Lefschetz Properties of Monomial Ideals

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Licentiate Thesis in Mathematics
Stockholm, Sweden 2018

TRITA-MAT-A 2018:08
ISRN KTH/MAT/A-18/08-SE
ISBN 978-91-7729-703-1

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Akademisk avhandling som med tillstånd av Kungl Tekniska högskolan framlägges till offentlig granskning för avläggande av teknologie licentiatexamen i matematik fredagen den 16 mars 2018 klockan 14.00 i sal F11, Lindstedtsvägen 24, KTH, Stockholm.

Tryck: Universitetsservice US-AB, 2018

Abstract

This thesis concerns the study of the Lefschetz properties of artinian monomial algebras. An artinian algebra is said to satisfy the strong Lefschetz property if multiplication by all powers of a general linear form has maximal rank in every degree. If it holds for the first power it is said to have the weak Lefschetz property (WLP).

In the first paper, we study the Lefschetz properties of monomial algebras by studying their minimal free resolutions. In particular, we give an affirmative answer to an specific case of a conjecture by Eisenbud, Huneke and Ulrich for algebras having almost linear resolutions.

Since many algebras are expected to have the Lefschetz properties, studying algebras failing the Lefschetz properties is of a great interest. In the second paper, we provide sharp lower bounds for the number of generators of monomial ideals failing the WLP extending a result by Mezzetti and Miró-Roig which provides upper bounds for such ideals. In the second paper, we also study the WLP of ideals generated by forms of a certain degree invariant under an action of a cyclic group. We give a complete classification of such ideals satisfying the WLP in terms of the representation of the group generalizing a result by Mezzetti and Miró-Roig.

Sammanfattning

Denna avhandling behandlar studiet av Lefschetzegenskaper hos artinska monomalgebror. En artinsk monomalgebra sägs ha den starka Lefschetzegenskapen om multiplikationen med alla potenser av en generell linjärform har maximal rang i alla grader. Om detta gäller för den första potensen sägs algebran ha den svaga Lefschetzegenskapen (WLP).

I den första artikeln studerar vi Lefschetzegenenskaper genom att studera minimala fria upplösningar. Speciellt ger vi ett positivt svar på ett specialfall av en förmodan av Eisenbud, Huneke och Ulrich för algebror som har en nästan linjär upplösning.

Eftersom många algebror förväntas ha Lefschetzegenskaperna är det mycket intressant att studera de algebror som inte har dessa egenskaper. I den andra artikeln bevisar vi en skarp undre gräns för antalet generatorer av monomideal som inte uppfyller WLP vilket utvidgar ett resultat av Mezzetti och Miró-Roig som ger en övre gräns för sådana ideal. I den andra artikeln studerar vi också WLP för ideal som genereras av former av en viss grad som är invarianta under verkan av cyklisk grupp. Vi ger en fullständig klassificering av sådana ideal i termer av representationen av gruppen, vilket generaliserar resultat av Mezzetti och Miró-Roig.

Contents

Part I

Introduction with summary of results

This thesis consists of two papers both dealing with the Lefschetz properties of artinian monomial algebras. In the first paper, we study the Lefschetz properties of artinian monomial algebras by studying their minimal free resolutions. In the second paper we provide sharp lower bounds for the Hilbert function in degree d of an artinian monomial algebra failing the weak Lefschetz property. We also study artinian ideals in the polynomial ring with n variables generated by homogeneous polynomials of degree d invariant under an action of a cyclic group $\mathbb{Z}/d\mathbb{Z}$, for $n \geq 3$ and $d \geq 2$. We give a complete classification of such ideals satisfying the WLP in terms of the action.

The purpose of this chapter is to give the mathematical background to the topics of this thesis and to summarize the main results of the papers.

1 Lefschetz properties

The weak and strong Lefschetz properties are strongly connected to many topics in algebraic geometry, commutative algebra and combinatorics. In this section we start with definitions and notations and we state some important results in this area and explain the approach of the work of this thesis. In the last part of this section we state some results of this thesis.

The weak and the strong Lefschetz properties

Let $S = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field $\mathbb{K}$ with all the variables of degree 1. An S-module M is *graded*, if it has a direct sum decomposition $M = \oplus_{d \in \mathbb{Z}} M_d$ as a $\mathbb{K}$-vector space and $S_i M_j \subseteq M_{i+j}$ for all $i, j \in \mathbb{Z}$. The $\mathbb{K}$-spaces M_d are called the *homogeneous components* of M. An element $m \in M$ is called *homogeneous* if $m \in M_d$ for some d and in this case we say that m has degree d and write $\deg(m) = d$. We may consider different types of grading on S, but when $\deg(x_i) = 1$ for every $1 \leq i \leq n$ we say S is *standard graded* and when we do not specify we consider the standard grading on S.
For finitely generated graded S-modules M and N, S-linear map $\varphi : M \longrightarrow N$ is said to be *homogeneous of degree a* for some $b \in \mathbb{Z}$ if linear if $\varphi(M_a) \subseteq N_{a+b}$ for all $a \in \mathbb{Z}$. We call φ *homogeneous* if it is homogeneous of degree 0.

Let us now define the weak and the strong Lefschetz properties.

Definition 1.1. Let $I \subset S$ be a homogeneous artinian ideal. We say that S/I has the *weak Lefschetz property* (WLP) if there is a linear form $\ell \in (S/I)_1$ such that, for all integers j, the multiplication map

$$\times \ell : (S/I)_j \longrightarrow (S/I)_{j+1}$$

has maximal rank, i.e. it is injective or surjective. In this case the linear form ℓ is called a *weak Lefschetz element* of S/I. If for the general form $\ell \in (S/I)_1$ and for an integer number j the map $\times \ell$ does not have the maximal rank we will say that

the ideal I fails the WLP in degree j.
We say that S/I has the *strong Lefschetz property* (SLP) if there is a linear form $\ell \in (S/I)_1$ such that, for all integers j and k the multiplication map

$$\times \ell^k : (S/I)_j \longrightarrow (S/I)_{j+k}$$

has maximal rank, i.e. it is injective or surjective. In this case the linear form i is called a *strong Lefschetz element*, we sometimes abuse the notation and say I has the WLP of SLP when we mean that S/I does so.

It may seem simple to determine whether an algebra satisfies the Lefschetz properties but it turns out to be rather difficult even for natural families of algebras. Also most artinian algebras are expected to have the WLP or SLP but many artinian algebras fail to have these properties and something interesting should be going on for the algebras failing the WLP or SLP.

In fact every ideal in the polynomial ring with one variable is principal so all artinian algebras in this case trivially satisfy SLP. In the polynomial ring with two variables there is the following result by Harima, Migliore, Nagel and Watanabe in [7].

Proposition 1.2. *If char*$(\mathbb{K}) = 0$ *and I is any homogeneous ideal in* $S = \mathbb{K}[x, y]$, *then S/I has the SLP.*

In a polynomial ring with more than two variables it is not true in general that every artinian monomial algebra has the SLP or WLP. The most general result in this case proved by Stanley in [14].

Theorem 1.3. *Let* $S = \mathbb{K}[x_1, \dots, x_n]$, *where char*$(\mathbb{K}) = 0$. *Let I be an artinian monomial complete intersection, i.e* $I = (x_1^{a_1}, \dots, x_n^{a_n})$. *Then S/I has the SLP.*

Let us now describe the *Macaulay duality* and *inverse systems.*

Macaulay Inverse Systems

Let $S = \mathbb{K}[x_1, \dots, x_n]$ and $R = \mathbb{K}[y_1, \dots y_n]$ be a new polynomial ring. Define an action of S on R by partial differentiation $x_j \circ y_i = \partial y_i / \partial y_j$. This action induces an exact pairing of $\mathbb{K}$-vector spaces:

$$\langle , \rangle : S \times R \longrightarrow \mathbb{K}$$
$$\langle f, g \rangle \to (f \circ g)(0)$$

This action makes S into a graded R-module. For homogeneous ideal $I \subset S$ we define the *inverse system* $I^{-1} \subset R$ as

$$I^{-1} := \{g \in R \mid f \circ g = 0, \text{ for all } f \in I\}.$$

The inverse system of a homogeneous ideal is a graded R-module, but in general I^{-1} is not an ideal. When I is a monomial ideal the inverse system module $(I^{-1})_d$

is generated by the monomials in R_d corresponding to the monomials in S_d but not in I_d.

Example 1.4. Let $I = (x_1^3, x_2^3, x_3^3, x_1x_2x_3)$ be an artinian monomial ideal in $S = \mathbb{K}[x_1, x_2, x_3]$. Then we have that $(I^{-1})_3 = (y_1^2y_2, y_1y_2^2, y_1^2y_3, y_1y_3^2, y_2^2y_3, y_2y_3^2)$ is the inverse system module of I in $R = \mathbb{K}[y_1, y_2, y_3]$.

There is a one-to-one correspondence between graded artinian algebras S/I and finitely generated graded S-submodules M of R, see [8].

Theorem 1.5. ***(Macaulay duality)*** *Let $S = \mathbb{K}[x_1, \ldots, x_n]$ be the n-dimensional polynomial ring over a field $\mathbb{K}$. There is an order-reversing bijection between the set of finitely generated sub-R-modules of $R = \mathbb{K}[y_1, \ldots, y_n]$ and the set of artinian ideals of R given by: if M is a submodule of R then $M^{-1} = (0 :_S M)$, and $I^{-1} = (0 :_R I)$ for an ideal $I \subset S$.*

Togliatti systems

In this part we describe a relation between a differential geometric notion, concerning varieties which satisfy certain Laplace equations and the weak Lefschetz property. Set $S = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring over an algebraically closed field of characteristic zero, $\mathbb{K}$.

In [2], Brenner and Kaid proved that every artinian ideal of the form $(x_1^3, x_2^3, x_3^3, f(x_1, x_2, x_3))$ where f is a form of degree 3, fails the WLP if and only if $f \in (x_1^3, x_2^3, x_3^3, x_1x_2x_3)$. Moreover they showed that the ideal $(x_1^3, x_2^3, x_3^3, x_1x_2x_3)$ is the only artinian monomial ideal in three variables which fails the WLP. On the other hand, Togliatti proved that the only non-trivial smooth surface $X \subset \mathbb{P}^5$ obtained by projecting the Veronese surface $V(2,3) \subset \mathbb{P}^9$ and satisfying a Laplace equation of order 2 is the image of $\mathbb{P}^2$ via the linear system $\langle x_1^2x_2, x_1x_2^2, x_1^2x_3, x_1x_3^2, x_2^2x_3, x_2x_3^2\rangle \subset |\mathcal{O}_{\mathbb{P}^2}(3)|$ see [16] and [15]. Note that the monomials in the linear system given by Togliatti is the inverse system module of the monomial ideal given by Brenner and Kaid.

In [12], Mezzetti, Miró-Roig and Ottaviani found this relation between artinian ideals $I \subset S$ generated by r homogeneous forms of degree d failing the WLP and projections of the Veronese variety $V(n-1, d) \subset \mathbb{P}^{\binom{n+d-1}{d}-1}$ in $X \subset \mathbb{P}^{\binom{n+d-1}{d}-r-1}$ satisfying at least one Laplace equation of order $d-1$.

Let us explain this relation in more details.

Definition 1.6. For an r dimensional variety $X \subseteq \mathbb{P}^m$ and $p \in X$ that $\mathcal{O}_{X,p}$ has local defining equations f_i, the d-th *osculating space* $T^d(X,p)$ is the linear subspace spanned by p and all $\dfrac{\partial(f_i)}{\partial x^\alpha}(p)$, with $|\alpha| \le d$.

At a general point $p \in X$, the expected dimension of $T^d(X,p)$ is $min\{m, \binom{r+d}{d} - 1\}$. If for some positive δ we have that $\dim T^d(X,p) = \binom{r+d}{d} - 1 - \delta < m$ for general p, then X is said to satisfy δ *Laplace equation* of order d.

Let $I \subset S$ be an artinian ideal generated by homogeneous polynomials $f_1, \ldots, f_r$ of degree d and I^{-1} be its inverse system module and consider the rational map $\varphi_{(I^{-1})_d} : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{\binom{n+d-1}{d}-r-1}$ associated to $(I^{-1})_d$. Denote its image by $X_{n-1,(I^{-1})_d} := \overline{\operatorname{Im}(\varphi_{(I^{-1})_d})} \subset \mathbb{P}^{\binom{n+d-1}{d}-r-1}$. Then $X_{n-1,(I^{-1})_d}$ is the projection of the Veronese variety $V(n-1,d)$ from the linear system of the vector space spanned by $f_1, \ldots, f_r$. Associated to I_d there is a morphism $\varphi_{I_d} : \mathbb{P}^{n-1} \longrightarrow \mathbb{P}^{r-1}$. Since I is artinian φ_{I_d} is regular. Denote $X_{n-1,I_d} := \overline{\operatorname{Im}(\varphi_{I_d})} \subset \mathbb{P}^{r-1}$ which is the projection of the Veronese variety $V(n-1,d)$ form the linear system of the vector space spanned by forms in $(I^{-1})_d$. The varieties $X_{n-1,(I^{-1})_d}$ and X_{n-1,I_d} are usually called *apolar*.

In [12], Theorem 3.2 Mezzetti, Miró-Roig and Ottaviani proved the following result. With the notations as above the theorem is as follows:

Theorem 1.7. *Let $I \subset S$ be an artinian ideal generated by r forms $f_1, \ldots, f_r$ of degree d. If $r \leq \binom{n+d-2}{n-2}$, then the following conditions are equivalent:*

(1) *The ideal I fails the WLP in degree $d-1$,*

(2) *The forms $f_1, \ldots, f_r$ become $\mathbb{K}$-linearly dependent on a general hyperplane H of $\mathbb{P}^{n-1}$,*

(3) *The variety $X_{n-1,(I^{-1})_d}$ of dimension $n-1$ satisfies at least one Laplace equation of order $d-1$.*

Definition 1.8. Let $I \subset S$ be an artinian ideal generated by r homogeneous polynomials in S of degree d, where $r \leq \binom{n+d-2}{n-2}$.

- I is said to be *Togliatti system* if, I satisfies the three equivalent conditions in Theorem 1.7.
- I is called *monomial Togliatti system* if, in addition I can be generated by monomials.
- I is a *smooth* Togliatti system if, in addition, the $(n-1)$-dimensional variety X is smooth.
- A monomial Togliatti system is *minimal* if there is no proper subset of the monomial set of generators I defining the Togliatti system.

Remark 1.9. By the definition if $I \subseteq S$ is an artinian ideal generated by r forms $f_1, \ldots, f_r$ of degree d such that $r \leq \binom{n+d-2}{n-2}$ and S/I fails WLP in degree $d-1$, I defines a *Togliatti system*. Note that the numerical hypothesis on the number of generators is equivalent to the condition that $\dim(S/I)_{d-1} \leq \dim(S/I)_d$ which means the condition that S/I fails the WLP in degree $d-1$ in only an assertion of failing injectivity of the multiplication by a general linear form from $(S/I)_{d-1}$ to $(S/I)_d$.

Later in Section 4 we state some of the results in the second paper concerning Togliatti systems.

In [10] Mezzetti and Miró-Roig determine a lower bound for the minimal number of generators $\mu(I)$ of any (resp. smooth) minimal monomial Togliatti system $I \subset \mathbb{K}[x_1, \dots, x_n]$ of forms of degree $d \geq 2$ and classify such systems which reach the bound. In [10, Theorem 3.9], Mezzetti and Miró-Roig prove the following theorem.

Theorem 1.10. *For an integer $n \geq 3$ and $d \geq 4$, if $I \subset \mathbb{K}[x_1, \dots, x_n]$ is a minimal (resp. smooth minimal) monomial Togliatti system of forms of degree d, then $\mu(I) \geq 2n - 1$.*

As we have noticed in Remark 1.9 an artinian ideal generated in degree d defining a Togliatti system is an ideal failing WLP by failing injectivity of the multiplication map by a general linear form in degree $d-1$. The work done in [10] motivated us to study monomial artinian ideals generated in degree d failing the WLP by failing the surjectivity in degree $d-1$, instead. In the second paper of this thesis we provide a sharp upper bound for the number of generators of such ideals. In fact our result in the polynomial ring with three variables is as follows:

Theorem 1.11. *Let $I \subset S = \mathbb{K}[x_1, x_2, x_3]$ be an artinian monomial ideal generated in degree d. If I fails the WLP, then*

$$\mu(I) \leq \begin{cases} \binom{d+2}{2} - (3d-3) & \text{if } d \text{ is odd} \\ \binom{d+2}{2} - (3d-2) & \text{if } d \text{ is even.} \end{cases}$$

Moreover, the bounds are sharp.

In the polynomial ring with more than three variables we have the different bound.

Theorem 1.12. *Let $I \subset S = \mathbb{K}[x_1, x_2, x_3]$ be an artinian monomial ideal generated in degree d. If I fails the WLP, then*

$$\mu(I) \leq \binom{n+d-1}{n-1} - 2d.$$

Moreover, the bound is sharp.

Remark 1.13. In Section 3 we see that the bounds given in the above theorems are actually the sharp lower bounds for the *Hilbert function* of S/I in degree d.

2 Free resolutions

In the first paper of this thesis we study the Lefschetz properties of artinian monomial ideal $I \subset S = \mathbb{K}[x_1 \dots, x_n]$ via Macaulay duality where the *minimal free resolution* of S/I is linear for at least $n-2$ steps. The goal of this section is to describe the minimal free resolutions and other related invariants and stating some of the important results in this area. At the end we explain the connection of the work done in this theses and we state our related results.

Minimal graded free resolutions

Minimal graded free resolutions are an important and central topic in algebra. They are useful tools for studying modules over finitely generated graded $\mathbb{K}$-algebras. Such a resolution determines the *Hilbert series*, *the Castelnuovo-Mumford regularity* and other invariants of the module.

Let $S = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring over a field $\mathbb{K}$. Let M be a graded S-module, for $a \in \mathbb{Z}$ the shifted module $M(a)$ is the graded module with $M(a)_b = M_{a+b}$. Note that if $\varphi : M \longrightarrow N$ is homogeneous of degree $b \in \mathbb{Z}$, then the induced map $\tilde{\varphi} : M(-b) \longrightarrow N$ is homogeneous.

Given homogeneous elements $m_i \in M$ of degree a_i that generate M as an S-module, we may define a map from the graded free module $F_0 = \oplus_i S(-a_i)$ onto M by sending the i-th generator to m_i (which is a homogeneous map). Let $M_1 \subset F_0$ be the kernel of this map $F_0 \longrightarrow M$. By the Hilbert Basis Theorem, see [4], M_1 is also a finitely generated module. The elements of M_1 are called *syzygies* of M. Choosing finitely many homogeneous syzygies that generate M_1, we may define a map from a graded free module F_1 to F_0 with image M_1. Continuing in this way we may construct a sequence of maps of graded free modules, called the *graded free resolution* of M:

$$\cdots \longrightarrow F_i \xrightarrow{\varphi_m} F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0.$$

It is an exact sequence of degree zero maps between graded free modules such that the cokernel of φ_1 is M. Since the φ_i preserve degrees, we get an exact sequence of finite-dimensional vector spaces by taking the degree d part of each module in this sequence, therefore we have

$$H_M(d) = \sum_i (-1)^i H_{F_i}(d).$$

In 1980 Hilbert showed that this sum is finite for finitely generated S-module M. See [4] and [3] for more details.

Theorem 2.1. ***(Hilbert Syzygy Theorem)*** *Any finitely generated graded S-module M has a finite graded free resolution*

$$0 \longrightarrow F_m \xrightarrow{\varphi_m} F_{m-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0.$$

Moreover, we may take $m \leq n$, the number of variables in S.

We denote $\mathfrak{m}$ to be the homogeneous maximal ideal $(x_1, \ldots, x_n) \subset S = \mathbb{K}[x_1, \ldots, x_n]$ and define a graded free resolution

$$\cdots \longrightarrow F_i \xrightarrow{\varphi_m} F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0.$$

to be *minimal* if for each i the image of φ_i is contained in $\mathfrak{m}F_{i-1}$. Equivalently, for each i the map φ_i maps a basis of F_i to a minimal set of generators of the image of φ_i.

More generally, a *complex* $\mathcal{F}$ is defined to be a collection of finitely generated S-modules $\{F_i \mid i \in \mathbb{Z}\}$ and homogeneous S-linear maps $\delta_i : F_i \longrightarrow F_{i-1}$ with $\operatorname{Im}(\delta_{i+1}) \subseteq \ker(\delta_i)$. We associate the *homology groups* $H_i(\mathcal{F}) = \ker(\delta_i)/\operatorname{Im}(\delta_{i+1})$ to every complex. A complex $\mathcal{F}$ is called *exact* if $H_i(\mathcal{F}) = 0$ for all $i \in \mathbb{Z}$.
For a complex $\mathcal{F}$ and finitely generated graded S-module M we have $\mathcal{F} \otimes_S M$, $M \otimes_S \mathcal{F}$ are also complexes with induced complex maps $\delta \otimes_S M$ and $M \otimes_S \delta$.

If M and N are finitely generated graded S-modules. Let $\mathbb{F}$ be a minimal free resolution of M and $\mathcal{G}$ be a minimal free resolution of N. Then we have

$$\operatorname{Tor}_i(M, N) \cong H_i(\mathcal{F}, N) \cong H_i(M, \mathcal{G})$$

where $\operatorname{Tor}_i(M, N)$ denotes the i-th Tor-group associated to M and N.

The following result shows that the minimal free resolution of a finitely generated S-module M is unique up to isomorophism which means it is only depend on M, see [4].

Theorem 2.2. *If $\mathcal{F} : \cdots \to F_1 \to F_0$ is the minimal free resolution of a finitely generated graded S-module M, then any minimal set of homogeneous generators of F_i contains exactly* $\dim \operatorname{Tor}_i^S(\mathbb{K}, M)_j$ *generators of degree j.*

Now suppose that $\mathcal{F} : \cdots \to F_1 \to F_0$ is the minimal free resolution of an S-module M, where $F_i = \oplus_j S(-j)^{\beta_{i,j}}$, that is F_i requires $\beta_{i,j}$ minimal generators of degree j. In fact we have $\beta_{i,j} = \dim \operatorname{Tor}_i^S(\mathbb{K}, M)_j$. The $\beta_{i,j}$ of $\mathcal{F}$ are called the *graded Betti numbers* of M, sometimes written $\beta_{i,j}(M)$.

Example 2.3. Let $S = \mathbb{K}[x_1, x_2]$ be the graded polynomial ring in two variables. Then $\mathbb{K} = S/\mathfrak{m}$ has the following minimal graded free resolution:

$$0 \longrightarrow S(-2) \longrightarrow S(-1)^2 \longrightarrow S \longrightarrow 0.$$

Therefore the nonzero graded Betti numbers are $\beta_{0,0}(\mathbb{K}) = 1$, $\beta_{1,1}(\mathbb{K}) = 2$ and $\beta_{2,2}(\mathbb{K}) = 1$.

We associate a module called *socle* to a finitely generated S-module M. Let I be an ideal in S we define $\operatorname{soc}(S/I) := \{f \in S/I \mid \mathfrak{m} f = 0\}$. It can be determined from the last syzygy module of S/I. In fact we have $\operatorname{soc}(S/I) \cong \operatorname{Tor}_n(S/I, \mathbb{K})$ an there for in particular we have $\dim_{\mathbb{K}}(\operatorname{soc}(S/I)) = \sum_j \beta_{n,j}(S/I)$.

Later we will see that some times the minimal free resolution of socle module is dual to the minimal free resolution of the module (complexes are dual to each other).

There are several invariants associated to a graded modules which can be obtained by the graded Betti numbers.

Definition 2.4. Let M be a finitely generated graded S-module. Then we define the *projective dimension* of M to be

$$\operatorname{pd}_S(M) := \sup\{i \in \mathbb{Z} \mid \beta_{i,j}(M) \neq 0 \text{ for some } i \in \mathbb{Z}\}$$

and the *Castelnuovo-Mumford regularity* of M to be

$$\operatorname{reg}_S(M) := \sup\{j \in \mathbb{Z} \mid \beta_{i,i+j}(M) \neq 0 \text{ for some } i \in \mathbb{Z}\}$$

Note that Hilbert syzygy theorem shows that for a finitely generated graded module M over $S = \mathbb{K}[x_1, \ldots x_n]$ we have $\operatorname{pd}_S(M) \leq n$ and $\operatorname{reg}_S(M) < \infty$.

Definition 2.5. Let M be a graded finitely generated S-module then M is said to have a *d-linear resolution* if $\beta_{i,i+j}(M) = 0$ for all $i \geq 0$ and all $j \neq d$.

By definition for an S-graded module M with a d-linear resolution we have $\operatorname{reg}_S(M) = d$. Also the module M is generated in degree d. We also say that the resolution of a graded finitely generated S-module M generated in a single degree d is linear for q steps if $\beta_{i,i+j}(M) = 0$ for all $0 \leq i \leq q$ and all $j \neq d$.

Lefschetz properties and minimal free resolutions

Set $S = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring over a field of characteristic zero $\mathbb{K}$. In [5] Eisenbud, Huneke and Ulrich study the minimal free resolution of artinian ideals in S. They prove that for artinian ideal $I \subset S$ generated in degree d where the minimal free resolution of S/I is linear for $p-1$ steps we have that $\mathfrak{m}^d \subset I + (l_p, \ldots, l_n)$ such that $l_p, \ldots, l_n$ are linearly independent forms. More generally, in [5, Corollary 5.2] they prove the following result:

Proposition 2.6. *Suppose $I \subset S$ is a homogeneous ideal, let p be an integer and set $m = \max\{j \mid \beta_{p,j}(S/I) \neq 0\}$. Let $L \subset S$ be any ideal generated by $n-p$ independent linear forms. If $I + L$ contains a power of $\mathfrak{m}$ then $I + L$ contains $\mathfrak{m}^{m-p+1}$, and more generally $\mathfrak{m}^{m-p+s} \subset I + L^s$.*
More specially, if I is an artinian ideal generated in degree d and the minimal free resolution of S/I is linear for $p-1$ steps, then

$$\mathfrak{m}^d \subset I + (l_p, \ldots, l_n)$$

where $l_p, \ldots l_n$ are linearly independent linear forms.

Then they also pose the following conjecture in [5, Conjecture 5.4].

Conjecture 2.7. *Let $I \subset S$ be an artinian ideal generated in a single degree d and the minimal free resolution of S/I is linear for $p-1$ steps. Then,*

$$\mathfrak{m}^d \subset I + (l_p, \ldots, l_n)^2$$

for sufficiently general linear forms $l_p, \ldots, l_n$.

We observe that Proposition 2.6 and Conjecture 2.7 are related to the Lefschetz properties of artinian ideal $I \subset S$ generated in degree d where the minimal free resolution of S/I is linear for $n-1$ steps. In fact we have the following corollary of Proposition 2.6.

Corollary 2.8. *Let $I \subset S$ be an artinian ideal generated in degree d. If the minimal free resolution of S/I is linear for $n-1$ steps then S/I satisfies the WLP.*

We also observe that Conjecture 2.7 for artinian ideals $I \subset S$ generated in degree d where the minimal free resolution of S/I is linear for $n-1$ steps is equivalent to the surjectivity of the multiplication map $\times \ell^2 : (S/I)_{d-2} \longrightarrow (S/I)_d$ for a general linear form ℓ.

In the first paper we prove the following result:

Theorem 2.9. *Let $I \subset S = \mathbb{K}[x_1, \dots, x_n]$ be an artinian monomial ideal generated in degree d. If there exist integers $1 \leq i < j \leq n$ such that for every monomial $m \in (S/I)_d$ we have $x_i^a x_j^b | m$ for some $a, b \geq 0$, then the multiplication map*

$$\times (x_i + x_j)^{a+b} : (S/I)_{k-a-b} \to (S/I)_k$$

has maximal rank for every k.

Using Theorem 2.9 and showing that any artinian monomial ideal with almost linear resolution satisfies the condition in the above theorem we prove the Conjecture 2.7 holds in this case.

Theorem 2.10. *Let $I \subset S = \mathbb{K}[x_1, \dots, x_n]$ be an artinian monomial ideal generated in degree d with almost linear resolution then Conjecture 2.7 holds.*

Note that an artinian ideal $I \subset S$ with linear resolution is a power of maximal ideal and therefore it satisfies the WLP. Also Corollary 2.8 proves artinian ideals with almost linear resolutions satisfy the WLP. In the next result we study artinian monomial ideal $I \subset S$ with less linear steps in its minimal free resolution.

Theorem 2.11. *Let $I \subset S = \mathbb{K}[x_1, \dots, x_n]$ be a monomial ideal generated in degree d and $\mathfrak{m}^{d+1} \subset I$. If the minimal free resolution of S/I is linear for $n-2$ linear steps, then S/I satisfies the WLP.*

Remark 2.12. The assumption $\mathfrak{m}^{d+1} \subset I$ is necessary. Considering Togliatti system $I = (x_1^3, x_2^3, x_3^3, x_1x_2x_3)$ (fails the WLP) where the minimal free resolution is linear for 1 step but note that $\mathfrak{m}^4 \not\subseteq I$.

3 Hilbert functions

The study of Lefschetz properties of artinian algebras is strongly connected to study the *Hilbert function* of the algebra. In this thesis we provide sharp lower bounds for the Hilbert functions of artinian algebra S/I where I is a monomial ideal generated in degree d and fails the WLP.

In this section we give the background to the Hilbert functions and we state some of the important results in this topic. We explain its connections with the

Lefschetz properties and in the last part we explain the connection of some of the results of this thesis with the Hilbert functions.

A famous and important numerical invariant of a graded module over S in the Hilbert function. It encodes important information about the module. For example, dimension, multiplicity of the module. One of the recent research areas is to classify different families of modules with the same Hilbert functions. One of the most important conjectures in this area is Fröberg's conjecture which many researchers have been studying this for a long time but it is still widely open.
Hilbert introduced free resolutions and in fact his motivation was to compute the Hilbert function of a finitely generated graded module using a resolution.

Definition 3.1. For $M = \oplus_{d\in\mathbb{Z}} M_d$ be a finitely generated graded S-module with d-th graded component M_d. Because M is finitely generated, each M_d is finite dimensional vector space, and we define the *Hilbert function* of M to be the generating function $d \mapsto \dim_{\mathbb{K}}(M_d)$ and we denote $H_M(d) := \dim_{\mathbb{K}}(M_d)$. The *Hilbert series* of M is defined by

$$Hilb_M(t) = \sum_{i\in\mathbb{N}} H_M(d)t^i.$$

The next Theorem shows that we can determine the Hilbert function of a module from its free resolution:

Theorem 3.2. *Suppose that $\mathcal{F}$ is a graded free resolution of a finitely generated graded S-module M with each F_i is a finitely generated free module $F_i = \oplus_j S(-j)^{c_{i,j}}$, then*

$$Hilb_M(t) = \frac{\sum_{i\geq 0}\sum_{j\in\mathbb{Z}}(-1)^i c_{i,j}t^j}{(1-t)^n}.$$

If the above resolution of M is minimal we have $\beta_{i,j}(M) = c_{i,j}$.

By the above theorem we write $Hilb_M(t) = \frac{h(t)}{(1-t)^s}$ where $s = n - r$ such that r is the largest power where $\sum_{i\geq 0}\sum_{j\in\mathbb{Z}}(-1)^i c_{i,j}t^j$ is divisible by $(1-t)^r$. The coefficients of the polynomial $h(t)$ is called the *h-vector* of M.

If M is an artinian finitely generated S-module we have that $h(t) = Hilb_M(t)$.

Example 3.3. Let $I = (x_1^3, x_1x_2, x_2^5)$ be an ideal of the polynomial ring $S = \mathbb{K}[x_1, x_2]$. The graded minimal free resolution of S/I is as follows

$$0 \longrightarrow S(-6)\oplus S(-4) \longrightarrow S(-5)\oplus S(-3)\oplus S(-2) \longrightarrow S \longrightarrow S/I.$$

Theorem 3.2 implies that

$$Hilb_{S/I}(t) = \frac{1-t^2-t^3-t^5+t^4+t^6}{(1-t)^2} = 1+2t+2t^2+t^3+t^4.$$

the h-vector of artinian S/I is equal to $(1, 2, 2, 1, 1)$.

In this thesis we work with modules with linear minimal free resolutions. In fact this class of modules are very important. For instance, in general the Hilbert function does not determine the graded Betti numbers of a module but we have the following result:

Proposition 3.4. *If a finitely generated graded S-module M has a d-linear minimal free resolution, then the graded Betti numbers of M are determined by its Hilbert series.*

One may ask about the characterization of series of non-negative integers that are the Hilbert series of standard graded $\mathbb{K}$ algebras. In fact a famous theorem due to Macaulay gives this characterization. To state this result we need the following notations and definitions:

Let h and $i > 0$ be integers, we can uniquely write h as

$$h = \binom{m_i}{i} + \binom{m_{i-1}}{i-1} + \cdots + \binom{m_j}{j}$$

where $m_i > m_{i-1} > \cdots > m_j \geq j \geq 1$. This expansion is called the *i-binomial expansion* of the integer h.

If $h > 0$ has i-binomial expansion as above then we set

$$h^{\langle i \rangle} = \binom{m_i + 1}{i+1} + \binom{m_{i-1} + 1}{i} + \cdots + \binom{m_j + 1}{j+1}$$

we also set $0^{\langle i \rangle} = 0$.

Definition 3.5. A sequence of non-negative integers $\underline{h} = (h_0, h_1, h_2, \dots)$ is called *O-sequence* if $h_0 = 1$ and $h_{i+1} \leq h_i^{\langle i \rangle}$ for all $i > 0$.

Theorem 3.6. *(Macaulay [9]) Let $\underline{h} = (h_0, h_1, h_2, \dots)$ be a sequence of integers, then the followings are equivalent:*

1) *$\underline{h}$ is the Hilbert function of a standard graded $\mathbb{K}$-algebra.*

2) *$\underline{h}$ is an O-sequence.*

Example 3.7. For the sequence of integers $(1, 3, 5, 4)$ we have that $5 \leq 3^{\langle 1 \rangle} = \binom{4}{2} = 6$ and $4 \leq 5^{\langle 2 \rangle} = \binom{4}{3} + \binom{3}{2} = 7$. By the definition it is an O-sequence and by Theorem 3.6, $(1, 3, 5, 4)$ is the Hilbert function of an standard graded $\mathbb{K}$-algebra.

Recall that a sequence of $a_0, \dots, a_r$ is called *unimodal* if there exists $0 \leq s \leq r$ such that $a_0 \leq a_1 \leq \cdots \leq a_s \geq a_{s+1} \geq \cdots \geq a_r$. Studying the Hilbert function of a module is a very interesting area. For instance Stanley in [14] conjectured the following:

Conjecture 3.8. *If S/I is Cohen-Macaulay integral domain, then its h-vector is unimodal.*

The following result in [7] proves that for algebras satisfying the WLP or SLP the Hilbert functions are unimodal.

Proposition 3.9. *Let $\underline{h} = (h_1, h_2, \ldots, h_r)$ be a finite sequence of positive integers. Then $\underline{h}$ is the Hilbert function of a graded artinian algebra with the WLP if and only if the positive part of the first difference is an $\mathcal{O}$-sequence (Definition 3.5) and after that the first difference is non-positive until $\underline{h}$ reaches 0. Furthermore, this is also a necessary and sufficient condition for $\underline{h}$ to be the Hilbert function of a graded artinian algebra with the SLP.*

One of the most important conjectures in commutative algebra is due to Fröberg in [6] in 1985, which conjectures the possible Hilbert functions of a set of general forms.

Conjecture 3.10. *Any ideal of general forms has the maximal rank property. More precisely, fix positive integers $a_1, \ldots, a_s$ for some $s > 1$. Let $F_1, \ldots, F_s \subset S$ be general forms of degrees $a_1, \ldots, a_s$ respectively and let $I = (F_1, \ldots, F_s)$. Then for each $2 \leq i \leq s$, and for all t, the multiplication by F_i on $S/(F_1, \ldots, F_{i-1})$ has maximal rank, from degree $t - a_i$ to degree t. As a result, the Hilbert function of S/I can be computed inductively.*

Fröberg showed the conjecture is true in the case of two variables. Observe that this can also be deduced from Theorem 1.2. In three variables Anick proved the conjecture in [1].

The following result by Migliore, Miró-Roig and Nagel in [13] explains the connection of Fröberg's conjecture and the weak Lefschetz property.

Proposition 3.11. *If Fröberg's conjecture is true for all ideals generated by general forms in n variables, then all ideals generated by general forms in $n+1$ variables have the WLP.*

Using Macaulay duality we may use the inverse system to compute the Hilbert function of homogeneous ideals. In fact if $I \subset S$ is a homogeneous polynomial for all $d \geq 0$ we have that

$$H_{S/I}(d) = \dim_{\mathbb{K}}(I^{-1})_d$$

Remark 3.12. The bounds given in Theorem 1.11 and Theorem 1.12 are the sharp lower bounds for the Hilbert function of S/I in degree d, $H_{S/I}(d)$, for artinian monomial ideal $I \subset S$ generated in degree d failing the WLP.

In the second paper in this thesis we also studied the Hilbert functions of monomial ideal $I \subset S$ generated in degree d where the the multiplication map by higher powers of a general linear form is not surjective. In the following result we provide a lower bound for $H_{S/I}(d)$ for such ideals.

Theorem 3.13. *Let $I \subset S = \mathbb{K}[x_1, \dots, x_n]$ be a monomial ideal generated in degree d. If a general linear form ℓ and integer a when $1 \leq a \leq d$, the multiplication map $\times \ell^a : (S/I)_{d-a} \longrightarrow (S/I)_d$ is not surjective then*

$$H_{S/I}(d) \geq d - a + 2.$$

Remark 3.14. If $I \subset S$ ia an artinian monomial ideal generated in degree d, the bound given in the above theorem is a bound for the Hilbert function of algebras failing WLP (if $a = 1$) or SLP by failing surjectivity in an specific degree.

We note that for artinian ideal $I \subset S$ and $a = 1$ bounds given in Theorem 1.11 and Theorem 1.12 are the better bounds for the Hilbert function of S/I.

4 Action of finite groups

We study the weak Lefschetz property of ideals generated by homogeneous polynomials in $\mathbb{C}[x_1, \dots, x_n]$ invariant by an action of the cyclic group $\mathbb{Z}/d\mathbb{Z}$. We give a complete classification of these ideals satisfying the WLP in terms of the representation of $\mathbb{Z}/d\mathbb{Z}$. As we have mentioned earlier finding examples of artinian algebras failing the WLP is of a great interest. Due to this classification we provide a family of examples failing the WLP.

Let us start with the definitions of *group actions* and *representation* of a group.

A group G is said to *act* on the set X if we have a map $G \times X \longrightarrow X$ defined by $(g, x) \mapsto gx$ satisfying $(gh)x = g(hx)$ and $ex = x$ for all $g, h \in G$ and e is the identity element of G.

If $X = V$ is a vector space over a field $\mathbb{K}$, we say that G acts *linearly* on V if in addition we have $g(u + v) = gu + gv$ and $g(rv) = r(gv)$ for every $u, v \in V$, $r \in \mathbb{K}$ and $g \in G$.

A *representation* α of G in V defines a linear action of G on V , by $gv = \alpha(g)v$ and every such action arises from a representation in this way.

Let integer $d \geq 2$ and $\xi = e^{2\pi i/d}$ be a primitive d-th root of unity. Define the action of the cyclic group $\mathbb{Z}/d\mathbb{Z}$ on the polynomial ring $S = \mathbb{C}[x_1, \dots, x_n]$ defined by $[x_1, \dots, x_n] \mapsto [\xi^{a_1} x_1, \dots, \xi^{a_n} x_n]$, for integers $a_1, \dots, a_n$. Therefore the representation of $\mathbb{Z}/d\mathbb{Z}$ on S_1 is given by the matrix

$$M_{a_1,\dots,a_n} := \begin{pmatrix} \xi^{a_1} & 0 & \cdots & 0 \\ 0 & \xi^{a_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \xi^{a_n} \end{pmatrix}$$

In [11] Mezzetti and Miró-Roig studied the ideals in $\mathbb{C}[x_1, x_2, x_3]$ generated by homogeneous polynomials invariant under an action of the cyclic group $\mathbb{Z}/d\mathbb{Z}$ represented by M_{a_1,a_2,a_3}. They show that these ideals can be generated by monomials. They prove that in the case where a_1, a_2, a_3 are distinct and $\gcd(a_1, a_2, a_3, d) = 1$ these ideals fail the WLP. In fact in [11, Proposition 3.4] they prove the following:

Theorem 4.1. *For $d \geq 3$ and let $I \subset \mathbb{C}[x_1, x_2, x_3]$ be the ideal generated by all monomials of degree d invariant under the action of M_{a_1,a_2,a_3} where a_1, a_2, a_3 are distinct and $\gcd(a_1, a_2, a_3, d) = 1$. Then I fails the WLP.*

Remark 4.2. In Theorem 4.1, Mezzetti and Miró-Roig prove that these ideals are minimal monomial Togliatti systems. In fact they prove that the WLP of these ideals fail by failing injectivity in degree $d-1$.

In the second paper we studied $I \subset S = \mathbb{C}[x_1, x_2, x_3]$ generated by forms of degree d invariant under M_{a_1,a_2,a_3} in the case where $\gcd(a_1, a_2, a_3, d) > 1$. We provide a formula to count the number of generators of such ideals, in fact we have:

Proposition 4.3. *For integers a_1, a_2, a_3 and $d \geq 2$, the number of monomials of degree d in $S = \mathbb{C}[x_1, \dots, x_n]$ fixed by the action of M_{a_1,a_2,a_3} is*

$$1+\frac{\gcd(a_2-a_1, a_3-a_1, d)\cdot d + \gcd(a_2-a_1, d) + \gcd(a_3-a_1, d) + \gcd(a_3-a_2, d)}{2}.$$

The formula shows that in the case where $\gcd(a_1, a_2, a_3, d) > 1$ the WLP of such algebras is an assertion of surjectivity in degree $d-1$. In fact we prove that in this case surjectivity in degree $d-1$ and therefore the WLP fails.

We also consider artinian ideals generated by forms of degree d fixed be the action $M_{a_1,\dots,a_n}$ in the polynomial ring $S = \mathbb{C}[x_1, \dots, x_n]$ for $n \geq 3$. We give the following classification of such ideals.

Theorem 4.4. *For integers $d \geq 2$, $n \geq 3$ and $0 \leq a_1, \dots, a_n \leq d-1$, let $M_{a_1,\dots,a_n}$ be a representation of cyclic group $\mathbb{Z}/d\mathbb{Z}$ and $I \subset S = \mathbb{K}[x_1, \dots, x_n]$ be the ideal generated by all forms of degree d fixed by the action of $M_{a_1,\dots,a_n}$. Then, I satisfies the WLP if and only if at least $n-1$ of the integers a_i are equal.*

As a consequence of the above theorem we have the following result which generalizes Theorem 4.1.

Corollary 4.5. *Let $d \geq 2$ and let $I \subset \mathbb{C}[x_1, x_2, x_3, x_4]$ be the ideal generated by all forms of degree d invariant under the action of M_{a_1,a_2,a_3,a_4} where at most two integers among the $a_1, \dots, a_4$ are equal and $\gcd(a_1, a_2, a_3, a_4, d) = 1$. Then I defines a monomial Togliatti system.*

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