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# THE WEAK LEFSCHETZ PROPERTY OF EQUIGENERATED MONOMIAL IDEALS

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ABSTRACT. We determine the sharp lower bound for the Hilbert function in degree  $d$  of a monomial algebra failing the WLP over a polynomial ring with  $n$  variables and generated in degree  $d$ , for any  $d \geq 2$  and  $n \geq 3$ . We consider artinian ideals in the polynomial ring with  $n$  variables generated by homogeneous polynomials of degree  $d$  invariant under an action of the cyclic group  $\mathbb{Z}/d\mathbb{Z}$ , for any  $n \geq 3$  and any  $d \geq 2$ . We give a complete classification of such ideals in terms of the WLP depending on the action.

## 1. INTRODUCTION

The weak Lefschetz property (WLP) for an artinian graded algebra  $A$  over a field  $\mathbb{K}$ , says there exists a linear form  $\ell$  that induces, for each degree  $i$ , a multiplication map  $\times \ell : (A)_i \rightarrow (A)_{i+1}$  that has maximal rank, i.e. that is either injective or surjective. Though many algebras are expected to have the WLP, establishing this property for a specific class of algebras is often rather difficult. In this paper we study the WLP of the specific class of algebras which are the quotients of a polynomial ring  $S = \mathbb{K}[x_1, \dots, x_n]$  over field  $\mathbb{K}$  of characteristic zero by artinian monomial ideals generated in the same degree  $d$ . For this class of artinian algebras, E. Mezzetti and R. M. Miró-Roig [8], showed that  $2n - 1$  is the sharp lower bound for the number of generators of  $I$  when the injectivity fails for  $S/I$  in degree  $d - 1$ . In fact they give the lower bound for the number of generators for the minimal monomial Togliatti systems in  $\mathbb{K}[x_1, \dots, x_n]$  of the forms of degree  $d$ . For more details see the original articles of Togliatti [14], [13]. In the first part of this article we establish the lower bound for the number of monomials in the cobasis of the ideal  $I$  in the ring  $S$  or equivalently, lower bound for the Hilbert function of  $S/I$  in degree  $d$ , which is  $H_{S/I}(d) := \dim_{\mathbb{K}}(S/I)_d$ , where surjectivity fails in degree  $d - 1$ . We observe that once multiplication by a general linear form on a quotient of  $S$  is surjective, then it remains surjective in the next degrees. This implies that all these algebras with the Hilbert function  $H_{S/I}(d)$  below our bound satisfy the WLP.

In the main theorems of the first part of this paper, we provide a sharp lower bound for  $H_{S/I}(d)$  for artinian monomial algebra  $S/I$ , where the surjectivity fails for  $S/I$  in degree  $d - 1$ . For the cases when the number of variables,  $n = 1$  and  $n = 2$  the bound is known. The first main theorem provides the bound when the polynomial ring has three variables.

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**Theorem 1.1.** *Let  $I \subset S = \mathbb{K}[x_1, x_2, x_3]$  be an artinian monomial ideal generated in degree  $d$ , for  $d \geq 2$  such that  $S/I$  fails to have the WLP. Then we have that*

$$H_{S/I}(d) \geq \begin{cases} 3d - 3 & \text{if } d \text{ is odd} \\ 3d - 2 & \text{if } d \text{ is even.} \end{cases}$$

*Furthermore, the bound are sharp.*

In the second theorem we provide a different sharp bound when the number of variables is more than three.

**Theorem 1.2.** *Let  $I \subset S = \mathbb{K}[x_1, \dots, x_n]$  be an artinian monomial ideal generated in degree  $d$ , for  $d \geq 2$  and  $n \geq 4$  such that  $S/I$  fails to have the WLP. Then we have that*

$$H_{S/I}(d) \geq 2d.$$

*Furthermore, the bound is sharp.*

In [9], Mezzetti and Miró-Roig construct a class of examples of Togliatti systems in three variables of any degree. More precisely, they consider the action on  $S = \mathbb{K}[x, y, z]$  of cyclic group  $\mathbb{Z}/d\mathbb{Z}$  defined by  $[x, y, z] \mapsto [\xi^a x, \xi^b y, \xi^c z]$ , where  $\xi$  is a primitive  $d$ -th root of unity and  $\gcd(a, b, c, d) = 1$ . They prove that the ideals generated by forms of degree  $d$  invariant by such actions are all defined by monomial Togliatti systems of degree  $d$ . In this note we generalize this result in the polynomial ring with four variables. In [10], Mezzetti, Miró-Roig and Ottaviani described a connection between projective varieties satisfying at least one Laplace equation and homogeneous artinian ideals generated by polynomials of the same degree  $d$  failing the WLP by failing injectivity of a multiplication map by a linear form in degree  $d - 1$ .

In this article, we consider an action of cyclic group  $\mathbb{Z}/d\mathbb{Z}$  defined by  $[x_1, \dots, x_n] \mapsto [\xi^{a_1} x_1, \dots, \xi^{a_n} x_n]$  and we consider the ideals generated by all forms of degree  $d \geq 2$  in  $S = \mathbb{K}[x_1, \dots, x_n]$  fixed by such actions. In Theorem 7.9, we prove that these ideals satisfy the WLP if and only if at least  $n - 1$  of the integers  $a_i$  are equal. In addition, in the polynomial ring with three variables we give a formula for the number of fixed monomials and we provide bounds for such numbers.

## 2. PRELIMINARIES

We consider standard graded algebras  $S/I$ , where  $S = \mathbb{K}[x_1, \dots, x_n]$ ,  $I$  is a homogeneous ideal of  $S$ ,  $\mathbb{K}$  is a field of characteristic zero and the  $x_i$ 's all have degree 1. Our ideal  $I$  will be an artinian monomial ideal generated in a single degree  $d$ . Given a polynomial  $f$  we denote the set of monomials with non-zero coefficients in  $f$  by  $\text{Supp}(f)$ .

Now let us define the weak and strong Lefschetz properties for artinian algebras.

**Definition 2.1.** Let  $I \subset S$  be a homogeneous artinian ideal. We say that  $S/I$  has the *Weak Lefschetz Property* (WLP) if there is a linear form  $\ell \in (S/I)_1$  such that, for all integers  $j$ , the multiplication map

$$\times \ell : (S/I)_j \longrightarrow (S/I)_{j+1}$$

has maximal rank, i.e. it is injective or surjective. In this case the linear form  $\ell$  is called a *Lefschetz element* of  $S/I$ . If for general linear form  $l \in (S/I)_1$  and for an integer  $j$  the map  $\times l$  does not have the maximal rank we will say that the ideal  $I$  fails the WLP in degree  $j$ .

We say that  $S/I$  has the *Strong Lefschetz Property* (SLP) if there is a linear form  $\ell \in (S/I)_1$  such that, for all integers  $j$  and  $k$  the multiplication map

$$\times \ell^k : (S/I)_j \longrightarrow (S/I)_{j+k}$$

has maximal rank, i.e. it is injective or surjective. We often abuse the notation and say that  $I$  fails or satisfies the WLP or SLP, when we mean that  $S/I$  does so.

In the case of one variable, the WLP and SLP hold trivially since all ideals are principal. Harima, Migliore, Nagel and Watanabe in [4], Proposition 4.4 proved the following result in two variables.

**Proposition 2.2.** *Every artinian ideal in  $\mathbb{K}[x, y]$  ( $\text{char } \mathbb{K} = 0$ ) has the Strong Lefschetz property (and consequently also the Weak Lefschetz property).*

In a polynomial ring with more than two variables, it is not true in general that every artinian monomial algebra has the SLP or WLP. Also it is often rather difficult to determine whether a given algebra satisfies the SLP or even WLP. One of the main general results in a ring with more than two variables is proved by Stanley in [12].

**Theorem 2.3.** *Let  $S = \mathbb{K}[x_1, \dots, x_n]$ , where  $\text{char}(\mathbb{K}) = 0$ . Let  $I$  be an artinian monomial complete intersection, i.e.  $I = (x_1^{a_1}, \dots, x_n^{a_n})$ . Then  $S/I$  has the SLP.*

Because of the action of the torus  $(\mathbb{K}^*)^n$  on monomial algebras, there is a canonical linear form that we have to consider. In fact we have the following result in [11], Proposition 2.2 proved by Migliore, Miró-Roig and Nagel.

**Proposition 2.4.** *Let  $I \subset S$  be an artinian monomial ideal. Then  $S/I$  has the weak Lefschetz property if and only if  $x_1 + x_2 + \dots + x_n$  is a Lefschetz element for  $S/I$ .*

Let us now recall some facts of the theory of the *inverse system*, or *Macaulay duality*, which will be a fundamental tool in this paper. For a complete introduction, we refer the reader to [2] and [5].

Let  $R = \mathbb{K}[y_1, \dots, y_n]$ , and consider  $R$  as a graded  $S$ -module where the action of  $x_i$  on  $R$  is partial differentiation with respect to  $y_i$ .

There is a one-to-one correspondence between graded artinian algebras  $S/I$  and finitely generated graded  $S$ -submodules  $M$  of  $R$ , where  $I = \text{Ann}_S(M)$  and is the annihilator of  $M$  in  $S$  and, conversely,  $M = I^{-1}$  is the  $S$ -submodule of  $S$  which is annihilated by  $I$  (cf. [2, Remark 1]), p.17). Since the map  $\circ \ell : R_{i+1} \longrightarrow R_i$  is dual of the map  $\times \ell : (S/I)_i \longrightarrow (S/I)_{i+1}$  we conclude that the injectivity (resp. surjectivity) of the first map is equivalent to the surjectivity (resp. injectivity) of the second one. Here by " $\circ \ell$ " we mean that the linear form  $\ell$  acts on  $R$ .

For a monomial ideal  $I$  the inverse system module  $(I^{-1})_d$  is generated by the corresponding monomials of  $S_d$  but not in  $I_d$  in the dual ring  $R_d$ .

Mezzetti, Miró-Roig and Ottaviani in [10] described a relation between existence of artinian ideals  $I \subset S$  generated by homogeneous forms of degree  $d$  failing the WLP and the existence of projections of the Veronese variety  $V(n-1, d) \subset \mathbb{P}^{\binom{n+d-1}{d}-1}$  in  $X \subset \mathbb{P}^N$  satisfying at least one Laplace equation of order  $d-1$ .

Let  $I \subset S$  be an artinian ideal generated by homogeneous polynomials  $f_1, \dots, f_r$  of degree  $d$  and  $I^{-1}$  be its inverse system module and consider the rational map  $\varphi_{(I^{-1})_d} : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{\binom{n+d-1}{d}-r-1}$  associated to  $(I^{-1})_d$ . Denote its image by  $X_{n-1, (I^{-1})_d} := \overline{\text{im}(\varphi_{(I^{-1})_d})} \subset \mathbb{P}^{\binom{n+d-1}{d}-r-1}$ . Then  $X_{n-1, (I^{-1})_d}$  is the projection of the Veronese variety  $V(n-1, d)$  from the linear system from the vector space spanned by  $f_1, \dots, f_r$ . Associated to  $I_d$  there is a morphism  $\varphi_{I_d} : \mathbb{P}^{n-1} \rightarrow \mathbb{P}^{r-1}$ . Since  $I$  is artinian  $\varphi_{I_d}$  is regular. Denote  $X_{n-1, I_d} := \overline{\text{im}(\varphi_{I_d})} \subset \mathbb{P}^{r-1}$  which is the projection of the Veronese variety,  $V(n-1, d)$ , forms the linear system of the vector space spanned by forms in  $(I^{-1})_d$ . The varieties  $X_{n-1, (I^{-1})_d}$  and  $X_{n-1, I_d}$  are usually called *apolar*.

With notations as above, in [10], Theorem 3.2 Mezzetti, Miró-Roig and Ottaviani proved the following theorem.

**Theorem 2.5.** *Let  $I \subset S$  be an artinian ideal generated by  $r$  forms  $f_1, \dots, f_r$  of degree  $d$ . If  $r \leq \binom{n+d-2}{n-2}$ , then the following conditions are equivalent:*

- (1) *The ideal  $I$  fails the WLP in degree  $d-1$ ,*
- (2) *The forms  $f_1, \dots, f_r$  become  $\mathbb{K}$ -linearly dependent on a general hyperplane  $H$  of  $\mathbb{P}^{n-1}$ ,*
- (3) *The  $n-1$ -dimensional variety  $X_{n-1, (I^{-1})_d}$  satisfies at least one Laplace equation of order  $d-1$ .*

If  $I$  satisfies the three equivalent conditions in the above theorem,  $I$  (or  $I^{-1}$ ) is called *Togliatti system*.

### 3. ON THE SUPPORT OF FORM $f$ ANNIHILATED BY $\ell$ AND ITS HIGHER POWERS

Let  $S = \mathbb{K}[x_1, \dots, x_n]$  be the polynomial ring where  $n \geq 3$  and  $\mathbb{K}$  is a field of characteristic zero. In this section we give some definitions and notations and prove some results about the number of monomials in the support of polynomials  $f \in (I^{-1})_d$  with  $(x_1 + \dots + x_n)^a \circ f = 0$  for some  $1 \leq a \leq d$ . Now let us define an specific type of well known integer matrices which we use them throughout this section.

**Definition 3.1.** For a non-negative integer  $k$  and positive integer  $m$ , where  $k \leq m$ , we define the Toeplitz matrix  $T_{k,m}$ , to be the following  $(k+1) \times (m+1)$  matrix

$$T_{k,m} = \begin{bmatrix} \binom{m-k}{0} & \binom{m-k}{1} & \binom{m-k}{2} & \cdots & \binom{m-k}{m-k} & 0 & \cdots & 0 \\ 0 & \binom{m-k}{0} & \binom{m-k}{1} & \cdots & \binom{m-k}{m-k-1} & \binom{m-k}{m-k} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{m-k}{m-k-3} & \binom{m-k}{m-k-2} & \binom{m-k}{m-k-1} & \binom{m-k}{m-k} \end{bmatrix}$$

where the  $(i, j)^{\text{th}}$  entry of this matrix is  $\binom{m-k}{j-i}$  and we use the convention that  $\binom{m}{i} = 0$  if  $i < 0$  or  $m > i$ .

We have the following useful lemma which proves the maximal minors of  $T_{k,m}$  are non-zero.

**Lemma 3.2.** *For each non-negative integer  $k$  and positive integer  $m$  where  $k \leq m$ , all maximal minors of the Toeplitz matrix  $T_{k,m}$  are non-zero.*

*Proof.* Let  $R = \mathbb{K}[x, y]$  be the polynomial ring in variables  $x$  and  $y$  and choose monomial bases  $\mathcal{A} := \{x^j y^{k-j}\}_{j=0}^k$  and  $\mathcal{B} := \{x^i y^{m-i}\}_{i=1}^m$  for the  $\mathbb{K}$ -vector spaces  $R_k$  and  $R_m$ , respectively. Observe that  $T_{k,m}$  is the matrix representing the multiplication map  $\times(x+y)^{m-k} : R_k \rightarrow R_m$  with respect to the bases  $\mathcal{A}$  and  $\mathcal{B}$ . Given any square submatrix  $M$  of size  $k+1$ , define ideal  $J \subset R$  generated by the subset of monomials in  $\mathcal{B}$ , called  $\mathcal{B}'$ , corresponding to the columns of  $T_{k,m}$  not in  $M$ . Therefore,  $\mathcal{A}$  and  $\mathcal{B} \setminus \mathcal{B}'$  form monomial bases for  $(R/J)_k$  and  $(R/J)_m$ , respectively and  $M$  is the matrix representing the multiplication map  $\times(x+y)^{m-k} : (R/J)_k \rightarrow (R/J)_m$  with respect to  $\mathcal{A}$  and  $\mathcal{B} \setminus \mathcal{B}'$ . Since by Proposition 2.2, any monomial  $R$ -algebra has the SLP, and by Proposition 2.4  $x+y$  is a Lefschetz element for  $R/J$ , the multiplication map by  $x+y$  is a bijection and therefore the matrix  $M$  has non-zero determinant. This implies that all the maximal minors of  $T_{k,m}$  are non-zero.  $\square$

Consider a non-zero homogeneous polynomial  $f$  of degree  $d$  in the dual ring  $R = \mathbb{K}[y_1, \dots, y_n]$  where we have  $(x_1 + \dots + x_n) \circ f = 0$ . We use the following notations and definitions to prove some properties of such polynomial  $f$ .

**Definition 3.3.** For an ideal  $I$  of  $S$ , we denote Hilbert function of  $S/I$  in degree  $d$  by  $H_{S/I}(d) := \dim_{\mathbb{K}}(S/I)_d$ , and the set of all artinian monomial ideals of  $S$  generated in a single degree  $d$  by  $\mathcal{I}_d$ . In addition, for an artinian ideal  $I$  we define  $\phi(I, d) : \times(x_1 + \dots + x_n) : (S/I)_{d-1} \rightarrow (S/I)_d$  and

$$\nu(n, d) := \min\{H_{(S/I)}(d) \mid \phi(I, d) \text{ is not surjective, for } I \in \mathcal{I}_d\}.$$

**Definition 3.4.** In a polynomial ring  $S = \mathbb{K}[x_1, \dots, x_n]$ , for any monomial  $m$  and variable  $x_i$ , we define

$$\deg_i(m) := \max\{e \mid x_i^e \mid m\}$$

Define the set  $\mathcal{M}_d$  to be the set of monomials of degree  $d$  in  $R$  and denote the set of monomials of degree  $k$  with respect to the variable  $y_i$  by,

$$\mathcal{L}_{i,d}^k := \{m \in \mathcal{M}_d \mid \deg_i(m) = k\} \subset \mathcal{M}_d.$$

**Lemma 3.5.** Let  $f$  be a form of degree  $d \geq 2$  in the dual ring  $R = \mathbb{K}[y_1, \dots, y_n]$  of  $S = \mathbb{K}[x_1, \dots, x_n]$  and let the linear forms  $\ell := x_1 + \dots + x_n$  and  $\ell' := \ell - x_j$  for  $1 \leq j \leq n$ . Write  $f = \sum_{i=0}^d y_j^i g_i$ , where  $g_i$  is a polynomial of degree  $d-i$  in the variables different from  $y_j$ , then for every  $0 \leq c \leq d$ , we have

$$(3.1) \quad \ell^c \circ f = \sum_{k=0}^{d-c} \sum_{i=0}^c \frac{(k+c-i)!}{k!} \binom{c}{i} y_j^k \ell'^i \circ g_{k+c-i}.$$

In particular,  $\ell^c \circ f = 0$  if and only if,

$$(3.2) \quad \sum_{i=0}^c \frac{(k+c-i)!}{k!} \binom{c}{i} \ell'^i \circ g_{k+c-i} = 0, \quad 0 \leq k \leq d-c.$$

*Proof.* We prove the lemma using induction on  $c$ . For  $c=0$  the equality (3.1) is trivial. For  $c=1$ , we have

$$(3.3) \quad \ell \circ f = \sum_{k=0}^d k y_j^{k-1} g_k + y_j^k \ell' \circ g_k = \sum_{k=0}^{d-1} (k+1) y_j^k g_{k+1} + y_j^k \ell' \circ g_k.$$

Assume the equality holds for  $c - 1$  then we have  $\ell^c \circ f = \ell \circ (\ell^{c-1} \circ f)$  and

$$\begin{aligned}
\ell \circ (\ell^{c-1} \circ f) &= \ell \circ \left( \sum_{k=0}^{d-c+1} \sum_{i=0}^{c-1} \frac{(k+c-1-i)!}{k!} \binom{c-1}{i} y_j^k \ell'^i \circ g_{k+c-1-i} \right) \\
&= \sum_{k=0}^{d-c+1} \sum_{i=0}^{c-1} \frac{(k+c-1-i)!}{k!} \binom{c-1}{i} \left( k y_j^{k-1} \ell'^i \circ g_{k+c-1-i} + y_j^k \ell'^{i+1} \circ g_{k+c-1-i} \right) \\
&= \sum_{k=0}^{d-c} \sum_{i=0}^{c-1} \frac{(k+c-i)!}{(k+1)!} \binom{c-1}{i} (k+1) y_j^k \ell'^i \circ g_{k+c-i} \\
&\quad + \frac{(k+c-1-i)!}{k!} \binom{c-1}{i} y_j^k \ell'^{i+1} \circ g_{k+c-1-i} \\
&= \sum_{k=0}^{d-c} \sum_{i=0}^c \left( \frac{(k+1)(k+c-i)!}{(k+1)!} \binom{c-1}{i} + \frac{(k+c-i)!}{k!} \binom{c-1}{i-1} \right) y_j^k \ell'^i \circ g_{k+c-i} \\
&= \sum_{k=0}^{d-c} \sum_{i=0}^c \frac{(k+c-i)!}{k!} \binom{c}{i} y_j^k \ell'^i \circ g_{k+c-i}.
\end{aligned}$$

□

Using the above lemma the following proposition gives properties about the form  $f$ .

**Proposition 3.6.** *Let  $f$  be a non-zero form of degree  $d$  in the dual ring  $R = \mathbb{K}[y_1, \dots, y_n]$  of  $S = \mathbb{K}[x_1, \dots, x_n]$  such that  $(x_1 + \dots + x_n) \circ f = 0$ . Then the following conditions hold:*

- (i) *If  $y_i^d \notin \text{Supp}(f)$ , then the sum of the coefficients of  $f$  corresponding to the monomials in  $\mathcal{L}_{i,d}^k \cap \text{Supp}(f)$  is zero; for each  $0 \leq k \leq d-1$ .*
- (ii) *If  $a = \max\{\deg_i(m) \mid m \in \text{Supp}(f)\}$ , then  $\mathcal{L}_{i,d}^k \cap \text{Supp}(f) \neq \emptyset$ ; for all  $0 \leq k \leq a$ .*

*Proof.* Write the form  $f$  as,  $f = \sum_{k=0}^d y_i^k g_k$ , where  $g_k$  is a degree  $d-k$  polynomial in variables different from  $y_i$ . Denote  $\ell = x_1 + \dots + x_n$  and  $\ell' = \ell - x_i$ . Since  $\ell \circ f = 0$ , Lemma 3.5 implies that

$$(3.4) \quad (k+1)y_i^k g_{k+1} + y_i^k \ell' \circ g_k = 0, \quad \forall 0 \leq k \leq d-1.$$

To show (i) we act each equation by  $(\ell')^{d-k-1}$  and we get that

$$(3.5) \quad (k+1)(d-k-1)!g_{k+1}(1, \dots, 1) + (d-k)!g_k(1, \dots, 1) = 0 \quad \forall 0 \leq k \leq d-1.$$

Since we assumed  $g_d = 0$  we get that  $g_k(1, \dots, 1) = 0$  for all  $0 \leq k \leq d-1$ , which implies that for all  $0 \leq k \leq d-1$  sum of the coefficients of  $f$  corresponding to the monomials in  $\mathcal{L}_{i,d}^k \cap \text{Supp}(f)$  is zero and proves part (i).

To show part (ii), note that  $a = \max\{\deg_i(m) \mid m \in \text{Supp}(f)\}$  implies that  $g_a \neq 0$ . Using Equation (3.4) recursively we get that  $g_j \neq 0$  for all  $0 \leq j \leq a$ , which means that  $\mathcal{L}_{i,d}^j \cap \text{Supp}(f) \neq \emptyset$  for all  $0 \leq j \leq a$ . □

In the following theorem we provide a bound for the number of monomials with non-zero coefficients in the non-zero form in the kernel of the map  $\circ(x_1 + \dots + x_n)^{d-a}: (I^{-1})_d \rightarrow$

$(I^{-1})_a$ . In particular it provides a bound on the number of generators for an equigenerated monomial ideal in  $S$  failing the WLP.

**Theorem 3.7.** *Let  $f \neq 0$  be a form of degree  $d$  in the dual ring  $R = \mathbb{K}[y_1, \dots, y_n]$  of the ring  $S = \mathbb{K}[x_1, \dots, x_n]$ . If for the linear form  $\ell := x_1 + \dots + x_n$  we have  $\ell^{d-a} \circ f = 0$  for some  $0 \leq a \leq d - 1$ , then  $|\text{Supp}(f)| \geq a + 2$ .*

*Proof.* For a variable  $y_j$  write  $f = \sum_{i=0}^d y_j^i g_i$  such that  $g_i$  is a polynomial of degree  $d - i$  in the variables different from  $y_j$ . Since for some  $1 \leq a \leq d - 1$  we have  $\ell^{d-a} \circ f = 0$  from Lemma 3.5 we have that

$$(3.6) \quad \sum_{i=0}^{d-a} \frac{(k+d-a-i)!}{k!} \binom{d-a}{i} \ell^i \circ g_{k+d-a-i} = 0, \quad 0 \leq k \leq a.$$

For every  $j$  with  $1 \leq j \leq a + 1$  act each equation in the above system by  $(\ell')^{j-k-1}$ , so we have

$$(3.7) \quad \sum_{i=0}^{d-a} \frac{(k+d-a-i)!}{k!} \binom{d-a}{i} \ell'^{i+j-k-1} \circ g_{k+d-a-i} = 0, \quad 0 \leq k \leq a,$$

equivalently for each  $j$  with  $1 \leq j \leq a + 1$  we have

$$(3.8) \quad \sum_{i=a-k}^{d-k} \frac{(d-i)!}{k!} \binom{d-a}{i-(a-k)} \ell'^{i+j-(a+1)} \circ g_{d-i} = 0, \quad 0 \leq k \leq j - 1.$$

Note that for  $k \geq j$  the equations in (3.7) are zero.

For each  $0 \leq j \leq a + 1$  the coefficient matrix of the system in (3.8) in the forms  $(d - i)! \ell'^{i+j-(a+1)} \circ g_{d-i}$  is the Toeplitz matrix  $T_{(j-1) \times (d-a+j-1)}$  up to multiplication of  $k$ -th row by  $\frac{1}{k!}$ . Using Lemma 3.2 we get that all the maximal minors of this coefficient matrix are non-zero. This implies that in each system of equations either all the terms are zero or there are at least  $j + 1$  non-zero terms.

Now we want to prove the statement by induction on the number of variables  $n$ . Suppose  $n = 2$  then each  $g_i$  is a monomial of degree  $d - i$  in one variable. In (3.8) consider the corresponding system of equations for  $j = a + 1$ . If for every  $0 \leq i \leq d$  we have that  $\ell^i \circ g_{d-i} = 0$  implies that for every  $0 \leq i \leq d$  we have  $g_{d-i} = 0$  which contradicts the assumption that  $f \neq 0$ . Therefore for at least  $a + 2$  indices  $0 \leq i \leq d$  we have  $\ell^i \circ g_{d-i} \neq 0$  which means  $|\text{Supp}(f)| \geq a + 2$ .

Now we assume that the statement is true for the forms  $f$  in polynomial rings with  $n - 1$  ( $n \geq 3$ ) variables and we prove it for the form with  $n$  variables.

We divide it into two cases, suppose in the system of equations for every  $1 \leq j \leq a + 1$  all terms are zero. In this case for each  $1 \leq j \leq a + 1$ , letting  $i = a - j + 1$  implies that  $(\ell')^{a-j+1+j-(a+1)} \circ g_{d-(a-j+1)} = g_{d-a+j-1} = 0$  for all  $1 \leq j \leq a + 1$ . Since we assume that  $f \neq 0$  there exists  $a + 1 \leq i \leq d$  such that  $g_{d-i} \neq 0$ , but considering  $j = 1$  in (3.8) with the assumption that all terms in this equation is zero we get that  $(\ell')^{i-a} \circ g_{d-i} = 0$ . Using the induction hypothesis on the polynomial  $g_{d-i}$  in  $n - 1$  variables we get that  $|\text{Supp}(f)| \geq |\text{Supp}(g_{d-i})| \geq d - (d - i) - (i - a) + 2 = a + 2$  as we wanted to prove.

Now we assume that there exists  $1 \leq j \leq a + 1$  such that there are at least  $j + 1$  indices  $0 \leq i \leq d$  such that  $\ell'^{i+j-(a+1)} \circ g_{d-i} \neq 0$  in the corresponding system of equations in 3.8.



We take the largest index  $j$  with this property and we get that for these  $j + 1$  indices we have that  $\ell^{i+j-(a+1)+1} \circ g_{d-i} = 0$ . Now using the induction hypothesis in these polynomials we get that  $|\text{Supp}(g_{d-i})| \geq d - (d - i) - (i + j - (a + 1) + 1) + 2 = a + 2 - j$ , therefore

$$|\text{Supp}(f)| \geq \sum_{i=0}^d |\text{Supp}(g_i)| \geq (j + 1)(a + 2 - j) \geq a + 2.$$

□

#### 4. BOUNDS ON THE NUMBER OF GENERATORS OF IDEALS WITH THREE VARIABLES FAILING WLP

In this section we consider artinian monomial ideals  $I \subset S = \mathbb{K}[x_1, x_2, x_3]$  generated in a single degree  $d$ . In [8], Mezzetti and Miró-Roig provided a sharp lower bound for the number of generators of such ideals failing the WLP by failing injectivity of the multiplication map on the algebra in degree  $d - 1$ . Here we prove a sharp upper bound for the number of generators of such ideals failing the WLP by failing surjectivity in degree  $d - 1$  equivalently we provide a sharp lower bound for the number of generators of  $(I^{-1})_d$  where the map  $\circ\ell: (I^{-1})_d \rightarrow (I^{-1})_{d-1}$  is not injective, where  $\ell = x_1 + x_2 + x_3$ .

First we prove an easy but interesting result. Recall that every polynomial in at most two variables factors as a product of linear forms over an algebraically closed field. Here we note that the same statement holds in three variables if the polynomial vanishes by the action of a linear form on the dual ring. This in some cases corresponds to the failure of WLP. Note that for the WLP, the assumption on the field to be algebraically closed is not necessary, but in order to factor the form as a product of linear forms we need to have this assumption on the field. In addition the statement does not necessarily hold in polynomial rings with more than three variables.

**Lemma 4.1.** *Let  $S = \mathbb{K}[x_1, x_2, x_3]$  and  $S/I$  be an artinian algebra over an algebraically closed field  $\mathbb{K}$ . Let  $f$  be a form in the kernel of the map  $\circ\ell: (I^{-1})_i \rightarrow (I^{-1})_{i-1}$  for a linear form  $\ell$  and integer  $i$ , then  $f$  factors as a product of linear forms each of which is annihilated by  $\ell$ .*

*Proof.* By a linear change of variables we consider  $S = \mathbb{K}[x'_1, x'_2, x'_3]$  and  $R = \mathbb{K}[y'_1, y'_2, y'_3]$  simultaneously in such a way that  $x'_1 = \ell$ . Then we have that  $\ell \circ f(y_1, y_2, y_3) = x'_1 \circ f(y'_1, y'_2, y'_3) = 0$  where this implies that  $f$  is a polynomial in two variables  $y'_2$  and  $y'_3$ . Using the fact that any polynomial in two variables over an algebraically closed field factors as a product of linear form we conclude that  $f$  factors as a product of linear forms in  $y'_2$  and  $y'_3$ . Hence all of them are annihilated by  $\ell = x'_1$ . □

The next proposition provides a bound for the number of non-zero terms in each homogeneous component with respect to one of the variables for a non-zero form  $f$ , where  $\ell \circ f = 0$ .

**Proposition 4.2.** *Let  $a_i = \max\{\deg_i(m) \mid m \in \text{Supp}(f)\}$ , then we have*

$$|\mathcal{L}_{i,d}^k \cap \text{Supp}(f)| \geq d - a_i + 1, \quad \forall 0 \leq k \leq a_i, \quad 1 \leq i \leq 3.$$

*Proof.* Write  $f = \sum_{k=0}^{a_i} y_i^k g_k$ , where  $g_k$  is a degree  $d - k$  polynomial in two variables different from  $y_i$ . Let  $\ell' = \ell - x_i$ , then we have

$$0 = \ell \circ f = \ell \circ \left( \sum_{k=0}^{a_i} y_i^k g_k \right) = \sum_{k=0}^{a_i} k y_i^{k-1} g_k + y_i^k \ell' \circ g_k$$

therefore

$$(4.1) \quad (k+1)g_{k+1} + \ell' \circ g_k = 0, \quad \forall 0 \leq k \leq a_i$$

after linear change of variables to  $u := y_\alpha + y_\beta$  and  $v := y_\alpha - y_\beta$ , Equation (4.1) implies that,  $(\partial/\partial u)^{a_i-k+1} \circ g_k = 0$  for any  $0 \leq k \leq a_i$ . Therefore, for each  $0 \leq k \leq a_i$  we have

$$g_k = \sum_{j=0}^{a_i-k} \lambda_j u^j v^{a_i-k+j} = v^{d-a_i} \sum_{j=0}^{a_i-k} \lambda_j u^j v^{a_i-k+j} \quad \lambda_j \in \mathbb{K}.$$

Rewriting  $g_k$  in the variables  $y_\alpha$  and  $y_\beta$  we get that

$$\begin{aligned} g_k &= (y_\alpha - y_\beta)^{d-a_i} \sum_{j=0}^{a_i-k} \lambda_j (y_\alpha - y_\beta)^j (y_\alpha + y_\beta)^{a_i-k-j} \\ &= \left( \sum_{s=0}^{d-a_i} (-1)^s \binom{d-a_i}{s} y_\alpha^s y_\beta^{d-a_i-s} \right) \left( \sum_{t=0}^{a_i-k} \lambda_t (y_\alpha - y_\beta)^t (y_\alpha + y_\beta)^{a_i-k-t} \right) \end{aligned}$$

where the second sum is a polynomial of degree  $a_i - k$  in the variables  $y_\alpha$  and  $y_\beta$ , and since any such polynomial is of the form  $\sum_{j=0}^{a_i-k} \mu_j y_\alpha^j y_\beta^{a_i-k-j}$  for some  $\mu_j \in \mathbb{K}$ . So we have

$$\begin{aligned} g_k &= \left( \sum_{s=0}^{d-a_i} (-1)^s \binom{d-a_i}{s} y_\alpha^s y_\beta^{d-a_i-s} \right) \left( \sum_{j=0}^{a_i-k} \mu_j y_\alpha^j y_\beta^{a_i-k-j} \right) \\ &= \sum_{s=0}^{d-a_i} \sum_{j=0}^{a_i-k} (-1)^s \mu_j \binom{d-a_i}{s} y_\alpha^{s+j} y_\beta^{d-k-s-j} \\ &= \sum_{j=0}^{a_i-k} \sum_{l=j}^{j+d-a_i} (-1)^{l-j} \mu_j \binom{d-a_i}{l-j} y_\alpha^l y_\beta^{d-k-l}. \end{aligned}$$

We claim that  $g_k$  has at most  $a_i - k$  coefficients that are zero. Suppose  $a_i - k + 1$  coefficients in the above expression of  $g_k$  are zero and consider the system of equations in the parameters  $\mu_j$  corresponding to these coefficients being zero. Observe that the coefficient matrix of this system of equations is the transpose of a square submatrix of maximal rank of the Toeplitz matrix  $T_{(a_i-k+1) \times (d-k+1)}$ , up to multiplication of every second row and every second column by negative one. Using Lemma 3.2 we get that the determinant of this coefficient matrix is non-zero and this implies that all the parameters  $\mu_j$  are zero hence  $g_k$  is zero. Therefore for all  $0 \leq k \leq a_i$  the polynomial  $g_k$  has at most  $(a_i - k + 1) - 1 = a_i - k$  zero terms. So we have  $|\mathcal{L}_{i,d}^k \cap \text{Supp}(f)| = |\text{Supp}(g_k)| \geq (d - k + 1) - (a_i - k) = d - a_i + 1$  for all  $0 \leq k \leq a_i$  and all  $1 \leq i \leq 3$ . □

Now we are able to state and prove the main theorem of this section.

**Theorem 4.3.** *For  $d \geq 2$  we have that*

$$\nu(3, d) = \begin{cases} 3d - 3 & \text{if } d \text{ is odd} \\ 3d - 2 & \text{if } d \text{ is even.} \end{cases}$$

*Proof.* First of all we observe that for  $f = (y_1 - y_2)(y_1 - y_3)(y_2 - y_3)^{d-2}$  we have  $\ell \circ f = 0$  and since  $|\text{Supp}(f)| = 3d - 3$  for odd  $d$  and  $|\text{Supp}(f)| = 3d - 2$ , we have  $\nu(3, d) \leq 3d - 3$  for odd  $d$ , and  $\nu(3, d) \leq 3d - 2$  for even  $d$ .

To prove the equality, we check that for any  $f \in (I^{-1})_d$  where,  $\ell \circ f = 0$ ,  $|\text{Supp}(f)| \geq 3d - 3$  for odd  $d$  and  $|\text{Supp}(f)| \geq 3d - 2$  for even  $d$ .

We start by showing that  $|\text{Supp}(f)| \geq 3d - 3$  for all  $d \geq 3$ . Set  $a_i = \max\{\deg_i(m) \mid m \in \text{Supp}(f)\}$  for  $1 \leq i \leq 3$ . Without loss of generality, we may assume that  $a_1 \leq a_2 \leq a_3$ . We can see  $a_1 \geq 2$ . In fact by using Proposition 4.2 we get that  $|\mathcal{L}_{1,d}^0 \cap \text{Supp}(f)| \geq d - a_1 - 1$ . On the other hand since  $I$  is an artinian ideal generated in degree  $d$  we have  $y_i^d \notin \text{Supp}(f)$  for each  $1 \leq i \leq 3$  and this implies that  $|\mathcal{L}_{1,d}^0 \cap \text{Supp}(f)| \leq d - 1$  and therefore,  $a_1 \geq 2$ .

Write  $f = \sum_{j=0}^{a_1} y_1^j g_j$ , where  $g_j$  is a polynomial of degree  $d - j$  in the variables  $y_2$  and  $y_3$ . Using Proposition 4.2 we get,  $|\mathcal{L}_{1,d}^j \cap \text{Supp}(f)| \geq d - a_1 + 1$  for all  $0 \leq j \leq a_1$ . Therefore,

$$|\text{Supp}(f)| \geq \sum_{j=0}^{a_1} |\mathcal{L}_{1,d}^j \cap \text{Supp}(f)| \geq (a_1 + 1)(d - a_1 + 1).$$

So  $|\text{Supp}(f)| \geq (a_1 + 1)(d - a_1 + 1) = 3(d - 1) + (a_1 - 2)(d - 2 - a_1) \geq 3d - 3$ , for  $2 \leq a_1 \leq d - 2$ . Furthermore, strict inequality holds for  $2 < a_1 < d - 2$ , which means  $|\text{Supp}(f)| \geq 3d - 2$ , for all  $2 < a_1 < d - 2$ . It remains to consider the cases where  $a_1 = 2$  and  $a_1 \geq d - 2$ .

If  $a_1 = 2$ , ideal  $J = (y_1^3, y_2^d, y_3^d) \subset S$  is an artinian monomial complete intersection and by Theorem 2.3,  $J$  has strong Lefschetz property. The Hilbert series of  $S/J$  shows that there is a unique generator for the kernel of the multiplication map  $\times(x_1 + x_2 + x_3) : (S/J)_{d-1} \rightarrow (S/J)_d$ . Since the polynomial  $(y_1 - y_2)(y_1 - y_3)(y_2 - y_3)^{d-2}$  is in the kernel of this map and has  $3d - 2$  non-zero terms for even degree  $d$  we have  $|\text{Supp}(f)| \geq 3d - 2$  for any homogeneous degree  $d$  form  $f$  where  $\ell \circ f = 0$ .

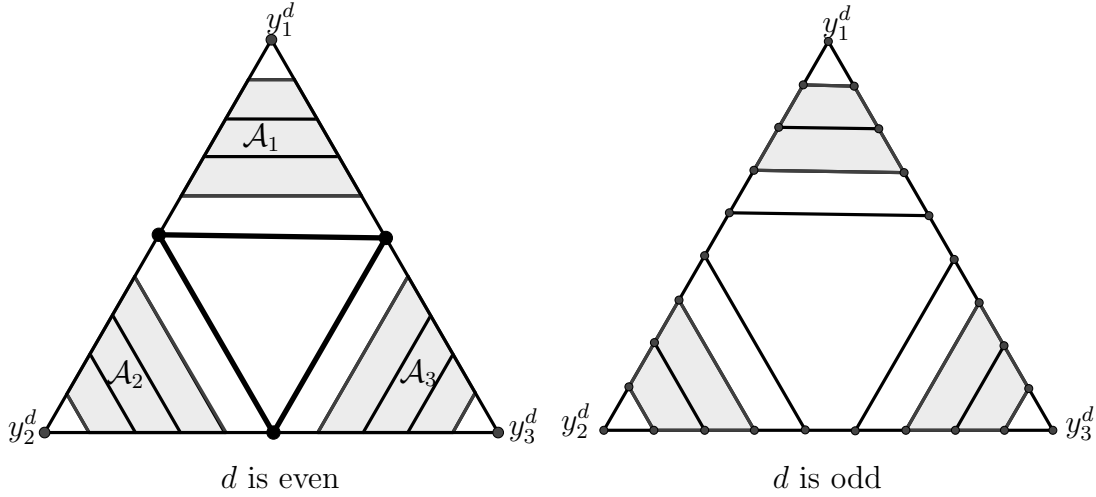
Now suppose that  $a_1 \geq d - 2$ , then since  $a_1 \leq a_2 \leq a_3 \leq d - 1$ , all possible choices for the triple  $(a_1, a_2, a_3)$  are  $(d - 1, d - 1, d - 1)$ ,  $(d - 2, d - 1, d - 1)$ ,  $(d - 2, d - 2, d - 1)$  and  $(d - 2, d - 2, d - 2)$ .

First, we consider the case  $(a_1, a_2, a_3) = (d - 1, d - 1, d - 1)$ . Proposition 4.2 implies that  $|\mathcal{L}_{i,d}^k \cap \text{Supp}(f)| \geq d - (d - 1) + 1 = 2$  for all  $0 \leq k \leq d - 1$  and  $1 \leq i \leq 3$ .

If  $d$  is odd, we consider  $\text{Supp}(f) = \sqcup_{i=1}^3 (\cup_{(d+1)/2}^{d-1} \mathcal{L}_{i,d}^k \cap \text{Supp}(f))$  as a partition for  $\text{Supp}(f)$ . Therefore,

$$\begin{aligned} |\text{Supp}(f)| &= |\cup_{(d+1)/2}^{d-1} \mathcal{L}_{1,d}^k \cap \text{Supp}(f)| + |\cup_{(d+1)/2}^{d-1} \mathcal{L}_{2,d}^k \cap \text{Supp}(f)| + |\cup_{(d+1)/2}^{d-1} \mathcal{L}_{3,d}^k \cap \text{Supp}(f)| \\ &\geq 3 \times 2 \times (d - 1 - (d + 1)/2 + 1) = 3d - 3. \end{aligned}$$

If  $d$  is even, we consider,  $\text{Supp}(f) = (\sqcup_{i=1}^3 \mathcal{A}_i) \sqcup \mathcal{B} \sqcup \mathcal{C}$  be a partition for  $\text{Supp}(f)$  where  $|\mathcal{A}_i \cap \mathcal{L}_{i,d}^k| = 2$ , for each  $1 \leq i \leq 3$  and  $(d - 2)/2 \leq k \leq d - 1$  and since each pair of the sets  $\mathcal{L}_{i,d}^{d/2}$  for  $1 \leq i \leq 3$  has intersection in two monomials we get  $|(\cup_{i=1}^3 \mathcal{L}_{i,d}^{d/2}) \cap \text{Supp}(f)| \geq 3 \times 2 - 3 = 3$ , so we can choose  $|\mathcal{B} \cap (\cup_{i=1}^3 \mathcal{L}_{i,d}^{d/2})| = 3$ . Note that  $|(\sqcup_{i=1}^3 \mathcal{A}_i) \sqcup \mathcal{B}| = 3 \times 2 \times (d - 2)/2 + 3 = 3d - 3$ .

FIGURE 1. For even  $d$ , the middle downward triangle is  $\mathcal{B}$ .

If  $|\mathcal{C}| = 0$ , the set  $\mathcal{B} \cap (\cup_{i=1}^3 \mathcal{L}_{i,d}^{d/2})$  contains exactly three monomials in the pairwise intersection of  $\mathcal{L}_{i,d}^{d/2}$  for  $1 \leq i \leq 3$ . Using Proposition 3.6 part (i), the sum of the coefficients of  $f$  corresponding to the monomials in  $\mathcal{L}_{i,d}^{d/2} \cap \text{Supp}(f)$  is zero for each  $1 \leq i \leq 3$ , which implies the sum of the coefficients of each pair of the monomials in  $\mathcal{B} \cap (\cup_{i=1}^3 \mathcal{L}_{i,d}^{d/2})$  is zero and this means all of them have to be zero. So  $|\mathcal{C}| \geq 1$ , so  $|\text{Supp}(f)| \geq 3d - 2$ . (see Figure 1)

For the three remaining cases where  $a_1 = d - 2$  we will show for even degree  $d$  we get  $|\text{Supp}(f)| \geq 3d - 2$ , see Figure 2. Note that when  $d = 4$  and  $a_1 = d - 2 = 2$  we have seen already that  $|\text{Supp}(f)| \geq 3d - 2$ . So we can assume  $d \geq 6$ . Using Proposition 4.2 we get  $|\mathcal{L}_{1,d}^j \cap \text{Supp}(f)| \geq d - (d - 2) + 1 = 3$  for each  $0 \leq j \leq d - 2$ , so we can partition  $\text{Supp}(f) = \mathcal{S}_1 \sqcup \mathcal{S}_2$ , where  $|\mathcal{S}_1 \cap \mathcal{L}_{1,d}^j| = 3$  for  $3 \leq j \leq d - 2$ . Then assume  $d \geq 6$  and consider the following cases.

If  $(a_1, a_2, a_3) = (d - 2, d - 2, d - 2)$  we apply Proposition 4.2 for the variables  $y_2$  and  $y_3$ , where  $d - 3 \leq j, k \leq d - 2$

$$\begin{aligned} |\text{Supp}(f)| &= |\mathcal{S}_1| + |(\cup_{j=d-3}^{d-2} (\mathcal{L}_{2,d}^j \cap \text{Supp}(f))) \cup (\cup_{k=d-3}^{d-2} (\mathcal{L}_{3,d}^k \cap \text{Supp}(f))) \setminus (\mathcal{S}_1 \cap \mathcal{L}_{1,d}^3)| \\ &\geq 3(d - 4) + (2 \times 3 - 1) + (2 \times 3 - 1) = 3d - 2. \end{aligned}$$

If  $(a_1, a_2, a_3) = (d - 2, d - 2, d - 1)$ , we apply Proposition 4.2 for the variables  $y_2$  and  $y_3$ , where  $d - 3 \leq j \leq d - 2$  and  $d - 3 \leq k \leq d - 1$

$$\begin{aligned} |\text{Supp}(f)| &= |\mathcal{S}_1| + |(\cup_{j=d-3}^{d-2} (\mathcal{L}_{2,d}^j \cap \text{Supp}(f))) \cup (\cup_{k=d-3}^{d-1} (\mathcal{L}_{3,d}^k \cap \text{Supp}(f))) \setminus (\mathcal{S}_1 \cap \mathcal{L}_{1,d}^3)| \\ &\geq 3(d - 4) + (2 \times 3 - 1) + (3 \times 2 - 1) = 3d - 2. \end{aligned}$$

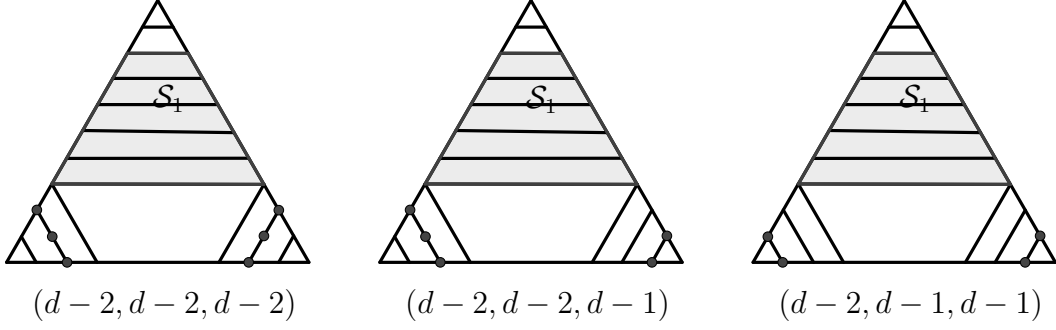


FIGURE 2. Three cases for the triple  $(a_1, a_2, a_3)$  when  $d$  is even.

If  $(a_1, a_2, a_3) = (d-2, d-1, d-1)$ , we apply Proposition 4.2 for the variables  $y_2$  and  $y_3$ , where  $d-3 \leq j, k \leq d-1$

$$\begin{aligned} |\text{Supp}(f)| &= |\mathcal{S}_1| + |(\cup_{j=d-3}^{d-1} (\mathcal{L}_{2,d}^j \cap \text{Supp}(f))) \cup (\cup_{k=d-3}^{d-1} (\mathcal{L}_{3,d}^k \cap \text{Supp}(f))) \setminus (\mathcal{S}_1 \cap \mathcal{L}_{1,d}^3)| \\ &\geq 3(d-4) + (3 \times 2 - 1) + (3 \times 2 - 1) = 3d - 2. \end{aligned}$$

□

## 5. BOUND ON THE NUMBER OF GENERATORS OF IDEALS WITH MORE THAN THREE VARIABLES FAILING WLP

In this section we consider artinian monomial ideals  $I \subset S = \mathbb{K}[x_1, \dots, x_n]$  generated in degree  $d$ , for  $n \geq 4$ . We provide a sharp lower bound for the number of monomials with non-zero coefficients in a non-zero form  $f \in (I^{-1})_d$  such that  $(x_1 + \dots + x_n) \circ f = 0$ . The next theorem provides such lower bound for the form  $f$  in terms of the maximum degree of the variables in  $f$ .

**Theorem 5.1.** *For  $n \geq 4$  and  $d \geq 2$ , let  $f$  be a non-zero form of degree  $d$  in the dual ring  $R = \mathbb{K}[y_1, \dots, y_n]$  of  $S = \mathbb{K}[x_1, \dots, x_n]$  such that  $(x_1 + \dots + x_n) \circ f = 0$ . Then we have  $|\text{Supp}(f)| \geq \max\{(a_i + 1)(d - a_i + 1) \mid a_i \neq 0\}$ , where  $a_i = \max\{\deg_i(m) \mid m \in \text{Supp}(f)\}$ .*

*Proof.* We show that for each  $1 \leq i \leq n$  we have  $|\text{Supp}(f)| \geq (a_i + 1)(d - a_i + 1)$ . Denote  $\ell = x_1 + \dots + x_n$  and  $\ell' = \ell - x_i$  and write  $f = \sum_{k=0}^{a_i} y_i^k g_k$ , where  $g_k$  is a polynomial of degree  $d - k$  in the variables different from  $y_i$ . Since we have  $\ell \circ f = 0$ , Lemma 7.9 implies that

$$(k+1)g_{k+1} + \ell' \circ g_k = 0, \quad \forall 0 \leq k \leq a_i$$

and acting on each equation by  $(\ell')^{a_i-k}$  we get that  $(\ell')^{a_i-k+1} \circ g_k = 0$  for all  $0 \leq k \leq a_i$ . By the definition of  $a_i$  we have  $g_{a_i} \neq 0$ , Proposition 3.6 part (ii) implies that for every  $0 \leq k \leq a_i$  we have  $g_k \neq 0$ . Now applying Theorem 3.7 we get that for each  $0 \leq k \leq a_i$ ,  $|\text{Supp}(g_k)| \geq d - k - (a_i - k + 1) + 2 = d - a_i + 1$ . Therefore,

$$|\text{Supp}(f)| = |\cup_{k=0}^{a_i} \mathcal{L}_{i,d}^k \cap \text{Supp}(f)| = \sum_{k=0}^{a_i} |\text{Supp}(g_k)| \geq (a_i + 1)(d - a_i + 1)$$

and we conclude that  $|\text{Supp}(f)| \geq \max\{(a_i + 1)(d - a_i + 1) \mid 1 \leq i \leq n\}$ . □

In general we can prove that the sharp lower bound is always  $2d$ .

**Theorem 5.2.** *For  $n \geq 4$  and  $d \geq 2$ , we have*

$$\nu(n, d) = 2d.$$

*Proof.* First of all, we observe that for  $f = (y_1 - y_2)(y_3 - y_4)^{d-1}$  we have  $\ell \circ f = 0$  and since  $|\text{Supp}(f)| = 2d$  we get  $\nu(n, d) \leq 2d$ . To show the equality, let  $I \subset S$  be an artinian monomial ideal. We check that for any  $f \in (I^{-1})_d$  where,  $\ell \circ f = 0$ ,  $|\text{Supp}(f)| \geq 2d$ .

Using Theorem 5.1 above, we get that for some  $1 \leq i \leq n$  where  $1 \leq a_i \leq d - 1$  we have  $|\text{Supp}(f)| \geq (a_i + 1)(d - a_i + 1)$ . Observe that since we have  $a_i \leq d - 1$  we get that  $(a_i + 1)(d - a_i + 1) = d(a_i + 1) - (a_i - 1)(a_i + 1) \geq 2d$ , which completes the proof.  $\square$

## 6. SIMPLICIAL COMPLEXES AND MATROIDS

In [1] Gennaro, Ilardi and Vallès describe a relation between the failure of the SLP of artinian ideals and the existence of special singular hypersurfaces. In particular, for the ideals we consider in this section they proved that in the following cases the ideal  $I$  fails the SLP at the range  $k$  in degree  $d + i - k$  if and only if there exists at any point  $M$  a hypersurface of degree  $d + i$  with multiplicity  $d + i - k + 1$  at  $M$  given given by a form in  $(I^{-1})_{d+i}$ , see [1] for more details. In [1, Theorem 6.2], they provide a list of monomial ideals  $I \subset S = \mathbb{K}[x_1, x_2, x_3]$  generated in degree 5 failing the WLP. Here we give the exhaustive list of such ideals.

**Definition 6.1.** Let  $I \subset S$  be an artinian monomial ideal and  $G = \{m_1, \dots, m_r\} \subset R_d$  be a monomial generating set of  $(I^{-1})_d$ . Assume that  $I$  fails the WLP by failing surjectivity in degree  $d - 1$  thus there is a non-zero polynomial  $f \in (I^{-1})_d$  with  $\text{Supp}(f) \subset G$  such that  $(x_1 + \dots + x_n) \circ f = 0$ . We say  $I$  fails the WLP *minimally* if the set  $G$  is minimal with respect to inclusion.

**Remark 6.2.** Note that for every artinian monomial ideal  $I \subset S$  where the WLP fails minimally, there is a unique form in the kernel of the map  $\circ(x_1 + \dots + x_n): (I^{-1})_d \rightarrow (I^{-1})_{d-1}$ . In fact, if there are two different forms with the same support we can eliminate at least one monomial in one of the forms and get a form where its support is strictly contained in the support of the previous ones, contradicting the minimality.

**Proposition 6.3.** *For an artinian monomial ideal  $I \subset S$  generated in degree 5 with at least 6 generators,  $S/I$  fails the WLP by failing surjectivity in degree 4 if and only if the set of generators for the inverse system module  $I^{-1}$  contains the monomials in the support of one of the following forms, up to permutation of variables:*

- $(y_2 - y_3)(y_1 - y_3)^2(y_1 - y_2)(2y_1 - y_2 - y_3)$
- $(y_2 - y_3)(y_1 - y_3)(y_1 - y_2)^2(2y_1 + y_2 - 3y_3)$
- $(y_2 - y_3)(y_1 - y_3)(y_1 - y_2)(y_1^2 + y_1y_2 + y_2^2 - 3y_1y_3 - 3y_2y_3 + 3y_3^2)$
- $(y_2 - y_3)(y_1 - y_3)(y_1 - y_2)(y_1^2 - y_1y_2 - y_2^2 - y_1y_3 + 3y_2y_3 - y_3^2)$
- $(y_2 - y_3)^2(y_1 - y_3)^2(y_1 - y_2)$
- $(y_2 - y_3)(y_1 - y_3)(y_1 - y_2)^3$
- $(y_2 - y_3)(y_1 - y_3)(y_1 - y_2)(y_1^2 - y_1y_2 + y_2^2 - y_1y_3 - y_2y_3 + y_3^2)$ .

*Moreover, the support of all the above forms define monomial ideals failing surjectivity minimally.*

*Proof.* We prove the statement using Macaulay2 and considering all artinian monomial ideals generated in degree 5 with at least 6 generators. There are 816 of such ideals but considering the ones failing the WLP by failing surjectivity in degree 4 and considering the forms in the inverse system module  $(I^{-1})_5$  there are only 25 distinct non-zero forms  $f \in (I^{-1})_5$  such that  $(x_1 + x_2 + x_3) \circ f = 0$ . Therefore, every ideal where  $I^{-1}$  contains the support of each polynomial fails WLP by failing surjectivity in degree 4. Permuting the variables we get only 7 equivalence classes which correspond to the forms given in the statement.  $\square$

**Remark 6.4.** The support of the last three forms in Proposition 6.3 consists of 12 monomials which is the same as  $\nu(3, 5) = 12$  given in 4.3. Therefore, the support of each form in the last three cases, up to permutations of the variables generates  $I^{-1}$  with least possible number of generators in degree 5 where  $I$  fails the WLP.

Using Proposition 4.1, each of the forms above factors in linear form over an algebraically closed field; e.g.  $\mathbb{K} = \mathbb{C}$ .

The next result completely classifies monomial ideals  $I \subset S = \mathbb{K}[x_1, x_2, x_3, x_4]$ , generated in degree 3, failing the WLP which extends Proposition 6.3 in [1].

**Proposition 6.5.** *For an artinian monomial ideal  $I \subset S$  generated in degree 3 with at least 10 generators, surjectivity of the multiplication map by the linear form in degree 2 of  $S/I$  fails if and only if the set of generators for inverse system module  $I^{-1}$  contains the monomials in the support of one of the following forms, up to permutation of variables:*

- $(y_2 - y_4)^2(y_1 - y_3)$
- $(y_2 - y_4)(y_1 - y_4)(y_1 - y_2)$
- $(y_2 - y_3)(y_1 - y_4)(y_1 - 2y_3 + y_4)$
- $(y_2 - y_3)(y_1 - y_4)(y_1 - y_2 - y_3 + y_4)$
- $(y_3 - y_4)(y_2 - y_4)(y_1 - y_3)$
- $(y_1 - y_4)(y_1y_2 + y_1y_3 - 2y_2y_3 - 2y_1y_4 + y_2y_4 + y_3y_4)$
- $(y_1 - y_2)(y_1y_2 - y_1y_3 - y_2y_3 + 2y_3y_4 - y_4^2)$
- $(y_3 - y_4)(y_2^2 - y_1y_3 + y_1y_4 - 2y_2y_4 + y_3y_4)$
- $(y_3 - y_4)(y_1^2 + y_2^2 - 2y_1y_3 - 2y_2y_4 + 2y_3y_4)$
- $2y_1^2y_2 - 3y_1y_2^2 + 2y_2^2y_3 - y_1y_3^2 - 2y_1^2y_4 + 2y_1y_2y_4 + y_2^2y_4 + 2y_1y_3y_4 - 4y_2y_3y_4 + y_3^2y_4$
- $y_1^2y_2 - y_1y_2^2 + y_2^2y_3 - y_1y_3^2 - y_1^2y_4 + 2y_1y_3y_4 - 2y_2y_3y_4 + y_3^2y_4 + y_2y_4^2 - y_3y_4^2$
- $y_1^2y_2 - y_1^2y_3 - 2y_1y_2y_3 + 2y_2y_3^2 + 4y_1y_3y_4 - 2y_2y_3y_4 - 2y_3^2y_4 - 2y_1y_4^2 + y_2y_4^2 + y_3y_4^2$
- $y_1^2y_2 - y_1^2y_3 - y_1y_2y_3 + y_2y_3^2 - y_1y_2y_4 + 3y_1y_3y_4 - y_2y_3y_4 - y_3^2y_4 - y_1y_4^2 + y_2y_4^2$ .

Moreover, the support of all the above forms define monomial ideals failing surjectivity minimally.

*Proof.* We prove it using the same method as the proof of Proposition 6.3 using Macaulay2. There are 8008 artinian monomial ideals generated in degree 3 with at least 10 generators. Considering the forms in the inverse system module  $(I^{-1})_3$  where  $(x_1 + x_2 + x_3 + x_4) \circ f = 0$  correspond to the ideals failing WLP with failing surjectivity in degree 2, there are 237 distinct non-zero forms. Thus any ideal  $I$  where its inverse system module  $I^{-1}$  contains the support of each of the forms fails WLP in degree 2. Also considering the permutation of the variables there are 13 distinct forms given in the statement.  $\square$

**Remark 6.6.** The first two forms have 6 monomials which is the same as  $\nu(4, 3) = 6$  given in 5.2. Therefore, each form in the last two cases, up to permutation of variables give the

minimal number of generators for the inverse system module  $I^{-1}$  where  $I$  fails the WLP. One can check that the factors in the forms given in Proposition 6.5 are irreducible even over the complex numbers (or any algebraically closed field of characteristic zero).

The above results lead us to correspond simplicial complexes to the class of ideals failing the WLP by failing surjectivity. Recall that Theorem 4.3 and Corollary 5.2 imply that in the polynomial ring  $S = \mathbb{K}[x_1, \dots, x_n]$  when the Hilbert function of an artinian monomial algebra generated in a single degree  $d$ ,  $H_{S/I}(d)$ , is less than  $\nu(n, d)$ , monomial algebra  $S/I$  satisfies the WLP. First we recall the following definitions:

**Definition 6.7.** A *matroid* is a finite set of elements  $M$  together with the family of subsets of  $M$ , called independent sets, satisfying,

- The empty set is independent,
- Every subset of an independent set is independent,
- For every subset  $A$  of  $M$ , all maximum independent sets contained in  $A$  have the same number of elements.

A *simplicial complex*  $\Delta$  is a set of simplices such that any face of a simplex from  $\Delta$  is also in  $\Delta$  and the intersection of any two simplices is a face of both. Note that every matroid is also a simplicial complex with independent sets as its simplices.

**Definition 6.8.** Recall  $\mathcal{M}_d$  from Definition 3.4 which is the set of monomials in degree  $d$  in the ring  $R$  and define  $\mathcal{M}'_d = \mathcal{M}_d \setminus \{y_1^d, \dots, y_n^d\}$ . We define *independent* set  $s \subset \mathcal{M}'_d$  to be the set of monomials such that the set  $\{(x_1 + \dots + x_n) \circ m \mid m \in s\}$  is a linearly independent set. A subset  $s \subset \mathcal{M}'_d$  is called *dependent* if it is not an independent set. Then define  $\Delta_{d,sur}$  to be the simplicial complex with the monomials in  $\mathcal{M}'_d$  as the ground set and all independent sets as its faces. Note that  $\Delta_{d,sur}$  forms a matroid.

Any proper subset of the support of each of the forms in 6.3 and 6.5 forms an independent set. Observe that for every independent set  $s$ , monomial ideal  $I \subset S$  generated by the  $d$ -th power of the variables in  $S$  and corresponding monomials of  $\mathcal{M}'_d \setminus s$  in  $S$  form an artinian ideal  $I$ , where  $S/I$  satisfies the WLP. Since the ground set of  $\Delta_{d,sur}$  is the subset of monomials  $R_d$  the size of an independent set is bounded from above with the number of monomials in  $R_{d-1}$ . Therefore we have  $\dim(\Delta_{d,sur}) \leq h_{d-1}(R) - 1$ .

**Example 6.9.** The support of each polynomial given in Proposition 6.3 is a minimal non-face of the simplicial complex  $\Delta_{5,sur}$  with the ground set  $\mathcal{M}'_5 = \mathcal{M}_5 \setminus \{y_1^5, y_2^5, y_3^5\}$ . This simplicial complex has 25 minimal non-faces (considering the permutations of variables).  $\Delta_{5,sur}$  has 7 minimal non-faces of dimension 11, 6 minimal non-faces of dimension 13 and 12 minimal non-faces of dimension 14.

Similarly we can construct another simplicial complex by the complement of dependent sets.

**Definition 6.10.** Define  $\Delta_{d,sur}^*$  to be the simplicial complex with the monomials in  $S_d \setminus \{x_1^d, \dots, x_n^d\}$  as its ground set and faces of  $\Delta_{d,sur}^*$  are the corresponding monomials of  $\mathcal{M}'_d \setminus s$  in  $S$  where  $s$  is a dependent set.

Observe that artinian algebra  $S/I$  where  $I$  is generated by the  $d$ -th power of the variables in  $S$  together with the monomials in a face of  $\Delta_{d,sur}^*$ , fails the WLP. In fact the multiplication



map on  $S/I$  from degree  $d - 1$  to  $d$  is not surjective. Theorem 4.3 and 5.2 imply that every subset  $s \subset \mathcal{M}'_d$  with  $|s| \leq \nu(n, d)$  is independent. Therefore we have  $\dim(\Delta_{d,sur}^*) = |S_d| - n - \nu(n, d) - 1$ , the equality is because the bound is sharp.

**Remark 6.11.** Recall that the Alexander dual of a simplicial complex  $\Delta$  on the ground set  $V$  is a simplicial complex with the same ground set and faces are all the subsets of  $V$  where their complements are non-faces of  $\Delta$ . Observe that  $\Delta_{d,sur}^*$  is a simplicial complex in  $S_d$  and  $\Delta_{d,sur}$  is a simplicial complex in the Macaulay dual ring  $R_d$ . Note that for any independent set  $s \subset \mathcal{M}'_d$  the corresponding monomials of the complement  $\mathcal{M}'_d \setminus s$  in the ring  $S$  is not a face of  $\Delta_{d,sur}^*$  which implies that  $\Delta_{d,sur}$  is Alexander dual to  $\Delta_{d,sur}^*$ .

We may construct simplicial complexes corresponding to artinian algebras failing or satisfying injectivity in a certain degree.

**Definition 6.12.** Define  $\Delta_{d,inj}$  to be a simplicial complex with the monomials in  $\mathcal{M}'_d$  as its ground set and faces correspond to generators of  $(I^{-1})_d$  where  $I$  fails injectivity in degree  $d - 1$ .

**Remark 6.13.** Recall that all minimal monomial Togliatti systems correspond to facets of  $\Delta_{d,inj}$ . In fact for minimal monomial Togliatti system  $I$  the inverse system module has the maximum number of generators where  $I$  fails injectivity in degree  $d - 1$ .

## 7. WLP OF IDEALS FIXED BY ACTIONS OF A CYCLIC GROUP

Mezzetti and Miró-Roig in [9] studied artinian ideals of the polynomial ring  $\mathbb{K}[x_1, x_2, x_3]$ , where  $\mathbb{K}$  is an algebraically closed field of characteristic zero generated by homogeneous polynomials of degree  $d$  invariant under an action of cyclic group  $\mathbb{Z}/d\mathbb{Z}$ , for  $d \geq 3$  and they proved that they are all monomial Togliatti systems. Here in this section we will consider such ideals in a polynomial ring with at least three variables. Throughout this section  $\mathbb{K} = \mathbb{C}$  and  $S = \mathbb{K}[x_1, \dots, x_n]$ , where  $n \geq 3$ . Let  $d \geq 2$  and  $\xi = e^{2\pi i/d}$  to be the primitive  $d$ -th root of unity. Consider diagonal matrix

$$M_{a_1, \dots, a_n} = \begin{pmatrix} \xi^{a_1} & 0 & \dots & 0 \\ 0 & \xi^{a_2} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \xi^{a_n} \end{pmatrix}$$

representing the cyclic group  $\mathbb{Z}/d\mathbb{Z}$ , where  $a_1, a_2, \dots, a_n$  are integers and the action is defined by  $[x_1, \dots, x_n] \mapsto [\xi^{a_1} x_1, \dots, \xi^{a_n} x_n]$ . Since  $\xi^d = 1$ , we may assume that  $0 \leq a_i \leq d - 1$ , for every  $1 \leq i \leq n$ . Let  $I \subset S$  be the ideal generated by all the forms of degree  $d$  fixed by the action of  $M_{a_1, \dots, a_n}$ . In [9, Theorem 3.1], Mezzetti and Miró-Roig showed that these ideals are monomial ideals when  $n = 3$ . Here we state it in general for all  $n \geq 3$  with a slightly different proof.

**Lemma 7.1.** *For integer  $d \geq 2$ , the ideal  $I \subset S = \mathbb{K}[x_1, \dots, x_n]$  generated by all the forms of degree  $d$  fixed by the action of  $M_{a_1, \dots, a_n}$  is artinian and generated by monomials.*

*Proof.* Since  $M_{a_1, \dots, a_n}$  is a monomial action in the sense that for every monomial  $m$  of degree  $d$  we have  $M_{a_1, \dots, a_n}^r m = cm$  for each  $0 \leq r \leq d - 1$  and for some  $c \in \mathbb{K}$ . Then if we have a form of degree  $d$  fixed by  $M_{a_1, \dots, a_n}$ , all its monomials are fixed by  $M_{a_1, \dots, a_n}$ . This implies that

$I$  is a monomial ideal. Note also that since  $\xi^d = 1$ , all the monomials  $x_1^d, x_2^d, \dots, x_n^d$  are fixed by the action of  $M_{a_1, \dots, a_n}$  which means  $I$  is artinian ideal. □

Using the above result, from now on we take the monomial set of generators for  $I$ . Observe that for two distinct primitive  $d$ -th roots of unity we get different actions, but the set of monomials fixed by both actions are the same. Also the action  $M_{a_n+r, \dots, a_n+r}$  which is obtained by multiplying the matrix  $M_{a_1, \dots, a_n}$  with a  $d$ -th root of unity defines the same action on degree  $d$  monomials in  $S$ . In [9], authors show that in the case that  $n = 3$  where  $a_i$ 's are distinct and  $\gcd(a_1, a_2, a_3, d) = 1$ , these ideals are all monomial Togliatti systems. In fact they show that the WLP of these ideals fails in degree  $d - 1$  by failing injectivity of the multiplication map by a linear form in that degree. In this section, we study the cases where WLP of such ideals fail by failing surjectivity in degree  $d - 1$ . Then we classify all such ideals in polynomial rings with more than 2 variables, in terms of their WLP.

We start this section by stating some results about the number of monomials of degree  $d$  fixed by the action  $M_{a_1, \dots, a_n}$  of  $\mathbb{Z}/d\mathbb{Z}$  in  $S$ . In fact we prove that this number depends on the integers  $a_i$ 's. In the next result we give an explicit formula computing the number of such monomials where  $n = 3$ .

**Proposition 7.2.** *For integers  $a_1, a_2, a_3$  and  $d \geq 2$ , the number of monomials in  $S = \mathbb{K}[x_1, \dots, x_n]$  of degree  $d$  fixed by the action of  $M_{a_1, a_2, a_3}$  is*

$$(7.1) \quad 1 + \frac{\gcd(a_2 - a_1, a_3 - a_1, d) \cdot d + \gcd(a_2 - a_1, d) + \gcd(a_3 - a_1, d) + \gcd(a_3 - a_2, d)}{2}.$$

*Proof.* From the discussion above, the number of monomials of degree  $d$  fixed by  $M_{0, a_2 - a_1, a_3 - a_1}$  and  $M_{a_1, a_2, a_3}$  are the same. Thus, we count the number of monomials of degree  $d$  fixed by  $M_{0, a_2 - a_1, a_3 - a_1}$ . Any monomial of degree  $d$  in  $S$  can be written as  $x^{d-m-n}y^mz^n$  with  $0 \leq m, n \leq d$  and  $m + n \leq d$  and it is invariant under the action of  $M_{0, a_2 - a_1, a_3 - a_1}$  if and only if  $(a_2 - a_1)m + (a_3 - a_1)n \equiv 0 \pmod{d}$ . In [7, Chapter 3], we find that the number of congruent solutions of  $(a_2 - a_1)m + (a_3 - a_1)n \equiv 0 \pmod{d}$  is  $\gcd(a_2 - a_1, a_3 - a_1, d) \cdot d$  but since the solutions  $(0, 0)$ ,  $(0, d)$  and  $(d, 0)$  (corresponding to the powers of variables) are all congruent to  $d$  and fixed by  $M_{0, a_2 - a_1, a_3 - a_1}$  we get two more solutions than  $\gcd(a_2 - a_1, a_3 - a_1, d) \cdot d$ . In order to count the monomials of degree  $d$  invariant under the action of  $M_{0, a_2 - a_1, a_3 - a_1}$  we need to count the number of solutions of  $(a_2 - a_1)m + (a_3 - a_1)n \equiv 0 \pmod{d}$  satisfying the extra condition  $m + n \leq d$ .

First we count the number of such solutions when  $m = 0$  and  $n \neq 0$ . So every  $1 \leq n < \gcd(a_3 - a_1, d)$  is a solution of  $(a_3 - a_1)n \equiv 0 \pmod{d}$ . Therefore there are  $\gcd(a_3 - a_1, d) - 1$  solutions in this case. Similarly, there are  $\gcd(a_2 - a_1, d) - 1$  solutions when  $n = 0$  and  $m \neq 0$ .

Counting the solutions when  $m + n = d$  is equivalent to counting the solutions of  $(a_3 - a_2)m \equiv 0 \pmod{d}$  which is similar to the previous case and is equal to  $\gcd(a_3 - a_2, d) - 1$ . There is also one solution when  $m = n = 0$ .

Now rest of the solutions (where  $m \neq 0$  and  $n \neq 0$  and  $m + n \neq d$ ) by [7, Chapter 3] is equal to  $\gcd(a_2 - a_1, a_3 - a_1, d) \cdot d - \gcd(a_3 - a_2, d) - \gcd(a_2 - a_1, d) - \gcd(a_3 - a_1, d) + 2$  but we need to count the number of those satisfying  $0 < m + n < d$ . Note that if  $0 < m_0 < d$  and

$0 < n_0 < d$  is a solution of  $(a_2 - a_1)m + (a_3 - a_1)n \equiv 0 \pmod{d}$  then  $0 < d - m_0 < d$  and  $0 < d - n_0 < d$  is also a solution but one and only one of the two conditions  $0 < m_0 + n_0 < d$  and  $0 < d - m_0 + d - n_0 < d$  satisfies. Therefore, there are

$$\frac{\gcd(a_2 - a_1, a_3 - a_1, d) \cdot d - \gcd(a_3 - a_2, d) - \gcd(a_2 - a_1, d) - \gcd(a_3 - a_1, d) + 2}{2}$$

solutions satisfying  $0 < m + n < d$ . Adding this with the solutions where  $m = 0$  or  $n = 0$  or  $m + n = d$  which we have counted them above together with two more pairs  $(0, d)$  and  $(d, 0)$  (explained in the beginning of the proof) we get what we wanted to prove.  $\square$

For a fixed integer  $d \geq 2$  Proposition 7.2 shows that how the number of fixed monomials of degree  $d$  depends on the integers  $a_1, a_2, a_3$ . In the following example we see how they are distributed.

**Example 7.3.** Using Formula (7.1) we count the number of monomials of degree 15 in  $\mathbb{K}[x_1, x_2, x_3]$  fixed by the action  $M_{0,a,b}$  for every  $0 \leq a, b \leq 14$ . We see the distribution of them in terms of  $\mu(I)$  in the following table:

$m$	10	11	12	13	17	28	34	46	51	136
$d_m$	24	72	24	48	24	12	12	2	6	1

where  $d_m = |\{(a, b) \mid \mu(I) = m\}|$ . Note that the last row of the table corresponds to the action  $M_{0,0,0}$  where we get  $\mu(I) = (\mathbb{K}[x_1, x_2, x_3])_{15} = 136$ . There are exactly 24 pairs  $(a, b)$  where either at least one of them is zero or  $a = b$ , which in these cases we get  $\mu(I) = 17$ . We have  $\gcd(a, b, d) \neq 1$  for all the cases with  $\mu(I) > 17$  and  $\gcd(a, b, d) = 1$  for all the cases with  $\mu(I) < 17$ .

As we saw in the above example the distribution of the number of monomials of degree  $d$  fixed by  $M_{a_1, a_2, a_3}$  is quite difficult to understand but we prove that such numbers are bounded from above depending on the prime factors of  $d$  in the case that  $a_i$ 's are distinct and  $\gcd(a_1, a_2, a_3, d) = 1$ .

**Proposition 7.4.** *For  $d \geq 3$  and distinct integers  $0 \leq a_1, a_2, a_3 \leq d-1$  with  $\gcd(a_1, a_2, a_3, d) = 1$ , let  $\mu(I)$  be the number of monomials of degree  $d$  fixed by  $M_{a_1, a_2, a_3}$ . Then*

$$\mu(I) \leq \begin{cases} \frac{(p+1)d+p^2+3p}{2p} & \text{if } p^2 \nmid d \\ \frac{(p+1)d+4p}{2p} & \text{if } p^2 \mid d \end{cases}$$

where  $p$  is the smallest prime dividing  $d$ . Moreover, the bounds are sharp.

*Proof.* Using Proposition 7.2 we provide an upper bound for  $\gcd(a_2 - a_1, d) + \gcd(a_3 - a_1, d) + \gcd(a_3 - a_2, d)$ . For some integer  $t$  we have  $d = \gcd(a_2 - a_1, d) \cdot \gcd(a_3 - a_1, d) \cdot \gcd(a_3 - a_2, d) \cdot t$ . Since  $\gcd(a_3 - a_1, d) \cdot \gcd(a_3 - a_2, d) = \frac{d}{\gcd(a_2 - a_1, d) \cdot t}$ , we have

$$\gcd(a_3 - a_1, d) + \gcd(a_3 - a_2, d) \leq 1 + \frac{d}{\gcd(a_3 - a_2, d) \cdot t},$$

therefore

$$\begin{aligned} \gcd(a_2 - a_1, d) + \gcd(a_3 - a_1, d) + \gcd(a_3 - a_2, d) &\leq \gcd(a_2 - a_1, d) + \frac{d}{\gcd(a_3 - a_1, d) \cdot t} + 1 \\ &\leq \gcd(a_2 - a_1, d) + \frac{d}{\gcd(a_2 - a_1, d)} + 1 \\ &\leq p + \frac{d}{p} + 1. \end{aligned}$$

Note that,  $\gcd(a_2 - a_1, d) + \gcd(a_3 - a_1, d) + \gcd(a_3 - a_2, d) = d + 2 > p + \frac{d}{p} + 1$  if and only if at least two integers  $a_i$  are the same which contradicts the assumption. Since for every  $q \geq p$  we have  $p + \frac{d}{p} + 1 \geq q + \frac{d}{q} + 1$  we get  $\gcd(a_2 - a_1, d) + \gcd(a_3 - a_1, d) + \gcd(a_3 - a_2, d) \leq p + \frac{d}{p} + 1$ . Now assume that  $p^2 \nmid d$ , to reach the bound we let  $a_2 - a_1 = p$  and  $a_3 - a_1 = \frac{d}{p}$ . In this case since  $\gcd(a_3 - a_2, d) = 1$ , Proposition 7.2 implies that  $\mu(I) \leq \frac{(p+1)d+p^2+3p}{2p}$ .

If  $p^2 \mid d$ , choosing  $a_2 - a_1 = p$  and  $a_3 - a_1 = \frac{d}{p}$  implies that  $\gcd(a_3 - a_2, d) = p$ . So the given bound can not be sharp. Observe that for  $q > p$  and  $q \mid d$  we have  $q + \frac{d}{q} + 1 \leq 1 + \frac{d}{p} + 1$ . Therefore in this case we have  $\gcd(a_2 - a_1, d) + \gcd(a_3 - a_1, d) + \gcd(a_3 - a_2, d) \leq 1 + \frac{d}{p} + 1$ , and equality holds for  $a_2 - a_1 = 1$  and  $a_3 - a_1 = \frac{d}{p}$  so by Proposition 7.2 we have  $\mu(I) \leq \frac{(p+1)d+4p}{2p}$ .  $\square$

In the proof of Proposition 7.2, we used the fact that the number of solutions  $(m, n)$  for  $(a_2 - a_1)m + (a_3 - a_1)n \equiv 0 \pmod{d}$  (corresponding to the action by  $M_{a_1, a_2, a_3}$ ) where  $m, n \neq 0$  and  $m + n \neq d$  is exactly twice the number of solutions of  $(b - a)m + (c - a)n \equiv 0 \pmod{d}$  satisfying  $0 < m + n < d$ . But in the polynomial ring with more than three variables this is no longer the case that the solutions of the corresponding equation of  $M_{a_1, \dots, a_n}$  are distributed in a nice way so we do not have the explicit formula as in Proposition 7.2 in higher number of variables. However, in Proposition 7.5 below we provide an upper bound for this number in the polynomial ring with four variables where  $\gcd(a_1, a_2, a_3, a_4, d) = 1$ . The bound implies  $H_{S/I}(d - 1) \leq H_{S/I}(d)$  and therefore the WLP in degree  $d - 1$  is an assertion of injectivity.

**Theorem 7.5.** *For  $d \geq 2$  and integers  $0 \leq a_1, a_2, a_3, a_4 \leq d - 1$ , where at most two of the integers among  $a_i$ 's are equal and  $\gcd(a_1, a_2, a_3, a_4, d) = 1$ . Let  $\mu(I)$  be the number of monomials of degree  $d$  in  $S = \mathbb{K}[x_1, x_2, x_3, x_4]$  fixed by  $M_{a_1, a_2, a_3, a_4}$ . Then*

$$\mu(I) \leq 1 + \frac{(d+2)(d+1)}{2}.$$

*Proof.* Any monomial of degree  $d$  in  $S$  can be written as  $x_1^{m_1} x_2^{m_2} x_3^{m_3} x_4^{d-m_1-m_2-m_3}$  with  $0 \leq m_1, m_2, m_3 \leq d$  and  $m_1 + m_2 + m_3 \leq d$ . Monomial  $x_1^{m_1} x_2^{m_2} x_3^{m_3} x_4^{d-m_1-m_2-m_3}$  is invariant under the action of  $M_{a_1, a_2, a_3, a_4}$  or equivalently  $M_{a_1 - a_4, a_2 - a_4, a_3 - a_4, 0}$  if and only if

$$(7.2) \quad (a_1 - a_4)m_1 + (a_2 - a_4)m_2 + (a_3 - a_4)m_3 \equiv 0 \pmod{d}, \quad m_1 + m_2 + m_3 \leq d.$$

In [7, Chapter 3] we find that the number of congruent solutions of  $(a_1 - a_4)m_1 + (a_2 - a_4)m_2 + (a_3 - a_4)m_3 \equiv 0 \pmod{d}$  is  $d^2$ . We first count the number of congruent solutions of 7.2 where at least one of  $m_1, m_2$  or  $m_3$  is zero. Suppose  $m_1 = 0$  then by [7, Chapter 3], the number of

congruent solutions of  $(a_2 - a_4)m_2 + (a_3 - a_4)m_3 \equiv 0 \pmod{d}$  is  $\gcd(a_2 - a_4, a_3 - a_4, d) \cdot d$ . Similarly, by [7, Chapter 3], the number of congruent solutions of 7.2 having two coordinates zero, for example  $m_1 = m_2 = 0$ , is  $\gcd(a_3 - a_4, d)$ . All together the number of congruent solutions of 7.2 where at least one of the coordinates  $m_1, m_2, m_3$  is zero is as follows

$$d(\gcd(a_1 - a_4, a_2 - a_4, d) + \gcd(a_2 - a_4, a_3 - a_4, d) + \gcd(a_1 - a_4, a_3 - a_4, d)) \\ - \gcd(a_1 - a_4, d) - \gcd(a_2 - a_4, d) - \gcd(a_3 - a_4, d) + 1.$$

Note that if  $(m_{10}, m_{20}, m_{30})$  is a solution of 7.2 such that  $m_{i0} \neq 0$  for  $i = 1, 2, 3$ , then  $(d - m_{10}, d - m_{20}, d - m_{30})$  is a solution of  $(a_1 - a_4)m_1 + (a_2 - a_4)m_2 + (a_3 - a_4)m_3 \equiv 0 \pmod{d}$  where  $3d - m_{10} - m_{20} - m_{30} \geq d$ . Therefore, the number of congruent solutions of 7.2 where no  $m_i$  is zero is bounded from above by

$$[d^2 - (d(\gcd(a_1 - a_4, a_2 - a_4, d) + \gcd(a_2 - a_4, a_3 - a_4, d) + \gcd(a_1 - a_4, a_3 - a_4, d)) \\ - \gcd(a_1 - a_4, d) - \gcd(a_2 - a_4, d) - \gcd(a_3 - a_4, d) + 1)]/2.$$

Using Proposition 7.2, we count the number of solutions 7.2 where at least one of the coordinates  $m_i$  is zero. If  $m_1 = 0$  then by Proposition 7.2 the number of solutions of  $(a_2 - a_4)m_2 + (a_3 - a_4)m_3 = 0 \pmod{d}$  where  $0 \leq m_2, m_3 \leq d$  and  $m_2 + m_3 \leq d$  is

$$\frac{\gcd(a_2 - a_4, a_3 - a_4, d) \cdot d + \gcd(a_2 - a_4, d) + \gcd(a_3 - a_4, d) + \gcd(a_2 - a_3, d) + 2}{2}.$$

Similarly we can count the number of such solutions when  $m_2 = 0$  or  $m_3 = 0$ . Now suppose that  $m_1 = m_2 = 0$  then we get  $\gcd(a_3 - a_4, d) + 1$  where  $0 \leq m_3 \leq d$ . All together the number of solutions of 7.2 where at least one  $m_i$  is zero is

$$[d(\gcd(a_1 - a_4, a_2 - a_4, d) + \gcd(a_2 - a_4, a_3 - a_4, d) + \gcd(a_1 - a_4, a_3 - a_4, d)) \\ + \gcd(a_1 - a_2, d) + \gcd(a_1 - a_3, d) + \gcd(a_2 - a_3, d) + 2]/2.$$

Therefore, the number of solutions of 7.2 is bounded from above by

$$(7.3) \quad \frac{d^2 + \sum_{i=1}^3 \gcd(a_i - a_4, d) + \sum_{1 \leq i < j \leq 3} \gcd(a_i - a_j, d) + 1}{2}.$$

To show the assertion of the theorem we need to show (7.3) is bounded from above by  $\frac{(d+2)(d+1)+2}{2}$  where at most 2 of integers among  $a_i$ 's are equal and  $\gcd(a_1, a_2, a_3, a_4, d) = 1$ . So we need to show that

$$(7.4) \quad \sum_{i=1}^3 \gcd(a_i - a_4, d) + \sum_{1 \leq i < j \leq 3} \gcd(a_i - a_j, d) = \sum_{1 \leq i \leq j \leq 4} \gcd(a_i - a_j, d) \leq 3d + 3.$$

To show this we consider the following cases:

- (1) Suppose at least two terms in the left hand side of 7.4 are equal to  $d$  then at least three integers among  $a_i$ 's are equal which contradicts the assumption.
- (2) Suppose that one of the terms in the left hand side is equal to  $d$ . By relabeling the indices we may assume that  $\gcd(a_1 - a_2, d) = d$ , this implies that  $a_1 = a_2$  then we need to show that

$$d + 2 \gcd(a_1 - a_3, d) + 2 \gcd(a_1 - a_4, d) + \gcd(a_3 - a_4, d) \leq 3d + 3$$

since we assume that  $\gcd(a_1, a_2, a_3, a_4, d) = 1$  we have that  $\gcd(a_1 - a_4, d), \gcd(a_3 - a_4, d)$  and  $\gcd(a_1 - a_3, d)$  are all distinct and strictly less than  $d$ . Thus we have

$$d + 2\gcd(a_1 - a_3, d) + 2\gcd(a_1 - a_4, d) + \gcd(a_3 - a_4, d) \leq d + 2\frac{d}{2} + 2\frac{d}{3} + \frac{d}{4} < 3d + 3.$$

(3) Suppose all the terms in the left hand side of (7.4) are strictly less than  $d$ . Then the assumption  $\gcd(a_1, a_2, a_3, a_4, d) = 1$  implies that at most two terms can be  $d/2$  and assuming the other terms are  $d/3$  we get

$$\sum_{1 \leq i < j \leq 4} \gcd(a_i - a_j, d) \leq 3d + 3 \leq 2(d/2) + 4(d/3) = d + d/2 < 3d + 3.$$

□

In the rest of this section we study the WLP of ideals in  $S = \mathbb{K}[x_1, \dots, x_n]$  for  $n \geq 3$  generated by all forms of degree  $d \geq 3$  invariant by the action  $M_{a_1, \dots, a_n}$  of  $\mathbb{Z}/d\mathbb{Z}$ . We classify all such ideals satisfying WLP.

**Lemma 7.6.** *For integer  $d \geq 2$  and distinct integers  $0 \leq a_1, a_2, a_3 \leq d - 1$ , let  $M_{a_1, a_2, a_3}$  be a representation of  $\mathbb{Z}/d\mathbb{Z}$ . Define the linear form*

$$L = \sum_{j=0}^l \xi^j x_1 + \sum_{j=l+1}^{l+k+1} \xi^j x_2 + \sum_{j=l+k+2}^{2d-1} \xi^j x_3,$$

where  $l$  and  $k$  are the residues of  $a_2 - a_3 - 1$  and  $a_3 - a_1 - 1$  modulo  $d$ . Then the support of the form  $F = L^d - \bar{L}^d$  are exactly the monomials of degree  $d$  in  $\mathbb{K}[x_1, x_2, x_3]$  which are not invariant under the action of  $M_{a_1, a_2, a_3}$ , where  $\bar{L}$  is the conjugate of  $L$  and  $\xi$  is a primitive  $d$ -th root of unity.

*Proof.* First, note that for a rational number  $j$  we let  $\xi^j = e^{j\frac{2\pi i}{d}}$ . We observe that for integers  $0 \leq p \leq q$  we have  $\sum_p^q \xi^i = \xi^{\frac{p+q}{2}} \sum_p^q \xi^{i-\frac{p+q}{2}}$ , where  $\sum_p^q \xi^{i-\frac{p+q}{2}} = \xi^{\frac{p-q}{2}} + \xi^{\frac{p-q}{2}+1} + \dots + \xi^{\frac{q-p}{2}-1} + \xi^{\frac{q-p}{2}}$  which is invariant under conjugation, so it is a real number. Therefore, we have

$$L = \sum_{j=0}^l \xi^j x_1 + \sum_{j=l+1}^{l+k+1} \xi^j x_2 + \sum_{j=l+k+2}^{2d-1} \xi^j x_3 = r_1 \xi^{\frac{l}{2}} x_1 + r_2 \xi^{\frac{2l+k+2}{2}} x_2 + r_3 \xi^{\frac{l+k+1}{2}} x_3$$

where  $r_1, r_2$  and  $r_3$  are non-zero real numbers. In fact, using the assumption that the  $a_i$ 's are distinct we get that  $0 \leq l, k \leq d - 2$  which implies that the  $r_i$ 's are all non-zero. The form  $F$  can be written as

$$\begin{aligned} F &= L^d - \bar{L}^d \\ &= \left( r_1 \xi^{\frac{l}{2}} x_1 + r_2 \xi^{\frac{2l+k+2}{2}} x_2 + r_3 \xi^{\frac{l+k+1}{2}} x_3 \right)^d - \left( r_1 \xi^{-\frac{l}{2}} x_1 + r_2 \xi^{-\frac{2l+k+2}{2}} x_2 + r_3 \xi^{-\frac{l+k+1}{2}} x_3 \right)^d. \end{aligned}$$

Consider monomial  $m = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}$  of degree  $d$  in  $\mathbb{K}[x_1, x_2, x_3]$ . The coefficient of  $m$  in  $F$  is zero if and only if the coefficients of  $m$  in  $L^d$  is real. The coefficient of  $m$  in  $L^d$  is real if and only if

$$\alpha_1 \frac{l}{2} + \alpha_2 \frac{2l+k+2}{2} + \alpha_3 \frac{l+k+1}{2} \equiv \alpha_1 \frac{-l}{2} + \alpha_2 \frac{-(2l+k+2)}{2} + \alpha_3 \frac{-(l+k+1)}{2} \pmod{d}$$

which is equivalent to have

$$\alpha_1 l + \alpha_2(2l + k + 2) + \alpha_3(l + k + 1) \equiv 0, \pmod{d}.$$

Therefore, the monomials with non-zero coefficients in  $F$  are exactly the monomials of degree  $d$  in  $\mathbb{K}[x_1, x_2, x_3]$ , which are not fixed by the action of  $M_{l, 2l+k+2, l+k+1}$ . Substituting  $l, k$  we get that  $M_{l, 2l+k+2, l+k+1}$  is equivalent to the action  $M_{a_2-a_3-1, 2a_2-a_3-a_1-1, a_2-a_1-1}$  and by adding the indices with  $a_1 - a_2 + a_3 + 1$  the last one is also equivalent to  $M_{a_1, a_2, a_3}$  which proves what we wanted.  $\square$

**Remark 7.7.** The assumption in Lemma 7.6 that  $a_i$ 's are distinct is necessary to have the form  $F$  non-zero. If at least two of the integers  $a_i$  are equal then in the linear form  $L$  at least the coefficient of one of the variables  $x_1, x_2$  and  $x_3$  is zero. Then we conclude that in  $L^d$  all the monomials have real coefficients which implies  $F = 0$ .

Lemma 7.6, can be extended to any polynomial ring with odd number of variables. In fact in this case we can find  $n - 1$  integers  $l_i$  in terms of the integers  $a_i$  defining the action  $M_{a_1, \dots, a_n}$  in such a way that a similar linear form as  $L$  in the lemma in  $n$  variables does the same.

In [9, Proposition 3.2], Mezzetti and Miró-Roig showed that the WLP of  $I$  fails by failing injectivity in degree  $d-1$  in the polynomial ring  $\mathbb{K}[x_1, x_2, x_3]$ . They provide the non-zero form in the kernel of the multiplication map by a linear form on artinian algebra  $\mathbb{K}[x_1, x_2, x_3]/I$  from degree  $d-1$  to degree  $d$ . Here we give a similar form in  $\mathbb{K}[x_1, \dots, x_n]$ .

**Lemma 7.8.** *For integers  $d \geq 2$  and  $0 \leq a_1, \dots, a_n \leq d-1$ , let  $M_{a_1, \dots, a_n}$  be a representation of  $\mathbb{Z}/d\mathbb{Z}$  and  $f$  be the form of degree  $d-1$*

$$f = \prod_{i=1}^{d-1} (\xi^{ia_1} x_1 + \dots + \xi^{ia_n} x_n).$$

*Then, all the monomials with non-zero coefficient in  $(x_1 + \dots + x_n)f$  are fixed by the action  $M_{a_1, \dots, a_n}$ .*

*Proof.* Since  $(x_1 + \dots + x_n)f$  is invariant under the action of  $M_{a_1, \dots, a_n}$  we have that  $(x_1 + \dots + x_n)f \in I$ . By 7.1, since  $I$  is a monomial ideal we conclude that all the monomials in  $(x_1 + \dots + x_n)f$  with non-zero coefficient are fixed by  $M_{a_1, \dots, a_n}$ .  $\square$

We can now state our main theorem which gives the complete classification of ideals in  $S = \mathbb{K}[x_1, \dots, x_n]$  generated by all forms of degree  $d$  fixed by the action of  $M_{a_1, \dots, a_n}$ , for every  $n, d \geq 3$ , in terms of their WLP.

**Theorem 7.9.** *For integers  $d \geq 2$ ,  $n \geq 3$  and  $0 \leq a_1, \dots, a_n \leq d-1$ , let  $M_{a_1, \dots, a_n}$  be a representation of cyclic group  $\mathbb{Z}/d\mathbb{Z}$  and  $I \subset S = \mathbb{K}[x_1, \dots, x_n]$  be the ideal generated by all forms of degree  $d$  fixed by the action of  $M_{a_1, \dots, a_n}$ . Then,  $I$  satisfies the WLP if and only if at least  $n-1$  of the integers  $a_i$  are equal.*

*Proof.* Suppose at least  $n-1$  of the integers  $a_i$ 's are equal and by relabeling the variables we may assume that  $a_1 = a_2 = \dots = a_{n-1}$ .  $I$  contains the ideal  $(x_1, x_2, \dots, x_n)^d$ , and then all the monomials in  $(S/I)_d$  are divisible by  $x_n$  which implies that the map  $\times x_n : (S/I)_{d-1} \rightarrow (S/I)_d$  is surjective. Since  $[(S/I)/x_n(S/I)]_d = 0$  we have that  $[(S/I)/x_n(S/I)]_j = 0$  for all

$j \geq d$  and then  $\times x_n : (S/I)_{j-1} \rightarrow (S/I)_j$  is surjective for all  $j \geq d$ . On the other hand, since  $I$  is generated in degree  $d$ , the map  $\times x_n : (S/I)_{j-1} \rightarrow (S/I)_j$  is injective, for every  $j < d$ . Therefore,  $I$  has the WLP.

To show the other implication, we assume that at most  $n - 2$  integers  $a_i$  are equal and we prove that  $I$  fails WLP by showing that map  $\times(x_1 + \cdots + x_n) : (S/I)_{d-1} \rightarrow (S/I)_d$  is neither injective or surjective.

By Lemma 7.8, for the non-zero form  $f = \prod_{i=1}^{d-1} (\xi^{ia_1} x_1 + \cdots + \xi^{ia_n} x_n)$  of degree  $d - 1$  we have that  $(x_1 + \cdots + x_n)f$  is a form of degree  $d$  in  $I$ . Therefore the map  $\times(x_1 + \cdots + x_n) : (S/I)_{d-1} \rightarrow (S/I)_d$  is not injective.

Now it remains to show the failure of surjectivity. To do so by Macaulay duality equivalently we show that the map  $\circ(x_1 + \cdots + x_n) : (I^{-1})_d \rightarrow (I^{-1})_{d-1}$  is not injective. Note that the inverse module  $(I^{-1})_d$  is generated by all the monomials of degree  $d$  in the dual ring  $R = \mathbb{K}[y_1, \dots, y_n]$  which are not fixed by the action  $M_{a_1, \dots, a_n}$ .

We consider two cases depending on  $a_i$ 's. First, assume that there are at least three distinct integers among  $a_i$ 's and by relabeling the variables we may assume that  $a_1 < a_2 < a_3$ .

By applying Lemma 7.6 on the ring  $R$ , we get the linear form

$$L = \sum_{j=0}^l \xi^j y_1 + \sum_{j=l+1}^{l+k+1} \xi^j y_2 + \sum_{j=l+k+2}^{2d-1} \xi^j y_3,$$

where  $l$  and  $k$  are the residues of  $a_2 - a_3 - 1$  and  $a_3 - a_1 - 1$  modulo  $d$  and  $\xi$  is a primitive  $d$ -th root of unity. Since  $a_1, a_2$  and  $a_3$  are distinct  $F = L^d - \bar{L}^d$  is non-zero form of degree  $d$ . The monomials with non-zero coefficients in  $F$  are exactly the monomials of degree  $d$  in  $\mathbb{K}[y_1, y_2, y_3]$  which are not fixed by the action  $M_{a_1, a_2, a_3}$ . Therefore, all the monomials of degree  $d$  in  $R$  fixed by the action  $M_{a_1, \dots, a_n}$  have coefficient zero in  $F$  and thus we get that  $F \in (I^{-1})_d$ . Moreover, sum of the coefficients in  $L$  is exactly  $2(1 + \xi^1 + \xi^2 + \cdots + \xi^{d-1}) = 0$ . Therefore,  $(x_1 + \cdots + x_n) \circ F = (x_1 + \cdots + x_n) \circ (L^d) - (x_1 + \cdots + x_n) \circ (\bar{L}^d) = 0$  and this implies that  $\times(x_1 + \cdots + x_n) : (S/I)_{d-1} \rightarrow (S/I)_d$  is not surjective in this case.

Now assume that there are only two distinct integers among  $a_i$ 's. Without loss of generality we may assume that  $a_1 = a_2 = \cdots = a_m < a_{m+1} = a_{m+2} = \cdots = a_n$ . Since we assume that at most  $n - 2$  of the integers  $a_i$ 's are equal, we have  $m, n - m \geq 2$  and so  $a_1 = a_2 \neq a_{n-1} = a_n$ . Consider the element  $H = (y_1 - y_2)(y_n - y_{n-1})^{d-1} \in R$ . Acting  $M_{a_1, \dots, a_n}^r$  on  $H$  we get that  $M_{a_1, \dots, a_n}^r (y_1 - y_2)(y_n - y_{n-1})^{d-1} = \xi^{ra_1 - a_n} (y_1 - y_2)(y_n - y_{n-1})^{d-1}$  for every  $0 \leq r \leq d - 1$ . So  $H$  is fixed by the action  $M_{a_1, \dots, a_n}$  if and only if  $a_1 = a_n$  which we assumed  $a_1 \neq a_n$ . This implies that  $H$  and none of the monomials in  $H$  are fixed by  $M_{a_1, \dots, a_n}$ , therefore  $H \in (I^{-1})_d$ . Moreover, we have that  $(x_1 + \cdots + x_n) \circ H = 0$  and then the map  $\times(x_1 + \cdots + x_n) : (S/I)_{d-1} \rightarrow (S/I)_d$  is not surjective.  $\square$

We illustrate Theorem 7.9 in the next example for the ideal in the polynomial ring with three variables failing the WLP.

**Example 7.10.** Let  $I \subset S = \mathbb{K}[x_1, x_2, x_3]$  be the ideal generated by forms of degree 10 fixed by the action of  $M_{0,2,4}$ . Theorem 7.1 implies that  $I$  is generated by all monomials of degree  $d$  fixed by the action of  $M_{0,2,4}$ . By Theorem 7.9 above we get that  $I$  fails WLP from degree 9 to degree 10. Since by Theorem 7.2 we have  $H_{S/I}(10) = 52 < 55 = H_{S/I}(9)$ , failing WLP is an assertion of failing surjectivity of the multiplication map  $\times(x_1 + x_2 + x_3) : (S/I)_9 \rightarrow (S/I)_{10}$ .



We equivalently show that the map  $\circ(x_1 + x_2 + x_3) : (I^{-1})_{10} \longrightarrow (I^{-1})_9$  is not injective. Using Lemma 7.6, we let  $L$  be the linear form  $L = \sum_{j=0}^7 \xi^j y_1 + \sum_{j=8}^{11} \xi^j y_2 + \sum_{j=12}^{19} \xi^j y_3$  for  $l = 7$  and  $k = 3$  in the dual ring  $R = \mathbb{K}[y_1, y_2, y_3]$ . Then we get the non-zero form  $F = L^{10} - \bar{L}^{10}$  in the kernel of the map  $\circ(x_1 + x_2 + x_3) : (I^{-1})_{10} \longrightarrow (I^{-1})_9$ . Computations by Macaulay 2 software, show that the kernel of this map has dimension 2. We can actually get the other form in the kernel by changing  $\xi$  with  $\xi' = \xi^3 = e^{6\pi i/d}$ . Therefore we have  $L' = \sum_{j=0}^7 \xi^{3j} y_1 + \sum_{j=8}^{11} \xi^{3j} y_2 + \sum_{j=12}^{19} \xi^{3j} y_3$  and then  $G = L'^d - \bar{L}'^d$  is another form of degree 10 in the kernel where  $(x_1 + x_2 + x_3) \circ G = 0$ .

**Remark 7.11.** Consider the ideals generated by the forms of degree  $d \geq 2$  in  $\mathbb{K}[x_1, x_2, x_3, x_4]$  invariant under the action  $M_{a_1, a_2, a_3, a_4}$  of the cyclic group  $\mathbb{Z}/d\mathbb{Z}$  where  $\gcd(a_1, a_2, a_3, a_4, a_d) = 1$  and at most two integers among the  $a_i$ 's are equal. Then by Proposition 7.5 and Theorem 7.9 we conclude that all such ideals are defined by monomial Togliatti systems generalizing the result in [9] to the polynomial rings with four variables.

## 8. DIHEDRAL GROUP ACTING ON $\mathbb{K}[x, y, z]$

In the previous section we have studied the WLP of ideals generated by invariant forms of degree  $d$  under an action of cyclic group of order  $d$ . In this section we study an action of dihedral group  $D_{2d}$  on the polynomial ring with three variables  $S = \mathbb{K}[x, y, z]$  where  $\mathbb{K} = \mathbb{C}$  and  $d \geq 2$ . Let  $\xi^{2\pi i/d}$  be a primitive  $d$ -th root of unity and

$$A_d = \begin{pmatrix} \xi & 0 & 0 \\ 0 & \xi^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, B_d = \begin{pmatrix} 0 & \xi^{-1} & 0 \\ \xi & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

be a representation of dihedral group  $D_{2d}$ . Let  $F = \prod_{j=0}^{d-1} (\xi^j x + \xi^{-j} y + z)(\xi^j x + \xi^{-j} y - z)$  which is a polynomial of degree  $2d$  invariant by the action  $A_d$  and  $B_d$  of dihedral group  $D_{2d}$ . We study the WLP of the artinian monomial ideal in  $S$  generated by all the monomials in  $F$  with non-zero coefficients. First we count the number of generators of such ideals.

**Proposition 8.1.** *For integer  $d \geq 2$ , let  $A_d$  and  $B_d$  be a representation of  $D_{2d}$  and let  $I \subset S$  be the artinian monomial ideal generated by all monomial with non-zero coefficients in  $F = \prod_{j=0}^{d-1} (\xi^j x + \xi^{-j} y + z)(\xi^j x + \xi^{-j} y - z)$ . Then  $\mu(I) = d + 3$ , if  $d = 2k + 1$ ; and  $\mu(I) = 2d + 5$ , if  $d = 2k$ .*

*Proof.* First, assume  $d = 2k + 1$  and consider the action of  $M_{2, 2d-2, d} = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & \omega^{2d-2} & 0 \\ 0 & 0 & \omega^d \end{pmatrix}$  of a cyclic group  $\mathbb{Z}/2d\mathbb{Z}$  where  $\omega = e^{2\pi i/2d}$  is a primitive  $2d$ -root of unity. Then consider the

form  $H = \prod_{j=0}^{2d-1} (\omega^{2j}x + \omega^{(2d-2)j}y + \omega^{dj}z)$ . We have that

$$\begin{aligned}
H &= \prod_{j=0}^{2d-1} (\xi^j x + \xi^{-j} y + (-1)^j z) \\
&= \prod_{j=0}^{d-1} (\xi^j x + \xi^{-j} y + (-1)^j z) (\xi^{j+d} x + \xi^{-j+d} y + (-1)^{j+d} z) \\
&= \prod_{j=0}^{d-1} (\xi^j x + \xi^{-j} y + (-1)^j z) (\xi^j x + \xi^{-j} y + (-1)^{j+d} z) \\
&= \prod_{j=0}^{d-1} (\xi^j x + \xi^{-j} y - z) (\xi^j x + \xi^{-j} y + z) = F.
\end{aligned}$$

Note that the monomials fixed by the action of  $M_{2,2d-2,d}$ ,  $M_{0,2d-4,d-2}$  and  $M_{0,1,a}$  are the same, where  $(2d-4)a = (d-2)$ , since  $d$  is odd such integer  $a$  exists. By Theorem 7.2 we get that the number of monomials fixed by any of those actions is  $d+3$ . On the other hand Theorem 2, in [6] implies that the number of terms with non-zero coefficient in  $H$  and then in  $F$  is exactly  $d+3$  which implies that  $\mu(I) = d+3$ .

Now assume that  $d = 2k$  and consider the action of  $M_{2,2d-2,0} = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & \omega^{2d-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$  of a cyclic group  $\mathbb{Z}/2d\mathbb{Z}$ . Consider the form  $G = \prod_{j=0}^{2d-1} (\omega^{2j}x + \omega^{(2d-2)j}y + z)$  then we have

$$\begin{aligned}
G &= \prod_{j=0}^{2d-1} (\xi^j x + \xi^{-j} y + z) \\
&= \prod_{j=0}^{d-1} (\xi^j x + \xi^{-j} y + z) (\xi^{j+d} x + \xi^{-j+d} y + z) \\
&= \prod_{j=0}^{d-1} (\xi^j x + \xi^{-j} y + z) (-\xi^j x - \xi^{-j} y + z) \\
&= (-1)^d \prod_{j=0}^{d-1} (\xi^j x + \xi^{-j} y + z) (\xi^j x + \xi^{-j} y - z) = F
\end{aligned}$$

also we have that  $F = G = (\prod_{j=0}^{d-1} (\xi^j x + \xi^{-j} y + z))^2$  and denote  $f := \prod_{j=0}^{d-1} (\xi^j x + \xi^{-j} y + z)$ . Theorem 2 in [6], implies that the monomials in  $f$  with non-zero coefficients are exactly

the monomials of degree  $d$  fixed by the action  $M_{1,d-1,0} = \begin{pmatrix} \xi & 0 & 0 \\ 0 & \xi^{d-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$  of a cyclic group

$\mathbb{Z}/d\mathbb{Z}$ . Therefore, using Theorem 7.2 we get that there are  $3 + d/2$  monomials with non-zero coefficients in  $f$ , and they are exactly the monomials of the form  $(xy)^\alpha z^{d-2\alpha}$  and  $x^d$  and  $y^d$ .

We now count the monomials in  $F = f^2$ . First we claim that the form  $f$  has alternating sign in the variable  $z$ . To show this we evaluate the form in  $x = y = 1$  then we get

$$\prod_{j=0}^{d-1} (\xi^j + \xi^{-j} + z) = \prod_{j=0}^{d/2-1} (\xi^j + \xi^{-j} + z)(\xi^{j+d/2} + \xi^{-j+d/2} + z) = (-1)^{d/2} \prod_{j=0}^{d/2-1} (w^2 + a_j^2)$$

where  $a_j = \xi^j + \xi^{-j}$  and  $z^2 = -w$ . Then this expression proves the claim.

Multiplying  $x^d$  and  $y^d$  in  $f$  with  $d/2 + 1$  monomials  $(xy)^\alpha z^{d-2\alpha}$  gives  $2(d/2 + 1) = d + 2$  monomials in  $F$ . Using the claim above we get that all  $d + 1$  monomials of degree  $2d$  of the form  $(xy)^\beta z^{2d-2\beta}$  have non-zero coefficients in  $F$ . Adding 2 corresponding to the monomials  $x^{2d}$  and  $y^{2d}$  we get that there are exactly  $2d + 5$  monomials in  $F$  with non-zero coefficients or equivalently  $\mu(I) = 2d + 5$ . □

**Proposition 8.2.** *For integer  $d \geq 2$  let  $I \subset \mathbb{K}[x, y, z]$  be the ideal generated by all the monomials with non-zero coefficients in  $F = \prod_{j=0}^{d-1} (\xi^j x + \xi^{-j} y + z)(\xi^j x + \xi^{-j} y - z)$ , introduced in Proposition 8.1. Then  $I$  fails WLP from degree  $2d - 1$  to degree  $2d$ .*

*Proof.* Suppose that  $d = 2k + 1$ , using Proposition 8.1 we have that

$$H_{S/I}(2d) = H_S(2d) - \mu(I) = (2d^2 + 3d + 1) - (d + 3) = 2(d^2 + d - 1) > d(2d + 1) = H_{S/I}(2d - 1).$$

Consider the form  $K = (x + y - z) \prod_{i=1}^{d-1} (\xi^i x + \xi^{-i} y + z)(\xi^{-i} x + \xi^i y - z)$  of degree  $2d - 1$ . Since we have  $(x + y + z)K = F$ , the map  $\times(x + y + z) : (S/I)_{2d-1} \rightarrow (S/I)_{2d}$  is not injective.

Now assume  $d = 2k$ , then Proposition 8.1 implies that

$$H_{S/I}(2d) = H_S(2d) - \mu(I) = (2d^2 + 3d + 1) - (2d + 5) = 2d^2 + d - 4 < d(2d + 1) = H_{S/I}(2d - 1).$$

Therefore, in order to prove  $S/I$  fails the WLP we need to prove that the multiplication map by  $x + y + z$  on the algebra from degree  $2d - 1$  to degree  $2d$  is not surjective. To do so we use the representation theory of the symmetric group  $S_2$  where 1 acts trivially and  $-1$  interchanges  $x$  and  $y$ . Note that this action fixes the form  $x + y + z$ . We look at the multiplicity of the alternating representation of  $S/I$  in degree  $2d - 1$  and  $2d$ . In degree  $2d - 1$  there are  $d(2d + 1)$  monomials in  $S/I$  that are fixed by the identity permutation and there are  $d$  monomials of the form  $(xy)^\alpha z^{2d-1-2\alpha}$  that are fixed by interchanging  $x$  and  $y$ . Therefore the multiplicity of the alternating representation in degree  $2d - 1$  is  $(d(2d + 1) - d)/2 = d^2$ . In degree  $2d$  there are  $(2d + 1)(d + 1) - (2d + 5) = 2d^2 + d - 4$  monomials in  $S/I$  that are all fixed by the identity permutation and there is no monomial of the form  $(xy)^\alpha z^{2d-2\alpha}$  in the algebra which is fixed by interchanging  $x$  and  $y$ , since they all belong to  $I$ . So the multiplicity of the alternating representation of  $S/I$  in degree  $2d$  is  $(2d^2 + d - 4)/2$ . Since  $(2d^2 + d - 4)/2 > d^2$  for  $d \geq 5$  the multiplication by  $x + y + z$  cannot be surjective by Schur's lemma.

For  $d = 4$  computations in Macaulay 2 show the multiplication map by  $x + y + z$  is not surjective on  $S/I$  from degree 7 to degree 8. □

**Remark 8.3.** In Proposition 8.2, we have proved that for odd integer  $d$  the monomial ideals generated by the monomials of degree  $2d$  with non-zero coefficients in  $F$  fail WLP by failing injectivity in degree  $2d - 1$ , therefore such ideals define minimal monomial Togliatti systems.

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