On Knots and DNA

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LiTH-MAT-EX–2017/17–SE

Credits: **16 hp**

Level: **G2**

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Linköping: **December 2017**

Abstract

Knot theory is the mathematical study of knots. In this thesis we study knots and one of its applications in DNA. Knot theory sits in the mathematical field of topology and naturally this is where the work begins. Topological concepts such as topological spaces, homeomorphisms, and homology are considered. Thereafter knot theory, and in particular, knot theoretical invariants are examined, aiming to provide insights into why it is difficult to answer the question "How can we tell knots appart?". In knot theory invariants such as the bracket polynomial, the Jones polynomial and tricolorability are considered as well as other helpful results like Seifert surfaces. Lastly knot theory is applied to DNA, where it will shed light on how certain enzymes interact with the genome.

Keywords:

Knot theory, Topology, Homology, Jones polynomial, Bracket polynomial, Tangles, DNA.

URL for electronic version:

`http://urn.kb.se/resolve?urn=urn:nbn:se:liu:diva-144294`

Acknowledgements

There are many people I would like to thank. First of all my family. Throughout my life they have always given me encouragement and love, and they have believed in me even when I did not believe in myself.

To my supervisor Milagros Izquierdo I would like to say thank you for your patience, support, feedback and guidance, and to my examiner Göran Bergqvist I wish to express my gratitude for your thorough review which has ensured the quality of this work. I am also thankful for all the valuable thoughts and feedback given to me by my opponent Erik Bråmå on how to improve my work.

Last but "knot" least I wish to thank my boyfriend Kalle Trenter for his understanding and unending support, as well as all of my friends and colleagues for hours of interesting discussions.

Nomenclature

$\mathbb{S}^n$	The n-sphere in $\mathbb{R}^m$, $n \leq m$.
B^n	The open n-ball R^n.
$\overline{A}$	The closure of a set A.
$A \times B$	The Cartesian product.
$G \simeq H$	Two isomorphic groups.
$\widetilde{\mathbf{0}}$	The Trivial group $\{0\}$.
$\mathbb{Z}$	The integers (infinite cyclic group).
$\mathbb{Z}_n$	The cyclic group of order n.
X, Y	Topological spaces.
$X^n, X^{\mathbb{N}}$	Finite and infinite product spaces.
$\langle K \rangle$	Kauffman polynomial of a link K.
(a, b)	The open interval $\{x \in \mathbb{R} : a < x < b\}$.
$[a, b]$	The closed interval $\{x \in \mathbb{R} : a \leq x \leq b\}$.
$\mathcal{T}$	The usual topology on $\mathbb{R}$.
$\hat{\mathcal{T}}$	The half-open topology on $\mathbb{R}$.
$A \setminus B$	The (relative) complement $\{x \in A :\notin B\}$.
$f^{-1}(A)$	The preimage of a set A, i.e. $\{\mathbf{x} \in X : f(\mathbf{x}) \in A\}$.
$X \cong Y$	Two homeomorphic spaces.
$Int(A)$	The interior of a set A.
$[z],\ \bar{z}$	The equivalence class of z.
$C_n(K)$	The group of n-chains.
$Z_n(K)$	The group of n-cycles.
$B_n(K)$	The group of n-boundaries.
$H_n(K)$	The n:th homology group.
K	A complex in homology and a knot in knot theory.
$\rightleftharpoons$	Equivalence of knot diagrams.

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Chapter 1

Introduction

Knot theory is the mathematical theory of knots. At this stage, you can think of a knot as a piece of string that you have tied and then glued the ends together. At first glance it might seem like a very unusual mathematical field. Does it really have any practical use? - Yes, it most certainly does! Knot theory can be applied to several areas, one of them being in the study of DNA.

DNA is the genetic code of every living thing and biologists study it with the purpose of better understanding life. Deeper knowledge about DNA would not only give more information about evolution, but it would also make it possible to find new and improved disease treatments. Because DNA is the instruction to how we function it is involved in several processes, processes that aim to decode or extract the information, or to improve or change the information. There are also processes that focus on keeping the DNA "tidy" to ensure that other processes work smoothly. The processes are conducted by enzymes. Enzymes facilitate reactions; some break down the food we eat, some help decode the genome and some are used in industry applications such as producing medicine. One group of enzymes called recombinase manipulate the genome by genetic recombinations. Recombinase can move one segment of DNA to another location on the genome or it can insert alien DNA into the genome. The latter is a key part of the life cycles of some viruses. Understanding the actions of some of these enzymes can be achieved by combining biology with mathematics. Tangle theory (a part of knot theory) can be used to understand how recombinase enzymes interact with DNA. [1]

To utilize the theory of tangles, one must first build a good basis in knot theory, looking for answers to questions like "What is a knot (mathematically)?" and "How do we tell knots apart?". The first question may seem trivial but it took some time before the definition of a knot was put on a firm mathematical

base. The second question is in fact one of the biggest unsolved questions in the field. Today, many knots can be distinguished from one another but there is still no invariant that can fully classify all knots. An **invariant** is a property of an object that never changes. Invariants help us tell objects apart. The invariant cannot change with the representation of the object but needs to remain the same. There are many thing that can be an invariant, e.g. a number, a polynomial, a group. Studying invariants will be a red thread throughout this work, not only invariants for knots but also for more general objects such as spaces.

The theory of knots is a subfield to the field known as topology. Topology can be described as the study of which properties of objects are left unchanged under continuous deformations. Topologists study ways to tell objects (often spaces) apart, and knot theory utilizes many of the tools and techniques found in topology.

This bachelor thesis will begin with a mathematical background, Chapter 2, that includes concepts from both algebra and topology. In the section on topology homeomorphisms, a concept central to this work, is described. In Chapter 3 the theory of homology is developed. Homology is a powerful tool used to study the structure of spaces. After Chapter 3 we have gained enough topological understanding to dive into Chapter 4: the theory of knots. This chapter covers many concepts in knot theory, however, it is still just a taste of all the wonders this field has to offer. This chapter is concluded by the Jones polynomial, a strong invariant for knots. In the final chapter, Chapter 5, tangles are described as well as some fantastic results they have given to the field of DNA-research.

Chapter 2

Background

This chapter covers preliminary concepts needed for the rest of this thesis. Both algebraic concepts (such as groups and rings) and topological concepts (such as topological spaces, continuity and compactness) are described. A big part of the section on topology are homeomorphisms, as they are essential to the rest of this thesis.

2.1 Algebra

In this section we shall go through some important concepts in algebra. A good account of algebraic concepts can be found in [18] and [11]. The key concept in this section will be Polynomial rings.

Definition 2.1.1. A *Group*, denoted $(G, *)$, is a set of elements G together with an operation $*$, that satisfy the following properties

1. Associativity: $*$ is associative, i.e. $(x * y) * z = x * (y * z)$ for all elements $x, y, z \in G$.
2. Identity element: There exists a unique identity element, $e \in G$ s.t. $e * x = x * e = x$ for all $x \in G$.
3. Inverse: For each element $x \in G$ exists an inverse x^{-1} s.t. $x * x^{-1} = x^{-1} * x = e$.

Moreover, if the operation is commutative ($x * y = y * x$) then the group $(G, *)$ is called an *abelian group*.

Example 2.1.2. The set $\mathbb{Z}$ of integers together with addition form a group, $(\mathbb{Z}, +)$. Addition with integers is associative, it has the identity element 0, and

$\forall x \in \mathbb{Z},\ \exists y \in \mathbb{Z} : x + y = 0$, y is the inverse and can also be denoted $-x$. $(\mathbb{Z}, +)$ is abelian. $\mathbb{N}$ with addition is an simple example of a set that fails to be a group.

Definition 2.1.3. A group G is called *cyclic* if it can be generated by a single element a, such that $G = \langle a \rangle = \{k * a : k \in \mathbb{Z}\}$.

Example 2.1.4. $\mathbb{Z}$ is a cyclic group $\mathbb{Z} = \langle 1 \rangle = \{1 * k : k \in \mathbb{Z}\}$.

The **order** of a finite group is the number of elements in the group. The set of (positive) integers modulo n form the finite cyclic group Z_n of order n. The **rank**, $rk(G)$, of a group G is the minimum number of generators needed to generate G. For example $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^3$ has rank 3, and every cyclic group (e.g. $\mathbb{Z}_6$) has rank 1.

Definition 2.1.5. Let $(G, *_G)$, $(H, *_H)$ be groups. A *homomorphism* $h : G \longrightarrow H$ is a function such that $\forall g_1, g_2 \in G$ it holds that $h(g_1 *_G g_2) = h(g_1) *_H h(g_2)$. An *isomorphism* is when we also require h to be a bijection. Two groups are therefore called isomorphic, denoted $G \simeq H$ if there is exists an isomorphism between them.

Example 2.1.6. The most general abelian group is a product of cyclic groups, finite or infinite, and can therefore be written as

$$\mathbb{Z}^r \times \mathbb{Z}_{n_1} \times ... \times \mathbb{Z}_{n_k}$$

where r is the rank. This a actually a very powerful result called the Structure Theorem and a proof can be found in [18].

From groups we now go on to create something called *rings*.

Definition 2.1.7. A *Ring*, denoted $(R, +, *)$, is a set of elements R together with two operations $+$ and $*$, called addition and multiplication respectively, that satisfy the following properties

1. $(R, +)$ is an abelian group.
2. $*$ is associative.
3. $*$ is distributive over $+$, meaning that $x * (by + z) = (x * y) + (x * z)$ and $(x + y) * z = (x * z) + (y * z)$ hold for all $x, y, z \in R$.

If $*$ is commutative we way that R is a *commutative ring*.

Example 2.1.8. Several of the most common sets of numbers $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ together with normal addition and multiplication are rings. As is $M_n(\mathbb{R})$, the set of all $n \times n$ matrices, under normal matrix addition and multiplication.

Now it is time to look at **polynomial rings** which basically are sets of polynomials with coefficients from a ring. It is called a polynomial ring because the set together with addition and multiplication will form a ring. Let's be precise.

Definition 2.1.9. Let R be a ring, then a polynomial p in x over R is a formal sum

$$p(x) = \sum_{k=0}^{\infty} a_k x^k$$

where $a_k \in R$ for all k and all but a finite number of $a_k = 0$. The set of all such polynomials is denoted $R[x]$.

The set of polynomials $R[x]$ is a ring with addition and multiplications defined as follows.

Definition 2.1.10. 1. The *sum* of two polynomials, $p_1(x) = \sum_{k=0}^{\infty} a_k x^k$ and $p_2(x) = \sum_{k=0}^{\infty} b_k x^k$ where a_k, $b_k \in R$, is defined as

$$p_1(x) + p_2(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k.$$

2. The *product* of two polynomials $p_1(x)$ and $p_2(x)$ is defined as

$$p_1(x)p_2(x) = \sum_{k=0}^{\infty} c_k x^k,$$

where $c_k = \sum_{j=0}^{k} a_j b_{k-j}$.

Looking at the definitions we see that addition and multiplication are done in the same fashion that we are used to. We add term by term and we multiply "crosswise".

Example 2.1.11. If $\mathbb{Z}_6$ is the set of integers modulo 6, with normal addition and multiplication, modulo 6, the $(\mathbb{Z}_6, +, *)$ forms a ring. Now, let $p_1(x) = x^2 + x + 3$ and $p_2(x) = 2x + 4$ be two polynomials in the ring $\mathbb{Z}_6[x]$, then

$$\begin{aligned} p_1(x) + p_2(x) &= (x^2 + x + 3) + (2x + 4) = \\ &= x^2 + (1 + 2)x + (3 + 4) = \\ &= x^2 + 3x + 1 \end{aligned}$$

and

$$\begin{aligned} p_1(x)p_2(x) &= (x^2+x+3)(2x+4) = \\ &= (1*2)x^3 + (1*4+1*2)x^2 + (1*4+3*2)x + (3*4) = \\ &= 2x^3 + 4x. \end{aligned}$$

Definition 2.1.12. A *field* F is a commutative ring R with identity 1, in which every nonzero element has a multiplicative inverse.

This means that if R is a field $(R, *)$ is an abelian group.

Definition 2.1.13. A *Laurent polynomial* with coefficients a_i in a field F, is the formal sum

$$p(x) = \sum_k a_k x^k, \ k \in \mathbb{Z},$$

with finitely many of the coefficients nonzero.

Simply put, a Laurent polynomial is a polynomial for which we allow negative powers of x. We denote the collection of polynomials over the field F by $F[x, x^{-1}]$. $F[x, x^{-1}]$ is a ring.

2.2 Topology

There are many concepts in knot theory that most people intuitively can understand. This allows for interesting and easy introductions in popular science (but oddly enough, most people are still unaware of knot theory). However, in order to dive deeper into knot theory, to be able to fully understand it, we first need to learn some topology. This section is meant to give readers who have not taken a course in topology enough background to be able to follow and understand the concepts of this thesis. We follow the books [2], [5], and [11] in this section.

In topology we rid ourselves of the concepts of distance when determining if two objects are near. Transforming spaces continuously is a cornerstone in this thesis. If you, by using for example stretching and bending, can transform A into B, and B into A, you can consider them to be the same, this is the notion of homeomorphisms. Homeomorphisms give equivalence between spaces and we wish to study properties that remain intact under homeomorphisms. We must be very restrictive with operations such as cutting and pasting. Combinations of these are often not continuous and can therefore yield non-homeomorphic spaces. See Figure 2.1. To emphasize the concept of homeomorphism we begin will the following definition.

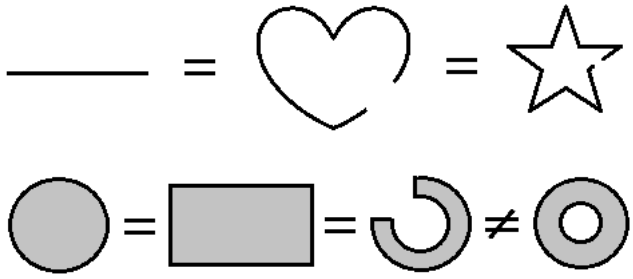


Figure 2.1: Examples of homeomorphisms.

Definition 2.2.1. Let A and B be topological spaces. Then A is topologically equivalent or *homeomorphic* to B if there is a continuous invertible function $f : A \to B$ with continuous inverse $f^{-1} : B \to A$. Such a function f is called a homeomorphism.

We need more knowledge to fully grasp this definition. In the following sections all needed concepts will be explained.

2.2.1 Topological Spaces

A topological space is a very general concept of a space. Later we will see that metric spaces are specializations of topological spaces where the topology is given by distances.

Basic concepts

Definition 2.2.2. A *topological space* is a set X with a collection $\mathcal{B}$ of collections $\mathcal{B}_x$, for all $x \in X$, of nonempty subsets $N \subseteq X$, called neighborhoods, such that

- every point is in some neighborhood, i.e.,

$$\forall x \in X,\ \exists N \in \mathcal{B}_x \text{ such that } x \in N$$

- the intersection of any two neighborhoods of a point contains a neighborhood of the point, i.e.,

$$\forall N_1, N_2 \in \mathcal{B}_x,\ \exists N_3 \in \mathcal{B}_x \text{ such that } x \in N_3 \subseteq N_1 \cap N_2$$

- for every neighborhood N of a point there is a smaller neighborhood N_0 such that each point in N_0 has a neighborhood contained in N, i.e.,

  for $N \in \mathcal{B}_x$, $\exists N_0 \in \mathcal{B}_x$ such that $\forall y \in N_0$ there is some $V_y \in \mathcal{B}_y$ with $V_y \subset N$.

The set $\mathcal{B}_x$ is called a neighborhood basis for x, and $\mathcal{B} = \cup\mathcal{B}_x$ generate a topology as follows: a subset $O \subseteq X$ is an open set if for each $x \in O$, there is a neighborhood $N \in \mathcal{B}$ such that $x \in N$ and $N \subseteq O$. The set $\mathcal{T}$ of all open sets is a topology on the set X, and the set $\mathcal{B}$ is called a basis for the topology on X.

Remark. By this definition $\emptyset$ is always open.

For a set X with topology $\mathcal{T}$ we will use the notation $(X, \mathcal{T})$ to refer to the topological space. However, when there is no ambiguity, this notation will sometimes be abused and X will stand for a topological space.

In the beginning of this section we mentioned that a metric space is a topological space, we shall now look at the Euclidean space.

Example 2.2.3. With the Euclidean distance metric in $\mathbb{R}^n$,

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + ... + (x_n - y_n)^2},$$

we can form n-dimensional balls, $B(\mathbf{x}, r) = \{y \in \mathbb{R}^n : d(\mathbf{x}, \mathbf{y}) < r\}$ of radius $r > 0$, around any point $\mathbf{x}$. Together these balls form a basis $\mathcal{B} = \{B(\mathbf{x}, r) : \mathbf{x} \in \mathbb{R}^n, r > 0\}$ for the topology on $\mathbb{R}^n$.

More generally: If X is a set with a metric d (metric space), then the set of balls $\mathcal{B} = \{B(\mathbf{x}, r), \mathbf{x} \in X, r > 0\}$ is a basis for a topology on X, where $B(\mathbf{x}, r) = \{y \in X : d(\mathbf{x}, \mathbf{y}) < r\}$ for $r > 0$.

Example 2.2.4. For $\mathbb{R}$ we construct $\mathcal{B} = \{(a, b) : x \in \mathbb{R},\ a < x < b\}$, the set of all open intervals on $\mathbb{R}$. The topology $\bar{\mathcal{T}}$ given by $\mathcal{B}$ is called *usual topology* on $\mathbb{R}$. The open neighborhoods are (a, b).

The construction of topology above is equivalent to the topology in Theorem 2.2.5 that is stated below. This theorem can be taken as the definition of a topology.

Theorem 2.2.5 (cf. [11])**.** *$\mathcal{T}$ is a topology on X iff*

1. *X and $\emptyset$ are elements of $\mathcal{T}$.*
2. *The union of any collection of elements in $\mathcal{T}$ is in $\mathcal{T}$.*
3. *The intersection of any finite collection of elements in $\mathcal{T}$ is in $\mathcal{T}$.*

Theorem 2.2.5 gives a new way to determine if $\mathcal{T}$ is a topology on a set X.

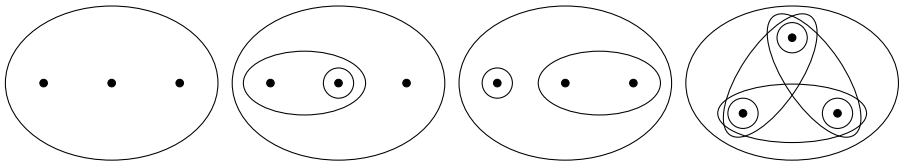


Figure 2.2: Four examples of topologies from Example 2.2.6.

Example 2.2.6. Let $X = \{x, y, z\}$ be three points. Four examples of topologies on X can be found in Figure 2.2. They are (from left to right):

- $\mathcal{T}_1 = \{\emptyset, X\}$,
- $\mathcal{T}_2 = \{\emptyset, \{y\}, \{x, y\}, X\}$,
- $\mathcal{T}_3 = \{\emptyset, \{x\}, \{y, z\}, X\}$, and
- $\mathcal{T}_4 = \{\emptyset, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, X\}$.

Non of these topologies are homeomorphic to one another. The first one $\{\emptyset, X\}$ is called the trivial topology, and the last one is called the discrete topology on X. There are several more topologies on X - simply find other combinations of subsets that satisfy Theorem 2.2.5.

Definition 2.2.7. Let C be a subset of a topological space X with topology $\mathcal{T}$, then we say that C is *closed* if $X \setminus C$ is open.

Example 2.2.8. Let $X = \{x, y, z\}$, $\mathcal{T} = \{\emptyset, \{y\}, \{x, y\}, \{y, z\}, X\}$, $C_1 = \{x\}$ and $C_2 = \{x, y\}$. Are C_1 and C_2 closed? Well, C_1 is closed if the complement is open (i.e. $X \setminus C_1 \subseteq \mathcal{T}$). $X \setminus C_1 = \{y, z\} \subseteq \mathcal{T}$ so $X \setminus C_1$ is open and C_1 is closed. Similarly, $X \setminus C_2 = \{z\} \not\subseteq \mathcal{T}$ so $X \setminus C_2$ is not open and therefore C_2 cannot be closed.

Open and closed sets depend on our choice of basis and topology. Changing them may change which sets we consider to be open and closed - altering all concepts that in turn depend on these, such as, compactness and continuity.

This makes things a little abstract. For instance, the trivial topology has only one neighborhood $N = X$. $\emptyset$ and X are, by definition, the only open *and* the only closed sets.

From the definition of neighborhoods we get that they can be any non-empty set, so one neighborhood of the point x could be $\{x\}$. For the discrete topology we have that any subset of X is both open and closed.

For $\mathbb{R}$ the standard neighborhoods are the open intervals (a, b) and the usual topology is the set of all of these open intervals, as seen in Example 2.2.4. On $\mathbb{R}$ there is a different, very interesting, topology called the *half-open topology*.

Example 2.2.9. Let the interval [a,b) with $a,\ b \in \mathbb{R}$, be a neighborhood $\forall x \in [a,b)$. There is a topology $\hat{\mathcal{T}}$ on $\mathbb{R}$ where set of intervals $\mathcal{B} = \{[a,b)\}$ form a basis for $\hat{\mathcal{T}}$. $\hat{\mathcal{T}}$ is called the half-open topology. The interval $[1,2)$ will be a neighborhood for the point 1, and is by definition an open set. Furthermore this half-open topology is fundamentally different form the usual topology with open neighborhoods (a,b), because there is no interval of the form (a,b) that is a neighborhood of 1 *and* also a subset of $[1,2)$.

The following definition gives us a property that makes spaces nice and informative to study.

Definition 2.2.10. A topological space is *Hausdorff* if, for any two distinct points in X, there are disjoint open sets, one containing one of the points, the other containing the other.

That a space is Hausdorff is a good property to have. It means that we can study each point individually, it is not hidden in a cluster of points. All spaces we will be working with are Hausdorff. In numerical analysis if you consider a situation where you have data from real numbers and an error margin, the resulting space will not be Hausdorff. Computers only allow a finite (in many cases fixed) number of decimals, and hence does not save enough information to study every point in $\mathbb{R}$ individually. There are several intervals in which you cannot tell points apart. This problem is common in real applications on data sets.

Example 2.2.11. Both $(\mathbb{R}, \bar{\mathcal{T}})$ and $(\mathbb{R}, \hat{\mathcal{T}})$ are Hausdorff.

Example 2.2.12. In Example 2.2.6 only $\mathcal{T}_4$ is Hausdorff. In $\mathcal{T}_3$, for example, we can never study just the point z, we must always consider y and z together.

From a topological space $(X, \mathcal{T})$ we can create a new space by taking a subset A of X and using $\mathcal{T}$ to induce a topology on A. The following definition explains how.

Definition 2.2.13. Let X be a topological space with topology $\mathcal{T}$ and let $A \subseteq X$. A neighborhood of a point $x \in A$ relative to A is of the form $N \cap A$ where N is a neighborhood of x in X. The topology $\mathcal{T}_A$ generated by this basis is called the *subspace topology* on A induced by the topology $\mathcal{T}$ on X.

From any topological space we can make new subspaces by using this methodology. The neighborhoods in the subspace topology $(N \cap A)$ are open relative to A, but not necessarily open relative to X. For the Euclidean space with the normal topology (see Example 2.2.3) we can take any subset and induce a subspace topology. For some subsets (e.g. closed ones) there will be neighborhoods in the subspace topology that are not a part of the normal topology.

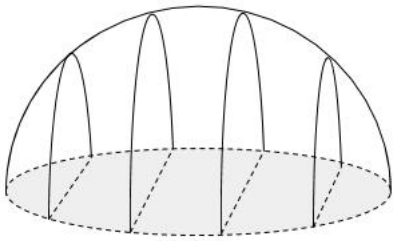


Figure 2.3: Unit disk to hemisphere.

Continuity, Connectedness & Compactness

Continuity is an important concept in many areas of mathematics and topology is the right setting for studying it. Several things depend on continuity, but first things first, we begin with the definition.

Definition 2.2.14. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A function $f : X \longrightarrow Y$ is *continuous* if for every $A \in \mathcal{T}_Y$, $f^{-1}(A) \in \mathcal{T}_X$.

This is equivalent to saying that for each $x \in X$, given a neighborhood $N_{f(x)}$ in $\mathcal{T}_Y$, there is a neighborhood N_X in $\mathcal{T}_X$ such that $x \in N_X$ which belongs to the preimage of $N_{f(x)}$. The inverse is given by $f^{-1}(A) = \{\mathbf{x} \in X : f(\mathbf{x}) \in A\}$.

Remark. We do not require f^{-1} to be a function, and in general, it is usually not one.

Example 2.2.15. The function $f : \mathbb{R}^2 \setminus \{(0,0)\} \longrightarrow \mathbb{S}^1$, given by $f(\mathbf{x}) = \frac{\mathbf{x}}{\|\mathbf{x}\|}$, is continuous. The inverse of f does not exist.

Example 2.2.16. Let f be a vertical projection from the open unit disk to the open hemisphere, see Figure 2.3. Then $f : (x, y) \longrightarrow (x, y, \sqrt{1 - x^2 - y^2})$, is a continuous function, as is f^{-1}.

Leaving continuity, our next step is connectedness. It is quite intuitive what connectedness is: connected spaces are in some sense "stuck together" - we could say that from any point in the set you can reach any other without ever leaving the set.

Definition 2.2.17. A topological space X is *connected* if X cannot be written as a union of two non-empty disjoint open sets.

We need to take care when applying this definition. All possible divisions of the space must be considered otherwise there is a possibility that a "gap" is hidden in one of the two open sets.

Example 2.2.18. The open interval $(-1, 1)$ is connected, as is the torus (Figure 2.4), the sphere and the ball.

Example 2.2.19. $(\mathbb{R}, \bar{\mathcal{T}})$ is connected but $(\mathbb{R}, \hat{\mathcal{T}})$ is not since the latter can be written as the disjoint union $(-\infty, 0) \cup [0, \infty)$.

The following theorem ties together the concepts of continuity and connectedness.

Theorem 2.2.20 (cf. [11])**.** *Let X and Y be topological spaces and $f : X \longrightarrow Y$ a continuous function onto Y. If X is connected, then Y is connected.*

Definition 2.2.21. Let A be a subset of a topological space X. An *open cover* of A is a collection $\mathcal{O}$ of open subsets of X so that A lies in the union of the elements of $\mathcal{O}$, i.e.,

$$A \subseteq \bigcup_{O \in \mathcal{O}} O$$

A subcover of $\mathcal{O}$ is a collection $\mathcal{O}' \subseteq \mathcal{O}$ so that A lies in the union of the elements in $\mathcal{O}'$. A finite cover (or subcover) is a cover $\mathcal{O}$ consisting of finitely many sets.

Definition 2.2.22. A topological space X is *compact* if every open cover of X has a finite subcover.

Example 2.2.23. The n-sphere is compact for $n \in \mathbb{N}$. Any finite topological space is compact, such as the spaces in Example 2.2.6.

Example 2.2.24. An infinite set with the discrete topology is not compact.

To conclude this part we will state two theorems concerning the compactness of subsets, and the connection between continuous functions and compact spaces. Thereafter we will move on to new ways of constructing spaces by using products and quotients.

Theorem 2.2.25 (cf. [11])**.** *If X is a compact topological space and A is a closed subset of X then A is compact.*

Theorem 2.2.26 (cf. [11])**.** *Let X be a compact topological space and $f : X \longrightarrow Y$ a continuous function from X onto a topological space Y. Then Y is compact.*

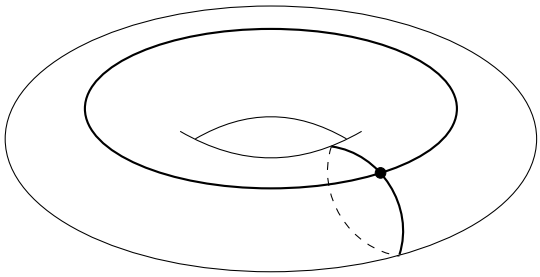


Figure 2.4: Torus: $S^1 \times S^1$.

Product & Quotient Spaces

This subsection starts by looking at the product of spaces and then it moves on to the quotients of spaces. The reader will have used the concept of a product space many times but perhaps without calling it so. For instance, $\mathbb{R}^3$ is the (Cartesian) product $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$. By taking the product of spaces we can create new, interesting, topological spaces for us to study.

Definition 2.2.27. Let X be a topological space with topology $\mathcal{T}$ and Y a topological space with topology $\mathcal{T}'$. A basis for the *product topology* on $X \times Y$ is given by $\mathcal{B}$ where $N \in \mathcal{B}$ if N is on the form $N = O \times O'$ for $O \in \mathcal{T}$ and $O' \in \mathcal{T}'$. Projection functions $p_X : X \times Y \longrightarrow X$ and $p_Y : X \times Y \longrightarrow Y$ are defined by

$$p_X(x, y) = x$$
$$p_Y(x, y) = y.$$

p_X and p_Y are continuous and $X \times Y$ is called a *product space*.

Example 2.2.28. If we take a circle, S^1 and an interval $[a, b] \subset \mathbb{R}$, the product will be a closed cylinder. The product of two circles, $S^1 \times S^1$, is a torus (see Figure 2.4).

For product spaces in general we have that for given $X = \prod X_i$ and $f_i : X \longrightarrow X_i$, the product topology is the minimal topology on X that makes all f_i continuous. Two valuable results about product spaces are that the product of connected topological spaces is connected, and the product of compact topological spaces is compact.

With product spaces and continuity we can study continuous deformation from one function to another. Imagine the function $f_0(x) = 2sin(x)$, if you continuously increase the amplitude you could obtain the function $f_1(x) =$

$10sin(x)$. We say that two functions are homotopic if one can be continuously transformed into the other. Formally we take the following definition.

Definition 2.2.29. Let X and Y be topological spaces and $I = [0, 1]$. Then $H : X \times I \longrightarrow Y$, given by $H(x, t) = H_t(x)$ for $x \in X$ and $t \in I$, is a *homotopy* between the continuous functions $f_0 : X \longrightarrow Y$ and $f_1 : X \longrightarrow Y$ if H continuous and satisfies

$$H(x, 0) = f_0$$
$$H(x, 1) = f_1.$$

Remark. It is important that the reader observes the fact that all $H_t(x)$ are required to be continuous for $t \in I$.

Example 2.2.30. In topology one usually talks about spaces being "elastic", we can now describe stretching in terms of homotopy. Take $\mathbb{R}^2$ and stretch it out so that each point is sent twice as far away from origo. We can construct a continuous function $H : \mathbb{R}^2 \times I \longrightarrow \mathbb{R}^2$ given by

$$H(x, 0) = H_0(x) = x$$
$$H(x, t) = H_t(x) = (1 + t)x$$
$$H(x, 1) = H_1(x) = 2x.$$

H is a homotopy between H_0 and H_1.

Now we move on to quotient spaces. Similarly to product spaces, quotient spaces will yield new spaces for us to study. They occur when we identify (or glue together) certain points in a space. If you would, for example, take the boundary of a disk to a single point (without shrinking the surface area), the result will be a sphere. We start with the definition of a quotient topology before we define a quotient space (also known identification space).

Definition 2.2.31. Let $(X, \mathcal{T})$ be a topological space and let $f : X \longrightarrow Y$ be a function onto the set Y. Define the *quotient topology* $\mathcal{T}'$ on Y by defining $U \subseteq Y$ to be open in Y if

$$U \in \mathcal{T}' \text{ iff } f^{-1}(U) \in \mathcal{T}$$

Remark. There is no demand for an initial topological structure on Y.

Instead of specifying a function that satisfies Definition 2.2.31 one can simply specify just the points that should be glued together according to an equivalence relation - saying that the points that are glued are equivalent.

Example 2.2.32. Take a rectangle and imagine gluing two of the opposite sides together to form a cylinder. This is the same as saying that the points on one edge are equivalent to those on the other.

Example 2.2.33. Let X be a topological space with $\sim$ an equivalence relation defined on X. The equivalence class of $x \in X$ is given by

$$[x] = \{y \in X : y \sim x\},$$

and the *identification space* $X/\sim$ is the set of equivalence classes of the relation $\sim$, equipped with the quotient topology:

$$X/\sim = \{[x] : x \in X\}.$$

The topology on X induces a topology in $X/\sim$. This is the quotient topology from Definition 2.2.31.

Example 2.2.34. To formalize Example 2.2.32; let $X = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$ and form the equivalence classes

$$[(x, y)] = \begin{cases} \{(x, y)\} & \text{if } x \neq 0, 1 \text{ and } 0 \leq y \leq 1, \\ (0, y) \sim (1, y) & \text{if } x = 0, 1 \text{ and } 0 \leq y \leq 1. \end{cases}$$

Then $X/\sim$ becomes the cylinder.

2.2.2 Homeomorphisms

Now we shall look at an updated version of the definition of homeomorphisms that was discussed in the beginning. Homeomorphisms gives us topologically equivalent spaces. It is thanks to the following definition that we can bend and stretch spaces and still view them as "the same space".

Definition 2.2.35. A function f, between topological spaces X and Y, is a *homeomorphism* if f is continuous and invertible and the inverse function f^{-1} is also continuous. The spaces X and Y are then topologically equivalent, also called homeomorphic, denoted $X \cong Y$.

The examples in Figure 2.1 are homeomorphic. Other examples are stretching one open interval in $\mathbb{R}$ to any other, and the very famous one of the torus which is homeomorphic to a coffee mug.

Homeomorphisms help us find unique properties that remain unchanged when during a continuous transformation. These properties lie "deeper" in the structure of the spaces and are therefore interesting to study. What characteristics can be found and how can two spaces be differentiated from one another?

Example 2.2.36. The stereographic projection $S : \mathbb{S}^3 \setminus \{N\} \longrightarrow \mathbb{R}^3$ is a homeomorphism. Therefore the 3-dimensional sphere ($\sum_{i=1}^{4} x_i^2 = 1$) minus the north pole, is the same space as $\mathbb{R}^3$.

Make the projection by sending a point $P = (x_1, x_2, x_3, x_4)$ on the sphere to a point $P' = (x, y, z, 0) \in \mathbb{R}^3$ as follows: draw a line from the north pole $N = (0, 0, 0, 1)$ of the sphere through P. This line is given by

$$\phi(t) = (0, 0, 0, 1) + (x_1, x_2, x_3, x_4 - 1)t$$

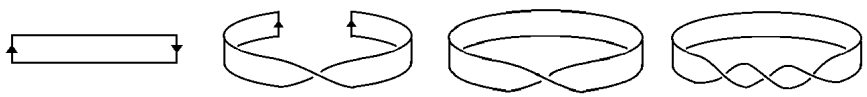
and it intersects $\mathbb{R}^3$ when $0 = 1 + (x_4 - 1)t \iff t = (1 - x_4)^{-1}$. S can now be written as

$$S(P) = \phi((1 - x_4)^{-1}) = \Big(\frac{x_1}{1 - x_4}, \frac{x_2}{1 - x_4}, \frac{x_3}{1 - x_4}\Big).$$

S is continuous, invertible and has a continuous inverse given by

$$S^{-1}(P') = \Big(\frac{2x}{\mu + 1}, \frac{2y}{\mu + 1}, \frac{2z}{\mu + 1}, \frac{\mu - 1}{\mu + 1}\Big), \text{ where } \mu = x^2 + y^2 + z^2.$$

S is therefore a homeomorphism, and $\mathbb{S}^3/\{N\} \cong \mathbb{R}^3$.



(a) Rectangle (b) Twist (c) One half twist (d) Three half twists

Figure 2.5: Möbius strip.

Example 2.2.37. This example is taken from Armstrong's book Basic Topology [2]. It is meant to illustrate that homeomorphisms are more than just stretching and bending. Hence, we cannot only think about topological spaces as being made of rubber. To, for example, construct a Möbius Strip, one starts with a rectangle that has 2 directed edges, as in Figure 2.5a, then identify the edges along the specified direction. In order to do so we induce a half twist,

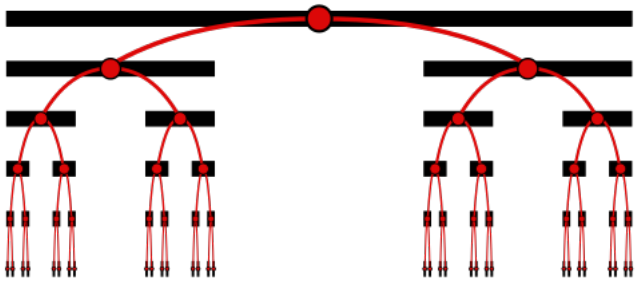


Figure 2.6: Cantor set binary tree - by Sam Derbyshire [16].

Figure 2.5b and 2.5c. However, nothing stops us from adding more twists. As long as we identify the edges according to the directions the same points on both edges will be made to coincide. This always gives an odd number of half twists. So all sub-figures in Figure 2.5 are homeomorphic.

Example 2.2.38. The famous Cantor set is constructed by iteratively removing the open middle third of line segments. Start with $A_0 = [0, 1]$, which is compact, remove the middle third and obtain $A_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, which also is compact. Again, remove the middle thirds and obtain $A_2 = [0, \frac{1}{3^2}] \cup [\frac{2}{3^2}, \frac{3}{3^2}] \cup [\frac{6}{3^2}, \frac{7}{3^2}] \cup [\frac{8}{3^2}, 1]$. Repeating this process gives

$$A_n = \bigcup_{j=0}^{3^{n-1}-1} \left(\left[\frac{3j}{3^n}, \frac{3j+1}{3^n} \right] \cup \left[\frac{3j+2}{3^n}, \frac{3j+3}{3^n} \right] \right), \text{ for } n \geq 1.$$

The Cantor set, defined as

$$\mathcal{C} := \bigcap_{n=1}^{\infty} A_n,$$

is a compact set. Look at Figure 2.6, in black we can see the first steps (A_0 to A_5) in the construction of $\mathcal{C}$. In red we can see a binary tree forming. In fact: each point in the Cantor set is determined by a unique (infinite) binary sequence. This means that there is a bijection from $\mathcal{C}$ to all infinite binary sequences.

Let 2 denote the topological space $\{0, 1\}$ with the discrete topology. From this we can create the space

$$2^{\mathbb{N}} = \{(x_n) : x_n \in \{0, 1\},\ n \in \mathbb{N}\},$$

of all infinite binary sequences with the product topology of the discrete topology in $\{0, 1\}$. By choosing 0 to represent the left interval and 1 to represent the right interval in each step of the Cantor set construction we can form a sequence (represented by the binary tree shown in Figure 2.6) that can uniquely lead us to a point in $\mathcal{C}$. It sounds plausible that $2^{\mathbb{N}}$ is homeomorphic to $\mathcal{C}$, which in fact it is. To prove this it is common in the literature to take the route through ternary expansions, but there is an approach closer to the material in this thesis which uses binary expansion and the Cantor Intersection Theorem to construct a homeomorphism between $\mathcal{C}$ and $2^{\mathbb{N}}$. The reader is referred to Landstedt's publication [12] for the details.

Concepts Depending on Homeomorphisms

The reader will notice that almost everything from now on will depend on homeomorphisms. Homeomorphisms are the foundation for the next chapter on homology, as well as the basis for the entire study of knots. Even so, this chapter will end with a few key concepts that are directly dependent on homeomorphic functions.

Homotopy has previously been defined (Definition 2.2.29) and an example of a dilation of $\mathbb{R}^2$ was given. If we put an additional requirement on the functions we get the following definition.

Definition 2.2.39. Let X and Y be topological spaces. A function $H : X \times I \longrightarrow Y$ is an *isotopy* if H is a homotopy and each $H_t(x)$ is a homeomorphism.

Demanding homeomorphic functions means that we not only need continuous functions (as for homotopy) but also continuous inverses. Again, looking back to Example 2.2.30. Stretching $\mathbb{R}^2$ has a logical inverse in shrinking it. It is common that to take $\hat{H}_0 = Id$ (the identity).

Example 2.2.40. From Example 2.2.30 we have a homotopy $H : \mathbb{R}^2 \times I \longrightarrow \mathbb{R}^2$. We can construct the following function for $x \in \mathbb{R}^2$ and $t \in I$: $\hat{H} : \mathbb{R}^2 \times I \longrightarrow \mathbb{R}^2$, where

$$\begin{aligned} H_0(x) &= x & \hat{H}_0(x) &= x \\ H_t(x) &= (1+t)x & \hat{H}_t(x) &= \frac{2-t}{2}x \\ H_1(x) &= 2x & \hat{H}_1(x) &= \frac{1}{2}x. \end{aligned}$$

All $\hat{H}_t$ are homeomorphisms, thus $\hat{H}$ is an isotopy between $\hat{H}_0 = Id$ and $\hat{H}_1$.

Definition 2.2.41. If $f : X \longrightarrow Y$ is a one-to-one map, and if $f : X \longrightarrow f(X)$ is a homeomorphism when we give $f(X)$ the induced topology from Y, we call f an *embedding* of X in Y.

Definition 2.2.42. An *n-dimensional manifold* (or n-manifold) is a topological space such that every point has a neighborhood homeomorphic to an open n-ball in $\mathbb{R}^n$.

Definition 2.2.43. An *n-manifold with boundary* is a topological space such that every point has a neighborhood homeomorphic to either an open n-ball in $\mathbb{R}^n$ or half of an open n-ball: $\{\mathbf{x} \in \mathbb{R}^n : \sum x_i < 1, x_1 \geq 0\}$.

By surface we will mean a 2-manifold. Surfaces with boundary will be an important part of our study of knots later, but for now we will just give a few examples. Boundary basically means that if you travel along the space you can come to a verge where you can "fall off" if you cross it.

Example 2.2.44. The sphere, torus, Klein bottle and the projective plane are all 2-manifolds. The punctured sphere (disk), punctured torus, a sheet of paper, and the Möbius band are 2-manifolds with boundary.

Chapter 3

Homology

In this chapter we will take a different approach - to study building blocks that together make up spaces. These blocks are called cells and cells can be glued together to form complexes. The essence of this chapter is to create algebraic structures that can help us distinguish different objects (spaces) from each other by identifying properties that are unique to each object. This chapter follows the excellent books of Armstrong [2], Fulton [5], and Kinsey [11].

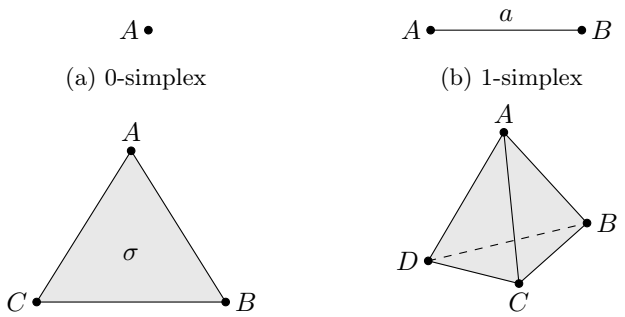
3.1 Complexes

Definition 3.1.1. An *n-cell* σ is a is a set whose interior $Int(\sigma)$ is homeomorphic to an open ball, with the additional property that its boundary must be divided into a finite number of lower-dimensional cells, called the faces of the n-cell. We write $\tau < \sigma$ if τ is a face of σ.

Example 3.1.2. For this example we will restrict ourselves to cells called simplexes, these are n-dimensional generalizations of triangles, see Figure 3.1.

1. A 0-simplex is a point A.
2. A 1-simplex is a line segment $a = AB$, and $A < a$, $B < a$.
3. A 2-simplex is a triangle, such as $\sigma = \Delta ABC$, and then AB, BC, $AC < \sigma$. Note that $A < AB < \sigma$, so $A < \sigma$.
4. A 3-cell is a tetrahedron, with triangles, edges, and vertices as faces.

Connecting back to all cells, notice from the definition that there are no requirements for a 1-cell to be a straight line, it suffices that its interior is



(a) 0-simplex (b) 1-simplex

(c) 2-simplex (d) 3-simplex

Figure 3.1: Examples of simplexes.

homeomorphic to $(-1, 1)$. A 2-cell is not necessarily a triangle rather its interior will be homeomorphic to an open 2-dimensional disk, and similarly for any n-cells.

The faces of an n-cell are cells of lower dimension. With these cells one can construct new objects: e.g., take a point, 0-cell, attach a 1-cell (string) to the point so it forms a loop, and then attach a 2-cell (surface) to the string. This might look a like flat disk with a marked edge and a point on the edge. This type of construction forms something called a complex - a union of cells that follow certain rules.

Definition 3.1.3. A *complex* K is a finite set of cells,

$$K = \bigcup \{\sigma : \sigma \text{ is a cell}\}$$

such that:

1. if σ is a cell in K, then all faces if σ are elements of K;
2. if σ and τ are cells in K, then $Int(\sigma) \cap Int(\tau) = \emptyset$.

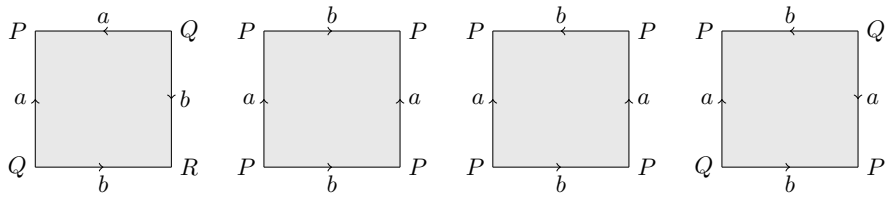
The *dimension of* K is the dimension of its highest-dimensional cell.

Remark. The second rule in the definition stops cells from overlapping or crossing each other. Cells can only meet at their boundary, i.e. they can share faces.

If a complex is constructed using only simplexes (the very nice cells that were shown in Example 3.1.2), we call it a simplicial complex. Simplexes usually make calculations more straightforward but also more tedious. Even if they are special cases of cells one does not lose any information when using them [6], and they are therefore very common in the literature.

Example 3.1.4. With a point, two strings and a rubber sheet you can make a torus. Start with the 0-cell, then attach the end of the strings (1-cells) to the point, forming two loops. Finally, bend and stretch the rubber sheet to align the edges of the sheet with the loops, see Figure 2.4.

Instead of giving instructions for how each cell should be glued together to form a complex we will represent complexes though *planar diagrams*. This means that we take a 2-dimensional disk and mark which cells are supposed to be glued together, as the examples in Figure 3.2. These show the four ways of gluing the sides of a square to form a surface, (a) the sphere, (b) the torus, (c) the Klein bottle, and (d) the projective plane.



(a) $\mathbb{S}^2$ (b) $\mathbb{T}^2$ (c) $\mathbb{K}^2$ (d) $\mathbb{P}^2$

Figure 3.2: Four surfaces: sphere, torus, Klein bottle, and projective plane.

Definition 3.1.5. The space underlying a complex K, or the *realization of* K, is the set of all points in the cells of K;

$$|K| = \{x : x \in \sigma \in K,\ \sigma \text{ a cell in } K\},$$

where x belongs to any ambient space.

So, what is the difference between a complex and a space? Well, a complex is a set of cells of various dimensions, while a space is a set of points. This means that $|K|$ is a space and K are the building blocks of that space.

There are many different ways to assign a complex to a space. Therefore, in examples we will not give a complex its own name (i.e. K) but rather use the notation of the space (i.e. $\mathbb{S}^2$) to mean the complex on that space. This is to emphasize that it is the structure of the space $|K|$ and not the complex itself that is important. By n-complex we will mean a complex K with cells of dimension n or lower, e.g. a 2-complex can only consist of 0-, 1-, and 2-cells.

The following definition is a special case of a complex, namely a simplicial complex on a surface consisting of only 2-simplexes.

Definition 3.1.6. A surface (with or without boundary) is *triangulable* if a simplicial 2-complex K can be found with $X = |K|$ and the 2-simplexes (or triangles) of K satisfy the additional property that any two triangles are identified along a single edge or a single vertex or are disjoint.

This definition deals with surfaces we divide into (or construct by) triangles (2-simplexes). We call these surfaces triangulable and we will base some of our study of knots on surfaces with this property.

3.2 The Algebra of Chains

In the previous section we gave 1-cells directions to elucidate which sides to glue together in the planar diagrams. Now we will use the fact that all cells can be given a orientation, see Figure 3.3, to form what is known as directed complexes.

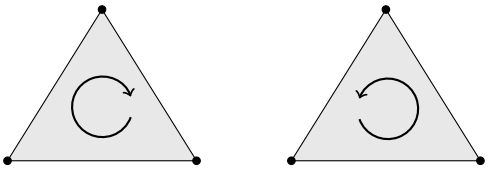


Figure 3.3: Orientations of 2-cells.

Giving a cell an orientation is the equivalent of saying if one is using the right-handed or the left-handed orientation of the basis in Euclidean three-space.

Definition 3.2.1. An n-cell σ is *oriented* if it has been assigned one of the two orientations: positive (σ) or negative ($-\sigma$).

The choice of which orientation is positive and negative is arbitrary. The orientation of an n-cell induces an orientation on its faces.

Definition 3.2.2. An n-complex is *directed* if all cells of dimension 1 or higher have been given an orientation.

Example 3.2.3. A 2-complex K is directed if each edge or 1-cell is given an orientation (from initial to terminal point) and each polygon or 2-cell is given an orientation (clockwise or counterclockwise).

In a directed complex the choice of orientation of the cells is arbitrary. It serves as a basis for calculations. If two cells share a face and the orientation of cells match up, the face can be removed without affecting the complex because

the cells induce opposite directions on the face and therefore cancel its influence. We say that the cells are oriented in a compatible manner. Compare the complex on the Klein bottle in Figure 3.2c with the one in Figure 3.4. The extra structure (c, d, and e) makes no real difference and can be added and removed at will because the orientations of the 2-cells are compatible (they induce opposite orientation on the edges c, d, and e). If all adjacent cells in a complex have compatible orientation we call the underlying space orientable.

Example 3.2.4. A surface is orientable when it is 2-sided, like the sphere and torus but unlike the Möbius strip and the Klein bottle. We will return to the concept of orientable surfaces in the chapter about knot theory.

Definition 3.2.5. Let K be a directed complex. An (integral) *k-chain* C in K is a formal sum

$$C = a_1\sigma_1 + a_2\sigma_2 + ... + a_n\sigma_n$$

where $\sigma_1, \sigma_2, ..., \sigma_n$ are k-cells in K and $a_1, a_2, ..., a_n$ are integers. Define $0\sigma = 0$.

Definition 3.2.6. Let $C = \sum_{i=1}^{n} a_i\sigma_i$ and $D \sum_{i=1}^{n} b_i\sigma_i$ be k-chains in a directed complex K. The *sum* $C + D$ is defined by

$$C + D = (a_1 + b_1)\sigma_1 + (a_2 + b_2)\sigma_2 + ... + (a_n + b_n)\sigma_n.$$

Remark. In addition to this definition we need to specify that 0 is a k-chain for each value of k.

This is a very intuitive definition, but a short example is in order. Let $C = d + 2e$ and $D = 3c - 2d + e$ be q-chains in a directed complex, then

$$C + D = (d + 2e) + (3c - 2d + e) = 3c - d + 3e.$$

This definition, addition of k-chains, have the normal properties one usually associates with addition; commutativity, associativity and the existence of both identity and inverse. One adds chains of the same dimension.

Definition 3.2.7. Let K be a directed complex. Denote the group of all k-chains on K, with addition, by $C_k(K)$, $k = 0, 1, ..., dim(K)$.

$C_k(K)$ is the kth chain group of K. $C_k(K)$ is an abelian group: the addition satisfies the definition and all chains in $C_k(K)$ are k-dimensional so adding them will only yield new k-chains. $C_k(K)$ is generated by the oriented k-cells of K. If K is a finite complex, $C_k(K)$ will be the free abelian group isomorphic $\mathbb{Z}^n$, for some n.

The following examples specify all chain groups for two different complexes, one on the sphere and the other on the Klein bottle.

Example 3.2.8. Look at the complex on the sphere in Figure 3.2. The 2-cell has no given name or orientation, call it σ and give it the clockwise orientation. Denote the whole complex K. Now we can calculate all the chain groups. The complex is constructed from three 0-cell: P, Q, and R, two 1-cells: a and b, and one 2-cell: σ. C_0 is the group of all 0-chains on $\mathbb{S}^2$. That means that all linear combinations of P, Q and R are, such as $3P + 2Q - R$, belong to C_0.

$$\begin{aligned}
C_0(\mathbb{S}^2) &= \{n_1P + n_2Q + n_3R : \ n_i \in \mathbb{Z}\} = \langle P, Q, R\rangle \simeq \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}^3 \\
C_1(\mathbb{S}^2) &= \{\emptyset, a, -a, ..., b, -b, ..., a + b, a - b, -3a + 4b, ...\} = \\
&= \{n_1a + n_2b : \ n_i \in \mathbb{Z}\} = \langle a, b\rangle \simeq \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}^2 \\
C_2(\mathbb{S}^2) &= \{n\sigma : n \in \mathbb{Z}\} \simeq \mathbb{Z}
\end{aligned}$$

Example 3.2.9. Look at the complex on the Klein bottle shown in Figure 3.4. It has two 0-cells: $\{P, Q\}$, five 1-cells: $\{a, b, c, d, e\}$, and three 2-cells: $\{\sigma, \tau, \rho\}$.

$$\begin{aligned}
C_0(\mathbb{K}^2) &= \{a_1P + a_2Q : a_i \in \mathbb{Z}\} \simeq \mathbb{Z}^2 \\
C_1(\mathbb{K}^2) &= \{a_1a + a_2b + a_3c + a_4d + a_5e : a_i \in \mathbb{Z}\} \simeq \mathbb{Z}^5 \\
C_2(\mathbb{K}^2) &= \{a_1\sigma + a_2\tau + a_3\rho : a_i \in \mathbb{Z}\} \simeq \mathbb{Z}^3
\end{aligned}$$

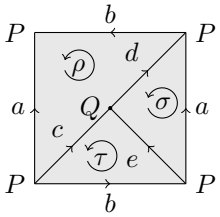
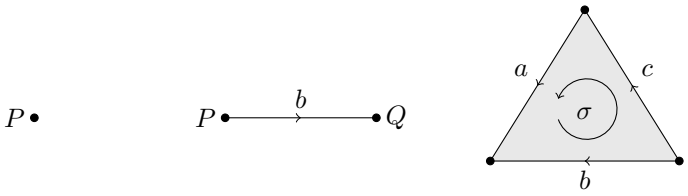


Figure 3.4: A complex on $\mathbb{K}^2$.

Definition 3.2.10. The *boundary* of a k-cell σ, denoted $\partial(\sigma)$, is the $(k-1)$-chain consisting of all the $(k-1)$-cells that are faces of σ, with orientation inherited from the orientation of σ.

Figure 3.5 shows some examples of boundaries for different k-cells. For a 0-cell P we define the boundary, $\partial(P) = 0$. For the directed 1-cell b the boundary is defined as the terminal point minus the initial point, i.e., $\partial(b) = Q - P$. The last one, 3.5(c), is an oriented 2-cell σ, whose boundary is a chain of 1-cells. The 1-cells a, b and c are positive if their directions are consistent with σ's and negative if not. Hence, for 3.5(c), we get the boundary $\partial(\sigma) = a - b + c$



(a) $\partial(P) = 0$ (b) $\partial(b) = Q - P$. (c) $\partial(\sigma) = a - b + c$.

Figure 3.5: Boundaries of cells.

Definition 3.2.11. Let C be a k-chain, $C = a_1\sigma_1 + a_2\sigma_2 + ... + a_m\sigma_m$. The boundary of C is

$$\partial_k(C) = a_1\partial(\sigma_1) + a_2\partial(\sigma_2) + ... + a_m\partial(\sigma_m).$$

So, ∂_k is a function that takes a k-chain and returns a $(k-1)$-chain, $\partial_k : C_k(K) \longrightarrow C_{k-1}(K)$. The boundary function ∂_k is a homomorphism, since, if C and D are both k-chains, then $\partial_k(C+D) = \partial_k(\sum_{i=1}^{n}(a_i+b_i)\sigma_i) = \sum_{i=1}^{n}(a_i + b_i)\partial_k(\sigma_i)$.

Example 3.2.12. Let C be the 2-chain $2\tau - 3\rho$ from the complex in Figure 3.4. Then the boundary of C is

$$\partial_2(C) = 2\partial(\tau) - 3\partial(\rho) = 2(b - c + e) - 3(b + c + d - a) = 3a - b - 5c - 3d + 2e.$$

Can we conclude something from this 1-chain that is the boundary? In this particular case there is not much to say, however, there is a special case when this boundary function gives rise to something interesting. Take another look at Figure 3.4. The boundary of b is $P - P = 0$. The same is true for $\partial(a - d - e) = 0$. So, what does it mean when the boundary terms of a chain cancel out? Well, it means that we have found a cycle!

Definition 3.2.13. If C is a k-chain in a directed complex K s.t. $\partial_k(C) = 0$, we say that C is a *k-cycle*. The set of all k-cycles in K, $Z_k(K)$, is a subgroup of $C_k(K)$.

Remark. $Z_k(K)$ is also the kernel of $\partial_k : C_k(K) \to C_{k-1}(K)$.

Definition 3.2.14. If C is a k-cycle in a directed complex K such that there exists a $(k+1)$-chain D with $\partial_{k+1}(D) = C$, we say C is a k-boundary. The set of all k-boundaries in K is $B_k(K) \subseteq C_k(K)$.

Remark. It is also the image of $\partial_{k+1} : C_{k+1}(K) \to C_k(K)$.

Not only is $B_k(K)$ a subgroup to $C_k(K)$ in fact it is true that $B_k(K) \subseteq Z_k(K) \subseteq C_k(K)$. This is a consequence of the following theorem.

Theorem 3.2.15 (cf. [2])**.** *The composition $\partial_{k+1} \circ \partial_k : C_{k+1} \longrightarrow C_{k-1}$ is the zero homeomorphism.*

If you apply the boundary function twice you will always get 0. A proof can be found in [2], but in short this means the the boundary of any cell is a cycle.

Example 3.2.16. Let us revisit Figure 3.4 again. In this complex, name the 1-cycle $a - d - e = C_1$. The definitions says that because the boundary of the 2-chain $D_1 = \sigma$ is precisely C_1, then C_1 is called a 1-boundary. The chains $C_2 = b - c + e$ and $C_3 = b + c + d - a$ are also 1-boundaries with $D_2 = \tau$ and $D_3 = \rho$ respectively. There exists no 2-boundaries because the complex contains no 3-cells.

3.3 Homology Groups

In this section we will reach our goal for this chapter - Homology groups. Their definition will be stated first and this whole chapter will be concluded by several examples that give a deeper understanding as well as illustrate why all of this has been worth doing, but first a short recap.

We have constructed chains from cells of the same dimension. On these chains we have defined orientation and boundary and noticed that chains with null-boundary form some kind of loop that we can think of as enclosing a k-dimensional hole. By studying sets of cycles we will distinguish spaces based on what type of cavities they have. For example, the normal sphere encloses a 3-dimensional cavity, as does the torus, but when we at the end of this section look at the combined picture, over all dimensions, we will be able to tell them apart. Homology groups $H_k(K)$ are used to describe structures in different dimensions. One can think of them as follows:

$H_0(K)$ describes the connectivity of the components,

$H_1(K)$ describes non-trivial loops, i.e. holes that look like a circle,

$H_2(K)$ describes holes which look like a sphere,

$\vdots$

Definition 3.3.1. Let K be a directed complex. The *kth homology group* of K is $H_k(K) = Z_k(K)/B_k(K) = \langle \bar{\sigma_k},\ k \in \mathbb{Z} \rangle$, the group of equivalence classes of elements of $Z_k(K)$ with the homology relation. In other words, $H_k(K)$ is $Z_k(K)$ with homology used instead of equality.

Remark. A general rule for calculating $H_k(K)$ is that we take the generators from $Z_k(K)$ and then the relations from $B_k(K)$

In this part we denote the equivalence class of σ as $\bar{\sigma}$ as this will make the calculations look tidier. The equivalence relation on chains is the following.

Definition 3.3.2. The k-chains C_1 and C_2 are *homologous*, written $C_1 \sim C_2$, if $C_1 - C_2 \in B_k(K)$; i.e., if $C_1 - C_2 = \partial(D)$ for some $(k+1)$-chain D.

Example 3.3.3. From Figure 3.5b, c: $\partial(b) = Q - P \iff P \sim Q$ therefor Q is in $\bar{P}$, and $\partial(\sigma) = a - b + c \iff a + c \sim b$, so $a + c \in \bar{b}$.

Example 3.3.4. A few examples from Figure 3.4:

- $\partial(a) = P - P \iff P \sim P$, naturally,
- $\partial(c) = Q - P \iff Q \sim P$,
- $\partial(\tau) = b + e - c \iff b + e \sim c$, or $b + e \in \bar{c}$ and
- $\partial(\rho) = b - a + c + d \iff a - b \sim c + d \iff a - b \in c \bar{+} d$.

Theorem 3.3.5 (cf. [11])**.** *Let K be a complex, with C_1, C_2, C_3, C_4 chains in $C_k(K)$:*

1. *$C_1 \sim C_2$.*
2. *If $C_1 \sim C_2$, then $C_2 \sim C_1$.*
3. *If $C_1 \sim C_2$ and $C_2 \sim C_3$, then $C_1 \sim C_3$.*
4. *If $C_1 \sim C_2$ and $C_3 \sim C_4$, then $C_1 + C_3 \sim C_2 + C_4$.*

Remark. This theorem shows that homology is an equivalence relation and that it works well with the chain addition (4).

Below are step-by-step directions for how to compute $H_k(K)$. These seem tedious and take a bit of practice to get the hang of. A reminder about notation: the trivial group is denoted by $\widetilde{\mathbf{0}}$.

Step-by-step Directions. First label and indicate orientation of all cells of the complex. Do the computations for one dimension at a time, starting with the highest.

1. Find $C_k(K)$, the group of all k-chains.
2. For each generating k-chain C from (1), compute $\partial(C)$.
3. Find $Z_k(K)$, using your computations from (2).
   (a) Note that if $Z_k(K) = \widetilde{\mathbf{0}}$, then $H_k(K) = \widetilde{\mathbf{0}}$.
4. Find $B_k(K)$.

   (a) When working in the highest dimension, note that $B_k(K) = \widetilde{\mathbf{0}}$, since there are no $(k+1)$-chains for which the k-chains can form boundaries.
   (b) Otherwise, look back to (2) in the next highest dimension, where you already computed which k-chains are boundaries of $(k+1)$-cells.
   (c) Note that if $B_k(K) = \widetilde{\mathbf{0}}$, then $H_k(K) = Z_k(K)$.
5. Compute $H_k(K)$, by taking $Z_k(K)$ from (3) and using any homologies found in (4).

Let's go through a few examples.

Example 3.3.6. We have already worked with the sphere in Figure 3.2a, now it is time to compute its homology groups. Call the 2-cell σ and give it a clockwise orientation. There are no cells of higher dimension than 2, therefore $H_k(\mathbb{S}^2) = \widetilde{\mathbf{0}}$, for $k > 2$.

$\mathbf{H_2}(\mathbb{S}^2)$:

1. There is only one 2-cell: σ, therefore $C_2(\mathbb{S}^2) = \langle\sigma\rangle \simeq \mathbb{Z}$.
2. $\partial(\sigma) = a - a + b - b = 0$. This means that σ is a 2-cycle.
3. The 2-cycle σ generates the whole group of 2-cycles: $Z_2(\mathbb{S}^2) = \langle\sigma\rangle \simeq \mathbb{Z}$.
4. Because there are no 3-cells $B_2(\mathbb{S}^2) = \widetilde{\mathbf{0}}$.
5. We get that $H_2(\mathbb{S}^2) = Z_2(\mathbb{S}^2) \simeq \mathbb{Z}$.

$\mathbf{H_1}(\mathbb{S}^2)$:

1. There are two 1-cells: a and b, therefore $C_1(\mathbb{S}^2) = \langle a, b\rangle \simeq \mathbb{Z} \oplus \mathbb{Z}$.
2. $\partial(a) = P - Q \neq 0$, $\partial(b) = R - Q \neq 0$. Neither a nor b are 1-cycles. The only 1-cycle is the trivial 0.
3. So $Z_1(\mathbb{S}^2) = \widetilde{\mathbf{0}}$.
4. The only 1-chain that is a 1-boundary (bounds σ) is $0 = a - a + b - b$, therefore $B_1(\mathbb{S}^2) = \widetilde{\mathbf{0}}$.
5. We get that $H_1(\mathbb{S}^2) = \widetilde{\mathbf{0}}$.

$\mathbf{H_0}(\mathbb{S}^2)$:

1. $C_0(\mathbb{S}^2) = \langle P, Q, R\rangle \simeq \mathbb{Z}^3$
2. $\partial(P) = \partial(Q) = \partial(R) = 0$
3. $Z_0(\mathbb{S}^2) = \langle P, Q, R\rangle \simeq \mathbb{Z}^3$
4. From the calculations of $H_1(\mathbb{S}^2)$, step 2, we have that $P \sim Q \sim R$ because $\partial(a) = P - Q \neq 0$, $\partial(b) = R - Q \neq 0$.
5. Combining $Z_0(\mathbb{S}^2)$ with the homology relations (Q, R are in $\bar{P}$) from step 4 gives $H_0(\mathbb{S}^2) = \langle\bar{P}\rangle \simeq \mathbb{Z}$.

Example 3.3.7. Take the complex on the torus shown in Figure 3.2b. Call the 2-cell σ and give it a clockwise orientation. There are no cells of higher dimension than 2.

$\mathbf{H_2}(\mathbb{T}^2)$:

1. There is only one 2-cell: σ, therefore $C_2(\mathbb{T}^2) = \langle\sigma\rangle \simeq \mathbb{Z}$.
2. $\partial(\sigma) = a + b - a - b = 0$. This means that σ is a 2-cycle.
3. The 2-cycle σ generates the whole group of 2-cycles: $Z_2(\mathbb{T}^2) = \langle\sigma\rangle \simeq \mathbb{Z}$.
4. Because there are no 3-cells $B_2(\mathbb{T}^2) = \widetilde{\mathbf{0}}$.
5. We get that $H_2(\mathbb{T}^2) = Z_2(\mathbb{T}^2) \simeq \mathbb{Z}$.

$\mathbf{H_1}(\mathbb{T}^2)$:

1. There are two 1-cells: a and b, therefore $C_1(\mathbb{T}^2) = \langle a, b\rangle \simeq \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}^2$.
2. $\partial(a) = \partial(b) = P - P = 0$. Both a and b, and any linear combination of them, are 1-cycles.
3. The 1-cycles a and b generate the group of 1-cycles: $Z_1(\mathbb{T}^2) = \langle a, b\rangle \simeq \mathbb{Z}^2$.
4. The only 1-chain that is a 1-boundary (bounds σ) is $0 = a + b - a - b$, therefore $B_1(\mathbb{T}^2) = \widetilde{\mathbf{0}}$.
5. We get that $H_1(\mathbb{T}^2) = Z_1(\mathbb{T}^2) \simeq \mathbb{Z}^2$.

$\mathbf{H_0}(\mathbb{T}^2)$:

1. $C_0(\mathbb{T}^2) = \langle P\rangle \simeq \mathbb{Z}$
2. $\partial(P) = 0$
3. $Z_0(\mathbb{T}^2) = \langle P\rangle \simeq \mathbb{Z}$
4. From the calculations of $H_1(\mathbb{T}^2)$, step 2, we have that $\partial(a) = \partial(b) = P - P = 0$. Which gives us that $B_0(\mathbb{T}^2) = \widetilde{\mathbf{0}}$.
5. Thus $H_0(\mathbb{T}^2) = Z_0(\mathbb{T}^2) \simeq \mathbb{Z}$.

Example 3.3.8. Take a look at the complex in Figure 3.6. Call this complex K. Observe that σ forms a sphere and that there are no cells of dimension 3 or higher.

$\mathbf{H_2}(\mathbf{K})$:

1. σ and τ are the only 2-cells and they generate the group of all 2-chains; $C_2(K) = \langle\sigma, \tau\rangle \simeq \mathbb{Z}^2$.
2. $\partial(\sigma) = 0$ and $\partial(\tau) = f + g - e$, so σ is a 2-cycle.
3. The group of 2-cycles is $Z_2(K) = \langle\sigma\rangle \simeq \mathbb{Z}$.
4. $B_2(K) = \widetilde{\mathbf{0}}$ since we have no 3-cells.

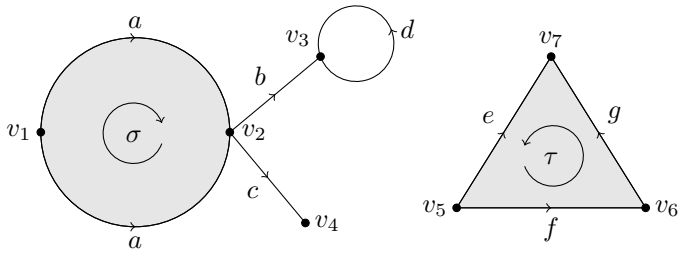


Figure 3.6: Complex K for Example 3.3.8.

5. $H_2(K) = Z_2(K)/B_2(K) = Z_2(K) \simeq \mathbb{Z}$.

$\mathbf{H_1(K)}$:

1. There are seven 1-cells; $a, ..., g$. These generate the $C_1(K)$, so $C_1(K) \simeq \mathbb{Z}^7$.
2. $\partial(d) = \partial(f + g - e) = 0$, so d and $f + g - e$ (and any multiple of these) are the only 1-cycles.
3. $\mathbb{Z}_1(K) = \langle d, f + g - e \rangle \simeq \mathbb{Z}^2$
4. Homology relations
   - From 2. we get that $e \sim f + g$. $f + g - e$ is a 1-boundary because it is the boundary of a 2-chain i.e. $\bar{e}$ is one of the generators of $B_1(K)$.
   - Looking at d we see that it is not the boundary of any 2-chain hence $\bar{d}$ is not a generator for $B_1(K)$.
   - $B_1(K) = \langle \bar{e} \rangle \simeq \mathbb{Z}$.
5. This gives us $H_1(K) = \langle \bar{d} \rangle \simeq \mathbb{Z}$.

$\mathbf{H_0(K)}$:

1. $C_0(K) = \langle v_1, ..., v_7 \rangle$
2. $\partial(v_i) = 0, i = \{1, ..., 7\}$
3. $Z_0(K) = \langle v_1, ..., v_7 \rangle \simeq \mathbb{Z}^7$
4. From the calculations of $H_1(K)$, step 2, we have that $v_1 \sim v_2 \sim v_3 \sim v_4$ and $v_5 \sim v_6 \sim v_7$, so there are only two equivalence classes $\bar{v_1}$ and $\bar{v_5}$. $B_0(K) = \langle \overline{v_1 - v_2}, \overline{v_1 - v_3}, \overline{v_1 - v_4}, \overline{v_5 - v_6}, \overline{v_5 - v_7} \rangle$
5. Combining $Z_0(K)$ with the homology relations found in step 4 gives $H_0(K) = \langle \bar{v_1}, \bar{v_5} \rangle \simeq \mathbb{Z}^2$.

Thees homology groups have given us the information that the complex has two connected components, one cavity similar to a sphere, and another cavity similar to a circle. The rest of the structure is disregarded, it does not add new information. We regard all structures containing two components, one circle, and one sphere, as the same.

In summary, homology is a powerful tool for studying and differentiating spaces. It helps us describe the structure of spaces in dimensions we cannot visualize. It can for instance be applied in the study of structures in data sets. Simplexes are cells with extra restriction. We could have chosen to work only with simplexes, this would have yielded a homology called simplicial homology. A powerful result is the fact that the homology groups obtained in simplicial homology actually coincides with the homology groups we have obtained. This is however not an easy feat to prove and the interested reader is refereed to [6] for a detailed account.

We will end this chapter by building a bridge into knot theory. In knot theory we will make use of something called polyhedral surfaces.

Definition 3.3.9. A *polyhedral surface* is a triangulable surface (or 2-manifold) possibly with boundary.

If we can construct a surface (with or without) boundary from triangles or 2-simplexes, following the rules of triangulation, we call this a polyhedral surface. A cylinder is an easy example of a polyhedral surface. Viewing the cylinder as a rectangle and imagine dividing the rectangle into two triangles gives a triangulation of the cylinder. By having complexes on a surface we know that there are tools that help us study these surfaces. In the chapter on knot theory the boundary of a surface will be the most important part, as it is actually a knot, and the techniques developed in this chapter will allow us to study the knot using this surface, but more on that later.

Orientation has been mentioned earlier and the following definition agrees with the one previously given but is stated only for polyhedral surfaces.

Definition 3.3.10. A polyhedral surface is *orientable* if it is possible to orient each triangle in such a way that that when two triangles meet along an edge, the two induced orientations run in opposite directions.

For polyhedral surfaces to be oriented we need that when two triangles are glued together along an edge the induced orientations on the edge run in opposite directions and therefore cancel out.

Definition 3.3.11. The *Euler characteristic* for a polyhedral surface S with a triangulation consisting of F triangles, E edges and V vertices, is given by $\chi(S) = F - E + V$.

The Euler characteristic is defined for simplexes but in fact we can use any complex on a triangulable surface to calculate it if we let F be 2-cells, E be 1-cells and V be 0-cells. This follows from the fact that homology and simplicial homology are equivalent.

Example 3.3.12. We have previously worked with the complexes in Figure 3.2, let us do so again. Both the sphere $\mathbb{S}^2$ and the torus $\mathbb{T}^2$ are triangulable and orientable. From the complexes in the figure we have that

$$\begin{aligned}\chi(\mathbb{S}^2) =& 1 - 2 + 3 = 2\\ \chi(\mathbb{T}^2) =& 1 - 2 + 1 = 0\end{aligned}$$

The reader can check that if a triangulation on the sphere is constructed, by adding say a 1-cell connecting P and R, the Euler characteristic does not change. Neither does adding a diagonal 1-cell in the complex on $\mathbb{T}^2$.

We have now studied both homology groups and the Euler characteristic and with the help of Betti numbers, see below, we will combine them. The Betti numbers are topological invariants that aid us in our study of spaces.

Definition 3.3.13. The *Betti numbers* of a complex K are $\beta_k = rk(H_k(K))$.

Theorem 3.3.14 (cf. [2])**.** *The Euler characteristic of a finite complex K is given by the formula*

$$\chi(K) = \sum_{k=0}^{n} (-1)^k \beta_k,$$

where n is the dimension of K.

Definition 3.3.15. The *genus* of a connected orientable surface S is given by

$$g(S) = \frac{2 - \chi(S) - B}{2},$$

where B is the number of boundary components of the surface.

Example 3.3.16. Neither the sphere nor the torus have boundary and therefore $g(\mathbb{S}^2) = 0$ and $g(\mathbb{T}^2) = 1$. The cylinder on the other hand has boundary, 2 boundary components to be exact, and can be triangulated with two triangles giving it Euler characteristic 0 and genus $\frac{2-0-2}{2} = 0$.

Chapter 4

Knot Theory

In the introduction a knot was described as something that is formed when we take a string, tie it, and then glue the ends of the string together. If you do not attach the ends together the string can be tied and untied over and over again in different ways. In this chapter a mathematical definition of a knot is given and several knot invariants are defined, some more helpful than others. The last two sections are dedicated to two knot polynomials: the bracket polynomial and the Jones polynomial. Both are invariants and have contributed greatly to this field.

4.1 Introduction to Knot Theory

In this secton we follow: [1] [3], [13] and [14]. In this work we study knots that are embedded in $\mathbb{S}^3$. One can project a knot onto a plane (or simply draw it on a paper). These projections are called **knot diagrams**, see Figure 4.1. To avoid ambiguity there are a few restrictions:

1. At each crossing it is clear which string segment passes over respectively under the other (this is usually done by drawing a gap in the bottom segment).
2. Each crossing involves exactly two segments of the string.
3. The segments must cross transversely.

Some intuitive terminology: At each **crossing** there is always one **overstrand** and one **understrand**. An **arc** is a piece of the knot that passes from one **undercrossing** to another with only **overcrossings** in between, i.e., an

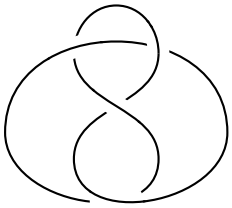


Figure 4.1: Figure 8 knot.

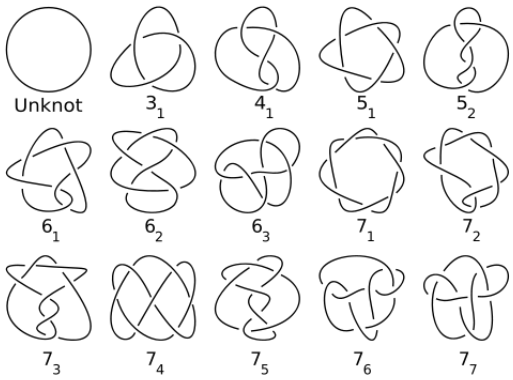


Figure 4.2: Knot table - by Wikipedia user Jkasd [22], published with permission.

"unbroken" line in the diagram. We will work extensively with the knot diagrams but it is important to remember that these diagrams are only projections of the knot.

Definition 4.1.1. K is a *knot* if there is an embedding $h : \mathbb{S}^1 \longrightarrow \mathbb{S}^3$ whose image is K. We say that $K \subseteq \mathbb{S}^3$ is homeomorphic to the unit circle $\mathbb{S}^1$

The actual knot is a smooth embedding of the unit circle in $\mathbb{S}^3$. The diagrams are, in fact, sufficient for showing all results that are covered in this work. Why this is true will be dealt with shortly. A knot can only consist of one component, a **link** on the other hand is a finite union of disjoint knots. Many of the following concepts can be generalized to links.

When studying knots one matter of great importance is being able to tell knots apart (as well as being certain when they are one and the same). It will later be apparent just how hard this actually is. For now you should know that

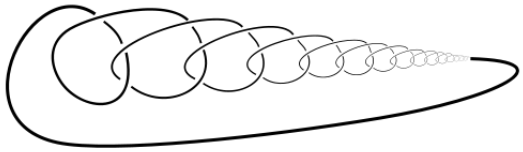


Figure 4.3: Wild knot - by Wikipedia user Jkasd [21], published with permission.

we are able to tell many knots apart, although, there exists no method that can fully distinguish all knots from each other. Finding methods for differentiating knots means finding knot invariants, that is, properties that remain unchanged no matter which projection or embedding of the knot we happen to be studying. We will work our way though several invariants all the way to the famous Jones polynomial which was revolutionary when discovered. To do this we must first specify what is meant by two knots being "the same".

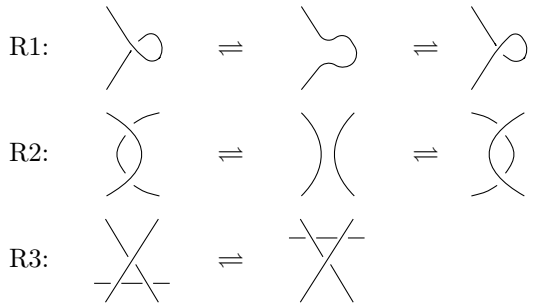
Definition 4.1.2. Knots K_1 and K_2 in $\mathbb{S}^3$ are *equivalent* if there exist a homeomorphism $h : \mathbb{S}^3 \longrightarrow \mathbb{S}^3$ such that $h(K_1) = K_2$.

We work with knots in the space $\mathbb{S}^3$ and we consider two knots to be two representations of the same knot if there is a homeomorphism of the whole space that takes one representation to the other. A **tame knot** is a knot that can be constructed with line segments. This ensures that the knots are "nice" to work with, e.g. shrinking a part of the knot to a point is not allowed, neither are knots that need an infinite construction. A knot that is not tame is called a **wild knot**, see Figure 4.3. All knots in this thesis are tame knots.

Definition 4.1.3. If H is an isotopy between ambient spaces, that is $H : \mathbb{S}^3 \times I \longrightarrow \mathbb{S}^3$, then H is an *ambient isotopy*.

Ambient isotopy is a stronger requirement than just the existence of a homeomorphism between knot embeddings. By looking at knots from this perspective we can utilize algebraic methods in our study and in fact no information is lost. Even though structure is gained it would still be very tedious and hard work to explicitly construct ambient isotopies between spaces whenever we wish to show that two knots are equivalent. Thankfully this is not necessary. Work done by Reidemeister integrated combinatorial techniques into knot theory and these are very easy to work with. The following definition describes what is called the Reidemeister moves. These operations work with the knot diagrams rather than the actual knot. The pictures in the definition only show a small part of the diagram, the rest remains unchanged.

Definition 4.1.4. The *Reidemeister Moves*.



So, what can we do with the Reidemeister moves? Well R1 allows us to remove (or introduce) a twist in a diagram. The result is that the knot will have one fewer (or one more) crossing. R2 lets us separate two strings that lie on top of each other or vice versa. This will add or remove two crossings. R3 allows us to move a strand from one side of a crossing to the other. This also works if the strand being moved is above the other two strands. R3 does not affect the number of crossings in the current projection. If a deformation of the diagram only uses R2 and R3 we call this a **Regular isotopy** (or planar isotopy). A word of caution: regular isotopy is an equivalence relation for the knot diagrams and is not defined for the knot embedding.

The Reidemeister moves only describe procedures performed on the diagrams. It is not intuitively clear that studying knot diagrams is a technique strong enough to allow us to draw conclusions about the actual knot. Fortunately it is a very strong technique, which is shown in the following theorem.

Theorem 4.1.5 (Reidemeister's theorem)**.** *If two knots K_1 and K_2 are ambient isotopic, then there is a sequence of Reidemeister moves taking a projection of K_1 to a projection of K_2.*

Proof by Kauffman in article [10]. Reidemeister not only defined these moves for knot diagrams he proved that one can, using only R1-R3, always transform two knot projections into one another if and only if they are projections of the same knot.

Example 4.1.6. The knot in Figure 4.4 is equivalent to the unknot. Use the Reidemeister Moves to convince yourself of this.

We are striving to find ways to tell knots apart. It feels intuitively obvious that there exists more knots than just the unknot and with the next result we can show that there exists (at least) two different knots. Another name for the

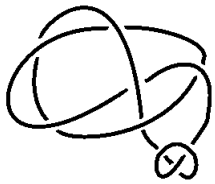


Figure 4.4: Unknot in disguise.

unknot is the trivial knot, and all knots distinguishable from the unknot are therefore called non-trivial knots.

Definition 4.1.7. A knot diagram is *tricolorable* if each arc in the knot digram can be assigned a single color such that

1. at least two colors are used, and
2. at each crossing all three incident arcs are either all different colors or the same color.

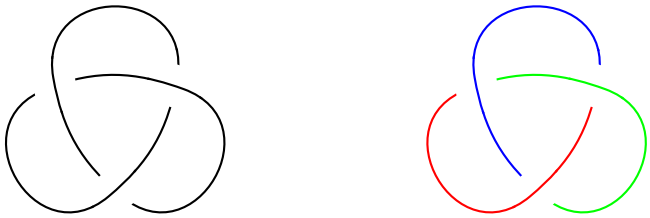


Figure 4.5: Tricoloring of trefoil.

In Figure 4.5 it is shown that the this projection of trefoil knot is tricolorable.

Theorem 4.1.8 (cf. [14])**.** *If a diagram of a knot, K, is tricolorable, then every diagram of K is tricolorable.*

Proof. (Outline) By tricoloring the Reidemeister Moves one can see that they do not affect the color of the rest of the diagram. This means that they cannot alter tricolorability for a knot diagram, and hence, if one diagram is tricolorable all are. □

This makes tricoloring a knot invariant. The unknot is not tricolorable, thus it is different from the trefoil. In the knot table, Figure 4.2, only knots 3_1, 6_1, 7_4 and 7_7 can be colored. Observe that we cannot yet distinguish the colorable (or uncolorable) knots from each other, at the moment we can only be certain that there exists two different knots - the trivial and a non-trivial.

The knots in the knot table, Figure 4.2, are labeled in the Alexander-Briggs notation. This organizes knots based on their crossing number, the subscript is just a counter to separate knots with the same crossing number.

Definition 4.1.9. The *crossing number*, $c(K)$ of a knot K, is the minimum number of crossings that occur in any diagram of K.

Example 4.1.10. For the unknot we have that $c(unknot) = 0$. The projections of the trefoil seen so far all have 3 crossings, so is $c(trefoil) = 3$? Yes, it is. Simply because there are no knots such that $c(K) = 1$ or 2. To show this first draw one crossing and connect the ends in all 4 possible ways: this only yields the unknot. Then draw two crossings and connect the ends in all possible ways: this gives the unknot, two unknots (which is a link), the Hopf link. Therefore 3 is the minimum number of crossings you need to draw the trefoil. The crossing number is not easy to find, you need to prove there are no representations of the knot with fewer crossings, and right now we cannot show that knot 6_2 is in fact not just a more complex diagram of any of the knots with fewer crossings. We will find a way to do so later. The crossing number for a knot is an invariant, but at the moment not a very helpful one.

Definition 4.1.11. A knot is *oriented* if it has an orientation assigned to it. This is illustrated by arrows.

Orientation is extra information that can be added to the knot diagram. If a knot K has a given orientation we take $-K$ to mean K with reverse orientation. A knot is called **invertible** if K and $-K$ are equivalent. All knots in Figure 4.2 are invertible. The first non-invertible knot is one with eight crossings (8_{17}) [20].

(a) $+1$ crossing. (b) -1 crossing.

Figure 4.6: Positive and negative crossing.

Definition 4.1.12. The *writhe* of an oriented knot, $w(K)$, is the sum of the crossings with signs as shown in Figure 4.6.

Example 4.1.13. The writhe of the trefoil in Figure 4.5 is -3 regardless of given orientation.

The writhe is a regular isotopy. It is not an invariant under R1, only for R2 and R3.

Definition 4.1.14. A knot K is called *alternating* if it has a diagram in which undercrossings and overcrossings alternate around K.

All knots with crossing number 7 or less are alternating. For knots with eight crossings there exists only three unaltering knots [19]. The first unalternating knot is 8_{19}.

Definition 4.1.15. The *knot sum* or connected sum $K_1 \# K_2$, is formed by placing two knots side by side, removing a small arc from each knot and then joining the knots together with two new arcs.

A knot is called **composite** if it can be written as the sum of K_1 and K_2, neither of which is the unknot and **prime** if it is not composite. Knot tables only show prime knots, see Figure 4.2.

Example 4.1.16. $K\#(unknot) = K, \forall$ knots K.

4.2 Prime Decomposition Theorem

For this next part we will develop the theory for prime decomposition. We have already defined what a prime knot is, but similarly to positive integers all knots have a unique factorization into prime knots. A knot is either prime or it can be decomposed into at least two non-trivial knots. If these factors are not prime we can continue. This will give a unique decomposition. This chapter will utilize results such as polyhedral surfaces, Euler characteristic and genus as defined in previous chapters.

Definition 4.2.1. An orientable surface with a knot as its boundary is called a *Seifert surface* for the knot.

Theorem 4.2.2 (cf. [14])**.** *Every knot is the boundary of an orientable surface.*

This was proven by Seifert in the 1930's. The proof relies of a construction now refereed to as the Seifert algorithm which produces a Seifert surface for a given knot projection.

The Seifert Algorithm

1. Fix an orientation for a projection of the knot.

2. Let x be a point on an arc, arbitrarily chosen.

3. Begin at x and travel in the direction of the arc. Whenever you reach a crossing change strands and follow the new strand in its direction. Do this until you end up back at x.

4. If there is an untraced part of the diagram, chose a new point x and start again. Repeat this until you have traveled along the entire diagram. The result is a collection of circles (Seifert circles) each of which is the boundary of a disk in the plane. If circles are nested "lift" the inner ones to a plane above, so you wary the heights of the different the disks.

5. Connect the disks by attaching bands at the points corresponding to the crossings in the original diagram. When attaching a band add a half-twist to it. The twist should reflect the crossing it represents, that is, it should follow the rotation of that crossing.

This construction yields an orientable surface with the knot as its boundary. A knot can have different Seifert surfaces, it depends on the knot projection one starts with.

Example 4.2.3. By following the Seifert algorithm for the star knot (5_1) one first gets two Seifert circles (on top of each other). After attaching twisted bands at all five crossing points (that match the orientation of the original crossings) one gets a surface that looks like the one in Figure 4.7.

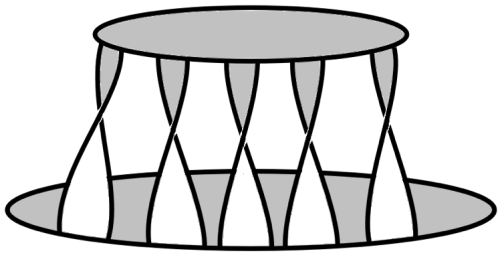


Figure 4.7: Seifert surface for 5_1.

Example 4.2.4. The trefoil knot has a surface quite similar to the star knot. It has two disks above each other, but with five connecting bands instead of three. All bands twist in the same direction.

Definition 4.2.5. The *genus of a knot*, $g(K)$ is the minimum possible genus of a Seifert surface for the knot.

For the minimal projection of the unknot the disk is its Seifert surface whose genus is zero. In fact, if $g(K) = 0$ for a knot K, then K is the unknot [13]. From the construction of the Seifert surfaces one gets the following relation: If the knot diagram had n crossings and s Seifert circles were obtained then $g(K) \leq \frac{1}{2}(n - s + 1)$. One of the most useful results about knot genus is its additivity.

Theorem 4.2.6 (cf. [13])**.** *For any two knots K_1 and K_2, $g(K_1\#K_2) = g(K_1)+ g(K_2)$.*

Theorem 4.2.7 (cf. [14])**.** *Every knot can be decomposed as the connected sum of non-trivial prime knots. If $K = K_1\#...\#K_n$, and $K = J_1\#...\#J_m$, with each K_i and J_i non-trivial prime knots, then $m = n$ and, after reordering, each K_i is equivalent to J_i.*

Theorem 4.2.8 (cf. [14])**.** *If K is nontrivial, there does not exist a knot J such that $K\#J$ is trivial.*

This is analogous to the fact that multiplying two positive integers can never yield 1.

4.3 Bracket Polynomial

This section is devoted to the bracket polynomial, also known as the Kauffman bracket after Louis Kauffman. Two of Kauffman's articles, [8] and [9], are the basis for this section and the next.

Definition 4.3.1. *The Bracket polynomial* Let K be an unoriented knot or link diagram. Let $\langle K \rangle$ be the element of the ring $\mathbb{Z}[A, B, d]$ defined by means of the rules:

$$\left\langle \bigcirc \right\rangle = 1 \tag{4.1}$$

$$\left\langle \bigcirc \cup K \right\rangle = d\left\langle K \right\rangle, \text{ } K \text{ not empty} \tag{4.2}$$

$$\left\langle \times \right\rangle = A\left\langle \asymp \right\rangle + B\left\langle)(\right\rangle \tag{4.3}$$

The last equation shows only parts of the diagram, and as with the Reidemeister moves knot and link diagrams may *only* differ in this one indicated crossing. A rule to help you remember: one gets A when turning right on the underpass (and similarly B when turning left).

We will start with an example - the bracket of the Hopf link.

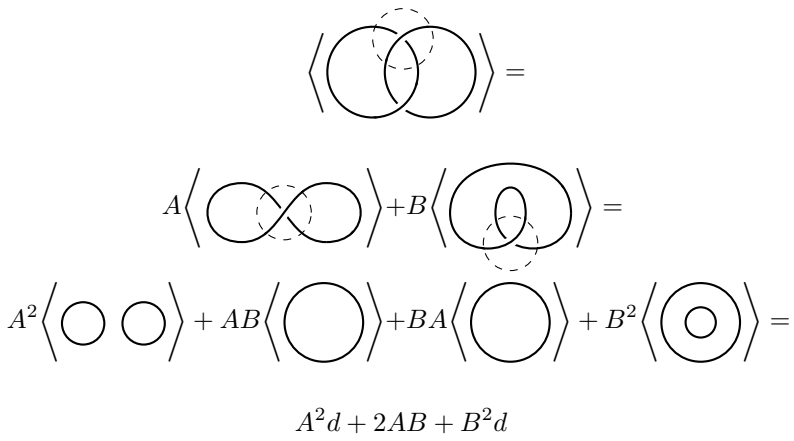
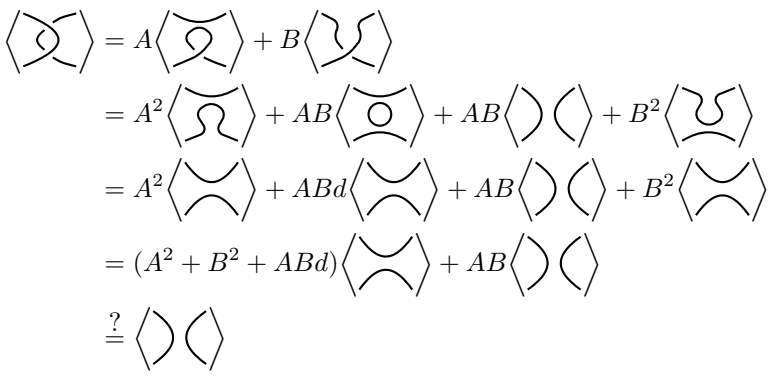


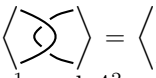
Figure 4.8: Bracket polynomial for the Hopf link.

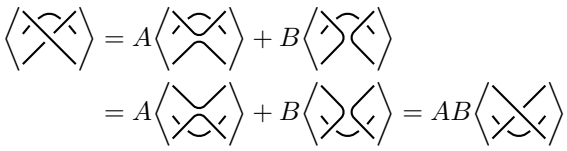
Example 4.3.2. Look at Figure 4.8. Starting with the Hopf link, we split the crossing marked with the dashed circle. This gives two outcomes according to rule 3 in the definition. Since there are still crossings left we continue the process and choose one new crossing in each of the diagrams and split again. The result is 4 diagrams with a total of 6 unknots. Using rules 1 and 2 from the definition we can complete the bracket polynomial.

Our goal is to investigate if the bracket polynomial is an invariant. To do this we must check it against the Reidemeister moves. As shown below, we start with one of the diagrams associated with R2 and examine how it is affected by the bracket polynomial.

Satisfying the Reidemeister moves, starting with R2:



R2 says that  , therefore we must have $AB = 1$, which gives that $B = A^{-1}$, and $A^2 + B^2 + ABd = 0$, which gives that $d = -A^2 - A^{-2}$. By accepting these restrictions on the variables we make sure that R2 does not change the polynomial. By using R2 we can now look at R3.



Remark. Between the first and second line we use R2.

By choosing $B = A^{-1}$ and $d = -A^2 - A^{-2}$ we have now made sure that the polynomial is unchanged by the Reidemeister moves R2 and R3. What a great result! This means that the bracket polynomial for the Hopf link in Figure 4.8 becomes $-A^4 - A^{-4}$.

Remembering the earlier definitions of regular and ambient isotopy, we have now shown that the bracket is at least an regular isotopy invariant. Is it also an ambient isotopy? Ambient is always the goal, since it is a far stronger invariant than regular. For the calculations on R1 both crossing directions must be looked at, this gives that

$$\left\langle \right\rangle = A\left\langle \right\rangle + A^{-1}\left\langle \right\rangle$$
$$= A\left\langle \right\rangle + A^{-1}(-A^2 - A^{-2})\left\langle \right\rangle$$
$$= -A^{-3}\left\langle \right\rangle,$$

and, with almost identical calculations, that

$$\left\langle \right\rangle = -A^3\left\langle \right\rangle.$$

Here we run into trouble. While R2 and R3 left the polynomial unchanged, we can clearly see that this is not true for R1. The bracket polynomial preserves regular isotopy (but not ambient). Moving the knot around like a ribbon flat on a table is permitted but lifting it from the surface to induce a twist is not. This is still a good result, we are able to study knots based on their respective polynomial but we did want an invariant that can handle all of the Reidemeister moves, i.e., we want ambient isotopy.

Remark. For the interested reader: In [9] Kauffman studies two invariants of regular isotopy. It is outside the scope of this paper but a very interesting read!

Example 4.3.3. The backet polynomial for trefoil in Figure 4.5 is $\langle K \rangle = A^{-7} - A^{-5} - A^3$. When splitting the first crossing one of the new knots it the Hopf link previously calculated. By following the procedure for the other knot one gets the stated polynomial.

4.4 Jones Polynomial

The bracket polynomial $\langle K \rangle \in \mathbb{Z}[A, A^{-1}]$ is a Laurent polynomial in A. To obtain an ambient invariant from this, we will use the writhe of the knot in order to "fix" the trouble that R1 caused. Recall that the writhe is the sum of the signs of crossings. The move R1 has the following effect.

$$w\left(\right) = 1 + w\left(\right)$$
$$w\left(\right) = -1 + w\left(\right)$$

.

Definition 4.4.1. Let $F(K) \in \mathbb{Z}[A, A^{-1}]$, a Laurent polynomial, and define

$$F(K) = (-A^{-3})^{w(K)}\langle K\rangle.$$

Theorem 4.4.2 (Kauffman, [8])**.** *$F(K)$ is an ambient isotopy invariant.*

Proof. $\langle K\rangle$ and $w(K)$ does not change under R2 and R3 $\iff$ regular isotopy. R1 changes the bracket by the factor $-A^{\pm 3}$, $w(K)$ is meant to nullify this.

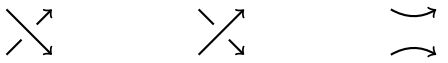
$$F\left(\right) = \left(-A^{-3}\right)^{w\left(\right)}\left\langle \right\rangle$$
$$= \left(-A^{-3}\right)^{w\left(\right)}\left\langle \right\rangle$$

So we get that

$$F\left(\right) = F\left(\right) = F\left(\right),$$

and this concludes the proof. □

The writhe can in fact be used to normalize any regular isotopy to obtain an ambient isotopy.



(a) L_+ (b) L_- (c) L_0

Figure 4.9: Crossings for the skein relation

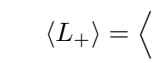
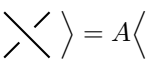
Definition 4.4.3. The *Jones polynomial*, $V_K(t)$, for a knot or link is a Laurent polynomial in $t^{1/2}$ satisfying the following properties:

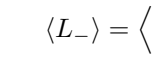
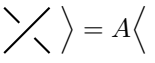
1. For an oriented knot or link K, $V_K(t) \in \mathbb{Z}[t, t^{-1}]$ is an ambient isotopy invariant of K.
2. $V_{\text{unknot}} = 1$.
3. The skein relation: $t^{-1}V_{L_+} - tV_{L_-} = (t^{1/2} - t^{-1/2})V_{L_0}$.

Jones first published this polynomial in [7]

Theorem 4.4.4 (Kauffman, [8])**.** *$V_K(t) = F(K)$ for $A = t^{-1/4}$.*

Proof. $V_{\text{unknot}}(t) = 1 = F(\text{unknot})$, property 2. from the definition holds.

$\langle L_+ \rangle =$  $= A$ $+ A^{-1}$ 

$\langle L_- \rangle =$  $= A$  $+ A^{-1}$, which gives

A $- A^{-1}$ $= (A^2 - A^{-2})$.

Adjusting the formula with the writhe of each diagram we get

$$A^4 F(L_+) - A^{-4} F(L_-) = (A^{-2} - A^2) F(L_0),$$

and substituting $A = t^{-1/4}$, gives the skein relationship in the definition and since $F(K)$ is ambient so is $V_K(t)$, which completes the proof. □

With the bracket polynomial Kauffman constructed another (easier) way to prove that the Jones polynomial is an ambient isotopy invariant.

Example 4.4.5. With this polynomial we can tell several different knots apart. All knots in the in Figure 4.2 have different polynomials. To give a few examples:

- The figure 8 knot in Figure 4.1 has polynomial $V_K(t) = t^2 - t + 1 - t^{-1} + t^{-2}$
- The star knot, 5_1, has the polynomial $V_K(t) = t^2 + t^4 - t^5 + t^6 - t^7$
- The 6_2 knot has the polynomial $V_K(t) = t - 1 + 2t^{-1} - 2t^{-2} + 2t^{-3} - 2t^{-4} + t^{-5}$
- The Hopf link Figure 4.8 has the polynomial $V_K(t) = -\sqrt{t}(1 + t^2)$

Chapter 5

Tangles and DNA

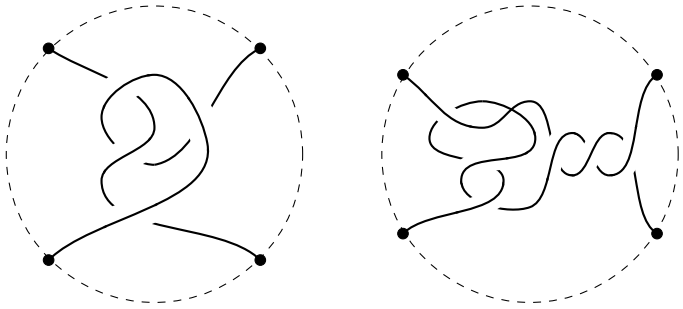
This chapter starts with an introduction to the theory of tangles. A tangle is a part of a knot (or link) and one only works with this part, rather than with the whole knot. Figure 5.1a depicts a tangle diagram, envision the knot continuing outside the circle. After the section on tangle theory we move on to DNA and apply what we have learned to the study of enzyme and DNA interaction. More specifically we target the recombination enzyme called resolvase.

5.1 Tangles

This section relies on the literature [1] and [15] as well as the article [4]. The closed 3-ball $\overline{B^3}$ is bounded by the sphere $\mathbb{S}^2$. Fix two points on the sphere (say the north and south pole) and attach a curve between them embedded in $\overline{B^3}$. The curve can be knotted but self-intersections are not permitted. This is called a $(1,1)$-tangle. Similarly, fixing 4 points on the sphere (preferably evenly spreed on a great circle) and connecting them with two curves embedded in B^3 gives $(2,2)$-tangle. So, 2n fixed points connected by n curves is called a (n,n)-tangle. A **tangle** is the set $(\overline{B^3}, T)$, where T is the collection of curves joining the points in pairs. This notation will be shortened and we let T denote the tangle.

There are several different type of tangles but all tangles henceforth will be $(2,2)$-tangles, and for simplicity refereed to as tangles. Some terminology for the fixed points on $\mathbb{S}^2$: denote these NW, NE, SW and SE (for north west, e.t.c.). A **tangle diagram** is a projection formed according to the rules of knot diagrams with the extra restrictions the points {NW, NE, SW, SE} should end up as shown in Figure 5.1.

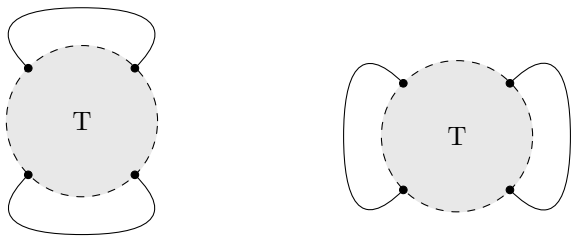
If you connect NW with NE, and SW with SE, outside of $\overline{B^3}$, as shown in



(a) (b)

Figure 5.1: Two examples of tangles.

Figure 5.2, you have formed the **numerator** of T, denoted $N(T)$. Connecting NW with SW, and NE with SE instead, give the **denominator** of T, $D(T)$. The numerator and the denominator gives us a natural connection between knots (or links) and tangles. This make it possible to view a tangle as a part of a knot, so many concepts for knots can be extended to tangles.



(a) $N(T)$ (b) $D(T)$

Figure 5.2: Numerator and denominator of a tangle.

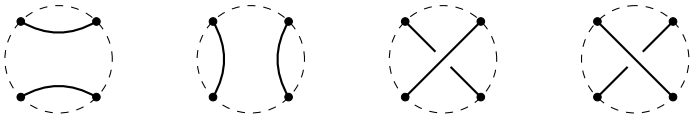
Definition 5.1.1. If an orientation-preserving homeomorphism $f : \overline{B^3} \longrightarrow \overline{B^3}$ satisfies the following conditions, then the tangles T_1 and T_2 are said to be *equivalent*:

1. f is the identity map for $\mathbb{S}^2$, i.e., the map keeps $\mathbb{S}^2$ fixed,
2. $f(T_1) = T_2$.

As for knots, Reidemeister moves can be used to manipulate the tangle diagrams. It is, however, important to remember that the strings can never be

(a) T_1 (b) T_2 (c) $T_1 + T_2$

Figure 5.3: Tangle sum



(a) (0)-type (b) (0,0)-type (c) (-1)-type (d) (1)-type

Figure 5.4: Exceptional tangles

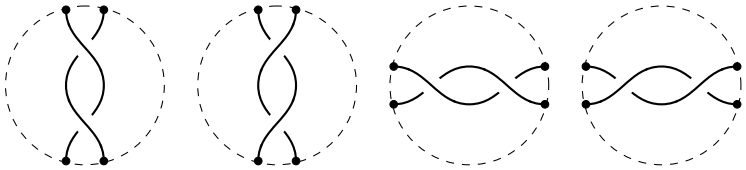
drawn outside the circle. This would represent a part of the tangle leaving $\overline{B^3}$ and in doing so creating more intersections in $\mathbb{S}^2$.

The formal **sum of tangles**, T_1+T_2 be formed by gluing the east hemisphere of $\overline{B_1^3}$ (containing T_1) to the west hemisphere of $\overline{B_2^3}$ (containing T_2) identifying NE and SE of T_1 to NW and SW of T_2 respectively. Lastly remove the gluing face to make one ball $\overline{B^3}$ containing T_1+T_2. This is illustrated diagrammatically in Figure 5.3. A vital observation is that even if $N(T_1)$ and $N(T_2)$ are non-trivial knots $N(T_1 + T_2)$ could still yield the unknot (or a trivial link).

Look at the tangles shown in Figure 5.4. These are four different types of tangles, the (0)-type, (0,0)-type also called the (∞)-type,(-1)-type and (1)-type respectively. Collectively we call these tangles the **exceptional tangles**. With knots we restricted ourselfs to tame knots, similarly we shall now restrict our study to what is called rational tangles.

Definition 5.1.2. Suppose f is a homeomorphism that maps the ball $\overline{B^3}$ to itself and maps the set of points {NW, NE, SW, SE} to itself, but not necessarily as the identity map (i.e. f need not map NW to NW e.t.c.). A *rational tangle* is a tangle that is the image of the $(0, 0)$-type tangle under this homeomorphism.

Rational tangles are in some literature refereed to as trivial tangles. All exceptional tangles are rational, by rotating $(0, 0)$ by $\frac{\pi}{2}$ we can obtain (0). Imagine holding the ball with $(0, 0)$ in it, induce a half twist by taking the points SW and SE and rotating them around each other so they change places, this will give you either (-1) or (1) depending on which direction you rotate. There are many rational tangles and one way to form them is by doing twists, just as the one just described. A **vertical twist** is a twist around a vertical



(a) V +2 (b) V -2 (c) H +2 (d) H -2

Figure 5.5: Orientations of vertical and horizontal twists

axis that keeps NW and NE fixed but interchanges SW and SE. A **horizontal twist** it a twist around a horizontal axis performed by keeping NW and SW fixed and rotating so that NE and SE interchange. Each twist has a natural orientation, we call them **positive twists** and **negative twists** (denoted by + and - respectively), see Figure 5.5. Look at the exceptional tangles: (-1) is obtained either by a negative vertical twist of (0) or a negative horizontal twist. Similarly the (1)-type comes from positive twist.

Description of **vertical and horizontal twists**. All trivial tangles can be made from (0) or (0,0) using only vertical and horizontal twists.

If we alternate between horizontal and vertical twists we get more complex looking tangles. A tangle constructed from twists will be denoted $T(a_1...a_n)$ where $(a_1...a_n)$ describes the sequence of twists in the following way:

- If n even, start with the $(0,0)$-type, make a_1 vertical twists followed by a_2 horizontal, and so on alternating the twists, until you finally perform a_n horizontal twists.
- If n odd, start with the (0)-type, make a_1 horizontal, a_2 vertical,..., a_n horizontal.

We allow $a_i = 0$, so we can have $T(3,0)$ for example. It is beneficial, or rather it will give us a shorter notation, if we require that $a_i \neq 0$, for $i \neq n$. The following example illustrates why this restriction works.

Example 5.1.3. The tangles is Figure 5.1 are $T_a(-2,1,1,0)$ and $T_b(2,-2,3)$.

Example 5.1.4. The tangle $T(1,5,0,-2)$ can also be described as $T(1,3)$. The five positive twists and the two negative twist end up next to each other (since $a_3 = 0$) and by the Reidemeister move R2 this is equivalent to just 3 positive twists.

Rational tangles can if fact be completely determined by the sequence of alternating twists [15]. The knots and link obtained by the denominator of a

rational tangle are actually a special kind of family of knots and links called 4-plats (also known as rational or 2-bridge knots and links). The set of 4-plats has been completely classified [15].

5.2 DNA

This section relies on extraordinary work done by C. Ernst and D. W. Sumners, [4] and [17]. DNA (deoxyribonucleic acid) is our genetic code. DNA is made form alternating sugar and phosphors and most of our DNA is organized on two strands (backbones) that twist around each other in a double helix. For the purpose of this work the double helix will be seen as one string in 3-dimensional space.

The genome is involved in several actions, there are for example, enzymes that perform transcription, replication, and, recombination of the DNA. We view an enzyme as a ball $\overline{B^3}$ and call this the **enzyme ball**. It is not possible for us to observe the actions taking place inside the enzymes, so in order to understand the action of an enzyme we must study the result (or product) of the action. In this work we will focus on the actions of the enzyme **recombinase**. Recombinase is a vital enzyme that moves segments or inserts new ones into the genome.

The topological approach to enzymology uses mathematical tools to study the geometrical (supercoiling) and topological (knotting and linking) properties of DNA. This is an experimental protocol where one tries to determine the enzyme mechanisms by combining laboratory experiments and topology. To do this one produces large quantities of circular DNA that can be genetically engineered to contain sites that the enzyme will react to. A solution with the enzyme is added and gel electrophoresis is used to determine the products. With the circular DNA as a starting point mathematical models can be made to study what actions the enzyme performed.

When recombinase acts on the genome it bounds two strand of juxtaposed DNA that both have sites recognized by the enzyme. These sites can be on two different DNA molecules or on the same one. The bounded DNA together with the enzyme is called **synaptosome**. The enzyme together with the DNA strand(s) (both inside and outside the enzyme ball) is called the **synaptic complex**.

We can model the actions of recombinase using mathematics, more specifically tangle theory will be used. Before beginning some biological assumptions must be made that we can build our mathematical model on.

Assumption 5.2.1 (cf. [17])**.** *The enzyme mechanism in a single recombination event is constant, independent of the geometry (supercoiling) and topology*

(knotting and linking) of the substrate population. Moreover, recombination takes place entirely within the domain of the enzyme ball, and the substrate configuration outside the enzyme ball remains fixed while the strands are being broken and recombined inside and on the boundary of the enzyme.

Assumption 5.2.2 (cf. [17]). *The synaptosome is a (2,2)-tangle and can be mathematically subdivided into the sum $O_b + P$ of two tangles.*

The tangle P in the assumption is called the **parental tangle**. It contains the sites where the breakage and reunion of the strands occur. The **outside bound tangle** O_b, is the part of the DNA strand bound by recombinase but not affected by the recombination. The recombination event is modeled as follows: we remove the tangle P and replace it by the **recombinant tangle** R. This gives

$$\begin{aligned} \text{pre-recombination synaptosome } &= O_b + P, \text{ and} \\ \text{post-recombination synaptosome } &= O_b + R. \end{aligned}$$

Assumption 5.2.3 (cf. [17]). *The entire synaptic complex is obtained from the tangle sum (O_f+ synaptosome) by the tangle closure construction (the nominator).*

From these assumptions we have four variables: R, P, O_b, and O_f. The assumptions give us two equations in these four variables:

$$\begin{aligned} N(O_f + O_b + P) &= \text{substrate, and} \\ N(O_f + O_b + R) &= \text{product.} \end{aligned}$$

Assumption 5.2.4 (cf. [17]). *In processive recombination, each round of recombination adds a copy of the recombinant tangle R to the synaptosome.*

If the enzyme, during a single bounding encounter, makes n recombinations we will obtain a system of equations that instead look like this:

$$\begin{aligned} N(O_f + O_b + P) &= \text{substrate, and} \\ N(O_f + O_b + iR) &= \text{product, for } i = 1, ..., n. \end{aligned}$$

There are still only four variables but by each new recombination a new equation is obtained. There are several different enzymes in the group recombinase and they will, naturally, have slightly different functions. So, with these assumptions and tangle theory as a basis, it is time to look more closely at one of the recombination enzyme: Tn3 resolvase. Using an electron microscope one can take pictures of the synaptic complex formed when resolvase interacts with

a DNA loop. From the obtained results it is clear that the DNA strands outside the synaptosome are untangled, thus $O_f = (0)$. This reduces our variables down to three: R, P, and O_b.

From experiment conducted on the products generated by resolvase one has found four different knots and links: the unknot, the Hopf link, the figure 8 knot, and the Whitehead link. Relying on the assumptions, a fair amount of tangle theory, and the experimental findings, the following theorem can be proved.

Theorem 5.2.5 (cf. [17])**.** *Suppose that the tangles R, P, and O_b, satisfy the following equations:*

1. $N(O_b + P) =$ *the unknot,*
2. $N(O_b + R) =$ *the Hopf link,*
3. $N(O_b + 2R) =$ *the figure 8 knot, and*
4. $N(O_b + 3R) =$ *the Whitehead link.*

Then $O_b = T(-3, 0)$, $R = T(1)$ *and* $N(O_b + 4R) = N(T(4, -3))$ *or the knot* 6_2.

The proof of this theorem relies on tangle theory that is outside the scope of this work but the the reader is refereed to the joint work of C. Ernst and D. W. Sumners [4] for a rigorous mathematical explanation of tangles and DNA recombination.

5.3 Summary

This thesis started with some thorough background in topology and the main concept was homeomorphic spaces. Homeomorphisms were then used several times throughout this work, for instance in the definition of a knot. In Chapter 3 we looked at the construction of homology theory by defining concepts such as cells and complexes. Thanks to homology we gained ways of understanding the structure of a space. After homology we moved into the field of knot theory where several invariants were described, the most prominent one was the Jones polynomial. Finally, in this last chapter, knot theory (or rather tangle theory) was used to shed light on how an enzyme acts on DNA.

Bibliography

[1] Colin Adams. *The Knot Book: An Elementary Introduction to the Theory of Knots*. New York: WH Freeman, 1994.

[2] Mark Anthony Armstrong. *Basic topology*. Springer-Verlag, New York, 1997.

[3] Richard H Crowell and Ralph H Fox. *Introduction to Knot theory*. Springer-Verlag, 1977.

[4] Claus Ernst and DeWitt Sumners. A calculus for rational tangles: applications to dna recombination. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 108, pages 489–515. Cambridge University Press, 1990.

[5] William Fulton. *Algebraic topology : a first course*. Graduate texts in mathematics: 153. New York ; Berlin : Springer-Vlg, cop. 1995, 1995.

[6] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, 2002.

[7] Vaughan F. R. Jones. A polynomial invariant for knots via von Neumann algebras. *Bull. Amer. Math. Soc. (N.S.)*, 12(1):103–111, 01 1985.

[8] Louis H Kauffman. State models and the Jones polynomial. *Topology*, 26(3):395–407, 1987.

[9] Louis H Kauffman. An invariant of regular isotopy. *Transactions of the American Mathematical Society*, 318(2):417–471, 1990.

[10] Louis H Kauffman. Knot Diagrammatics. *ArXiv Mathematics e-prints, math/0410329*, October 2004.

[11] Christine Kinsey. *Topology of surfaces*. Springer-Verlag, New York, 1993.

[12] Erik Landstedt. Fractals and billiard orbits on Sierpinski carpets, 2017. Bachelor thesis, Linköping University, Department of Mathematics, ISRN: LiTH-MAT-EX–2017/05–SE.

[13] WB Raymond Lickorish. *An Introduction to Knot Theory*, volume 175. Springer Science & Business Media, 1997.

[14] Charles Livingston. *Knot theory*, volume 24. The Mathematical Association of America, 1993.

[15] Kunio Murasugi. *Knot theory and its applications*. Birkhäuser, 1996.

[16] Sam Derbyshire. Cantor set binary tree, 2016. Own work, licensed under [CC BY-SA 4.0 (http://creativecommons.org/licenses/by-sa/4.0)], via Wikimedia Commons.

[17] De Witt Sumners. Lifting the curtain: using topology to probe the hidden action of enzymes. *Notices of the American Mathematical Society*, 42(5):528–537, 1995.

[18] Per-Anders Svensson. *Abstrakt algebra*. Studentlitteratur, Lund, 2001.

[19] Eric W. Weisstein. Alternating knot. From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/AlternatingKnot.html (2017-07-21).

[20] Eric W. Weisstein. Invertible knot. From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/AlternatingKnot.html (2017-11-14).

[21] Wikipedia user Jkasd. Wild knot, 2008. Own work, released to the Public Domain, via Wikimedia Commons - https://commons.wikimedia.org/wiki/File:Wild_knot.svg.

[22] Wikipedia user Jkasd. Knot table, 2011. Own work, released to the Public Domain, via Wikimedia Commons - https://commons.wikimedia.org/wiki/File:Knot_table.svg.

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