\[
\|M_1(\lambda)\|_{L_p \to L_p} \leq c \sup_{\eta \in \mathbb{R}} \int \frac{\mu}{\nu} \left| q(\eta) + \lambda + ir(\eta) \right| \left| \frac{\prod_{k=1}^3 m_k(\eta)}{\prod_{k=1}^3 m_k(x)} - \frac{\prod_{k=1}^3 m_k(\eta)}{\prod_{k=1}^3 m_k(x)} \right| dx.
\]

Further, if we use conditions (2), (3), (5), then we find that

\[
\|M_1(\lambda)\|_{L_p \to L_p} \leq c \sup_{\eta \in \mathbb{R}} \int \frac{\mu}{\nu} \left| q(\eta) + \lambda + ir(\eta) \right| \left| \frac{\prod_{k=1}^3 m_k(\eta)}{\prod_{k=1}^3 m_k(x)} - \frac{\prod_{k=1}^3 m_k(\eta)}{\prod_{k=1}^3 m_k(x)} \right| dx.
\]

We assume that \(\eta - x = \frac{1}{\sigma b_\lambda(\eta)} z\). Then

\[
\|M_1(\lambda)\|_{L_p \to L_p} \leq
\]

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Hence, calculating the integrals and using conditions (2), (3), (4), we obtain

\[ c \left| \frac{q(\eta) + \lambda + i\eta}{(cb_\lambda(\eta))^{n+3}} \right| \nu + c \left| \frac{q(\eta) + \lambda + i\eta}{(b_\lambda(\eta))^{n+3}} \right| \nu = c \left( \frac{|q(\eta) + \lambda + i\eta|}{\prod_{k=1}^{3} m_k(\eta)} \right)^{\nu - \frac{1}{3}}. \]

Moreover, according to the condition (2), we have \( \frac{|q(\eta) + \lambda + i\eta|}{\prod_{k=1}^{3} m_k(\eta)} \geq \sqrt{1 + \lambda} \).

Therefore the estimate (13) follows from the previous inequality.

We now prove the estimate (14). In accordance with conditions (2) - (4) of Theorem 1, we have that

\[
\|M_2(\lambda)\|_{L^p_{\eta}} \leq \sup_{\eta \in R} \int_{R} \left[ |M_2(x, \eta, \lambda)| + |M_2(\eta, \lambda, \lambda)| \right] dx \leq \]

\[
\leq c \sup_{\eta \in R} \int_{R} \left( \frac{3}{3} \prod_{k=1}^{m_k(\eta)} \frac{m_k(\eta)}{m_k(x)} \frac{e^{-\sigma(x-\eta) b_\lambda(x)}}{b_\lambda(x)} \right. + \]

\[ + \frac{3}{3} \prod_{k=1}^{m_k(\eta)} \frac{m_k(\eta)}{m_k(x)} \frac{e^{-\sigma(x-\eta) b_\lambda(\eta)}}{b_\lambda(\eta)} \left. \right) dx. \]

Hence, calculating the integrals and using conditions (2), (3), (4), we obtain that

\[
\|M_2(\lambda)\|_{L^p_{\eta}} \leq c \sup_{\eta \in R} \int_{R} \left[ e^{-\sigma(x-\eta) b_\lambda(\eta)} + \frac{e^{-\sigma(x-\eta) b_\lambda(\eta)}}{b_\lambda(\eta)} + \frac{e^{-\sigma(x-\eta) b_\lambda(\eta)}}{b_\lambda^2(\eta)} \right] dx = \]

\[
= c \sup_{\eta \in R} \left[ \frac{1 - e^{-\sigma b_\lambda(\eta)}}{\sigma b_\lambda(\eta)} + \frac{(1 - e^{-\sigma b_\lambda(\eta)})}{\sigma b_\lambda^2(\eta)} + \frac{(1 - e^{-\sigma b_\lambda(\eta)})}{\sigma b_\lambda(\eta)} \right] \leq \frac{c}{b_\lambda(\eta)}. \]
so (14) is proved. Now we will prove (15). Applying the estimate (21) it yields that

\[ \|M_3(\lambda)\|_{L^p \to L^p} \leq \sup_{\eta \in \mathbb{R}} \int_{\mathbb{R}} \left[ |M_3(x, \eta, \lambda)| + |M_3(\eta, x, \lambda)| \right] dx \leq \]

\[ \leq \sup_{\eta \in \mathbb{R}} \int_{\eta^{-1}}^{\eta+1} \left( \frac{2}{3 \prod_{k=1}^{3} m_k(x)} e^{-\sigma(x-\eta)b_\lambda(x)} + \frac{2}{3 \prod_{k=1}^{3} m_k(\eta)} e^{-\sigma(\eta-x)b_\lambda(\eta)} \right) dx. \]

Next, we take into account the condition (3) and obtain that

\[ \|M_3(\lambda)\|_{L^p \to L^p} \leq \]

\[ \leq c_3 \sup_{\eta \in \mathbb{R}} \int_{\eta^{-1}}^{\eta+1} \left( \frac{2}{3(c_1^{-1})^2 \prod_{k=1}^{3} m_k(\eta)} \exp[-\sigma(x-\eta)\tilde{c}b_\lambda(\eta)] + \right. \]

\[ + \frac{2}{3 \prod_{k=1}^{3} m_k(\eta)} \exp[-\sigma(\eta-x)b_\lambda(\eta)] \left. \right) dx \leq \]

\[ \leq 2c_3 \sup_{\eta \in \mathbb{R}} \left( \frac{c}{(c_1^{-1})^2 \prod_{k=1}^{3} m_k(\eta)b_\lambda(\eta)} + \frac{c}{\prod_{k=1}^{3} m_k(\eta)b_\lambda(\eta)} \right) \leq \frac{c_4}{\prod_{k=1}^{3} m_k(\eta)b_\lambda(\eta)}. \]

Thus also (15) is proved and the proof is complete. □

**Proof of Lemma 3.** It is obvious that

\[ (L + \lambda E) [M_3(\lambda)f](\eta) = \]

\[ = -m_1(\eta) \left( m_2(\eta) \int m_3(\eta) \left( \int_{-\infty}^{\eta} M_0(x, \eta, \lambda) \theta(\eta-x) f(x) dx \right) + \right. \]

\[ \left. \right. \]

\[ \]

\[ 12 \]
\[
\begin{align*}
+ \int_{\eta}^{+\infty} M_0(x, \eta, \lambda) \theta(\eta - x) f(x) dx \right) \left( \int_{\eta}^{+\infty} M_0(x, \eta, \lambda) \theta(\eta - x) f(x) dx \right)(\eta).
\end{align*}
\]

We calculate the derivatives and then by using (10)-(12), we find that (16) holds. The proof is complete. □

**Proof of Theorem 1.** By condition (5), the expressions on the right-hand sides of inequalities (13) and (14) tend to zero as \( \lambda \to +\infty \). Therefore, there exists a number \( \lambda_0 > 0 \) such that when \( \lambda \geq \lambda_0 \) the inequality

\[
\|M_1(\lambda)\|_{L_p \to L_p} + \|M_2(\lambda)\|_{L_p \to L_p} \leq \frac{1}{2}
\]

holds. Hence when \( \lambda \geq \lambda_0 \) the operator \( G(\lambda) := E + M_1(\lambda) + M_2(\lambda) \) has a bounded inverse \( G^{-1}(\lambda) \) in \( L_p \). Suppose that \( h = [E + M_1(\lambda) + M_2(\lambda)] f \).

Then from the relation (16) we obtain that

\[
(L + \lambda E) [M_3(\lambda)G^{-1}(\lambda)h](\eta) = h.
\]

From this and from Definition 1 it follows that the function \( y = M_3(\lambda)G^{-1}(\lambda)f \) is a solution of equation (1). The proof is complete. □

**Proof of Lemma 4.** The Lemma can be proved similarly as the proof of Theorem 1. So we leave out the details.

**Proof of Theorem 2.** Lemma 4 implies that the operator \((L + \lambda E)\)' acting in the space \( L_{p'}(R) \), at \( \lambda \geq \lambda_1 \) has a right inverse, which is defined on \( L_p(R) \). Then, according to the general theory of linear closed operators \( ker((L + \lambda E)')^* = \{0\} \), where \(((L + \lambda E)')^*\) is the operator adjoint to \((L + \lambda E)'\). It is easy to see that \(((L + \lambda E)')^*\) is an extension of the operator \( L + \lambda E \). Hence, \( ker(L + \lambda E) = \{0\} \) for all \( \lambda \geq \lambda = \max(\lambda_0, \lambda_1) \). Thus, the operator \( L + \lambda E \) is a boundedly invertible operator in the space \( L_{p'}(R) \) \((1 < p' < \infty)\) and

\[
(L + \lambda E)^{-1} = M_3(\lambda)G^{-1}(\lambda), \quad \lambda \geq \lambda = \max(\lambda_0, \lambda_1).
\]

(22)
Let $y$ be a solution of equation (1), where $\lambda \geq \tilde{\lambda} = \max(\lambda_0, \lambda_1)$. We shall prove the estimate (7). By using the relation (22), Lemma 1 and the conditions (2) - (5) we obtain that

$$\| (q + \lambda + ir)(L + \lambda E)^{-1} \|_{L_p \to L_p} = \| (q + \lambda + ir)M_3(\lambda)G^{-1}(\lambda) \|_{L_p \to L_p} \leq$$

$$\leq c \sup_{\eta \in R} \int_{\eta-1}^{\eta+1} b_\lambda^4(\eta)b_\lambda^{-2}(x) \exp[-\sigma|x - \eta|b_\lambda(x)]dx \leq$$

$$\leq c_1 \sup_{\eta \in R} b_\lambda(\eta) \int_{\eta-1}^{\eta+1} \exp[-\sigma|x - \eta|b_\lambda(x)]dx < \infty.$$  

Then, in view of (1), we have that

$$\left\| m_1(x) (m_2(x) (m_3(x)y')')' \right\|_p \leq c \left( \|f\|_p + \|y\|_p \right).$$

By combining the last two estimates we obtain (7). The theorem is proved. □

References


