Two-point functions of SU(2)-subsector and length-two operators in dCFT

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\textbf{A B S T R A C T}

We consider a particular set of two-point functions in the setting of \( \mathcal{N} = 4 \) SYM with a defect, dual to the fuzzy-funnel solution for the probe D5-D3-brane system. The two-point functions in focus involve a single trace operator in the SU(2)-subsector of arbitrary length and a length-two operator built out of any scalars. By interpreting the contractions as a spin-chain operator, simple expressions were found for the leading contribution to the two-point functions, mapping them to earlier known formulas for the one-point functions in this setting.

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1. Introduction

Integrable structures in \( \mathcal{N} = 4 \) SYM have been explored extensively since they were first noted in [1] and have provided a useful tool for both deeper field theoretic understanding and numerous tests of the AdS/CFT correspondence. For a pedagogical overview of the first decade, see [2]. Among other directions, the work has lead on to look for, and to employ, surviving integrability in similar theories, departing in different ways from \( \mathcal{N} = 4 \) SYM. One particular branch of this focus is the study of various CFTs with defects (dCFTs).

The setting for these notes is \( \mathcal{N} = 4 \) SYM with a codimension-one defect residing at the coordinate value \( z = 0 \). The theory is the field theory dual of the probe D5-D3-brane system in \( AdS_5 \times S^5 \), in which the probe-D5-brane has a three-dimensional intersection (the defect) with a stack of \( N \) D3-branes. We will study the dual of the so called fuzzy-funnel solution [3–6], in which a background gauge field has \( k \) units of flux through an \( S^2 \)-part of the D5-brane geometry, meaning that \( k \) D3-branes dissolve into the D5-brane. These parameters appear on the field theory side as the rank \( N \) of the gauge group which is broken down to \( N - k \) by the defect.

The dCFT action is built out of the regular \( \mathcal{N} = 4 \) SYM field content plus additional fields constrained to the three dimensional defect. These additional fields interplay both within themselves and with the bulk\textsuperscript{1} fields. However, only the six scalars from \( \mathcal{N} = 4 \) SYM will play a role within these notes.

The defect breaks the 4D conformal symmetry down to those transformations that leave the boundary intact (i.e. that map \( z = 0 \) onto itself). Its presence thus changes many of the general statements about CFTs, such as allowing for non-vanishing one-point functions and two-point functions between operators of different conformal dimensions. These new features were first studied in [7,8] and within the described setting, they have been the topic of a series of recent works. Tree-level one-point functions in the SU(2)- and SU(3)-subsectors where considered in [9–11] while bulk propagators and loop corrections to the one-point functions where worked out in [12–14]. Two-point functions were very recently addressed in [15] and earlier in [16].\textsuperscript{2}

The underlaying idea of all this business is to interpret single-trace operators as states in a spin-chain and employ the Bethe ansatz from within this context. The one-point functions where in this spirit found to be expressible in a compact determinant formula, making use of a special spin-chain state, called the Matrix Product State (MPS), and Gaudin norm for Bethe states. The end result for the tree-level one-point functions of operators

\[ O_L \sim \text{Tr} \left( \hat{Z} \ldots \hat{XZ} \ldots \hat{XZ} \ldots \right) \]

in the SU(2)-subsector was

\textsuperscript{1} Meaning the region \( z > 0 \).

\textsuperscript{2} Wilson loops in these settings with a defect have also attracted attention, see e.g. [17–19].

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\[ \langle \mathcal{O}_1 \rangle_{\text{tree}} = \frac{2^{L-1}}{2^L} C_2 (u) \]
\[ \times \sum_{j=1}^{L-1} \prod_{j=1}^{L-1} \left[ \frac{u_j^2 (u_j^2 + \frac{k^4}{4})}{u_j^2 + (j - \frac{1}{2})^2} \right] \left[ u_j^2 + (j + \frac{1}{2})^2 \right] \]

under the condition that both the length \( L \) and the number of excitations \( M \) are even and that the set of \( M \) Bethe rapidities has the special form \( u = (u_1, -u_1, u_2, -u_2, \ldots) \). The parameter \( k \) can be any positive integer and

\[ C_2 (u) = 2 \left( \frac{2\pi^2}{\lambda} \right)^{L/2} \frac{1}{L} \prod_{j=1}^{L} \frac{u_j^2 + 1}{u_j^2 + \frac{1}{4}} \text{det} G^+ - \text{det} G^- \]

where \( G^\pm \) are \( \frac{M}{2} \times \frac{M}{2} \) matrices with matrix elements

\[ G^\pm_{jk} = \left( \frac{L}{u_j^2 + \frac{1}{4}} - \sum_n K^+_j n \right) \delta_{jk} + K^\pm_{jk}, \]

within which, in turn,

\[ K^\pm_{jk} = \frac{2}{1 + (u_j - u_k)^2} \pm \frac{2}{1 + (u_j + u_k)^2}. \]

The expression for \( C_2 \) was obtained from the spin-chain overlap

\[ C_2 = \left( \frac{8\pi^2}{\lambda} \right)^{L/2} \frac{1}{\sqrt{L}} \langle \text{MPS} | \Psi \rangle \]

which is the form we will mostly refer to here. \( |\Psi\rangle \) is the spin-chain Bethe state corresponding to the operator \( \mathcal{O}_1 \); the MPS will be defined below in equation \((2)\).

1.1. The goal of the present notes

These notes consider the leading contribution, in the ’t Hooft coupling \( \lambda \), to the specific two-point function \( \langle \mathcal{O}_1 \mathcal{O}_2 \rangle_{\text{contr.}} \), where

- both \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) are single-trace scalar operators of length \( L \) and \( 2 \), respectively, and
- \( \mathcal{O}_1 \) is restricted to the SU(2)-subsector while \( \mathcal{O}_2 \) can be built out of any pair of scalars.

We do this by interpreting the contraction as a spin-chain operator \( Q \) acting on the Bethe state corresponding to \( \mathcal{O}_1 \), whence re-expressing the two-point function in terms of the previously known one-point functions.

2. The particular two-point functions

We define the complex scalar fields as

\[ Z = \phi_1 + i\phi_4, \quad X = \phi_2 + i\phi_3, \quad W = \phi_3 + i\phi_5, \]
\[ \bar{Z} = \phi_1 - i\phi_4, \quad \bar{X} = \phi_2 - i\phi_3, \quad \bar{W} = \phi_3 - i\phi_5, \]

which in the dual fuzzy-funnel solution each has the non-zero classical expectation value

\[ \phi_i^1 = \frac{1}{2} t_i \otimes 0_{(N-k)}, \quad i = 1, 2, 3; \quad \phi_4^1 = 0, \quad j = 4, 5, 6, \]

where \( (t_1, t_2, t_3) \) forms a \( k \times k \) unitary representation of SU(2) and the \( 0_{(N-k)} \) pads the rest of the matrix to the full dimensions \( N \times N \).

For definiteness, we choose \( Z \sim |\uparrow\rangle \) and \( X \sim |\downarrow\rangle \) as the SU(2)-subsector.

We now set out to calculate

\[ \langle \mathcal{O}_1 \mathcal{O}_2 Y_1 Y_2 \rangle_{\text{contr.}} = \sum_{\ell=1}^{L} \langle \Psi^{1, \ldots, \ell} | \text{Tr} (X_1^{\ell} \cdots X_{\ell-1}^{\ell} | Y_1 Y_2^{\ell} \rangle \text{Tr} (Y_1 Y_2^{\ell}) \]
\[ + (Y_1 \leftrightarrow Y_2), \quad i_\ell = \uparrow, \downarrow \]

where \( X_1 = Z, X_j = X, Y_1, Y_2 \) can be any complex scalar and the coefficients \( \Psi^{1, \ldots, \ell} \) of \( \mathcal{O}_2 \) are chosen such that they map to a Bethe state \(|\Psi\rangle\) in the spin-chain picture.

We will express it by help of the MPS, which is the following state in the spin-chain Hilbert space:

\[ \langle \text{MPS} \rangle = \text{Tr} \left[ \left( \uparrow |t_1 + (\downarrow |t_2 \right)^{\otimes L} \right], \]

where the trace is over the resulting product of \( t \)'s.

2.1. Scalar propagators

The defect mixes the scalar propagator in both color and flavor indices, explained in detail in [13]. However, since the contracted fields are multiplied by classical fields from both sides we will only need the upper (\( k \times k \))-block. The propagator diagonalization involves a decomposition of these components in terms of fuzzy spherical harmonics \( \hat{Y}_\ell^m \):

\[ [\phi_1^{s_2}] = \sum_{\ell=1}^{k} \sum_{m=-\ell}^{\ell} \phi_{\ell,m} [\hat{Y}_\ell^m]_{s_2}, \quad s_{1,2} = 1, \ldots, k. \]

Translating back to the \( s \)-indices, the relevant propagators for \( l, j = 1, 2, 3 \) read

\[ (i|\phi_i(x)|^{s_1} \langle \phi_j(y)|^{s_2} \rangle_{\ell_1, \ell_2} = \delta_{ij} \sum_{\ell, m} \hat{Y}_\ell^m [\hat{Y}_\ell^m]_{s_1} \hat{Y}_\ell^m [\hat{Y}_\ell^m]_{s_2} K_\ell^m(x, y) \]
\[ - i\epsilon_{jkl} \sum_{\ell, m, m'} \hat{Y}_\ell^m [\hat{Y}_\ell^m]_{s_1} \hat{Y}_\ell^m [\hat{Y}_\ell^m]_{s_2} K_\ell^{m'}(x, y) \]

where \( K_\ell^{m'}(x, y) \) is in the \((2\ell + 1)\)-dimensional representation. The remaining scalars \( l, j = 4, 5, 6 \) have the diagonal propagator

\[ (i|\phi_i^{s_1} |^{s_2} \langle \phi_j^{s_2} |^{s_1} \rangle_{\ell_1, \ell_2} = \delta_{ij} \sum_{m=-\ell}^{\ell} \hat{Y}_\ell^m [\hat{Y}_\ell^m]_{s_1} \hat{Y}_\ell^m [\hat{Y}_\ell^m]_{s_2} K_\ell^{m^2}(x, y) \]

The spacetime dependent factors are

\[ K_\ell^m(x, y) = \frac{\ell + 1}{2 \ell + 1} [K_\ell^{m^2}(x, y) + \ell \ell (\ell + 1) K_\ell^{m^2}(x, y)]. \]

\[ K_\ell^{m^2}(x, y) = \frac{1}{2 \ell + 1} \left[ (m^2 + \ell + 1) K_\ell^{m^2}(x, y) \right]. \]

\[ K_\ell^{m^2}(x, y) \]

is related to the scalar propagator in AdS and reads

\[ K_\ell^{m^2}(x, y) = \frac{\ell^2}{2 \ell + 1} \int \frac{d^2 k}{(2\pi)^2} \left( \sum_{\ell=1}^{k} \sum_{m=-\ell}^{\ell} \left| \hat{Y}_\ell^m \right|^2 \right) \left( \sum_{\ell=1}^{k} \sum_{m=-\ell}^{\ell} \left| \hat{Y}_\ell^m \right|^2 \right) \]

in which \( I \) and \( K \) are modified Bessel functions with \( x_3^2 \) (\( x_3^2 \)) the smaller (larger) of \( x_3 \) and \( y_3 \), and lastly where \( \nu = \sqrt{m^2 + \frac{\ell}{2}}. \)

We will from now on suppress all spacetime dependence.

\[ \text{See appendices in [13,20]. We use the normalization of [13].} \]
3. The contraction as a spin-chain operator

With the expressions of the propagators, we can now view the contraction in equation (1) as a \((k \times k)\)-matrix

\[
[T_{X_0Y_1 Y_2}]^{s_1 s_2} = \left( [X_{i1} s_1 s_2 [Y_1 s_1 s_2 [Y_2 s_1 s_2]], [Y_2 s_1 s_2 s_1], [Y_1 s_1 s_2 s_1] \right],
\]

replacing the field at site \(i\) in the first trace while absorbing the second trace completely.

It turns out that this matrix always is proportional to either \(t_1\), \(t_2\) or \(t_3\). To see this, first use that the fuzzy spherical harmonics are tensor operators, such that

\[
\sum_m \hat{Y}^m \langle t_K^{2l+1} \rangle_{l=m+1, l-m+1} = \hat{t}_K^{(k)}, \quad \hat{Y}^m \rangle = m \hat{Y}^m \langle m'.
\]

Then use the orthogonality of the fuzzy spherical harmonics\(^4\) in the trace by decomposing the \(t\) in \(Y^{cl}_2\) as

\[
t_j = d_j \hat{Y}_j^{-1} + (-1)j \hat{Y}_j^1, \quad j = 1, 2,
\]

\[
t_3 = \sqrt{2}d_j \hat{Y}_j^0, \quad d_j = 1 + \frac{1}{2}(-1)^{k+1} k(k^2 - 1)/6.
\]

To get the total contribution to \(Y^{cl}_j\) as a function of \(S\) and \(d_j\) with coefficients \(d_j\) only for indices 1, 2 and 3 (4, 5, and 6). Taking into account both the sums in the two-point function (1), we can then write the contractions in the spin-chain picture as

\[
Q_{Y_1 Y_2}(\Psi) = \sum_{i=1}^L \Psi \otimes \cdots \otimes \hat{Q}_{Y_1 Y_2}^{(j)} \otimes \cdots \otimes \Psi |\Psi\rangle,
\]

i.e. a linear combination of the spin-chain operators \(\{\Psi \otimes \cdots \otimes \hat{Q}_{Y_1 Y_2}^{(j)} \otimes \cdots \otimes \Psi |\Psi\rangle\}.\)

The result arranges itself in the two cases \(Y^{cl}_1 = Y^{cl}_2\) and \(Y^{cl}_1 \neq Y^{cl}_2\), for which\(^5\)

\[
Q^{(j)} = \begin{pmatrix} c^+ & 0 \\ 0 & c^+ \end{pmatrix}, \quad Q^{(j)}_{\neq} = \begin{pmatrix} 0 & c^+ \\ c^- & 0 \end{pmatrix},
\]

and the various coefficients \(c\) implicitly depend on \(Y_1, Y_2\). They are listed in Appendix A.

- Case \(Y^{cl}_1 = Y^{cl}_2\). The action of \(Q_{\neq}\) is trivial on any Bethe state. Still denoting the total number of spin-down excitations as \(M\), we immediately get

\[
Q_{\neq}(\Psi) = (c^+ (L - M) + c^+ M) |\Psi\rangle.
\]

Combining this with the one-point function formula implies

\[
\langle O_L O_{-1} \rangle_{\text{contr.}} = (c^+(L - M) + c^+ M) |O_L\rangle_{\text{tree}}.
\]

As an example, the Konishi operator has the two-point function \(2Km^2 = 0|O_L\rangle_{\text{tree}}\) with any SU(2)-subsector operator.

- Case \(Y^{cl}_1 \neq Y^{cl}_2\). In this case we have the spin-flipping operator

\[
Q_{\neq} = c^+ S^+ + c^- S^-.
\]

Its action simplifies significantly when acting on a Bethe state. First of all, Bethe states with non-zero momenta are highest weight states implying that \(S^+ |\Psi\rangle = 0\). Secondly, we have that

\[
S^- |\Psi_M\rangle = \lim_{p_{M+1} \rightarrow 0} |\Psi_{M+1}\rangle.
\]

meaning that acting on a Bethe state with the lowering operator creates a new Bethe state with one more excitation but with the corresponding momentum \(p_{M+1} = 0\). All other momenta are the same. These states are called (Bethe) descendants.

It was shown in [9] that only states with \(L\) and \(M\) both even can have a non-zero overlap with the MPS. Furthermore, by studying the action of \(Q_3\), the third conserved charge in the integrable hierarchy, it was proven that only unpaired\(^7\) states yield finite overlaps. This is true since \(Q_3 |\Psi_M\rangle = 0\) and since \(Q_3\) is non-zero on states that are not invariant under parity.

That \(Q_{3}\) alters the number of excitations now makes it possible to have non-zero overlaps with states with odd \(M\). However, since

\[
[L, S^+] = 0
\]

the requirement of an unpaired state is still imposed. Hence, the only possible way for the overlap

\[
\langle MPS | Q_{\neq} | \Psi_M \rangle
\]

to be non-vanishing is that \(M\) is odd and that the Bethe state is a descendant.

The general expression for such a state is

\[
|\Psi_M = M + n\rangle = (S^-)^n |\Psi_M\rangle, \quad n \text{ odd}.
\]

The two-point function (1) then follows from the commutation relation of the spin-operators, the action of \((S^-)^n\) on the MPS and the norm of the descendants [15,21]:

\[
\langle MPS | (S^-)^n | \Psi_M \rangle = \frac{n! \left( \frac{L}{2} - M \right)!}{\left( \frac{L}{2} \right)! \left( \frac{L}{2} - 2M - n \right)!} |\Psi_M\rangle.
\]

\[
\langle MPS | (S^-)^n | \Psi_M \rangle = \frac{n! (L - 2M)!}{(L - 2M - n)!} |\Psi_M\rangle.
\]

We find

\[
\langle O_L O_{M+n} O_{\neq} \rangle_{\text{contr.}} =
\]

\[
\left( c^+ n (L - 2M - n + 1) c^+_L M_n + c^- c^- c^+_L M_n \right) |O_L M\rangle_{\text{tree}}
\]

with

\[
{	ext{c}^+_L M_n = \frac{(n + 1)! (\frac{L}{2} - M)!}{(n + 1)! \left( \frac{L}{2} - 2M - n + 1 \right)!} \sqrt{\frac{(L - 2M - n)!}{n! (L - 2M)!}.}
\]

\(^4\) \(T \hat{Y}^m \langle \hat{Y}^m \rangle = \delta_{r c} \delta_{m m'}\).

\(^5\) This does not explicitly cover the case of \(T \propto t_3\). However, this case eventually yields zero and will be addressed below.

\(^6\) We will denote both the \(\text{dDFT}\) operator and its spin-chain correspondent with subscripts \(\neq\) and \(=\) for these two cases.

\(^7\) "Unpaired" refers to states which are invariant under parity transformation, implying momenta of the form \(\{p_1, -p_1, \cdots\}\).
3.2. Remark on $T \propto t_3$

When one of $Y_1$ or $Y_2$ is either $W$ or $W$, $T$ is proportional to $t_3$ and the corresponding $Q_{i,j}^{(i)}$ is no longer a proper spin-chain operator. Insisting on a spin-chain interpretation would describe it as a flip of site $l+1$ followed by a removal of the site $l$, thus shrinking the length $L$ by one. $Q_{i,j}^{(i)}$ always appears preceded by a projection $\Pi_{i,j}$ on either spin-up or spin-down, depending on the $Y$ which does not involve $W$. It is straightforward to show by explicit calculation that

$$\langle MPS|_{L-1}\rangle \sum_{l=1}^{L} \Pi_{i,j}^{(i)} \Pi_{i,j}^{(j)} \otimes \cdots \otimes \Pi_{i,j} = 0$$

for any basis vector $|\psi_L\rangle$ of length $L$.

4. Conclusion

We have studied the $\mathcal{N} = 4$ SYM theory with a defect, dual to the probe D5-D3-brane system. Within this theory, the two-point function between a length $L$ operator $O_L$ in the SU(2)-subsector and any operator $O_{Y_1}O_{Y_2}$ of two scalars can, in the leading order, be written as a spin-chain operator insertion in the scalar product between a matrix product state $\langle MPS\rangle$ and the Bethe state $|\Psi\rangle$ corresponding to the operator $O_L$.

$$\langle O_L O_{Y_1} O_{Y_2} |_{\text{contr.}} \rangle \propto \langle MPS|O_{Y_1}O_{Y_2}|\Psi\rangle.$$

The operation of $Q$ depends on the two fields $Y_1, Y_2$ but is simple for any choice of scalar fields:

- For $Y_1^{cl} = Y_2^{cl}$ we get

$$\langle O_L O_{Y_1} O_{Y_2} \rangle = \langle (c^+ L + c^+(L-M)) (O_L)_{\text{tree}} \rangle$$

where both $L$ and the number of excitations $M$ need to be even and the Bethe state needs to be unpaired.

- For $Y_1^{cl} \neq Y_2^{cl}$, the two-point function is zero for any $O_L$ mapping to a highest weight Bethe state. For operators $O_{L,M+n}$ mapping to (Bethe) descendants, however, the two-point function is non-vanishing, under the condition that $n$ is odd and that the corresponding Bethe state descends from an unpaired state $|\Psi_{L,M}\rangle$. The result is

$$\langle O_{L,M+n} O_{Y_1} O_{Y_2} |_{\text{contr.}} \rangle = \langle c^+ n (L - 2M - n + 1) c^+_{L,M,n} + c^- c^+_{L,M,n} \rangle (O_L)_{\text{tree}}$$

where the combinatorial factors $c^\pm_{L,M,n}$ can be found in equation (3).

The coefficients $c$ with various indices depend on $Y_1, Y_2$ and are all spacetime-dependent since they contain expressions of the propagator. See Appendix A below for the full list of coefficients. These results hold for any $k$.

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Appendix A. List of coefficients

Here follows the list of coefficients for the considered two-point functions, written in the form $Q_{Y_1 Y_2} : (c^+ c^+)$.

$$Q_{zz} : \begin{pmatrix} \frac{2}{3} (2K m^2 - 3K m^2 - 9 + K m^2 - 6) & 0 \\ 0 & -\frac{2}{3} (K m^2 - 0 - K m^2 - 6) \end{pmatrix}$$

$$Q_{zz} : \begin{pmatrix} \frac{2}{3} (2K m^2 + 3K m^2 - 6) & 0 \\ 0 & -\frac{2}{3} (K m^2 - 0 - K m^2 - 6) \end{pmatrix}$$

$$Q_{zx} : \begin{pmatrix} \frac{2}{3} (2K m^2 - 3K m^2 - 9 + K m^2 - 6) & 0 \\ 0 & -\frac{2}{3} (K m^2 - 0 - K m^2 - 6) \end{pmatrix}$$

$$Q_{zx} : \begin{pmatrix} \frac{2}{3} (2K m^2 + 3K m^2 - 6) & 0 \\ 0 & -\frac{2}{3} (K m^2 - 0 - K m^2 - 6) \end{pmatrix}$$

$$Q_{xx} : \begin{pmatrix} \frac{2}{3} (2K m^2 - 3K m^2 - 9 + K m^2 - 6) & 0 \\ 0 & -\frac{2}{3} (K m^2 - 0 - K m^2 - 6) \end{pmatrix}$$

$$Q_{xx} : \begin{pmatrix} \frac{2}{3} (2K m^2 + 3K m^2 - 6) & 0 \\ 0 & -\frac{2}{3} (K m^2 - 0 - K m^2 - 6) \end{pmatrix}$$

$$Q_{WW} = Q_{WW} : \begin{pmatrix} \frac{2}{3} (K m^2 - 0 - K m^2 - 6) & 0 \\ 0 & -\frac{2}{3} (K m^2 - 0 - K m^2 - 6) \end{pmatrix}$$

References


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