Validity of Decentralized Control Configurations with Respect to Model Uncertainty

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Abstract

This paper proposes a method to indicate the minimum uncertainty on nominal system gains for which a decentralized control configuration is invalid. The outset for the method is that a decentralized control configuration is selected based on the nominal system model using the relative interaction array (RIA) methodology. Thereafter for the uncertain system, the minimum relative uncertainty parameter ($\alpha_{\text{min}}$) which invalidates the current control configuration decision is determined. Hence, a powerful tool is provided to indicate to what extent a certain configuration of a decentralized controller is still recommended in the presence of the uncertainty. Examples to illustrate the proposed method for a $2 \times 2$ distillation column and a $5 \times 5$ stock-preparation process are given.

Keywords: Decentralized Control Configuration, Model Uncertainty, Relative Interaction Array (RIA).

1. Introduction

Interaction Measures (IMs) are widely used for Control Configuration Selection (CCS) since the 1960s, when the celebrated relative gain array (RGA) was proposed by Bristol \cite{1}. Ever since, decentralized controllers are often the first hand choice for control of industrial processes, despite the progress that has been made in the design and deployment of multivariable control strategies \cite{2}. This is especially true for the chemical industry sector, due to the interaction nature processes such as distillation columns \cite{3} and wastewater treatment plants \cite{4} are often subject of investigation. Decentralized controllers also called Single-Input-Single-Output (SISO) controllers, are usually appreciated for the simplicity in design, implementation and tuning, and are commonly apprehended as being robustness against loop failures.

The selection of sensor-actuator pairings for closing SISO loops targeting low loop interaction is known as the input-output pairing problem. It is well known that poor pairings may lead to significant performance degradation \cite{5} or even system instability \cite{6}, which motivates the need for design tools such as the Interaction Measures (IMs). The existing IMs include the relative gain array (RGA) \cite{1} and its variants, as well as many other methods.\cite{7,8,9} Aided by pairing rules, the RGA has been used extensively for CCS since it

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characterizes the closed-loop interaction and integrity properties in square multivariable systems using only
the steady-state gain.

Most of the available IMs require the use of estimated linear process models, which are inherently un-
certain due to the presence of e.g. nonlinearities, sensor noise, process disturbances or time varying-aspects
such as actuator wear. The inherent model uncertainty motivates the study of robust control strategies,
which for the input-output problem led to the study of RGA bounds using different approaches such as
statistical, geometrical, optimization-based or based on the traditional robust control framework. In these
previous approaches, the configuration is first decided based on the RGA values derived from the nominal
 gains. Then, the validity of the selected configuration for different gain variations is judged based on generated RGA bounds. However, for new gain variations, the RGA bounds have to be recalculated and the configuration revalidated. As a result, the user has to determine the tolerable
uncertainty in a tedious iteration.

The main contribution and novelty of this paper is to address this challenge and to propose a method
that directly reveals the minimum gain perturbation of nominal system gains such that the input-output
pairing decision becomes invalid. Essentially, the method determines the minimum relative uncertainty
parameter of the perturbation that invalidates the pairing decision. Thereby, the practitioner is provided
with a guaranteed validity region of the input-output pairing that is based on the nominal system model.

The proposed method uses the RGA variant known as relative interaction array (RIA), since the RIA
provides an overall measure of loop interaction, which is closer to linear. Representing the RIA of the
perturbed system as an upper linear fractional transform (LFT) and using the structured singular value
(µ) facilitate the calculation of a relative uncertainty parameter αmin that characterizes the uncertainty
invalidating the configuration obtained based on the nominal model.

The article is organized as follows. The preliminaries on the RIA are given in Section 2. Then the
notation and the problem description are introduced in Section 3. The proposed method is derived in
section Section 4 where also step-by-step instructions are given. Two illustrative examples from chemical
industry are used in Section 5 to evaluate and discuss the method. Finally, conclusions and future outlook
are given in Section 6. The appendix includes a new derivation for the calculation of the RIA for perturbed
systems using LFT which is necessary for the proposed method.

2. RIA Preliminaries

It is well known that in the RGA approach, the interaction has been evaluated by the distance of its
elements from one, but the closeness of RGA elements to one is rather qualitative. Also, ambiguity of
input-output pairing based on RGA may arise when several alternatives satisfy the RGA pairing rules. This
problem has been addressed since the 90’s and a number of solutions to identify a particular pairing
which results in minimum interaction have been introduced. Zhu in 1996 introduced RIA as an IM which provides better measure of interaction in comparison with RGA. Zhu showed that RIA
provides information about closed-loop stability, integrity and robustness as well as suggests the optimal input-output pairing by introducing the overall interaction measure. Now, an $n \times n$ linear multivariable system $G(s)$ with steady-state gain matrix $G(0)$ is considered. Using the loop decomposition approach [21] a decentralized control system is decomposed into equivalent SISO loops with explicit loop interaction effect. By assuming that the closed-loop system has integrity against loop failure and that each SISO controller fulfills the perfect control condition, the $(ij)$-element of the RIA matrix $(\Phi)$ can be calculated as [18] (for the sake of simplicity, $(0)$ of the steady-state is here dropped)

$$
\phi_{ij} = \frac{1}{G_{ij}[G^{-1}]_{ji}} - 1 = \frac{1}{\lambda_{ij}} - 1
$$

(1)

where $\lambda_{ij}$ is the $(ij)$-element of the RGA matrix $(\Lambda)$. The RGA matrix is calculated using the nominal system steady-state gain matrix $G$ as

$$
\Lambda = G \times (G^{-1})^T
$$

(2)

with $\times$ and $T$ denote element-by-element multiplication and matrix transpose, respectively.

Finally, the RIA pairing rules state that the input-output pairing is selected such that

(a) Niederlinski Index $(NI) > 0$, where $NI = \frac{\det(G)}{\prod_{i=1}^{n}(G_{ii})}$ (stability rule);

(b) $\phi_{ij} > -1$ (integrity rule);

(c) $\phi_{ij}$ close to $-1$ is avoided (robustness rule);

(d) $\min \sum |\phi_{ij}^r|$ (overall interaction rule);

where $\phi_{ij}^r$ represents the pairing elements corresponding to the $r^{th}$ possible configuration that satisfies the first three rules [18]. The configuration that satisfies the above mentioned pairing rules is henceforth denoted as the optimal configuration. It is worth mentioning that the proposed method is based on pairing rule (d) as will be shown in Section 4.

3. Notation and problem definition

In this article, the perturbed $(ij)$-element of the steady-state gain matrix $G_p$ of the $n \times n$ perturbed system $G_p(s)$ is written as

$$
[G_p]_{ij} = G_{ij} + \alpha \cdot \delta_{ij} \cdot |G_{ij}|, \ |\delta_{ij}| \leq 1, \ \alpha \in \mathbb{R}^+
$$

(3)

where $(i, j) \in [1, n]$ and $G_{ij}$ is the $(ij)$-element of the steady-state gain matrix $G$ of the nominal model of the system $G(s)$.

The problem can now be defined as follows: Determining the maximum relative uncertainty parameter $\alpha$ such that the optimal control configuration for any perturbed plant $G_p$ is the same as for the nominal $G$. 

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While this definition clearly sets the validity region for a control configuration, the solution to this problem is complex. Instead, it is more feasible to determine the minimum value of $\alpha$, denoted $\alpha_{\text{min}}$, for which there exists a perturbed plant $G_p$ such that the optimal configuration can either not be decided or is different than the one for the nominal $G$. As a result, $\alpha_{\text{min}}$ provides a characterization of the validity region, but is not included in it. The uncertainty representation of (3) is adopted in order to give a more realistic solution to the problem since $\alpha$ represents a relative magnitude to absolute $G_{ij}$.

However, in order to facilitate the derivations, $G_p$ is to be written in an additive norm-bounded description as

$$G_p = G + W \Delta V : \bar{\sigma}(\Delta) \leq 1$$

where the weighting matrices $W$ and $V = \alpha \cdot \dot{V}$ are to be selected in such a way that each perturbed $(ij)$-element in $G_p$ matrix is as (3). Moreover, it is essential that $\Delta$ is formulated as a structured uncertainty block where the diagonal elements are the $\delta'$s of the perturbed elements. Hence, for a system with a single perturbed element, $\Delta = \delta$. In this context, the unknown $\alpha$ will be accompanied with $\Delta$ where it can be understood as a scaling factor to the normalized $\Delta$, $\bar{\sigma}(\Delta) \leq 1$.

Hence, based on the RIA pairing rules, the problem can now be interpreted in the following way. Obtaining the minimum $\alpha$ by equalizing the sum of the RIA elements, corresponding to the configuration given by the nominal model $G$, with the sum of the RIA elements, corresponding to any other configurations. Therefore, there will be as many $\alpha'$s as there are alternative configurations, and obviously, $\alpha_{\text{min}}$ is the minimum $\alpha$ in the set of $\alpha'$s. As a consequence, the calculation of $\alpha_{\text{min}}$ requires writing the RIA elements based on $G_p$ in terms of a linear fractional transform (LFT) form and to apply the structured singular value ($\mu$) framework.

To conclude, the decentralized controller based on the optimal configuration for $G$ is not recommended for uncertain system $G_p$ with a perturbation level resultant from a value equals or exceeds $\alpha_{\text{min}}$ when $\bar{\sigma}(\Delta) = 1$.

4. Method

First, the concept of the proposed method will be introduced using a $2 \times 2$ system. Thereafter the method is stated for general $n \times n$ systems and the steps of the method will be summarized at the end of the section.

4.1. Determining $\alpha_{\text{min}}$ for a $2 \times 2$ system

Consider a steady-state gain matrix $G_p$ of $2 \times 2$ system is written as in (4). Obviously, there are two potential decentralized configurations which are either the diagonal pairing $(y_1 - u_1, y_2 - u_2)$ or the off-diagonal pairing $(y_1 - u_2, y_2 - u_1)$. It is now assumed that the diagonal pairing is the optimal configuration in accordance to the nominal $G$ with the RIA methodology as outlines above. The basic idea is now to
calculate the minimum value for $\alpha$ of (3) before a $G_p$ shifting the optimal configuration decision to an off-diagonal pairing. Essentially, this means the diagonal pairing obtained from the nominal $G$ becomes invalid for that specific $G_p$. Since only one alternative configuration is available in the $2 \times 2$ case, the sought value of $\alpha$ is the $\alpha_{\text{min}}$.

According to the above assumption and to the pairing rule (d), the following inequality is true

$$|\phi_{12}| + |\phi_{21}| > |\phi_{11}| + |\phi_{22}|$$

(5)

where the $\phi'$s are to be obtained from (1).

In order to maintain the diagonal configuration as the optimal pairing for $G_p(s)$, the sum of $\phi'$s correspond to the diagonal configuration has to be always less than the sum of the $\phi'$s correspond to the off-diagonal configuration, and thus the inequality

$$|\phi_{p12}| + |\phi_{p21}| - |\phi_{p11}| - |\phi_{p22}| > 0$$

(6)

is satisfied. Notice that $\phi'$s in (6) are quantified based on $G_p$ unlike those of (5) which are quantified based on $G$.

Following pairing rule (d), if $(>)$ in (6), under the uncertainty effect, changes to $(<)$ it would promote the off-diagonal configuration and the diagonal one is no longer valid. Thus, the value of $\alpha$ that changes (6) to

$$|\phi_{p12}| + |\phi_{p21}| - |\phi_{p11}| - |\phi_{p22}| = 0$$

(7)

represents the minimum $\alpha$ just before $G_p$ in (4) returns an off-diagonal configuration and invalidates the diagonal one.

Provided that, $\phi_{pij}$ is represented as an upper-LFT as (See the details in Appendix A)

$$\phi_{pij} = F_u(M^{ij}, \alpha \cdot \bar{\Delta})$$

(8)

where

$$M^{ij} = \begin{bmatrix} M_{11}^{ij} & M_{12}^{ij} \\ M_{21}^{ij} & M_{22}^{ij} \end{bmatrix}$$

(9)

and

$$\bar{\Delta} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix}$$

(10)
with $e_i$ is the unit column vector with $i$-th element being 1 and the remaining elements being 0 and $E_i = e_i e_i^T$.

Similarly, $e_j$ and $E_j$ are defined corresponding to $j$-th element.

Then, (7) can be rewritten as

$$-|F_u(M^{11}, \alpha \cdot \bar{\Delta})| + |F_u(M^{12}, \alpha \cdot \bar{\Delta})| + |F_u(M^{21}, \alpha \cdot \bar{\Delta})| - |F_u(M^{22}, \alpha \cdot \bar{\Delta})| = 0$$

(11)

Since the addition and subtraction operations of the LFT’s result in another LFT [22], (11) is written as an augmented LFT as follows (See Appendix B for the details)

$$F_{u, Aug}(M_{Aug}, \alpha \cdot \bar{\Delta}) = 0$$

(12)

where

$$M_{Aug} = \begin{bmatrix} M_{11, Aug} & M_{12, Aug} \\ M_{21, Aug} & M_{22, Aug} \end{bmatrix}$$

$$= \begin{bmatrix} M_{11}^{11} & 0 & 0 & 0 & M_{12}^{11} \\ 0 & M_{11}^{12} & 0 & 0 & M_{12}^{12} \\ 0 & 0 & M_{11}^{21} & 0 & M_{12}^{21} \\ 0 & 0 & 0 & M_{11}^{22} & M_{12}^{22} \\ -M_{21}^{11} & M_{21}^{12} & M_{21}^{21} & -M_{21}^{22} & -|\phi_{11}| + |\phi_{12}| + |\phi_{21}| - |\phi_{22}| \end{bmatrix}$$

(13)

and

$$\bar{\Delta} = \begin{bmatrix} \bar{\Delta} & 0 & 0 & 0 \\ 0 & \bar{\Delta} & 0 & 0 \\ 0 & 0 & \bar{\Delta} & 0 \\ 0 & 0 & 0 & \bar{\Delta} \end{bmatrix}$$

(14)

The $\alpha \cdot \bar{\Delta}$ that satisfies (12) is the same $\alpha \cdot \bar{\Delta}$ that satisfies

$$(F_{u, Aug}(M_{Aug}, \alpha \cdot \bar{\Delta}))^{-1} = F_{u_{inv}}(N, \alpha \cdot \bar{\Delta}) = \infty$$

(15)

where [23]

$$N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}$$

$$= \begin{bmatrix} M_{11, Aug} & -M_{12, Aug}M_{22, Aug}^{-1}M_{21, Aug} & M_{12, Aug}M_{22, Aug}^{-1} \\ -M_{21, Aug}^{-1}M_{22, Aug} & M_{21, Aug}^{-1} \end{bmatrix}$$

(16)

This step is adopted in the calculation of $\alpha_{min}$ in order to fit the definition of the structural singular value
Consequently, $F_{u_{inv}}(N, \alpha \cdot \tilde{\Delta})$ can be written as (the $\cdot$ is dropped for simplicity)

$$F_{u_{inv}}(N, \alpha \tilde{\Delta}) = N_{22} + N_{21} \alpha \tilde{\Delta}(I - N_{11} \alpha \tilde{\Delta})^{-1} N_{12}$$

and the $\alpha \tilde{\Delta}$ that satisfies (15) is the $\alpha \tilde{\Delta}$ that makes $(I - N_{11} \alpha \tilde{\Delta})$ singular as shown from (17). In other words, finding $\mu_{\tilde{\Delta}(N_{11})}^{-1}$ as

$$\mu_{\tilde{\Delta}(N_{11})}^{-1} = \min \alpha : \det(I - \alpha N_{11} \tilde{\Delta}) = 0 \text{ for structured } \tilde{\Delta}, \bar{\sigma}(\tilde{\Delta}) \leq 1$$

(18)

gives the minimum value of $\alpha$ with $\bar{\sigma}(\tilde{\Delta}) = 1$ that satisfies (15) and subsequently (12).

From the block structure of $\tilde{\Delta}$ in (14) and applying the properties of the largest singular value of block diagonal matrices yields

$$\bar{\sigma}(\tilde{\Delta}) = \bar{\sigma}(\bar{\Delta}) = \bar{\sigma}(\Delta)$$

(19)

Hence $\mu_{\tilde{\Delta}(N_{11})}^{-1}$ characterizes the amount of the gain uncertainty tolerated before the off-diagonal configuration becomes valid.

Since for the studied case, the off-diagonal pairing is the only alternative configuration, $\mu_{\tilde{\Delta}(N_{11})}^{-1}$ in (18) is equal to $\alpha_{\min}$. Subsequently, the decentralized controller with diagonal configuration is not recommended for uncertain system with value equals to or exceeds $\alpha_{\min}$, provided that $\bar{\sigma}(\Delta) = 1$.

4.2. Generalization for $n \times n$ systems

Now we consider a general $n \times n$ uncertain system with steady-state gain matrix $G_p$ written as in (4), where $G$ is the nominal steady-state gain matrix, $W$ and $V$ are weighting matrices selected such that each of the perturbed elements is in accordance with (3) and $\Delta$ is a structured uncertainty block.

In general, there will be $n!$ possible control configurations including the optimal one. Therefore, the proposed method needs to be repeated $k = n! - 1$ times in order to determine $\alpha_{\min}$ that invalidates the optimal configuration. Defining the alternative configurations manually for large-scale systems is tedious, hence the approach proposed by Kadhim et al. (24) can be used to recover the alternatives automatically.

As an example, suppose that the RIA values obtained from $G$ and the pairing rules show that the diagonal pairing $(y_1 - u_1, y_2 - u_2, \cdots, y_n - u_n)$ is the optimal configuration. The minimum value of $\alpha$ which would change the pairing decision to the first out of the $k$ alternative pairings will be denoted $\alpha_1$. Let’s say, the first examined alternative pairing is the anti-diagonal configuration $(y_1 - u_n, y_2 - u_{n-1}, \cdots, y_n - u_1)$ then $\alpha_1$ satisfies

$$\phi's \ of \ the \ diagonal \ pairing \ -|\phi_{p11}| - |\phi_{p22}| - \cdots - |\phi_{pnn}| + |\phi_{p1n}| + |\phi_{p2n-1}| + \cdots + |\phi_{pnn}| = 0$$

(20)

$$\phi's \ of \ the \ off-diagonal \ pairing \ (1^{st} \ alternative)$$
By representing the $\phi_i$'s as LFT’s using (9), (20) can be written as an augmented LFT as

$$\mathcal{F}_{u,\text{Aug}}(M_{\text{Aug}}, \alpha \cdot \tilde{\Delta}) = 0$$  \hspace{1cm} (21)

where

$$M_{\text{Aug}} = \begin{bmatrix} M_{11,\text{Aug}} & M_{12,\text{Aug}} \\ M_{21,\text{Aug}} & M_{22,\text{Aug}} \end{bmatrix}$$


\begin{equation}
\begin{pmatrix}
M_{11}^{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_{11}^{12} \\
0 & M_{11}^{22} & 0 & 0 & 0 & 0 & 0 & 0 & M_{12}^{12} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & M_{11}^{1n} & 0 & 0 & 0 & 0 & M_{12}^{1n} \\
0 & 0 & 0 & 0 & M_{11}^{2n-1} & 0 & 0 & 0 & M_{12}^{2n-1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & M_{11}^{n-1} & M_{12}^{n-1} \\
-M_{21}^{11} & -M_{21}^{22} & \cdots & -M_{21}^{nn} & M_{21}^{2n-1} & \cdots & M_{21}^{n1} & -|\phi_{11}| & \cdots & -|\phi_{nn}| + |\phi_{1n}| + \cdots + |\phi_{n1}| \\
\end{pmatrix}
\end{equation}

and

$$\tilde{\Delta} = \begin{bmatrix} \Delta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \Delta & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \Delta & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \Delta & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \Delta \\
\end{bmatrix}$$  \hspace{1cm} (23)

Thereafter, $\alpha_1$ is obtained using the definition of $\mu$ as in (18) after calculating $N_{11}$ through (16). Repeating these steps for the remaining $(n! - 2)$ alternative configurations, the remaining $(\alpha_2 \ldots \alpha_k)$ will be obtained.

Eventually, the optimal configuration is judged based on the uncertainty corresponding to the smallest $\alpha$ as

$$\alpha_{\text{min}} = \min(\alpha_1, \alpha_2, \ldots, \alpha_k)$$  \hspace{1cm} (24)

For a general $n \times n$ system the procedure can now be summarized as follows in Table 1.

In many cases and based on a prior knowledge of the studied plant, the practitioner may discard many configurations from the $k = n! - 1$ alternatives as they might be infeasible [3], e.g. actuator that has no influence on a specific variable. Thus, the optimal configuration, in such cases, is to be compared with less number of alternative configurations when obtain $\alpha_{\text{min}}$ as will be seen in example 2.
Table 1: Steps of the proposed method

<table>
<thead>
<tr>
<th>Description</th>
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<tbody>
<tr>
<td><strong>step 1.</strong></td>
</tr>
<tr>
<td><strong>step 2.</strong></td>
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<tr>
<td><strong>step 3.</strong></td>
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<tr>
<td><strong>for $i = 1 \rightarrow k$ do</strong></td>
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<tr>
<td><strong>step 4.</strong></td>
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<td><strong>step 5.</strong></td>
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<td><strong>step 6.</strong></td>
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<td><strong>step 7.</strong></td>
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<tr>
<td><strong>end for</strong></td>
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<td><strong>step 8.</strong></td>
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5. Examples

In this section, examples are given for $2 \times 2$ distillation column and $5 \times 5$ stock-preparation process.

For the $2 \times 2$ system two cases are considered; first: when one element in the gain matrix is susceptible to change and hence the result of the proposed method can be validated against a mathematical derived expression. The second case is for all elements of the gain matrix changing independently. In example 2, three alternative configurations are examined, i.e $k$ in **step 3** equals 3.

**Example 1. Wood-Berry distillation column.**

For this process, the nominal steady-state gain matrix is given as

$$G = \begin{bmatrix} 12.8 & -18.9 \\ 6.6 & -19.4 \end{bmatrix}$$

(25)

**Case 1:**

The uncertainty is defined only in $G_{11}$ where it is described as

$$[G_p]_{11} = G_{11} + \alpha \cdot \delta \cdot |G_{11}|, \ |\delta| \leq 1$$

**step 1** involves selecting the weighting matrices, thus $W$ and $V$ are selected as

$$W = \begin{bmatrix} |G_{11}| & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 12.8 & 0 \\ 0 & 0 \end{bmatrix}, \ V = \alpha \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
with
\[ \Delta = \delta \]

**step 2:** The RIA is calculated as
\[
\Phi = \begin{bmatrix} -0.5023 & -1.9907 \\ -1.9907 & -0.5023 \end{bmatrix}
\]

**step 3:** Following the RIA pairing rules, the optimal configuration of this system is the diagonal pairing \((y_1 - u_1, y_2 - u_2)\). More specifically, the stability, integrity and robustness rules are fulfilled by selecting the diagonal configuration as \(NI_{\text{diagonal}} = 0.4977 > 0\) and \(\phi_{11} = \phi_{22} = -0.5023 > -1\) while the stability and integrity rules are not fulfilled with the off-diagonal pairing as \(NI_{\text{off-diagonal}} = -0.9907 < 0\) and \(\phi_{12} = \phi_{21} = -1.9907 < -1\). Thus, the alternative configuration, \(k = 1\), in this case is the off-diagonal pairing.

Then \(M^{11}, M^{12}, M^{21}\) and \(M^{22}\) are calculated as in **step 4** to form the augmented \(M_{\text{Aug}}\) according to **step 5**. **Steps 6 and 7** result in \(\alpha_1\) equals to 0.4977. As \(k = 1\), **step 8** gives \(\alpha_{\text{min}} = \alpha_1 = 0.4977\).

In the case when a single gain element in \(2 \times 2\) systems is susceptible to change, \(\alpha_{\text{min}}\) is mathematically tractable (see Appendix C) and is equal to
\[
\alpha_{\text{min}} \cdot \delta = \sqrt{\phi_{11}} - 1 = -0.4977
\]
where it is observed that \(\delta = -1\) invalidates the optimal pairing. The value obtained from (26) confirms the value of \(\alpha_{\text{min}}\) of the proposed method. Thus, when substituting \(\alpha_{\text{min}}\) in \(V, W\) and \(\Delta = \delta = -1\) in \(G_p\) as
\[
G_p = G + W \Delta V = \begin{bmatrix} 6.4299 & -18.9 \\ 6.6 & -19.4 \end{bmatrix}
\]
the perturbed RIA matrix (\(\Phi_p\)) becomes
\[
\Phi_p = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}
\]
which promotes neither the diagonal nor the off-diagonal configuration. In this article, such matrix is referred to as boundary RIA matrix. Any increment beyond \(\alpha_{\text{min}} = 0.4977\) the off-diagonal pairing is promoted, provided that \(\delta = -1\).
Case 2:

The uncertainty is defined in all the elements of the steady-state gain matrix as

\[ (G_p)_{ij} = G_{ij} + \alpha \cdot \delta_{ij} \cdot |G_{ij}|, \quad |\delta_{ij}| \leq 1 \]

where \( i, j = 1, 2 \). In order to calculate the value of \( \alpha \) that invalidates the diagonal configuration, the weighting matrices \( W \) and \( V \) are defined following step 1 as

\[
W = \begin{bmatrix} |G_{11}| & |G_{12}| & 0 & 0 \\ 0 & 0 & |G_{21}| & |G_{22}| \end{bmatrix} = \begin{bmatrix} 12.8 & 18.9 & 0 & 0 \\ 0 & 0 & 6.6 & 19.4 \end{bmatrix},
\]

\[
V = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}^T
\]

with

\[ \Delta = \text{diag}(\delta_{11}, \delta_{12}, \delta_{21}, \delta_{22}) \]

The results of steps 2 and 3 are the same as in the previous case since they depend on \( G \) in (25).

Applying the remaining steps, \( \alpha_{\text{min}} \) is calculated to be \( \alpha_{\text{min}} = 0.1704 \). In order to confirm this result, the perturbed system \( G_p \) is obtained by inserting the value of \( \alpha_{\text{min}} \) in \( V \), and substituting \( V, W \), when \( \delta_{11}, \delta_{12} = -1 \) and \( \delta_{21}, \delta_{22} = 1 \) being the worst-case.

\[
G_p = G + W \Delta V = \begin{bmatrix} 10.6183 & -22.1214 \\ 7.7249 & -16.0934 \end{bmatrix}
\]

This specific \( G_p \) renders the boundary RIA matrix as in (27), which invalidates the diagonal configuration.

The result of \( \alpha_{\text{min}} \) obtained in this case is aligned with the result shown by Chen and Seborg\cite{11} and Kariwala et al.\cite{15}. However, there this result is reached in a reversed approach by tracking the effect of several given values of \( \alpha \) on the configuration selection. In such way, it is computationally inefficient to check different \( \alpha \)'s before finding the minimum value that invalidates the configuration of the nominal model.

In conclusion, the user needs to be aware that the optimal configuration is going to be revoked under the effect of uncertainty quantified by \( \alpha_{\text{min}} \), when the worst-case of \( \Delta \) occurs.

**Example 2. Stock Preparation Plant.** In the stock preparation section of pulp & paper mills, the pulp is refined to improve important properties which relate to the strength of the paper web and the facility to dry it\cite{26}. The transfer function of the stock-preparation process in SCA Obbola AB, Sweden is
From $G(s)$, it is shown that $u_1$ could only control $y_1$ so the pair $(y_1 - u_1)$ will keep being coupled despite the perturbation. This result can be anticipated directly from RIA matrix as will be shown later. For that reason, the change in $G_{11}, G_{12}$ and $G_{13}$ will be excluded from the analysis since it does not affect selecting $(y_1 - u_1)$ as a pair. With the perturbed elements are defined as

$$[G_p]_{ij} = G_{ij} + \alpha \cdot \delta_{ij} \cdot |G_{ij}|, \quad |\delta_{ij}| \leq 1$$

where $i,j$ correspond to the shaded elements in (28). Following step 1, the steady-state gain matrix $G_p$ of the perturbed system $G_p(s)$ is written as

$$G_p = G + W \Delta V$$

with

$$G = \begin{bmatrix} 2.8961 & -0.5431 & -0.8799 & 0 & 0 \\ 0 & 1.536 & 0.4055 & 0 & 0 \\ 0 & 0.3522 & 1.898 & 0 & 0 \\ 0 & 0 & 0 & 0.2484 & -0.0198 \\ 0 & 0 & 0 & -0.0425 & 0.202 \end{bmatrix}$$

$$W = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1.5359 & 0.4055 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(30)
\[ V = \alpha \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}^T \]

and \( \Delta = \text{diag}(\delta_{22}, \delta_{23}, \delta_{32}, \delta_{33}, \delta_{44}, \delta_{45}, \delta_{54}, \delta_{55}) \).

**step 2:** The RIA is calculated as

\[
\Phi = \begin{bmatrix} 0 & \infty & \infty & \infty & \infty \\ \infty & -0.0490 & -20.4074 & \infty & \infty \\ \infty & -20.4074 & -0.0490 & \infty & \infty \\ \infty & \infty & \infty & -0.0167 & -59.7793 \\ \infty & \infty & \infty & -59.7793 & -0.0167 \end{bmatrix}
\]

It can be noticed from the RIA matrix that the pair \((y_1 - u_1)\) is fixed, which confirms our previous analysis. Additionally, following step 1, the \(\infty\) values in the elements of the RIA matrix that correspond to \(G_{ij} = 0\) means that \(u_j\) does not influence \(y_i\) and the pair \((y_i - u_j)\) is to be discarded. As a result, the number of the possible configurations for this particular plant is reduced tremendously below 5!. Moreover, in order to find the possible configurations two separated blocks are observed in the RIA matrix beside the fixed pair. The observed blocks can be treated as two separate \(2 \times 2\) systems and consequently for this specific stock-preparation system only 4 possible configurations are available.

Applying the RIA pairing rules as in **step 3**, the diagonal configuration \((y_1 - u_1, y_2 - u_2, y_3 - u_3, y_4 - u_4, y_5 - u_5)\) is selected to be the optimal pairing since it satisfies the stability (\(NI_{\text{diagonal}} = 0.9351 > 0\)), integrity and robustness conditions (see the corresponding \(\phi_{ij}'s\) in the RIA matrix) while all other configurations do not fulfill the integrity condition (see the corresponding \(\phi_{ij}'s\) in the RIA matrix). The 3 alternative configurations are \((k = 3)\)

1st alter. \( : y_1 - u_1, y_2 - u_2, y_3 - u_3, y_4 - u_4, y_5 - u_5, \)

2nd alter. \( : y_1 - u_4, y_2 - u_2, y_3 - u_3, y_4 - u_5, y_5 - u_2, \)

3rd alter. \( : y_1 - u_1, y_2 - u_3, y_3 - u_2, y_4 - u_5, y_5 - u_4. \)

and thus there are three \(\alpha's\) to be found.

Finding \(\alpha_1\) implicitly means that the sum of the \(\phi's\) that correspond to the diagonal configuration is equal to the sum of \(\phi's\) that correspond to the 1st alternative configuration. Therefore, **step 4** requires calculating \(M_{22}^2, M_{33}^3, M_{23}^2\) and \(M_{32}^3\) (since \(M_{11}^1, M_{44}^4\) and \(M_{55}^5\) are the same in both configurations and

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will cancel out each other). By following steps 5-7, \( \alpha_1 \) is found to be equal to 0.6375. By substituting \( \alpha_1 \) in \( V \) and \( \Delta \) with \( \delta_{22} = \delta_{33} = \delta_{45} = -1 \) and \( \delta_{23} = \delta_{32} = \delta_{44} = \delta_{54} = \delta_{55} = 1 \), \( G_{p1} \) obtained from (29) gives boundary RIA matrix as

\[
\Phi_{p1} = \begin{bmatrix}
0 & \infty & \infty & \infty & \infty \\
\infty & -1 & -1 & \infty & \infty \\
\infty & -1 & -1 & \infty & \infty \\
\infty & \infty & \infty & -0.0037 & -269.3871 \\
\infty & \infty & \infty & -269.3871 & -0.0037
\end{bmatrix}
\]

which promotes neither selecting the diagonal configuration nor the 1st alternative.

Finding \( \alpha_2 \) through a second run, \( k = 2 \), an equalization is assumed between the optimal configuration and the 2nd alternative, thus step 4 requires calculating \( M_{44}, M_{55}, M_{45} \) and \( M_{54} \) (since \( M_{11}, M_{22} \) and \( M_{33} \) are same in both configurations and hence will cancel out each other). Thereafter, steps 5-7 return \( \alpha_2 \) as 0.7707. \( G_{p2} \) obtained from (29) by substituting \( \alpha_2 \) in \( V \) and \( \Delta \) with \( \delta_{22} = 1 \) and \( \delta_{33} = \delta_{45} = \delta_{23} = \delta_{32} = \delta_{44} = \delta_{54} = \delta_{55} = -1 \), results in a boundary RIA matrix as

\[
\Phi_{p2} = \begin{bmatrix}
0 & \infty & \infty & \infty & \infty \\
\infty & -0.0063 & -157.608 & \infty & \infty \\
\infty & -157.608 & -0.0063 & \infty & \infty \\
\infty & \infty & \infty & -1 & -1 \\
\infty & \infty & \infty & -1 & -1
\end{bmatrix}
\]

which neither favour selecting the diagonal configuration nor the 2nd alternative.

Finding \( \alpha_3 \): Provided that the worst-case \( \Delta \) occurs, \( \delta_{22} = \delta_{33} = \delta_{45} = \delta_{44} = \delta_{54} = \delta_{55} = -1 \) and \( \delta_{23} = \delta_{32} = 1 \), and since \( \alpha_1 < \alpha_2 \), thus any value greater than \( \alpha_2 \) will shift the optimal configuration to the 3rd alternative. As a result, there is no need to perform a new run as \( \alpha_3 \) is any value greater that \( \alpha_2 \), i.e. \( \alpha_1 < \alpha_2 < \alpha_3 \), as will be shown further in the example.

Three values of \( \alpha \) are obtained; \( \alpha_1 = 0.6375, \alpha_2 = 0.7707 \) and \( \alpha_3 > \alpha_2 \). Finally, as in step 8, the \( \alpha_{\min} \) is the smallest value among the obtained \( \alpha \)'s as

\[
\alpha_{\min} = \min(\alpha_1, \alpha_2, \alpha_3) = 0.6375
\]

In conclusion, the user has to be aware that when \( \bar{\sigma}(\Delta) = 1 \), the uncertainty corresponds to \( \alpha_{\min} = 0.6375 \) may invalidate the selection of the diagonal configuration and thus a decentralized controller with diagonal pairing is not recommended for that uncertainty amount.

In order to confirm this result, a decentralized controller consists of five SISO PI controllers in diagonal configuration with parameters depicted in Table 2 is applied on the nominal system described by (28).
Table 2: PI controller parameters \((K_p + K_i s)\) of the diagonal controller

<table>
<thead>
<tr>
<th>Controller</th>
<th>(K_p)</th>
<th>(K_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(C_1)</td>
<td>0.277</td>
<td>0.0927</td>
</tr>
<tr>
<td>(C_2)</td>
<td>0.679</td>
<td>0.0481</td>
</tr>
<tr>
<td>(C_3)</td>
<td>0.549</td>
<td>0.0556</td>
</tr>
<tr>
<td>(C_4)</td>
<td>4.2</td>
<td>2.8</td>
</tr>
<tr>
<td>(C_5)</td>
<td>5.16</td>
<td>7.67</td>
</tr>
</tbody>
</table>

The step responses of \(y_1, \cdots, y_5\) of the multivariable system depicted in Figure 1 show very satisfactory agreement with the responses of the diagonal elements \(G_{ii}\) with \(i = 1, \cdots, 5\) without any interaction.

Thereafter, the same decentralized controller is applied on a specific perturbed system \(G_p\) as described by \((29)\) with \(\alpha = 0.5875 < \alpha_1\) and worst-case \(\Delta\). The step responses depicted in Figure 1 (first column) show, on the one hand, sluggish yet stable responses for both \(y_2\) and \(y_3\). On the other hand, the responses of \(y_4\) and \(y_5\) have changed compared to the responses of the nominal system in \((28)\) but they are still acceptable. Thus, the decentralized controller with the diagonal configuration could handle that amount of uncertainty which coincides with the obtained results since \(\alpha < \alpha_1\).

Another perturbed system \(G_p\) is defined based on \((29)\) with \(\alpha > \alpha_1\) but less than \(\alpha_2\), \(\alpha_1 < \alpha = 0.6875 < \alpha_2\), and worst-case \(\Delta\). The step responses of the new \(G_p\) under the same decentralized controller are depicted in Figure 1 (second column). The responses show that \(y_2\) and \(y_3\) become unstable while in the responses of \(y_4\) and \(y_5\) some sluggishness occurs yet they are still stable. These results again coincide with previous outcomes as the pairs \((y_2 - u_2, y_3 - y_3)\) are not valid for this uncertain system \(G_p\) in contrast to the pairs \((y_4 - u_4, y_5 - y_5)\).

Lastly, a perturbed system \(G_p\) is defined following \((29)\) with \(\alpha > \alpha_2\) as \(\alpha_1 < \alpha_2 < \alpha = 0.8207\) and worst-case \(\Delta\). Again, the step responses of the perturbed system \(G_p\) are observed under the same decentralized controller. The responses depicted in Figure 1 (third column) show that the responses of all outputs, except \(y_1\), become unstable. This result is predicted in the previous analysis as the perturbed system \(G_p\) with \(\alpha > \alpha_2\) may lead to invalidating the diagonal pairing selection and thus the decentralized controller with the diagonal configuration is not preferable for that amount of uncertainty.

In order to support this result, the \(G_p\) with \(\alpha = 0.8207\) is used to calculate the perturbed RIA matrix \(\Phi_p\) as

\[
\Phi_p = \begin{bmatrix}
0 & \infty & \infty & \infty & \infty \\
\infty & -5.0515 & -0.1980 & \infty & \infty \\
\infty & -0.1980 & -5.0515 & \infty & \infty \\
\infty & \infty & \infty & -1.7291 & -0.5783 \\
\infty & \infty & \infty & -0.5783 & -1.7291 \\
\end{bmatrix}
\]

which shows that for this specific \(G_p\), the 3rd alternative becomes the optimal configuration rather than diagonal one as expected from the previous results. Moreover, the instability of the system of \(G_p\) with the
diagonal decentralized controller is expected since the selected pairings are correspondents to $[\phi_{\text{pair}}]_{i}=2.5 < -1$ which violates the integrity condition in the RIA pairing rules. As a conclusion, the decentralized controller may not be able to handle an uncertainty quantified by $\alpha_{\text{min}} = 0.6375$ provided that worst-case $\Delta$ occurs.

### 6. Conclusions

Following the RIA methodology, the article presented a method to quantify the uncertainty amount in the uncertain system $G_p$ that leads to invalidating the optimal control configuration obtained from the nominal model $G$. At worst-case uncertainty, the minimum value of the relative uncertainty parameter $\alpha$ ($\alpha_{\text{min}}$) that indicates the configuration change, is calculated using the structure singular value $\mu$. Hence, the proposed method provides a valuable tool informing the user to which extent the selected decentralized control configuration is recommended under the effect of uncertainty.

Two examples, a $2 \times 2$ distillation column and a $5 \times 5$ stock-preparation process, were used to illustrate the proposed method outcome. The value of $\alpha_{\text{min}}$ obtained in the examples rendered an RIA matrix that did not favor any configuration (boundary matrix) indicating a state where the configuration based on the nominal model has just been invalidated. A mathematical expression limited to $2 \times 2$ system when a change occurred in only one element is used to confirm the value of $\alpha_{\text{min}}$ obtained for a similar case in the distillation column example.

The method presented here is not limited to system size and it can be extended to large-scale systems. However, for such systems the number of alternative configurations is expected to be very high and finding $\alpha_{\text{min}}$ manually seems to be inefficient. By automating the selection of the weighting matrices $W$ and $V$, the proposed method can be made fully automated. This step is planned as a future work. Moreover, knowing the value of $\alpha_{\text{min}}$ reveals only the possibility to a change in the configuration selection. However, in order to investigate whether this change affects the stability, calculating the perturbed RIA matrix for the worst-case is required, as shown in example 2. Deriving a direct relation between the value of $\alpha_{\text{min}}$ and system instability is also planned as a future work.

### Appendices

#### Appendix A. RIA for perturbed systems

The RIA for the uncertain system is formulated as an upper-LFT following the robust control framework. This upper-LFT form of the RIA is itself of interest to the research community as it allows to to quantify the effect of gain uncertainty on the RIA.

Consider a perturbed system $G_p(s)$ with a steady-state gain matrix $G_p$ written as

$$G_p = G + W \Delta V : \gamma(\Delta) \leq 1 \quad (A.1)$$
Figure 1: Step response of full nominal system $G$, the diagonal elements of $G$ (without interaction) and $G_p$ systems for different $\alpha$. Blue: responses of the SISO diagonal systems. Red: responses of the full nominal system. Black: responses of $G_p$. First column: $G_p$ corresponds to $\alpha = 0.5875$. Second column: $G_p$ corresponds to $\alpha = 0.6875$. Third column: $G_p$ corresponds to $\alpha = 0.8207$. Notice that the time scales of the plots are not identical.
where $G$ in the steady-state gain matrix of the nominal model of the system $G(s)$, $W$ and $V$ are weighting matrices and $\Delta$ is a structured uncertainty block.

The perturbed RIA element $\phi_{pij}$ can be written aided by [1] as

$$\phi_{pij} = \frac{1 - \lambda_{pij}}{\lambda_{pij}} \quad (A.2)$$

where $\lambda_{pij}$ is the $(ij)$-perturbed RGA element. By assuming the existence of $G_p^{-1}$, the $\lambda_{pij}$ in (A.3)

$$\lambda_{pij} = [G_p]_{ij}[G_p]_{ji}^{-1} \quad (A.3)$$

is written employing (A.1) as [15]

$$\lambda_{pij} = e_i^T(G + W\Delta V)e_j e_j^T(G + W\Delta V)^{-1} e_i \quad (A.4)$$

where $e_i$ is the unit column vector with $i$-th element being 1 and the remaining elements being 0 and same definition is applied for $e_j$ with respect to $j$-th element.

With some manipulations, $\lambda_{pij}$ is rewritten as [15]

$$\lambda_{pij} = e_i^T(G + W\Delta V)e_j e_j^T(I + G^{-1}W\Delta V)^{-1} G^{-1} e_i \quad (A.5)$$

where the term $(I + G^{-1}W\Delta V)^{-1} G^{-1}$ is considered as an inverse additive uncertainty representation for $G^{-1}$.

After representing $\lambda_{pij}$ in (A.5), the inner dotted area in Figure A.2, the $\phi_{pij}$ in (A.2) can be formulated as a signal-based representation, $y = \phi_{pij}u$. The term $(1 - \lambda_{pij})$ in (A.2) is obtained by adding a unity feed-forward path to the $(-\lambda_{pij})$. Thereafter, the $\phi_{pij}$ is achieved by introducing a positive unity feedback around the term $(1 - \lambda_{pij})$ as shown in Figure A.2.

![Figure A.2: Signal-based representation of perturbed relative interaction element $\phi_{pij}$](image-url)
Analyzing $\phi_{pij}$ signal-based representation shown in Figure A.2 gives the following relationships (notice that $E_i = e_i e_i^T$ and $E_j = e_j e_j^T$)

\[
\begin{bmatrix}
  y_A^T \\
  V_A^T \\
  u_A^T \\
  \psi_A^T
\end{bmatrix}
= M^{ij}
\begin{bmatrix}
  u_A^T \\
  \psi_A^T \\
  u
\end{bmatrix}
\tag{A.6}
\]

\[
\begin{bmatrix}
  u_A^T \\
  \psi_A^T
\end{bmatrix}
= \Delta
\begin{bmatrix}
  y_A^T \\
  V_A^T
\end{bmatrix}
\tag{A.7}
\]

where

\[
M^{ij} =
\begin{bmatrix}
  M_{11}^{ij} & M_{12}^{ij} \\
  M_{21}^{ij} & M_{22}^{ij}
\end{bmatrix}
= \begin{bmatrix}
  -VE_j G^{-1} E_i W \\
  -VG^{-1} E_i W & -\frac{E_{ij} G^{-1} e_i}{\lambda_{ij}}\\
  \frac{V G^{-1} e_i}{\lambda_{ij}} & \frac{V G^{-1} e_i}{\lambda_{ij}} & \phi_{ij}
\end{bmatrix}
\tag{A.8}
\]

and

\[
\Delta =
\begin{bmatrix}
  \Delta & 0 \\
  0 & \Delta
\end{bmatrix}
\tag{A.9}
\]

Combining (A.8) and (A.9), the $\phi_{pij}$ in Figure A.2 can form as an upper-LFT as shown in Figure A.3 and hence $\phi_{pij}$ can be written as

\[
\phi_{pij} = F_u(M^{ij}, \Delta)
= M_{22}^{ij} + M_{21}^{ij} \Delta(I - M_{11}^{ij} \Delta)^{-1} M_{12}^{ij}
\tag{A.10}
\]
Appendix B. Addition and subtraction of LFT’s

Suppose two upper LFT’s, $F_u(A, \Delta_1)$ and $F_u(B, \Delta_2)$, the addition and subtraction operations $F_u(A, \Delta_1) \pm F_u(B, \Delta_2)$ as in Figure B.4 renders another LFT as shown below. We can write

\begin{align*}
y_{\Delta_1} &= A_{11}u_{\Delta_1} + A_{12}u \\
y_A &= A_{21}u_{\Delta_1} + A_{22}u \\
u_{\Delta_1} &= \Delta_1 y_{\Delta_1}
\end{align*}

and

\begin{align*}
y_{\Delta_2} &= B_{11}u_{\Delta_2} + B_{12}u \\
y_B &= B_{21}u_{\Delta_2} + B_{22}u \\
u_{\Delta_2} &= \Delta_2 y_{\Delta_2}
\end{align*}

The addition and subtraction of the two LFT’s ($y$) is written as

\begin{align*}
y &= y_A \pm y_B \\
&= \left[ A_{21} \pm B_{21} \right] \begin{bmatrix} u_{\Delta_1} \\ u_{\Delta_2} \end{bmatrix} + (A_{22} \pm B_{22})u
\end{align*} \tag{B.1}

and thus, the result can be formulated as a new LFT as

Figure B.4: AdditionSubtraction of two upper-LFT
\[
\begin{bmatrix}
  y_{\Delta_1} \\
  y_{\Delta_2} \\
  y
\end{bmatrix} =
\begin{bmatrix}
  A_{11} & 0 & A_{12} \\
  0 & B_{11} & B_{12} \\
  A_{21} & \pm B_{21} & A_{22} \pm B_{22}
\end{bmatrix}
\begin{bmatrix}
  M \\
  u_{\Delta_1} \\
  u_{\Delta_2} \\
  u
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
  u_{\Delta_1} \\
  u_{\Delta_2}
\end{bmatrix} =
\begin{bmatrix}
  \Delta_1 & 0 \\
  0 & \Delta_2
\end{bmatrix}
\begin{bmatrix}
  y_{\Delta_1} \\
  y_{\Delta_2}
\end{bmatrix}
\]  

(B.2)

Appendix C. Determining \(\alpha_{\text{min}}\) for \(2 \times 2\) system when single element is changing

Consider a \(2 \times 2\) system with nominal steady-state gain matrix \(G\) as

\[
G =
\begin{bmatrix}
  G_{11} & G_{12} \\
  G_{21} & G_{22}
\end{bmatrix}
\]

The RIA matrix \(\Phi\) can be defined following the RGA definition for \(2 \times 2\) systems \([8]\) and \([1]\) as

\[
\Phi =
\begin{bmatrix}
  \phi_{11} & \phi_{12} \\
  \phi_{21} & \phi_{22}
\end{bmatrix}
\]

where

\[
\phi_{11} = \phi_{22} = -\frac{G_{12}G_{21}}{G_{11}G_{22}}, \quad \text{and} \quad \phi_{12} = \phi_{21} = -\frac{G_{11}G_{22}}{G_{12}G_{21}}
\]  

(C.1)

Suppose that, the diagonal configuration is the optimal configuration thus the following inequality is true

\[
|\phi_{12}| + |\phi_{21}| > |\phi_{11}| + |\phi_{22}|
\]

following to the pairing rule (d).

Let’s assume a perturbed system as

\[
G_p =
\begin{bmatrix}
  G_{11} + \alpha \cdot \delta \cdot G_{11} & G_{12} \\
  G_{21} & G_{22}
\end{bmatrix}
\]

where \(|\delta| \leq 1\). Thus, \(\phi_{p11}\) and \(\phi_{p22}\) can be written as in \((C.1)\) as

\[
\phi_{p11} = \phi_{p22} = -\frac{G_{12}G_{21}}{G_{11}(1 + \alpha \cdot \delta)G_{22}} = \frac{\phi_{11}}{(1 + \alpha \cdot \delta)}
\]  

(C.2)

Similarly

\[
\phi_{p12} = \phi_{p21} = \phi_{12}(1 + \alpha \cdot \delta)
\]  

(C.3)
In order to find $\alpha_{\text{min}}$ that invalidates the diagonal pairing, the above inequality has to be equality as

$$2|\phi_{p21}| - 2|\phi_{p11}| = 0$$

Substituting (C.2) and (C.3) in the equality, gives

$$2\phi_{12}(1 + \alpha_{\text{min}} \cdot \delta) - 2\phi_{11}(1 + \alpha_{\text{min}} \cdot \delta) = 0$$

Notice that, the $\Phi$ elements in $2 \times 2$ case have the same sign and thus remove the absolute will not effect the result. After short manipulations, the following result is obtained

$$\alpha_{\text{min}} \cdot \delta = \sqrt{\frac{\phi_{11}}{\phi_{12}}} - 1$$  \hspace{1cm} (C.4)

With assumption that $|\delta| = 1$ for the worst-case uncertainty, (26) is obtained.

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