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Philosophy of mathematics in “La Science et l’Hypothèse”, from Henri Poincaré.

Mathilde Aboud

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Handledare: Gunnar Berg
Examinator: Martin Herschend
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A large, faint watermark of the Uppsala University seal is visible in the bottom right corner of the page. It features a sunburst and the Latin text "ALERE FLAMMAM VERITATIS".

Department of Mathematics
Uppsala University

Philosophy of mathematics in “La Science et l’Hypothèse”, from Henri Poincaré.

Bachelor thesis in mathematics

Uppsala University
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Mathilde Aboud

Supervisor
Gunnar Berg, Lecturer
Uppsala University
Department of Mathematics

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Abstract

La Science et l'Hypothèse, written by Poincaré at the beginning of the 20th century, became a reference in the philosophy of science. In his book, Poincaré aims at popularizing science. He addresses several scientific topics, such as numbers, magnitude, space and geometries, and physics and mechanics. This essay mainly investigates Poincaré's point of view on mathematics, and the way his work was influenced by other intellectuals. Therefore, the first part of the essay explores the historical background to Poincaré's work. Next, the development of geometry before, during, and after Poincaré's era is studied in detail; and finally the contribution of Poincaré to the sciences of the 20th and 21st century is also explored.

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1- Biography

1.1 Childhood, family and early education

Jules Henri Poincaré was a French mathematician, physicist, engineer and philosopher of science. He was born in 1854 in Nancy (France), and died in 1912 in Paris (France), at 58 years old. He was part of an important family: his father Léon, was a professor of medicine at the University of Nancy, his sister Aline Poincaré married the well-known philosopher Emile Boutroux, and his cousin Raymond Poincaré was President of France from 1913 to 1920, and was a member of the Académie Française¹. Henri Poincaré was raised in a catholic family, but when he grew up he got himself away from religion. Some said he was an atheist.



Figure 1 Henri Poincaré. Source : [1]

Henri went to school for the first time in 1862, where he prospered in all subjects. It was in the 4th class that his talent for mathematics was first recognized. [2] He then had a break from school during the Franco-Prussian War in 1870, because he served with his father in the Ambulance Corps.

Next, in high school, Henri Poincaré was the best student in every topic, as much for the written essays as for the mathematics exercises. He won the first prizes in the Concours Général in 1872, a competition between the best students in France. Henri was also very good at languages. His family liked to travel, and this seems to have encouraged his ability with languages. He learned Latin, Greek, German and later English. [3] It is at the period of high-school that the stories about Poincaré's genius began. People said that Poincaré did not take notes, and did not give the impression of being a serious student. But when an older pupil asked him to explain a particularly complex point in the course, Poincaré could reply immediately. [2] Appell² was another excellent student in Henri's school. Appell and Poincaré were the only students from Nancy to sit the final exam. Poincaré was not really satisfied by his performance during the exam, but it turned out that he came first, with Appell second. "Rollier, the inspector general who had marked the papers, said Poincaré was an extraordinary pupil who would go far, with incredible broad knowledge and aptitudes. [4]

¹ The Académie Française, known in English as the French Academy, is the pre-eminent French council for matters related to the French language. The Académie was officially established in 1635 by Cardinal Richelieu.

² Paul Appell (1855-1930) was a French mathematician and Rector of the University of Paris.

1.2 Polytechnique and Academic Life

After having passed the final exam of high school, Appell and Poincaré had to sit the entrance examinations for the Ecole Polytechnique³ and the Ecole Normale Supérieure⁴. It finally turned out that Appell was admitted at the Ecole Normale Supérieure, and Poincaré at the Ecole Polytechnique in 1873. There Poincaré studied mathematics, mechanics, astronomy, physics, chemistry, history and literature, and he published his first paper, “*Démonstration nouvelle des propriétés de l’indicatrice d’une surface*”⁵ in 1874. Henri Poincaré graduated second from the Ecole polytechnique in 1875. Then he went to the Ecole des Mines, which is the most prestigious of the engineering schools that the Ecole Polytechnique trained students for. [4] At the Ecole des Mines, he studied mathematics and mining engineering from 1875 to 1878. During his time there, he spent about four months in Norway and Sweden on an educational trip, where he also looked for traces of Abel⁶. [2]

1.3 Early Career

After graduating from the Ecole des Mines, Henri Poincaré joined the Corps des Mines as a mining inspector for a region in the northeast of France, in Vesoul. While he was a mining inspector, Henri found time to begin a novel and prepare his doctorate in science of mathematics. His doctoral thesis was about differential equations, “*Sur les propriétés des fonctions définies par les équations aux différences partielles*”⁷. Poincaré graduated from the University of Paris and received the title of docteur ès sciences mathématiques in 1879 for his thesis. His thesis was published for the first time in the first volume of his *Oeuvres* (1928). [4] He then considered for a time pursuing two careers: engineer and mathematician. But he finally focused on mathematics instead of engineering, and decided to enter the University of Caen.

He then married Mlle Louise Poulain d’Andecy in 1881, was appointed a Maître des Conférences at the University of Paris, and became a member of the Association Française pour l’Avancement des Sciences⁸ in the same year. [4] In 1883, Poincaré was appointed teacher at the Ecole Polytechnique, and

³ Ecole Polytechnique (also known by the nickname “X”) is a French public institution of higher education and research in Palaiseau near Paris. It was established in 1794 by the mathematician Gaspard Monge during the French Revolution and became a military academy under Napoleon 1 in 1804.

⁴ The Ecole Normale Supérieure (ENS) is a French public institution of higher education in Paris. It was initially conceived during the French Revolution and was intended to provide the Republic with a new body of professors.

⁵ “*New demonstration of the properties of an indicator of a surface*”

⁶ Niels Abel (1802-1829) was a Norwegian mathematician who made pioneering contributions in a variety of fields.

⁷ “*On properties of functions defined by partial differential equations*”

⁸ French Association for the Progress of Science.

he became a Swedish Knight of the Polar Star⁹. He was elected in the section of geometry at the Académie des Sciences¹⁰ in 1887, after several failed attempts over the years¹¹. 1888 was a very busy year, since Poincaré was elected president of the Comité d'Organisation du Congrès International de Bibliographie des Sciences Mathématiques (Exposition Universelle Internationale de 1889)¹², won in 1889 the Swedish Prize Competition (with Appell second), and was elected a few weeks later Chevalier de la Légion d'honneur. All the honours received by Henri Poincaré over the years won't be specified here, but he was part of around forty Academies and scientific societies around the world. Among them was the Royal Societies of Sciences of Göttingen, Uppsala, Haarlem, London, Edinburgh, Copenhagen, Stockholm, and also the Academies of Sciences of Italy, Russia, America, Netherlands, Belgium. In 1901 he also became president of the Société astronomique de France, founded by Camille Flammarion¹³. Poincaré became a member of the French Academy of Sciences in 1887 and was elected as its president in 1906. In 1908 he joined the Académie Française alongside his cousin Raymond Poincaré.

Poincaré was a father of four children, named Jeanne, Yvonne, Henriette, and Léon, respectively born in 1887, 1889, 1891, and 1893.

1.4 The Dreyfus Affair

The Dreyfus affair (in French: L'affaire Dreyfus), was a political scandal that divided the Third French Republic from 1894 to 1906. The affair began in 1894, when a serving maid working in the German Embassy came into possession of a note, torn into six pieces, that suggested the Germans had a spy in the French Army. Captain Alfred Dreyfus, a young French artillery officer of Alsatian and Jewish decent, was then arrested and found guilty by a closed military tribunal, despite the fact that the only evidence against him was the note, which was said to be in his handwriting. Alfred Dreyfus was then sentenced to life imprisonment for having communicated French military secrets to the German Embassy in Paris, and imprisoned on Devil's Island in French Guiana.

Two years later in July 1896, Lieutenant Colonel Picquart¹⁴ -who was then head of counter-espionage- looked at the documents that had been presented to the court, and concluded that Dreyfus was innocent.

⁹ The Order of the Polar Star was a Swedish order of chivalry created by King Frederick I in 1748, which was until 1975 intended as a reward for Swedish and foreign "civic merits, for devotion to duty, for science, literary, learned and useful works and for new and beneficial institutions."

¹⁰ French Academy of Sciences.

¹¹ Poincaré received many honours. For a full list, see his *Oeuvres*.

¹² Organisational Comity of the International Congress of Mathematical Sciences (Universal International Exhibition of 1889).

¹³ Camille Flammarion (1842-1925) was a French astronomer and author. He published more than 50 books: astronomy books, psychical essays and science fiction novels.

¹⁴ Marie-Georges Picquart (1854-1914) was a French army officer and Minister of War.

He identified instead a French Army major named Ferdinand Walsin Esterhazy as the real culprit. However, the army attempted to silence the Lieutenant Colonel Picquart by transferring him to Tunisia and by suppressing the new evidence blaming F.W. Esterhazy. Nevertheless, Colonel Picquart was able to gain the support of influential Senators, and in January 1898 Esterhazy was arrested. But the military tribunal acquitted him. This is when Emile Zola published his famous open letter “*J’accuse!*” to Emile Loubet¹⁵, the president of the Republic. The letter, which was published in a Paris newspaper in January 1898, claimed Dreyfus’s innocence and put pressure on the government to reopen the case. But Zola was fined and sent to prison. However, he managed to escape to London. In 1899, Dreyfus was brought back to France for another trial. At that point, France was divided between those who supported Dreyfus (called “Dreyfusards”), such as Sarah Bernhardt¹⁶, Anatole France¹⁷, Emile Boutroux¹⁸, Henri Poincaré and Georges Clémenceau¹⁹, and those who condemned him (the “Anti-dreyfusards”), such as Edouard Drumont²⁰ and Charles Hermite²¹. For this new trial, Poincaré was asked to give his scientific opinion on the paternity of the note and on the main pieces of accusation. This was a matter of comparing stylistic mannerisms and computing probabilities. Poincaré had just published a lecture course on probability theory and was the most eminent theorist in the field in France. [4] He did not pronounce on the guilt or innocence of Dreyfus but he maintained that if he was to be condemned it had to be on another evidence. The handwriting on the note did not prove anything. However, the new trial still resulted in another conviction and a 10-year sentence, but Dreyfus was then given a pardon and set free.

1.5 Poincaré’s commitment to disseminate science

During his career, Henri Poincaré was very interested in the diffusion of scientific knowledge, and in the internationalisation of science. He was very involved in communicating his ideas to a general audience, and he remains unique among scientists and mathematicians in the way he presented his ideas to the public: he tried to make science popular. Poincaré was indeed engaged in many international scientific organizations, and also actively participated in the organisation of the first international congresses of mathematics and physics.

Henri Poincaré published four popular scientific books during his life:

La Science et l’Hypothèse (1902)

¹⁵ Emile François Loubet (1838-1929) was the eighth President of France.

¹⁶ Sarah Bernhardt (1844-1923) was a very famous and talented French stage and early film actress.

¹⁷ Anatole France (1844-1924) was a successful French poet, journalist and novelist.

¹⁸ Emile Boutroux (1845-1921) was a French philosopher of science and religion, and a historian of philosophy.

¹⁹ Georges Clémenceau (1841-1929) was a French politician, physician and journalist who served as Prime Minister of France during the First World War.

²⁰ Edouard Drumont was the director and publisher of the antisemitic newspaper *La Libre Parole*.

²¹ Charles Hermite (1822-1901) was a French mathematician who did research on number theory, quadratic forms, invariant theory, orthogonal polynomials, elliptic functions and algebra.

La Valeur de la Science (1905)

Science et Méthode (1908)

Dernières Pensées (posthumous) (1913) Most chapters of this last book are based on Poincaré's conferences, so they are independent one from another.

In his books, Poincaré tries to present a very large and wide panorama of science, and reflects on the role of theory in the structure of scientific reasoning, while assigning an important role to experience and experiments. All of his books are written in an easy language, accessible to everyone, which is pretty special. Poincaré wanted his book to be readable by everybody: kids, and people without a scientific background.

Poincaré also contributed to the international journal of mathematics *Acta mathematica*, founded by the Swedish mathematician Gösta Mittag-Leffler²².

Finally, modern mathematics arrived approximately at Poincaré's time, and we can say that Henri Poincaré was the father of several new theories and opened the scientific world to previously unknown fields of research. Some of his hypothesis were only verified many years after his disappearance, which shows his modernity (see 5.1 The conjecture of Poincaré).

1.6 Poincaré's main scientific contributions, and his many publications

Poincaré worked on many different scientific topics: he was able to switch continuously from one topic to another with an extraordinary rapidity. He published articles or books on many different subjects of mathematics, physics, and philosophy. Few scientists are able to excel at several different subjects like Henri Poincaré.

Poincaré devoted three long memoirs between 1890 and 1895 to the partial differential equations of mathematical physics. He invented the sweeping method to solve the Dirichletian problem, proved for the first time the existence of infinitely many eigenvalues²³ for the same problem, and introduced some inequalities which are still the cornerstones of the modern theory of partial differential equations. [5]

“Poincaré brought together the mathematical subjects of complex function theory and complex differential equations with an entirely unexpected use of non-Euclidean geometry, to create the theory of automorphic functions”. [4] “His most lasting achievement is his creation of the subject of algebraic topology, but he was one of the few to advance the subject of complex function theory in several variables in the 1900s, and he made important contributions to algebraic geometry and Sophus Lie's

²² Gösta Mittag-Leffler (1846-1927) was a Swedish mathematician. He mainly worked on the theory of functions.

²³ Eigenvalues are a special set of scalars associated with a linear system of equations. They are also known as characteristics roots, characteristic values, or proper values.

theory of transformation groups, and even to number theory.” [4] Indeed, at the beginning of the 20th century Poincaré published volumes on algebraic topology, and created the basic tools of algebraic topology. When working on curves defined by a particular type of differential equation, he also showed that the number and types of singular points are determined purely by the topological nature of the curve.

When it comes to mathematical physics, Poincaré worked a lot on trying to demonstrate the stability of the solar system. This means he had to show that equations of motion for the planets could be solved, and the orbits of the planets shown to be curves that stay in a bounded region of space for all time. Poincaré discovered that the motion could be chaotic, and that even small changes in the initial conditions could produce large, unpredictable changes in the resulting orbit. (see 5.2 Poincaré and the Three Body Problem: establishing the stability of the solar system) He then summarized his new mathematical methods in astronomy in *Les Méthodes nouvelles de la mécanique céleste*, 3 vol.²⁴ [6] We can also say that Poincaré’s main achievement in mathematical physics was his magisterial treatment of the electromagnetic theories of Hermann von Helmholtz, Heinrich Hertz, and Hendrik Lorentz. His work on this topic led him to write a paper in 1905 on the motion of the electron, and came close to anticipating Albert Einstein’s discovery of the theory of special relativity. [6] Finally, one of Poincaré’s last papers, published in 1912, was about quantum theory.

Besides his work in applied or theoretical mathematics and physics, Poincaré also published many papers in scientific philosophy (especially in *Science and Hypothesis* and in the three other volumes of the series previously presented), in which he discusses the role of logics in mathematics, the birth of set theory, the foundations of arithmetic. In those books, Poincaré expressed among other things his opinion on the fact that the aim of science is the prediction more than the explanation, and expressed strong feelings about the freedom of science.

Most of Poincaré’s papers are published in the ten volumes of his *Oeuvres*, and each one has some remarkable content. The first two volumes carry the theory of automorphic functions, the fourth volume is about Abelian functions and complex functions of several variables, volume 6 is about his work on the invention of algebraic topology, and volume 10 about his work on the partial differential equations of mathematical physics.

²⁴ “*The New Methods of Celestial Mechanics*”: 1892, 1893, 1899.

2. Poincaré's scientific influencers and opponants

2.1 His relationship with Charles Hermite

Charles Hermite²⁵ taught analysis at the Ecole Polytechnique²⁵, where Henri Poincaré was one of his students. [4]. When Poincaré stopped being his student, they both maintained a strong relationship. After joining the University of Caen and becoming a renowned mathematician, Poincaré conducted some studies and published a couple of essays on different subjects, some of which reflected direct influence from Hermite: mainly the theory of algebraic forms and arithmetic and the theory of linear differential equations in the complex domain. [7]

Hermite strongly supported Poincaré in the election for the geometry section of the Académie des sciences²⁶. [4] He notably praised Poincaré's work on the theory of Fuchsian functions, on number theory, on Abelian functions, and on functions of several complex variables. Hermite said of Poincaré that *"the talent and spirit of adventure of the young geometer [...] were worth the attention of the Académie."* [4] Later on, Hermite also wrote to Mittag-Leffler²¹, who was creating a journal for mathematics, and said that although *"in great fear of being overheard by Madame Hermite"*, he considered Poincaré to be the most brilliant of his three mathematical stars (the others were Appell² and Picard²⁷), and that he was *"a charming young man, who comes from Lorraine, like me, and knows my family very well."* [4]

Besides, Henri Poincaré published a famous article in which he wrote about Hermite, stating: *"I have never known a more realistic mathematician"*. Finally, it appears that one of the only times Hermite disagreed with Poincaré was during the Dreyfus Affair: Hermite was convinced of Dreyfus's guilt, while Poincaré delivered a scientific opinion on the case without condemning Dreyfus. [4] (see 1.4 The Dreyfus affair).

As a mathematician, Hermite had opinions and showed interest in several areas of mathematics, including pathological functions²⁸. He was strongly opposed to pathological functions, which were also

²⁵ Ecole Polytechnique (also known by the nickname "X") is a French public institution of higher education and research in Palaiseau near Paris. It was established in 1794 by the mathematician Gaspard Monge during the French Revolution and became a military academy under Napoleon 1 in 1804.

²⁶ The French Academy of Sciences (French: Académie des sciences) is a learned society founded in 1666 by Louis XIV to encourage and protect the spirit of French scientific research.

²⁷ Emile Picard (1856-1941) was a French mathematician. He had a seat at the Académie française.

²⁸ Pathological functions: functions continuous everywhere, but differentiable nowhere. A function is continuous when its graph is a single unbroken curve. Differentiable means that the derivative exists and it must exist for every value in the function's domain.

rejected by important mathematicians at the time, including Poincaré. It is what we call the Pathological Controversy. Hermite once said: *“I turn away with fear and horror from the lamentable plague of continuous functions which do not have derivatives...”*. [8] Perhaps Hermite and Poincaré were trying to protect the simplicity and elegance of analysis, and did not see any value in those functions, which came at the cost of the universal applicability of their theorems. [9]

2.2 Georg Cantor and set-theory

Georg Cantor²⁹ was a German mathematician who invented the set-theory. He introduced it to the public in a paper published in 1874: *“On a Property of the Collection of All Real Algebraic Numbers”*. Set theory is a branch of mathematical logic that studies sets, which informally are collections of objects. The language of set theory can be used in the definitions of nearly all mathematical objects. “Cantor began with the insight that we can distinguish the size of two collections of objects by paring them off one by one. If we find that one collection can be paired exactly in this way with the other collection so

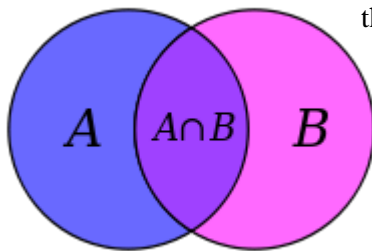


Figure 2 A Venn diagram illustrating the intersection of two sets. Source: [10]

that no object of either collection is left out or used more than once, then we say that the collections have the same number of objects. This paring, called a one-to-one correspondence, is the fundamental insight into the nature of number. It does not depend on which object in one collection is paired with which object in the other. We can use it to sort every collection of objects we come across into boxes, where the collections in each box are in one-to-one correspondence with each other and with no collection in any other box.” [4]

Cantor’s work polarized the mathematicians of his days. Karl Weierstrass, David Hilbert³⁰ and Richard Dedekind³¹ supported him, whereas Charles Hermite²¹, Leopold Kronecker³² and Henri Poincaré did not.

Indeed, according to Poincaré there were two schools: the pragmatists (to which he belonged) and the Cantorians. “The formers will only speak of objects that can be defined in a finite number of words, *“the others, on the other hand, think that objects exist in a sort of large store, independently of any mankind or any divinity that could talk or think about them... And from this initial misunderstanding*

²⁹ Georg Cantor (1845-1918) was a German mathematician. He invented set theory, which became a fundamental theory in mathematics.

³⁰ David Hilbert (1862-1943) was a German mathematician, recognized as one of the most influential and universal mathematicians of the 19th and 20th centuries.

³¹ Richard Dedekind (1831-1916) was a German mathematician. He made important contributions to abstract algebra, algebraic number theory, and the definition of real numbers.

³² Leopold Kronecker (1823-1891) was a German mathematician who worked on number theory, algebra and logic.

all sorts of divergences in details result". [4] The pragmatists consider that only one infinite cardinal is possible, the smallest, Aleph-zero³³, and doubt the existence of Aleph-one³⁴. "So they do not accept a definition that purports to define a whole set of objects if they cannot also examine the objects in the set, a restriction Cantorians find artificial and meaningless." [4] Cantorians don't need to see members of a set individually in order to know the set. "The main characteristic of Cantorians, as opposed to pragmatists, is that given a set they believe that they know *all* its members, something the pragmatists believe requires a construction for each object. Only after its construction does an object exist." [11] "For Poincaré, pragmatists were idealists who believed that a mathematical object exists only when it is conceived by the mind, and Cantorians were realists who believed in the existence of objects whether known or not." [4] [12] In a word, "Pragmatists adopt the point of view of extension. Cantorians adopt the point of view of comprehension." [11]

However, we could think that Poincaré was not always strongly supporting this separation between Cantorians and pragmatists. According to him, "Real mathematicians are more complicated", and he gave the example of Hermite, who he said, often claimed "I am anti-Cantorian because I am a realist." [11]

A controversy arose from one of Poincaré's interventions about set Theory. Even though there is no strict proof, it appears that he once referred to set theory as an interesting "*pathological case*", and predicted that "*Later generations will regard [Cantor's] Mengenlehre*³⁵ *as a disease from which one has recovered*" at the International Mathematical Congress of 1908 in Rome. [13] However, according to Jeremy Gray³⁶ the popular belief might give the impression that Poincaré was much more opposed to set theory than was indeed the case. "In Rome, Poincaré pointed out that there are certain problems, but one has the joy of the doctor called to a beautiful pathological case. Set theory, by implication, has a disease, but Poincaré did not say that set theory is itself a disease. Moreover, a doctor's attitude to a disease may be interesting, and Poincaré spoke of one's joy at being called to the problem." [13] So as a conclusion we can say that Poincaré might not have had such a strong position against set theory as sometimes attributed to him.

Moreover, Poincaré began to change his mind after he met Cantor in person in 1884. He started showing interest in Cantor's ideas, and applied them to his own work on automorphic functions. He used Cantor's work when trying to solve the Three Body Problem. (see 5.2 Poincaré and the Three Body Problem: establishing the stability of the solar system). "Following Cantor, Poincaré defined the derived set of P , noted P' . Poincaré showed that P' is what Cantor called a perfect set. [...] And Poincaré used

³³ Aleph-zero is the cardinality of the set of all natural numbers, and is an infinite cardinal.

³⁴ Aleph-one is the cardinality of the set of all countable ordinal numbers.

³⁵ Set theory (in German).

³⁶ Jeremy Gray (1947-) is an English mathematician primarily interested in the history of mathematics.

what Cantor said: “the point set [*Punktmenge*] formed by the different points of the trajectory, is everywhere dense [*überalldicht*] in the interior of the annulus.” [4] When secretary of the SMF (Société Mathématique de France), Poincaré even proposed Cantor for membership in 1885 (and Cantor was unanimously elected).

2.3 Emile Boutroux³⁷ and the conventionalist philosophy

We can notice direct influence on Poincaré’s philosophy from Boutroux.

“His [Poincaré] overall philosophy of mathematics is Kantian³⁸ because he believes that intuition provides a foundation for all of mathematics, including geometry.” [14] (see 3.2.3 The role of intuition in science). “It is often speculated that some of the Kantian aspects of Poincaré’s thought derive from Boutroux. [...] On Boutroux’s formulation, as on Poincaré’s, the mathematical laws of nature cannot be made to apply except by treating them as free choices of the mind, taken with as much pragmatism as is worthwhile.” [4] Indeed, Poincaré’s philosophical ideas corresponded to what we call conventionalist philosophy. “Conventionalist philosophy asserted that fundamental scientific principles are not reflections of the “real” nature of the universe but are convenient ways of describing the natural world insofar as they are not contradicted by observation or experiment.” Conventionalist philosophy “began to flourish in the 1870s with the return of Emile Boutroux from Germany and the philosophical inquiries shared by members of the Boutroux Circle.” [15] Poincaré’s philosophical position fits into a group of thinkers among which are Jules Tannery³⁹, Paul Tannery⁴⁰, Benjamin Baillaud⁴¹, and Emile Boutroux, who was his brother-in-law.

Boutroux addressed a number of important scientific and philosophical subjects. He drew up a critique of scientism⁴², defended positivist philosophy⁴³, and also discussed recent developments in thermodynamics, biological evolutionism, and psychophysics⁴⁴. “He [Boutroux] demonstrated the extent to which natural laws (including scientific laws and historical laws) are man’s freely reasoned

³⁷ Emile Boutroux (1845-1921) was a famous French philosopher of science and religion, and an historian of philosophy. He married Aline Poincaré, the sister of Henri Poincaré.

³⁸ Kantianism is the philosophy of Immanuel Kant (1724-1804), a German philosopher.

³⁹ Jules Tannery (1848-1910) was a French mathematician.

⁴⁰ Paul Tannery (1843-1904) was a French mathematician and historian of mathematics, older brother of Jules Tannery.

⁴¹ Benjamin Baillaud (1848-1934) was a French astronomer.

⁴² Scientism is a belief in the universal applicability of the scientific method and approach, and the view that the empirical method constitutes the most authoritative worldview or the most valuable part of human learning.

⁴³ Positivism is the view that the only authentic knowledge is scientific knowledge, and that such knowledge can only come from positive affirmation of theories through strict scientific method (techniques for investigating phenomena based on gathering observable, empirical and measurable evidence, subject to specific principles of reasoning).

⁴⁴ Psychophysics investigates the relationship between physical stimuli and the sensations and perceptions they produce.

creations and not nature's necessity [...] The idea that nature's laws themselves change in time was a concept which Boutroux based on evolutionary views of nature, which he combined with ideas of spontaneity and finality in nature." [15] Poincaré and Boutroux believed that "since the scientist relies closely on observation and experimentation, his understanding is linked intimately to the phenomenal world via his intuition." [15] "Like Boutroux, Poincaré argued that experimental laws are only approximate, and if some appear exact to us, it is because we have transformed them artificially into principles." [15] Indeed, experience cannot establish the truth of geometrical principles or arithmetical principles. [16] It seems natural to think that the physician uses his intuition to conduct experiments and establish new laws. But according to Poincaré, the mathematician also uses his intuition to translate observations into mathematical principles. We can therefore say that "intuition brings closer mathematical truth and physical truth." [16]

In the introduction of *La Science et L'Hypothèse*, Poincaré writes:

«De chaque expérience, une foule de conséquences pourront sortir par une série de déductions mathématiques, et c'est ainsi que chacune d'elles nous fera connaître un coin de l'Univers. » [La Science et l'Hypothèse, pages 1-2] « L'expérience nous laisse notre libre choix, mais elle le guide en nous aidant à discerner le chemin le plus commode. » [La Science et l'Hypothèse, page 3]

"From each experiment a crowd of consequences will follow by a series of mathematical deductions and thus each experiment will make known to us a corner of the universe." [Science and Hypothesis, pages 1-2] "Experiment leaves us our freedom of choice, but it guides us by aiding us to discern the easiest way." [Science and Hypothesis, page 3]

Therefore, there are no facts for which we can say that they are absolutely real, and no theories for which we can say they are absolutely true: the general scientific concepts slip from any verification. [16] Finally, Poincaré writes that scientists (mathematicians, physicians, biologists, etc.) all try to appeal to the most easy-going solution, which can let us think that they don't always go toward the « most true » laws or principles, but go instead toward the most convenient. [16]

However, Poincaré and Boutroux did not agree on every subject. Boutroux recognized that Poincaré did not share his view on the evolution of the "intrinsic laws of nature". Finally, Boutroux's philosophy did not achieve explicit recognition and admiration in most scientific circles, "perhaps because of its links with the Catholicism to which the French administration of the Third Republic was intractably opposed." [15]

3. “La Science et l’Hypothèse” : Poincaré’s thoughts

3.1 The book

3.1.1 Presentation

La Science et L’Hypothèse (*Science and Hypothesis*) was first published in 1902 by the editor Ernest Flammarion, in the collection “*Bibliothèque de philosophie scientifique*”⁴⁵. This collection was created by a collaboration between Gustave Le Bon⁴⁶ and Ernest Flammarion. Indeed, Gustave Le Bon contacted Flammarion to propose him to publish a series of books about popular science. “The aim was to produce a series of largely philosophical books that would also appeal to a scientific and an educated general audience. Le Bon commissioned the titles, and intervened actively to persuade authors to reflect his own views in their work. In the case of Poincaré, the decision was taken to reprint a selection of articles and prefaces to some of his more technical works, and Poincaré modified some of the texts.” [4] So this is how Poincaré was led to publish his book “*Science and Hypothesis*”. The book was remarkably successful. The first printing issued 1650 copies and sold out within weeks, and was rapidly reissued. A second corrected edition was published in 1906. By 1914, 20 900 copies of the book had been sold, and it still continues to sell a century later. [4]

The book aims at giving a very general and complete overview of science to the general public and non-professional readers. It is a popularisation of Poincaré’s philosophical point of view, and of his position in the scientific debates at the time. The book shows Poincaré’s strong involvement in the philosophical and scientific community. Poincaré writes about mathematics and physics. For each subject, he discusses open and general scientific questions, by comparing several laws or conventions previously established by scientists and by detailing his own arguments.

The book had a huge impact and was read by many people around the world. The very general aspect of the book might have raised some negative critics, but readers were generally satisfied about it. Besides, Wilson (1905), said: “Not logical enough for the logician, not mathematical enough for the mathematician, not physical enough for the physicist, not psychological enough for the psychologist, not metaphysical enough for the metaphysician, Poincaré’s *Science and Hypothesis* can hardly give the satisfaction of finality to anyone; and yet it probably comes nearer to satisfying the requirements of all these classes of investigators than any single book of our acquaintance.” [11] Moreover, since the chapters of the book are taken from published lectures of Poincaré and adapted or simplified, the readers

⁴⁵ “*Library of Scientific Philosophy*”

⁴⁶ Gustave Le Bon (1841-1931) was a French doctor, anthropologist, sociologist and psychologist. He wrote several books.

had the opportunity to elucidate what Poincaré meant in a chapter by turning to the much fuller academic versions.

Poincaré's book was among the first books to state that the theory of an absolute time in the universe should maybe be abandoned. Albert Einstein was inspired by this book when he wrote the *Annus Mirabilis papers* published in 1905, which included an article about the theory of special relativity.

The first translations of the book were into:

- German in 1904 and 1906 by F. and L. Lindermann
- English in 1905 by Walter Scott in London and by G. B. Halsted in New-York
- Spanish in 1907 by Gonzales Quijano
- Hungarian in 1908 by Szilard Béla
- Japanese in 1909 by Tsuruiche Hayashi
- Swedish in 1910 by Anna Sundqvist

Many other translated versions of the book have been published up to now. This shows that the book had quickly an important international impact and influence, and that it raised the interest of numerous readers around the world.

3.1.2 Structure

The book opens with an introduction highlighting the role of hypothesis in science, and calling the nature of mathematical reasoning into question. It contains four parts and has approximately 280 pages. The first part of the book focuses mainly on mathematics, and the second half of the book focuses on physics. The four parts are respectively about mathematics, space (including non-Euclidean geometry), physics (mechanics, relativity of movements, energy, thermodynamics) and nature (role of probabilities, optics, electricity, electrodynamics). In this thesis, we will focus on the two first parts of the book, about maths.

The first part, "*Number and Magnitude*", contains two chapters: "*On the Nature of Mathematical Reasoning*" and "*Mathematical Magnitude and Experience*". The two chapters are revised versions of articles respectively published in 1894 and 1893. [4]

The second part, entitled "*Space*", contains three chapters: "*The Non-Euclidean Geometries*", "*Space and Geometry*", and "*Experience and Geometry*". Those three chapters are revised versions of articles respectively published in 1891, 1895 and 1898. [4]

The third part, entitled “*Force*”, contains three chapters: “*The Classic Mechanics*”, “*Relative Motion and Absolute Motion*”, and “*Energy and Thermodynamics*”.

The fourth and last part, “*Nature*” contains five chapters: “*Hypothesis in Physics*”, “*The Theories of Modern Physics*”, “*The Calculus of Probabilities*”, “*Optics and Electricity*”, and “*Electrodynamics*”.

The chapter “*On the Nature of Mathematical Reasoning*” explores the precision, discernment and logics of mathematics. Poincaré also explains in detail mathematical induction, by studying the properties of addition and multiplication. In this chapter, Poincaré uses Leibniz’s work concerning additions as an example.

The chapter “*Mathematical Magnitude and Experience*” studies the notion and creation of continuum in mathematics and in physics, in one and several dimensions. It also explores the measurable magnitude, and how does the continuum become a measurable magnitude.

The chapter “*The Non-Euclidean Geometries*” aims to defend the idea that the axioms of geometry are not theoretical truths, neither experimental truths, but conventions. Experience helps us to choose the most convenient conventions. This chapter is divided into two parts. In the first one, Poincaré presents non-Euclidean geometries, including Lobachevskii’s geometry (hyperbolic geometry), and Riemann’s geometry (spherical geometry). In the second part, Poincaré explains his mathematical and philosophical interpretation of non-Euclidean geometries, and introduces what he calls the “fourth geometry”. [16]

In the chapter “*Space and Geometry*”, Poincaré tries to explain that experience has a very important role in geometry, but that it does not make geometry an experimental science. He compares the representative space and the geometrical space, and tries to explain what is the notion of space due to. Finally, he describes a hypothetical world with imaginative beings, and wonders what kind of geometry would those beings adopt, given the fact that their experience of space is modified.

In the chapter “*Experience and Geometry*”, Poincaré draws on the two previous chapters, on his exchanges with Russell, and on his articles in *The Monist* and the *Revue de Métaphysique et de Morale*. He highlights the importance of experience in the perception of the world, and he explores the influence experience can have on our understanding of Euclidean and non-Euclidean geometry. Finally, he concludes by saying that over the decades our spirit adapted to the conditions of the external world, and that we adopted the geometry the most advantageous -or the most convenient- to the human species; which in our case is Euclidean geometry. It is what Poincaré calls the “ancestral experience”.

3.2 Poincaré's mathematical reasoning

3.2.1 The role of hypothesis

In the introduction of his book *La Science et l'Hypothèse*, Poincaré briefly studies the role of hypothesis.

« On a aperçu la place tenue par l'hypothèse ; on a vu que le mathématicien ne saurait s'en passer et que l'expérimentateur ne s'en passe pas davantage. [...] nous devons donc examiner avec soin le rôle de l'hypothèse ; nous reconnâtrons alors, non seulement qu'il est nécessaire, mais que le plus souvent il est légitime. Nous verrons aussi qu'il y a plusieurs sortes d'hypothèses, que les unes sont vérifiables et qu'une fois confirmées par l'expérience, elles deviennent des vérités fécondes ; que les autres, sans pouvoir nous induire en erreur, peuvent nous être utiles en fixant notre pensée, que d'autres enfin ne sont des hypothèses qu'en apparence et se réduisent à des définitions ou à des conventions déguisées. »

[*La Science et l'Hypothèse*, Introduction]

« It was perceived how great a place hypothesis occupies; that the mathematician cannot do without it, still less the experimenter. [...] We ought therefore to examine with care the rôle of hypothesis; we shall then recognize, not only that it is necessary, but that usually it is legitimate. We shall also see that there are several sorts of hypotheses; that some are verifiable, and once confirmed by experiment become fruitful truths; that others, powerless to lead us astray, may be useful to us in fixing our ideas; that others, finally, are hypotheses only in appearance and are reducible to disguised definitions or conventions. » [Science and Hypothesis, Introduction]

A hypothesis is an idea or explanation for something that is based on known facts but has not yet been proved (*Cambridge dictionary*). It has yet to be tested through study and experimentation. Once the hypothesis has been carefully tested, and validated, it can then be called a theory. In a non-scientific context, the word hypothesis is used with less precision, and could be used to replace the words “theory” or “guess” in some cases. For instance, a policeman could have a hypothesis about who committed a robbery, which in this situation would be like making a guess.

According to Poincaré, the mathematician and the experimenter cannot work without making hypothesis. The following quote from Claude Bernard⁴⁷ backs up Poincaré's opinion about hypothesis:

“An hypothesis is, then, the necessary starting point for all experimental reasoning. Without it, we could not make any investigation at all nor learn anything; we could only pile up sterile observations.”

⁴⁷ Claude Bernard (1813-1878) was a French physiologist, and a great scientist.

But then, if the work of the scientists always depends on hypothesis, can we really rely on the results? Poincaré says that any hypothesis should be verified. But how can we verify a hypothesis? First of all, we can say that a hypothesis is not a blind guess. A hypothesis is an “educated guess”, based on what is already known or what has previously been observed. Therefore the hypothesis should be stated before the experiments are conducted.

Poincaré also pointed out the fact that André-Marie Ampère⁴⁸ made some irresponsible hypothesis in his book *Théorie des phénomènes électrodynamiques, uniquement fondée sur l'expérience*⁴⁹. The title of the book means that Ampère's work is only founded on experiments, and not at all on hypothesis. But the following scientists that showed some interest in Ampère's work, did notice some weaknesses and pointed out some hypothesis Ampère had made without being aware of it. Ampère's mistake was not that he used hypothesis, but that he used hypothesis without expressing them explicitly. Expressing explicitly the hypothesis contributes to the strictness and precision of the scientific reasoning.

“Nous verrons [...] quelles hypothèses inconscientes faisaient Ampère et les autres fondateurs de l'électrodynamique. » [La Science et l'Hypothèse, page 7] « Il [Ampère] s'imaginait donc qu'il n'avait fait aucune hypothèse ; il en avait fait pourtant, [...] ; seulement il les avait faites sans s'en apercevoir. » [La Science et l'Hypothèse, page 261]

« We shall see [...] what unconscious hypotheses were made by Ampère and the other founders of electrodynamics. [Science and Hypothesis, page 7] “He [Ampère] therefore imagined that he had made no hypothesis, but he had made them, [...] ; only he made them without being conscious of it.” [Science and Hypothesis, page 261]

We can wonder how a scientist can be sure that he is not making hypothesis without realising it? And what makes a good hypothesis? Is it possible to assess the reliability of a hypothesis? Maybe a hypothesis is reliable when the person formulating it has been studying it as much as possible, by asking many questions. *“According to Poincaré, no concept is admitted without a way of evaluating it and deciding upon its correctness. [...] He would always ask: How do you know?” [4]*

Finally, we can conclude with this quote from Claude Bernard⁵⁰:

“Scientific invention lies in the creation of a successful and fertile hypothesis; it is given by the genius of the savant who created it.” Claude Bernard, Introduction to the study of experimental medicine.

⁴⁸ André-Marie Ampère (1775-1836) was a French physicist and mathematician who was one of the founders of the science of classical electromagnetism, which he referred to as “electrodynamics”.

⁴⁹ *Theory of the electrodynamical phenomena, only founded on experience.*

3.2.2 Mathematical induction

a. History and definition

Mathematical induction is a mathematical proof technique used to prove a given statement about any well-ordered set. Most commonly, it is used to establish statements for the set of all natural numbers. The earliest implicit traces of mathematical induction may be found in Euclid⁵¹'s proof that the number of primes is infinite, and in Bhaskara's "cyclic method"⁵². "To demonstrate that any number A includes among its divisors a prime number B, Euclid showed that otherwise the number A would have had an infinity of divisors, each smaller than the one before." [17] An implicit proof by mathematical induction for arithmetic sequences was also introduced in the al-Fakhri written by al-Karaji⁵³ around 1000 AD, who used it to prove the binomial theorem and properties of Pascal's triangle⁵⁴. However none of these ancient mathematicians explicitly stated the inductive hypothesis. The first demonstration explicitly using the principle of induction was given by Francesco Maurolico⁵⁵ in the 16th century. [17] From then on, the method became more or less well-known, and was used by several mathematicians like Bernoulli⁵⁶, Fermat, etc. However, the modern rigorous treatment of the principle came only in the 19th century, with Henri Poincaré, George Boole⁵⁷, Richard Dedekind⁵⁸, Giuseppe Peano⁵⁹, etc. [18]

The principle of mathematical induction originally comes from a physical principle. There are two different kinds of mathematical induction. The first one, called "simple induction", worked on the assumption that when a few statements came from experience, then all the statements did. This "simple induction" principle was the first one used by mathematicians. The second principle, called the "complete induction", involves the infinity, and is the one used by mathematicians today. Poincaré

⁵¹ Euclid (300 BCE) was a Greek mathematician, often referred to as the "father of geometry".

⁵² Bhaskara II (1114-1185) was an Indian mathematician and astronomer. He was called the greatest mathematician of medieval India. The chakravala method, also called the cyclic method, is a cyclic algorithm to solve indeterminate quadratic equations. It is commonly attributed to Bhaskara II. Chakra means "wheel" in Sanskrit, and is a reference to the cyclic nature of the algorithm. This method contains traces of mathematical induction.

⁵³ Al-Karaji (c.953-c.1029) was a 10th century Persian mathematician and engineer. His three principal surviving works are mathematical: Al-Badi fi'l-hisab (Wonderful on calculation), Al-Fakhri fi'l-jabr wa'l-muqabala (Glorious on algebra) and Al-Kafi fi'l-hisab (Sufficient on calculation).

⁵⁴ Pascal's triangle is a triangular array of the binomial coefficients. It is named after the French mathematician, physicist and philosopher Blaise Pascal (1623-1662), although other mathematicians studied it centuries before him.

⁵⁵ Francesco Maurolico (1494-1575) was a mathematician and astronomer from Sicily.

⁵⁶ Jacob Bernoulli (1655-1705) was a Swiss mathematician, and one of the many prominent mathematicians in the Bernoulli family.

⁵⁷ George Boole (1815-1864) was an English mathematician, educator, philosopher and logician.

⁵⁸ Richard Dedekind (1831-1916) was a German mathematician.

⁵⁹ Giuseppe Peano (1858-1932) was an Italian mathematician.

“defended the view that the principle of complete induction is both necessary to the mathematician and not reducible to logic.” [11]

Definition of the mathematical induction principle:

Assume you want to prove that for some statement P , $P(n)$ is true for all n starting with $n=1$. The *Principle of Mathematical Induction* states that, to this end, one should accomplish just two steps:

- 1) Prove that $P(1)$ is true.
- 2) Assume that $P(k)$ is true for some k . Derive from here that $P(k+1)$ is also true.

The idea of Mathematical Induction is that a finite number of steps may be needed to prove an infinite number of statements $P(1), P(2), P(3), \dots$

In the first chapter of *Science and Hypothesis*, Poincaré defines mathematical induction and explains it to his readers, using a non-mathematical vocabulary:

“On voit donc que dans les raisonnements par récurrence, on se borne à énoncer la mineure du premier syllogisme, et la formule générale qui contient comme cas particuliers toutes les majeures. Cette suite de syllogismes qui ne finirait jamais se trouve ainsi réduite à une phrase de quelques lignes. [...] [le raisonnement par récurrence] est un instrument qui permet de passer du fini à l’infini. [...] il nous dispense de vérifications longues, fastidieuses et monotones qui deviendraient rapidement impraticables. Mais il devient indispensable dès qu’on vise au théorème général, dont la vérification analytique nous rapprocherait sans cesse, sans nous permettre de l’atteindre. [...] » [La Science et l’Hypothèse, pages 20-22]

“We see, then, that in reasoning by recurrence we confine ourselves to stating the minor of the first syllogism, and the general formula which contains as particular cases all the majors. This never-ending series of syllogisms is thus reduced to a phrase of a few lines. [...] [reasoning by recurrence] is an instrument which enables us to pass from the finite to the infinite. [...] it spares us verifications, long, irksome and monotonous, which would quickly become impracticable. But it becomes indispensable as soon as we aim at the general theorem, to which analytic verification would bring us continually nearer without ever enabling us to reach it. [...]” [Science and Hypothesis, pages 20-22]

The above definition written by Poincaré matches the mathematical definition of reasoning by recurrence given previously. In a few words, Poincaré says that reasoning by recurrence is a method

that allows to reduce an infinite series of syllogisms⁶⁰ into a few lines. It is a method that helps establishing general theorems without having to go through extremely long demonstrations. This method, which enables the mathematician to pass from the finite to the infinite, is often called induction. However, the word “induction” should be used carefully. It does not refer to the everyday meaning of induction (going from specific to general). This mathematical method is in any case about induction, because it does not go from specific to general: it goes from finite to infinite. [17] Moreover, for Poincaré, “an infinite set consists of a finite but growing set of elements along with all the objects specified by any method that can be invented, “And it is ‘that can’ which is the infinity”.” [11] Poincaré’s main advices regarding infinite sets are:

- “1. Never consider any objects but those capable of being defined in a finite number of words.
2. Never lose sight of the fact that every proposition concerning infinity must be the translation, the precise statement of propositions concerning the finite;
3. Avoid non-predicative classifications and definitions.” [11]

b. Discussion on the nature of reasoning by recurrence

In the first chapter of “*Science and Hypothesis*”, called “*On the Nature of Mathematical Reasoning*”, Poincaré analyses the reasoning by recurrence. According to him, the mathematical induction is obtained on the basis of two logical principles: the hypothetical syllogism⁶⁴:

« Le caractère essentiel du raisonnement par récurrence c’est qu’il contient, condensés pour ainsi dire en une formule unique, une infinité de syllogismes. [...] Ce sont bien entendu des syllogismes hypothétiques. [...] Si le théorème est vrai de $n-1$, il l’est de n . » [La Science et l’Hypothèse, page 20]

“The essential characteristic of reasoning by recurrence is that it contains, condensed, so to speak, in a single formula, an infinity of syllogisms. [...] These are of course hypothetical syllogisms. [...] If the theorem is true of $n-1$, so it is of n .” [Science and Hypothesis, page 20]

... and the principle of universal instantiation⁶¹ (UI, also called universal specification):

⁶⁰ A syllogism is a deductive scheme of a formal argument consisting of a major and a minor premise and a conclusion. Example: Every virtue is laudable; kindness is a virtue; therefore kindness is laudable. An hypothetical syllogism is a syllogism consisting wholly or partly of hypothetical propositions. Example: If I do not wake up, then I cannot go to work. If I cannot go to work, then I will not get paid. Therefore, if I do not wake up, then I will not get paid.

⁶¹ The universal instantiation is a valid rule of inference from a truth about each member of a class of individuals to the truth about a particular individual of that class. It is one of the basic principles used in quantification theory. Example: All dogs are mammals. Fido is a dog. Therefore Fido is a mammal.

« [...], il faut qu'on trouve quelque avantage à considérer la construction plutôt que ses éléments eux-mêmes. [...] C'est qu'il y a des propriétés qu'on peut démontrer pour les polygones d'un nombre quelconque de côtés et qu'on peut ensuite appliquer immédiatement à un polygone particulier quelconque. Le plus souvent, au contraire, ce n'est qu'au prix des plus longs efforts qu'on pourrait les retrouver en étudiant directement les rapports des triangles élémentaires. La connaissance du théorème général nous épargne ces efforts. » [La Science et l'Hypothèse, page 27]

“[...], there must be some advantage in considering the construction rather than its elements themselves. [...] It is because there are properties appertaining to polygons of any number of sides and that may be immediately applied to any particular polygon. Usually, on the contrary, it is only at the cost of the most prolonged exertions that they could be found by studying directly the relations of the elementary triangles. The knowledge of the general theorem spares us these efforts.” [Science and Hypothesis, page 27]

Then, in the following quote from the first chapter, Poincaré mainly wonders whether the induction principle (also called reasoning by recurrence) could be reduced to simpler logical principles.

« Le jugement sur lequel repose le raisonnement par récurrence peut être mis sous d'autres formes ; on peut dire par exemple que dans une collection infinie de nombres entiers différents, il y en a toujours un qui est plus petit que tous les autres. [...] Mais on sera toujours arrêté, on arrivera toujours à un axiome indémontrable [...] On ne peut donc se soustraire à cette conclusion que la règle du raisonnement par récurrence est irréductible au principe de contradiction. Cette règle ne peut non plus nous venir de l'expérience ; [...]. c'est seulement devant l'infini que ce principe [de contradiction] échoue, c'est également là que l'expérience devient impuissante. Cette règle [le raisonnement par récurrence], inaccessible à la démonstration analytique et à l'expérience, est le véritable type du jugement synthétique à priori. » [La Science et l'Hypothèse, pages 22-23]

“The judgment on which reasoning by recurrence rests can be put under other forms; we may say, for example, that in an infinite collection of different whole numbers there is always one which is less than all the others. [...] But we shall always be arrested, we shall always arrive at an undemonstrable axiom [...] We can not therefore escape the conclusion that the rule of reasoning by recurrence is irreducible to the principle of contradiction. Neither can this rule come to us from experience; [...] it is only before the infinite that this principle [of contradiction] fails, and there too, experience becomes powerless. This rule [reasoning by recurrence], inaccessible to analytic demonstration and to experience, is the veritable type of the synthetic a priori judgement.” [Science and Hypothesis, pages 22-23]

The notions of analytical and synthetic judgment are borrowed from Kant⁶². According to Kant, a statement is analytical if its truth is logically deduced from the non-contradiction principle, and a statement is synthetic if its truth does not only depend on the principle of non-contradiction. Also according to Kant, a statement is *a priori* if it does not depend on experience, but structures experience. Otherwise, it is said *a posteriori*. Kant believes all the arithmetical statements are *a priori*, because independent from experience. [19] In the quote above, Poincaré wrote that the mathematical induction involves an infinity of statements and can therefore not be reduced to the contradiction principle. Poincaré gives also an argument in favour of the *a priori* characteristic of the induction principle. He says that if the principle was *a posteriori* (if it was from experience) its validity would depend on a generalization made based on a finite number of cases. In other words, it would be equivalent to the same kind of inductive generalization that leads us from the observation of a sample of black crows to the generalisation that “all crows are black”. But the trust we have in the induction principle is much higher than the one we have in a generalisation based on experience. In short, the induction principle is for Poincaré a principle that is neither demonstrable from logic principles more elementary nor based on experience. Thus, Poincaré concludes that the mathematical induction, beyond reach of analytical demonstration and experience, is the “*true type of synthetic judgement a priori*”.

Finally, Poincaré concludes that the induction principle:

« n'est que l'affirmation de la puissance de l'esprit qui se sait capable de concevoir la répétition indéfinie d'un même acte dès que cet acte est une fois possible. L'esprit a de cette puissance une intuition directe et l'expérience ne peut être pour lui qu'une occasion de s'en servir et par là d'en prendre conscience. » [La Science et l'Hypothèse, page 23-24]

“is only the affirmation of the power of the mind which knows itself capable of conceiving the indefinite repetition of the same act when once this act is possible. The mind has a direct intuition of this power, and experience can only give occasion for using it and thereby becoming conscious of it.” [Science and Hypothesis, page 23-23]

The previous quote shows perfectly the importance of intuition in Poincaré's conception, since the founding principle he offers is based on an intuition of the mind. [19] (see 3.2.3. The role of intuition in science).

3.2.3 The role of intuition in science

Intuition is the ability to understand or know something without needing to think about it (Cambridge Dictionary). This definition might not be perfectly accurate, or might not communicate what intuition

⁶² Immanuel Kant (1724-1804) was a German philosopher.

exactly is about. Etymologically, intuition comes from the Latin words “intuition” -image reflected in a mirror-, and “intueri” -look attentively inside. [20] Therefore, intuition is a form of spontaneous understanding, which does not rely on experience nor on sensory indications: it is the access to a knowledge that comes from inside us. This means that there is inside us knowledge that we usually look for in the outside world. For a scientist, intuition can be like an internal wisdom, or a revelation. In fact, Poincaré expressed at several occasions how intuition was essential for him, and he once said that “*even the next generation of leading mathematicians will need intuition, for if it is by logic that one proves, it is by intuition that one invents*”⁶³. [4] Intuition can also explain why some notions are indefinable or some principles are indemonstrable.

Intuition can indeed become a tool of orientation in science, and lead the scientist toward new ideas, and toward the conclusion he is looking for. We could therefore say that intuition is not only a tool to facilitate reasoning, but a tool of discovery. “The role of intuition would be to grasp the real, as it escapes the settings science has prepared [...]. Perhaps intuition expresses the deep progress of objects, the continual variation, and maybe even, the final destiny of Universe.” [17] “Intuition, in its original meaning, is the apprehension of an object with the eyes. The mathematical knowledge will be intuitive, [...] as it will be about notions that are accompanied by images. Therefore geometry, as Greeks created it, is an intuitive science; mathematicians who used the geometrical representation to solve problems of abstract mathematics, [...] are considered as intuitive.” [17]

Furthermore, there can be several kinds of intuition. For instance, Pascal opposed rational (Cartesian) intuition which relies on a clear and distinct notion of area, to intuition of feeling which “principles remind implicit, but which, cleared from logical scruples, is even more agile for the conquest of the infinite.” [17]

However, Poincaré also said, at a Conference for which he studied the role of intuition and logic in mathematics, that if we include into intuition any knowledge which is not the outcome of a logical reasoning, we expose ourselves to inextricable uncertainties. [17] Thus we can conclude that intuition is essential to mathematics and science, but that it has to be carefully manipulated, because intuition does not act as a guarantor for truth. The scientist and the mathematician should always be aware that intuition is a precious asset, but is not infallible. Intuition should therefore be backed up with other scientific facts, demonstrations, observations, etc. But even though intuition has to be manipulated with care, it is still a great tool that can be the cause of great discoveries, and therefore should be used by scientists. As Einstein said: “*The intuitive mind is a sacred gift and the rational mind is a faithful servant. We have created a society that honors the servant and has forgotten the gift.*” Einstein’s quote

⁶³ Poincaré (1899d). La logique et l’intuition dans la science mathématique et dans l’enseignement. *Enseignement mathématique* 1, 157-162. In *Œuvres* 11, 129-133.

shows that intuition is not relied upon enough. Maybe too many people don't have enough the audacity to rely on their intuition, or don't have enough confidence to trust their intuition. It could be therefore important and useful to educate in universities future scientists about intuition.

3.2.4 Mathematical Magnitude

a. Physical and mathematical continuum

In the second chapter of the book "Science and Hypothesis", called "Mathematical Magnitude and Experience", Poincaré analyses the concept of continuum. He defines the mathematical continuum, the physical continuum, and compares them. Poincaré strongly contributed to the advancement of the notion of continuum, mainly elaborated by logician mathematicians in the 19th century, like Cantor⁶⁴. According to Poincaré, physics have given mathematics the concept of the continuum, without which little mathematics could have been done. [4] However, Poincaré also writes that the mathematical and physical continua are very different:

« le véritable continu mathématique est tout autre chose que celui des physiciens et celui des métaphysiciens. » [La Science et l'Hypothèse, page 30]

"the veritable mathematical continuum is a very different thing from that of the physicists and that of the metaphysicians." [Science and Hypothesis, page 30]

The difference between mathematical continuum and physical continuum is explained in the following quotes from Poincaré, in which he defines mathematical continuum and physical continuum. The first definition of mathematical continuum given in the book is the following:

« Partons de l'échelle des nombres entiers ; entre deux échelons consécutifs, intercalons un ou plusieurs échelons intermédiaires, puis entre ces échelons nouveaux d'autres encore, [...]. Nous aurons ainsi un nombre illimité de termes, ce seront les nombres que l'on appelle fractionnaires, rationnels ou commensurables. [...] entre ces termes qui sont pourtant déjà en nombre infini, il faut encore en intercaler d'autres, que l'on appelle irrationnels ou incommensurables. [...] Le continu ainsi conçu n'est plus qu'une collection d'individus rangés dans un certain ordre, [...], mais extérieurs les uns aux autres. Ce n'est pas là la conception ordinaire, où l'on suppose entre les éléments du continu une sorte de lien intime qui en fait un tout [...]. Le continu est l'unité dans la multiplicité. » [La Science et l'Hypothèse, page 30]

⁶⁴ Georg Cantor (1845-1918) was a German mathematician who invented the set theory.

“Let us start from the scale of whole numbers; between two consecutive steps, intercalate one or more intermediary steps, then between these new steps still others, [...]. Thus we shall have an unlimited number of terms; these will be the numbers called fractional, rational or commensurable. [...] between these terms, which, however, are already infinite in number, it is still necessary to intercalate others called irrational or incommensurable. [...] The continuum so conceived is only a collection of individuals ranged in a certain order, [...] but exterior to one another. This is not the ordinary conception, wherein is supposed between the elements of the continuum a sort of intimate bond which makes them a whole [...] ‘the continuum is unity in multiplicity’.” [Science and Hypothesis, page 30]

So according to Poincaré, the mathematical continuum is an infinite series of numbers, intercalated on a scale of integers. All those numbers are equidistant. This definition incites Poincaré to suggest that our knowledge of fractional and incommensurable⁶⁵ numbers only comes from our intuition and perception of a matter infinitely divisible, like a continuum. He therefore believes that our use of fractional and incommensurable numbers does not originate from experience, but from a pure creation of the mind.

Poincaré then defines the physical continuum in the following way, quoting Fechner⁶⁶'s experiment:

« On a observé, qu'un poids A de 10 grammes et un poids B de 11 grammes produisaient des sensations identiques, que le poids B ne pouvait non plus être discerné d'un poids C de 12 grammes, mais que l'on distinguait facilement le poids A du poids C. Les résultats bruts de l'expérience peuvent donc s'exprimer par les relations suivantes : $A=B$, $B=C$, $A<C$ qui peuvent être regardées comme la formule du continu physique. » [La Science et l'Hypothèse, pages 34-35]

« un système d'éléments formera un continu, si l'on peut passer d'un quelconque d'entre eux à un autre également quelconque, par une série d'éléments consécutifs tels que chacun d'eux ne puisse se discerner du précédent. » [La Science et l'Hypothèse, page 45]

“It has been observed, that a weight A of 10 grams and a weight B of 11 grams produce identical sensations, that the weight B is just as indistinguishable from a weight C of 12 grams, but that the weight A is easily distinguished from the weight C. Thus the raw results of experience may be expressed by the following relations: $A=B$, $B=C$, $A<C$, which may be regarded as the formula of the physical continuum.” [Science and Hypothesis, pages 34-35]

⁶⁵ Two numbers are incommensurable if their ratio is irrational (cannot be written as a rational number).

⁶⁶ Gustav Fechner (1801-1887) was a German philosopher, physicist and experimental psychologist.

“a system of elements will form a continuum if we can pass from any one of them to any other, by a series of consecutive elements such that each is indistinguishable from the preceding.” [Science and Hypothesis, page 45]

The first definition of physical continuum above, based on Fechner’s experiment, shows that the notion of physical continuum is dependent on our sensations. The physical continuum is a group of impressions that can be compared, but not correctly distinguished. Since our sensations are imperfect and subject to error, the physical continuum is characterized by a certain vagueness. And the approximations we make with our sensations are inevitable. We can for instance improve our perception of temperature with a thermometer, or improve our vision with a microscope, but we will always use those instruments with our sensations. Even what we observe with the help of those instruments will be falsified by our human sensations. Therefore, the physical continuum is not strictly rigorous, and cannot be applied to mathematics, which require more rigorous rules and principles. In the mathematical continuum, one has $A < B < C$. Poincaré concludes that the definition of mathematical continuum cannot rely upon our sensations, otherwise it would very quickly lead to contradictions with mathematical principles. For instance, the relation $A = B$, $B = C$, $C < A$ is a contradiction with which a mathematician has to free himself from. And the mathematician will free himself from this kind of contradiction by not relying upon the definition/principle of physical continuum.

b. Definition of the dimensions of a continuum

Poincaré then defines and makes a distinction between the different dimensions of a mathematical continuum and of a physical continuum.

First, the mathematical continuum of second order is defined as an infinite series of numbers containing incommensurable numbers:

« Qu’on me permette, [...] d’appeler continu mathématique du premier ordre tout ensemble de termes formés d’après la même loi que l’échelle des nombres commensurables. Si nous y intercalons ensuite des échelons nouveaux d’après la loi de formation des nombres incommensurables, nous obtiendrons ce que nous appellerons un continu du deuxième ordre. » [La Science et l’Hypothèse, page 38]

“Let me call [...] a mathematical continuum of the first order every aggregate of terms formed according to the same law as the scale of commensurable numbers. If we afterwards intercalate new steps according to the law of formation of incommensurable numbers, we shall obtain what we will call a continuum of the second order.” [Science and Hypothesis, page 38]

According to Poincaré, the creation of a continuum of the second order comes from the need to have incommensurable numbers. He explains that it was necessary to invent the incommensurable numbers, because if we had not, only the geometrical points which have commensurable coordinates would have been considered as real:

« si les points dont les coordonnées sont commensurables étaient seuls regardés comme réels, le cercle inscrit dans un carré et la diagonale de ce carré ne se couperaient pas, puisque les coordonnées du point d'intersection sont incommensurables. » [La Science et l'Hypothèse, page 39]

“if the points whose coordinates are commensurable were alone regarded as real, the circle inscribed in a square and the diagonal of this square would not intersect, since the coordinates of the point of intersection are incommensurable.” [Science and Hypothesis, page 39]

Therefore, mathematicians need the continuum of the second order without which many arithmetic and geometric phenomena would have been impossible to be studied, clarified and explained. Poincaré then adds that the mathematical continuum of third order is too small, and is therefore not considered seriously or used by mathematicians and geometers.

In order to define the physical continuum with several dimensions, Poincaré introduces the notion of “cut”, to divide the continuum:

« Envisageons un continu C et enlevons-lui certains de ses éléments [...]. L'ensemble des éléments ainsi enlevés s'appellera une coupure. [...] grâce à cette coupure, C [sera] subdivisé en plusieurs continus distincts [...]. Si on peut subdiviser un continu physique C par une coupure se réduisant à un nombre fini d'éléments tous discernables les uns des autres [...], nous dirons que C est un continu à une dimension. [...] S'il suffit de coupures qui soient des continus à une dimension, nous dirons que C a deux dimensions, [...], et ainsi de suite. » [La Science et l'Hypothèse, pages 45-46]

“Consider a continuum C and remove from it certain of its elements [...]. The aggregate of the elements so removed will be called a cut. [...] thanks to this cut, C may be subdivided into several distinct continua [...]. If a physical continuum C can be subdivided by a cut reducing to a finite number of elements all distinguishable from one another [...], we shall say C is a one-dimensional continuum. [...] If cuts which are continua of one dimension suffice, we shall say C has two dimensions, [...], and so on.” [Science and Hypothesis, pages 45-46]

What this means is that to divide a space (a volume), we need cuts in the form of surfaces. To divide surfaces, we need cuts in the form of lines. And to divide lines, we need cuts in the form of points. Points are not a continuum. Lines are a continuum of one dimension, surfaces are a continuum of two

dimensions, and volumes are a continuum of three dimensions. However, Poincaré notices that experience does not prove us that space has three dimensions. It only proves us that it is convenient to assign three dimensions to it. Finally, Poincaré notices a difference between the representative space, which is a physical continuum, and the geometrical space, which is a mathematical continuum. [21]

4. The evolution of geometry, and Poincaré's contribution to it

"Poincaré is regarded as one of the great geometers." [4]

4.1 Poincaré and non-Euclidean geometry

According to the dictionary of Cambridge, geometry is the area of mathematics relating to the study of space and the relationships between points, lines, curves and surfaces.

4.1.1 Introduction

Poincaré published in 1891 an essay called "Les géométries non euclidiennes" (*Non-euclidean geometries*) in the second volume of the *Revue Générale des sciences pures et appliquées*. [4] He also wrote about non-Euclidean geometry in the second part of his book "La Science et l'Hypothèse" (*Science and Hypothesis*). Poincaré became a well-known mathematician thanks to his contribution to the three-body problem, but his work a couple of years later on non-Euclidean geometry marked the start of a long controversy. Non-Euclidean geometry was one of Poincaré's main interests over his scientific career. His philosophy of geometry was rooted in his analysis of how we experience the world around us and construct our sense of geometrical space. [11]



Figure 3 Documents where Poincaré's essay on non-Euclidean geometry can be found: *Science and Hypothesis* book on the left and *Revue générale des sciences pures et appliquées* on the right. Source : exposé du CNRS sur Henri Poincaré et le monde non euclidien (diap)

4.1.2 Previous to Poincaré and to non-Euclidean geometry: Euclidean geometry

During the fourth and third centuries B.C.E⁶⁷, Euclid⁶⁸ wrote *The Elements*, in which he laid down the foundations for working with various two- and three- dimensional shapes. Most of what he wrote had been discovered before by other mathematicians, but Euclid was the first systemizing all the previous observations into a single coherent system. The geometry he established was called Euclidean Geometry, and is also sometimes called Plane Geometry (because it studies the geometric properties of objects that exist in a flat two-dimensional plane). There were more than a thousand different printed editions of Euclid's *The Elements* since 1482. It is the most studied and edited work of all times, after the Bible. [23]

Euclidean geometry is the study of plane and solid figures on the basis of axioms and theorems employed by Euclid, and it is the geometry nowadays commonly taught in secondary schools. Five main axioms are based upon this geometry.

1. A straight line segment can be drawn joining any two points.
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as centre.
4. All right angles are congruent.
5. If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough. This postulate is what we call the parallel postulate.

Euclid used only the first four postulates for the first 28 propositions of the *Elements*, but was forced to invoke the parallel postulate (fifth postulate) for the 29th proposition. For years, many mathematicians attempted to prove the fifth postulate as a theorem, but no one succeeded. [24] Poincaré even wrote about it:

« On a longtemps cherché en vain à démontrer également le troisième axiome, connu sous le nom de postulatum d'Euclide. Ce qu'on a dépensé d'efforts dans cet espoir chimérique est vraiment inimaginable. Enfin au commencement du siècle à peu près en même temps, deux savants, un Russe et un Hongrois, Lowatschewsky et Bolyai établirent d'une façon irréfutable que cette démonstration est impossible ; ils nous ont à peu près débarrassés des inventeurs de géométries sans postulatum ; depuis

⁶⁷ Before Christ Existed

⁶⁸ Euclid (300 BCE), sometimes called Euclid of Alexandria was a Greek mathematician, often referred to as the "father of geometry".

lors l'Académie des Sciences ne reçoit plus guère qu'une ou deux démonstrations nouvelles par an. »
[La Science et l'Hypothèse, page 50]

“It was long sought in vain to demonstrate likewise the third axiom, known as Euclid's Postulate. What vast effort has been wasted in this chimeric hope is truly unimaginable. Finally, in the first quarter of the nineteenth century, and almost at the same time, a Hungarian and a Russian, Bolyai and Lobachevski, established irrefutably that this demonstration is impossible; they have almost rid us of inventors of geometries ‘sans postulat’; since then the Académie des Sciences receives only about one or two new demonstrations a year.” [Science and Hypothesis, page 50]

Indeed, in 1823 Janos Bolyai⁶⁹ and Nicolai Lobachevskii⁷⁰ independently realized that entirely self-consistent “non-Euclidean geometries” could be created in which the parallel postulate did not hold. [24] (see 3.1.3.1)

Several famous theorems, like Pythagora's Theorem follow on from Euclidean Geometry.



Figure 4 Oxyrhynchus papyrus showing fragment of Euclid's Elements, AD 75-125 (estimated) [23]

4.1.3 Arrival of the new geometry: the non-Euclidean geometry

However, our world is not a two-dimensional world and thus the rules of Euclidean geometry do not apply to it. The term non-Euclidean geometry refers to any kind of geometry that is not Euclidean, thus which does not exist in a flat world. Several geometries are not Euclidean. In the non-Euclidean geometry, we redefine the properties of points, lines and several shapes are redefined. For instance, spherical geometry (which is a kind of plane geometry warped onto the surface of a sphere) is one example of non-Euclidean geometry.

⁶⁹ Janos Bolyai (1802-1860) was a Hungarian mathematician, one of the founders of non-Euclidean geometry.

⁷⁰ Nicolai Ivanovich Lobachevskii (1792-1856) was a Russian mathematician, mainly known for his work on non-Euclidean geometry, also known as Lobachevskian geometry.

a. Lobachevskii's and Bolyai's geometry

Lobachevskii and Bolyai's geometry is a non-Euclidean geometry. It is also called hyperbolic geometry. Lobachevskii and Bolyai started studying non-Euclidean geometry quite early, by the mid-1820s. But they did not manage to spread their work, and the scientific community was not ready to listen to them. So unfortunately, they died without having received support for their work, and without suspecting their theory would become accepted a couple of years later. The only recognition Bolyai and Lobachevskii ever received "came from Carl Friedrich Gauss⁷¹, who was widely recognized as one of the greatest mathematicians ever, but who confined his enthusiasm for their work to his correspondence with friends and the Göttingen Scientific Society⁷²." [4] However, when Gauss died mathematicians were able to access his unpublished scientific papers, and they discovered his work on non-Euclidean geometry. [4] Due to Gauss's influence in the scientific community, this discovery was enough for mathematicians to start accepting Lobachevskii and Bolyai's new geometry.

Lobachevskii's geometry (hyperbolic geometry) is based on the same fundamental postulates as Euclidean geometry, except for the fifth one, the axiom of parallelism, which is replaced by its negation. In Euclidean geometry, according to the fifth axiom, in a plane through a point P not lying on a straight line $A'A$ there passes precisely one line $B'B$ that does not intersect $A'A$. The line $B'B$ is called parallel to $A'A$. It is sufficient to require that there is at most one straight line, since the existence of a non-intersecting line can be proved by successively drawing lines PQ perpendicular to $A'A$ and PB perpendicular to PQ . [25]

But in Lobachevskii's geometry (hyperbolic geometry), the axiom of parallelism requires that through a point P there passes more than one line not intersecting $A'A$: there exist a line $A'A$ and a point P not on $A'A$ such as at least two distinct lines parallel to $A'A$ pass through P . This can be seen on figure 5, and is called the Universal Hyperbolic Theorem.

⁷¹ Carl Friedrich Gauss (1777-1855) was a German mathematician who contributed significantly to any fields, including number theory, algebra, statistics, analysis, differential geometry, matrix theory, optics.

⁷² The Göttingen Scientific Society is the second oldest of the seven academies of sciences in Germany. Its purpose is to promote research and collaboration with academics in and outside Germany.

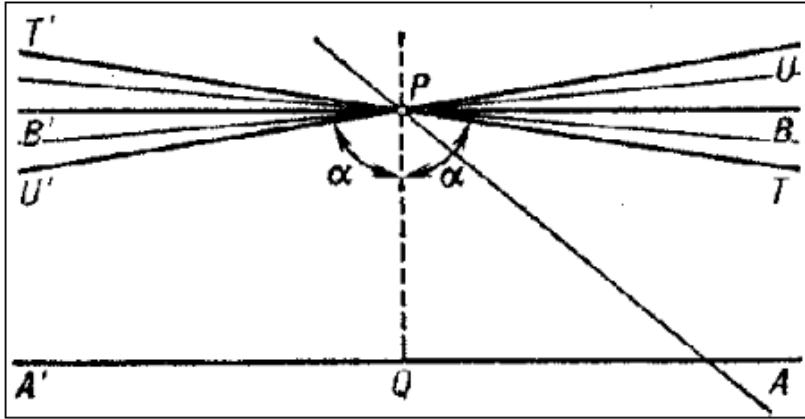


Figure 5 Intersecting lines parallel to $A'A$. Source: [25]

The first property that can be obtained from this axiom is the following lemma:

Lemma 1: *Rectangles don't exist in hyperbolic geometry.*

This lemma can be used to prove the Universal Hyperbolic Theorem (which won't be proved here).

Universal Hyperbolic Theorem (theorem 1): *In hyperbolic geometry, for every line $A'A$ and every point P not on $A'A$ there pass at least two distinct parallels to $A'A$ through P . Moreover there are infinitely many parallels to $A'A$ through P .*

Lemma 1 also leads to the following theorems:

Theorem 2: *In hyperbolic geometry, all triangles have angle sum inferior to 180 degrees. And all convex quadrilaterals have angle sum less than 360 degrees.*

Theorem 3: *In hyperbolic geometry if two triangles are similar, they are congruent.* [26]

i) The upper half-plane model and Poincaré plane model

There are more popular models for the hyperbolic plane: the upper half-plane model and the Poincaré plane model. There are two seemingly different types of hyperbolic lines in the half-plane model, both defined in terms of Euclidean objects in \mathbb{C} (the complex plane). One is the intersection of the half-plane with a Euclidean line in the complex plane perpendicular to the real axis \mathcal{R} . The other is the intersection of \mathcal{H} with a Euclidean circle centered on the real axis \mathcal{R} . [26]

The functioning of the upper half-plane model won't be explained in detail here, since this model is not very good to visualise objects like polygons or circles. However, the Poincaré plane model is more

appropriate for the visualisation of those objects. The underlying space of Poincaré's plane model is an open unit disk \mathcal{D} in the complex plane.

$$\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$$

In the Poincaré plane model, lines are given by diameters of the unit circle $C = \{z \in \mathbb{C} \mid |z|=1\}$ or of circles that intersect C at right angles. The hyperbolic distance in \mathcal{D} is easier to calculate than in the half-space model (see the following property). Therefore since distances can be easily measured, it is now possible to define circles and polygons, and to calculate areas.

Property 1 For each pair x and y of points in \mathbb{D} we have that:

$$\frac{|x - y|^2}{(1 - |x|^2)(1 - |y|^2)} = \frac{1}{2} \left(\frac{e^{d(x,y)} + e^{-d(x,y)}}{2} - 1 \right)$$

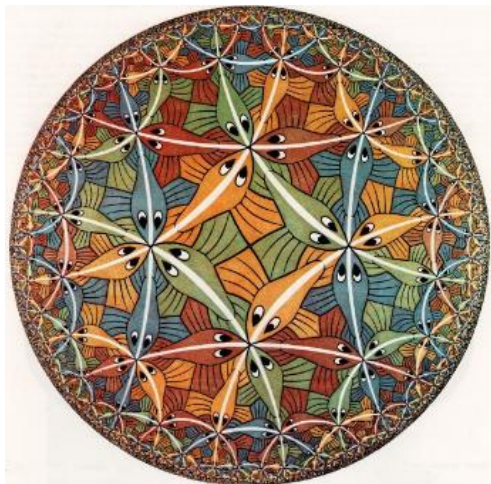
where $d(x, y)$ is the hyperbolic distance between the two points.

[25]

ii) Visualization of objects with Poincaré's disk model

Like said before, the Poincaré disk model is very good to visualize objects. Thanks to Escher⁷³, we can see how "beings" look in the hyperbolic plane (figure 6).

Although the fish appear smaller the closer we get to the boundary of the disk, they are the same size in the geometry of the hyperbolic plane. All the white segments going from the fishes' tails to their noses are the same length (and they are, in fact, straight lines). This tells that in a way, we can imagine this plane as a kind of sphere, that has a point at infinity. [27]



"Circle Limit III"

Figure 6 Drawing illustrating how "beings" look in the hyperbolic plane.

⁷³ Maurits Cornelis Escher (1898-1972) was a Dutch graphic artist who made mathematically inspired woodcuts and other drawings.

b. Riemann's geometry

Riemann⁷⁴ wrote a paper “*On the hypotheses that lie at the foundations of geometry*” in 1854 about a new kind of geometry. Riemann knew Gauss personally and had attended his lectures at the University of Göttingen. [28] His paper generalises the hyperbolic geometry of Bolyai and Lobachevskii. Riemannian geometry “unified and vastly generalized the three types of geometry, as well as the concept of manifold or mathematical space, which generalized the ideas of curves and surfaces.” [28] “With his Riemann metric, Riemann completely broke away from all the limitations of 2 and 3-dimensional geometry, [...] extending the differential geometry of surfaces into n dimensions.” [28] Indeed, Riemann developed a conception of a multi-dimensional space (known as Riemannian space or Riemannian manifold or simply “hyperspace”). Thus Riemannian geometry is the branch of differential geometry that studies Riemannian manifolds.

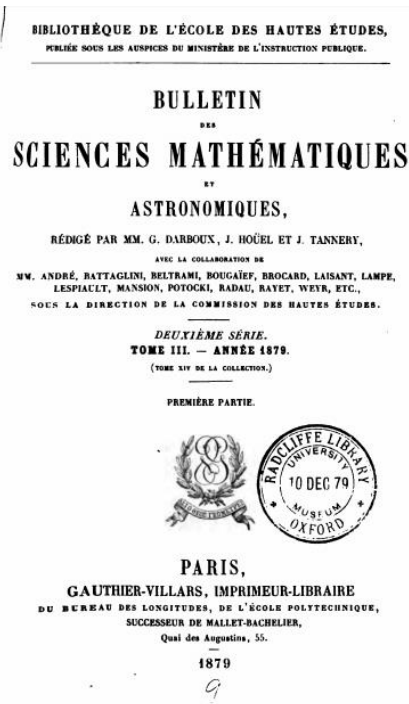
Riemannian geometry can be a synonym for geometry on a sphere. It also refers to “all the geometries of constant or variable curvature, in which the free motion of rigid bodies is either constrained or impossible.” [4] Riemannian geometry is a non-Euclidean geometry in which straight lines are geodesics and in which the parallel postulate is replaced by the postulate that every pair of straight lines intersects. “The Riemannian (spherical) geometry was in some ways the opposite of Lobachevskian geometry: in spherical geometry there are no parallels; in Lobachevskii’s geometry there are infinitely many. In spherical geometry the angle sums of triangles exceed two right angles, in Lobachevskii’s geometry it is always less.” [4] Finally, we can say that it is Riemannian geometry that allowed mathematicians to construct rigorous models of non-Euclidean geometry.

c. Dissemination of the non-Euclidean geometry across Europe

One year after Riemann’s death, his paper “*On the hypotheses that lie at the foundations of geometry*” was published. It reached Beltrami¹⁶, a young Italian mathematician, who had also come up with the same kind of conclusions. Riemann’s paper gave Beltrami the confidence he needed to publish his own very convincing work on the new non-Euclidean geometry in 1868. [4] Slowly but surely, non-Euclidean geometry started spreading across Europe.

Figure 7 Bulletin des Sciences Mathématiques, 1879.
Source : [29]

⁷⁴ Bernhard Riemann (1826-1866) was a German mathematician who made contributions to analysis, number theory and differential geometry.



Non-Euclidean geometry mainly came to France through the *Bulletin des sciences mathématiques*, a journal set up in 1870 by two young Frenchmen: Gaston Darboux⁷⁵ and Jules Hoüel⁷⁶. Their objective with this journal was to bring mathematical discoveries from abroad to the attention of French mathematicians. One of Hoüel's first accomplishments for this journal was to translate some publications from Beltrami, Lobachevskii, and Bolyai. [4] As a result, French mathematicians showed interest in non-Euclidean geometry, but did not really adopt it, since there was no evident reason to prefer non-Euclidean geometry over Euclidean geometry, and since Euclidean geometry was much easier to use. However, this attitude towards non-Euclidean geometry changed when Poincaré started getting interested in the matter.

Finally, the following quote shows the development of the three geometries, from Euclid to Riemann, and before Poincaré's involvement: "In Euclidean geometry, the curvature of the space is everywhere non-existent [...]. Riemann therefore concluded: "the multiplicities whose curvature is everywhere equal to zero, can be regarded as a special case of multiplicities with a curvature everywhere constant.". In those spaces the sum of the angles of a rectilinear triangle will not be equal to two rights; [...]. If the constant curvature is negative, the sum of the angles of the triangle is smaller than two rights, and we thus find again, just like Beltrami demonstrated, the geometry of Lobatschewsky and Bolyai. If the curvature is positive, the sum is bigger than two rights, we obtain a geometry whose triangles have properties analogous to spherical triangles, where geodesic lines have two shared points, like the arc of big circles on a sphere, where space is unlimited without being infinite; and this will be Riemann's geometry." [17]

4.1.4 Poincaré's work and opinion on non-Euclidean geometry

"In 1880, Poincaré showed that the new non-Euclidean geometry was the natural geometry for treating a range of problems in mathematics, and that in that setting Euclidean geometry was a small, singular case." [4]

⁷⁵ Gaston Darboux (1842-1917) was a French mathematician.

⁷⁶ Jules Hoüel (1823-1886) was a French mathematician.

It was hard for many people to understand non-Euclidean geometry. So Poincaré tried to make non-Euclidean geometry understandable by everyone, by giving an explanation full of imagery in the chapter “*Les géométries non euclidiennes*” in his book “*La Science et l’Hypothèse*”:

“Imaginons un monde uniquement peuplé d’êtres dénués d’épaisseur ; et supposons que ces animaux ‘infinitement plats’ soient tous dans un même plan et n’en puissent sortir. [...] Mais supposons maintenant que ces animaux imaginaires, tout en restant dénués d’épaisseur, aient la forme d’une figure sphérique, et non d’une figure plane et soient tous sur une même sphère sans pouvoir s’en écarter. Quelle géométrie pourront-ils construire ? Il est clair d’abord qu’ils n’attribueront à l’espace que deux dimensions ; ce qui jouera pour eux le rôle de la ligne droite, ce sera le plus court chemin d’un point à un autre sur la sphère, c’est-à-dire un arc de grand cercle, en un mot leur géométrie sera la géométrie sphérique. [...] Et bien, la géométrie de Riemann, c’est la géométrie sphérique étendue à trois dimensions. Pour la construire, le mathématicien allemand a dû jeter par-dessus bord, non seulement le postulat d’Euclide, mais encore le premier axiome : Par deux points on ne peut faire passer qu’une droite. [...] Il y a une sorte d’opposition entre la géométrie de Riemann et celle de Lowatchewski. [...] Ajoutons que l’espace de Riemann est fini, quoique sans limite, au sens donné plus haut à ces deux mots. » [La Science et l’Hypothèse, pages 53-54]

“Imagine a world uniquely peopled by beings of no thickness (height); and suppose these ‘infinitely flat’ animals are all in the same plane and can not get out. [...] But suppose now that these imaginary animals, while remaining without thickness, have the form of a spherical, and not of a plane, figure, and are all on the same sphere without power to get off. What geometry will they construct? First it is clear they will attribute to space only two dimensions; what will play for them the rôle of the straight line will be the shortest path from one point to another on the sphere, that is to say, an arc of a great circle; in a word, their geometry will be the spherical geometry. [...] Well, Riemann’s geometry is spherical geometry extended to three dimensions. To construct it, the German mathematician had to throw overboard, not only Euclid’s postulate, but also the first axiom: Only one straight can pass through two points.[...] There is a sort of opposition between Riemann’s geometry and that of Lobachevski.[...] Add that Riemann’s space is finite, although unbounded, in the sense given above to these two words.” [Science and Hypothesis, pages 53-54]

Properties of non-Euclidean geometry could also be explained with balloons: balloons can be very good at demonstrating some amazing properties of non-Euclidean geometry. Those properties can be approached by laying down on a flat surface an uninflated balloon, and drawing a triangle on it. Once the triangle is drawn, the balloon can be blown up and inflated. Is the once-perfect triangle still perfect? Do the angles still add up to 180 degrees? What is testified here is equivalent to the difference between Euclidean and non-Euclidean geometry. The flat balloon respected the rules of Euclidean geometry, but

the inflated balloon does not. With the inflated balloon, the interior angles of the triangle add up to more than 180 degrees.

To try to popularize and reach as many readers as possible, Poincaré also created a small dictionary with which one could translate every statement in Lobachevskian geometry into a statement in Euclidean geometry.

« Espace: Portion de l'espace située au-dessus du plan fondamental.

Plan : Sphère coupant orthogonalement le plan fondamental.

Droite : Cercle coupant orthogonalement le plan fondamental.

Sphère : Sphère.

Cercle : Cercle.

Angle : Angle.

Distance de deux points : Logarithme du rapport anharmonique de ces deux points et des intersections du plan fondamental avec un cercle passant par ces deux points et le coupant orthogonalement. » [La Science et l'Hypothèse, page 57]

“Space: Portion of space situated above the fundamental plane.

Plane: Sphere cutting the fundamental plane orthogonally.

Straight: Circle cutting the fundamental plane orthogonally.

Sphere: Sphere.

Circle: Circle.

Angle: Angle.

Distance between two points: Logarithm of the cross ratio of these two points and the intersections of the fundamental plane with a circle passing through these two points and cutting it orthogonally.”
[Science and Hypothesis, page 57]

With this dictionary, Poincaré shows that “*Were two mutually contradictory statements to arise in Lobachevskian geometry, their translations would be two mutually contradictory statements in Euclidean geometry, and no one doubts that Euclidean geometry is free from contradiction.*”, and therefore proves that Lobachevskian geometry can be relied upon. [4]

The new non-Euclidean geometry was accepted by professional mathematicians, but was harder to accept by scientists, engineers, and teachers. “If indeed there were two geometries [Euclidean and non-Euclidean] with different theorems then only one could be true. [...] Which one is true, the old Euclidean or the new non-Euclidean geometry?” [4] Poincaré was aware of the fact that there is no certain way of telling which geometry (Euclidean or non-Euclidean) is true, because it all depends on chosen conventions. “Throughout his working life he [Poincaré] was going to take the side of multiple interpretations and remain reluctant to say that any particular statement was true. It is a position with some philosophical advantages”, [4] but it could also annoy some scientists. In *La Science et l'Hypothèse*, Poincaré wrote:

« Dès lors, que doit-on penser de cette question : La géométrie euclidienne est-elle vraie ? Elle n'a aucun sens. [...] Une géométrie ne peut pas être plus vraie qu'une autre ; elle peut seulement être plus commode. Or la géométrie euclidienne est et restera la plus commode :

1- Parce qu'elle est la plus simple [...]

2- Parce qu'elle s'accorde assez bien avec les propriétés des solides naturels, ces corps dont se rapprochent nos membres et notre œil et avec lesquels nous faisons nos instruments de mesure. »
[La Science et l'Hypothèse, page 67]

“Then what are we to think of that question: Is the Euclidean geometry true? It has no meaning. [...] One geometry cannot be more true than another; it can only be more convenient. Now, Euclidean geometry is, and will remain, the most convenient:

1- Because it is the simplest; [...]

2- Because it accords sufficiently well with the properties of natural solids, those bodies which our hands and our eyes compare and with which we make our instruments of measure.” [Science and Hypothesis, page 67]

However, not all scientists agreed with Poincaré. G.Mouret (engineer at Ponts et Chaussées⁷⁷) replied to Poincaré and explained in his article *« L'égalité mathématique »* published in the *Revue philosophique* in 1891 that for him Euclidean geometry was the only true geometry. He dismissed non-Euclidean geometry as being “purely speculative, factitious, and akin to doing thermodynamics with negative temperatures”. [4] Poincaré answered in the same volume and explained that Riemann and

⁷⁷ The Ecole Nationale des Ponts et Chaussées is a university-level institution of higher education and research in the field of science, engineering and technology in the south of Paris. Founded in 1747, it is the oldest and one of the most prestigious French schools.

Beltrami's geometries were valuable because they "helped the imagination to break the bounds of habits created by daily experience". [4] He also wrote in *La Science et l'Hypothèse*:

« Supposons, par exemple, un monde renfermé dans une grande sphère et soumis aux lois suivantes : La température n'y est pas uniforme ; elle est maximale au centre, et elle diminue à mesure qu'on s'en éloigne, pour se réduire au zéro absolu quand on atteint la sphère où ce monde est renfermé. [...] Soit R le rayon de la sphère limite ; soit r la distance du point considéré au centre de cette sphère. La température absolue sera proportionnelle à $R^2 - r^2$. [...] Je ferai encore une autre hypothèse ; je supposerai que la lumière traverse des milieux diversement réfringents et de telle sorte que l'indice de réfraction soit inversement proportionnel à $R^2 - r^2$. Il est aisé de voir que, dans ces conditions, les rayons lumineux ne seraient pas rectilignes, mais circulaires. [...] Si un être sentant se trouve dans le voisinage, ses impressions seront modifiées [...] S'ils [les êtres pensants] fondent une géométrie, ce ne sera pas comme la nôtre, [...] ce sera la géométrie non euclidienne. Ainsi des êtres comme nous, dont l'éducation se ferait dans un pareil monde, n'auraient pas la même géométrie que nous. [La Science et l'Hypothèse, pages 84-87]

"Suppose, for example, a world enclosed in a great sphere and subject to the following laws: The temperature is not uniform; it is greatest at the center, and diminishes in proportion to the distance from the center, to sink to absolute zero when the sphere is reached in which this world is enclosed. [...] Let R be the radius of the limiting sphere; let r be the distance of the point considered from the center of this sphere. The absolute temperature shall be proportional to $R^2 - r^2$. [...] I will make still another hypothesis; I will suppose light traverses media diversely refractive and such that the index of refraction is inversely proportional to $R^2 - r^2$. It is easy to see that, under these conditions, the rays of light would not be rectilinear, but circular. [...] If a sentient being happens to be in the neighbourhood, his impressions will be modified [...] If they [the sentient beings] construct a geometry, it will not be, as ours is, [...] it will be non-Euclidean geometry. Thus beings like ourselves, educated in such a world, would not have the same geometry as ours." [Science and Hypothesis, pages 84-87]

Therefore, "Such creatures [invented by Poincaré in the quotes above] would believe that their world was infinite, and what they called straight lines would be what we call arcs of circles perpendicular to the boundary sphere." [4] And as a result those creatures would naturally adopt the non-Euclidean geometry.

The previous quote shows that experience and imagination are very important aspects of Poincaré's scientific reasoning. For instance, one of the purposes of the fifth chapter of *La Science et l'Hypothèse* (*Experience and Geometry*) is to show that experience guides the scientists in their choices of conventions. Poincaré believes that experience has an important role in the choice of geometry we use.

4.1.5 Poincaré and the fourth geometry

The axiom saying that we can apply a straight line on itself by a reversal is implicitly admitted in the Euclidean and non-Euclidean geometries. However, Poincaré said that by dropping this axiom, it is possible to construct a fourth geometry, as coherent and structurally very similar to the geometries of Euclid, Lobatchevsky and Riemann. [16]

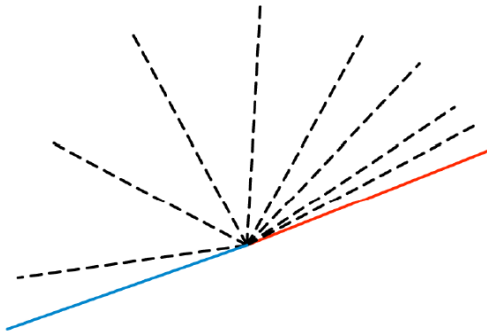


Figure 8 Illustration of the axiom stating that we can apply a straight line on itself by a reversal. Source: [16]

In Poincaré's fourth geometry, a straight line can be orthogonal to itself. Poincaré concedes that this fourth geometry is surprising. However, it is not contradictory.

« Pour démontrer que l'on peut toujours élever en un point A une perpendiculaire à une droite AB, on considère une droite AC mobile autour du point A et primitivement confondue avec la droite fixe AB ; et on la fait tourner autour du point A jusqu'à ce qu'elle vienne dans le prolongement de AB. On suppose ainsi deux propositions : d'abord qu'une pareille rotation est possible, et ensuite qu'elle peut se continuer jusqu'à ce que les deux droites viennent dans le prolongement l'une de l'autre. Si l'on admet le premier point et que l'on rejette le second, on est conduit à une suite de théorèmes encore plus étranges que ceux de Lowatchewski et de Riemann, mais également exempts de contradiction. Je ne citerai qu'un de ces théorèmes et je ne choisirai pas le plus singulier : une droite réelle peut être perpendiculaire à elle-même. » [La Science et l'Hypothèse, pages 61-62]

“To prove that a perpendicular may always be erected at a point A to a straight AB, we consider a straight AC movable around the point A and initially coincident with the fixed straight AB; and we make it turn about the point A until it comes into the prolongation of AB. Thus two propositions are presupposed: First, that such a rotation is possible, and next that it may be continued until the two straights come into the prolongation one of the other. If the first point is admitted and the second rejected, we are led to a series of theorems even stranger than those of Lobachevski and Riemann, but

equally exempt from contradiction. I shall cite only one of these theorems and that not the most singular: A real straight may be perpendicular to itself.” [Science and Hypothesis, pages 61-62]

With this fourth geometry, it appears that Poincaré wants to show that geometers have admitted many non-neutral implicit hypothesis, which explains the forgetting of the fourth geometry. However, there is a finite number of conceivable geometries. [5]

4.1.6 Poincaré’s opinion on axioms in geometry

All geometries are based on some common presuppositions in the axioms, postulates or definitions. Very early in the history of mathematics, Euclid used axioms in his book *Eléments*. In this book, “axioms could seem justified by their inherent obviousness, and like the conditions of the intellectual activity.” [17] Non-Euclidean geometries can be constructed by substituting alternative versions of Euclid’s parallel axiom, and what differentiates Riemannian geometry from Lobachevskian geometry and from Euclidean geometry are different axioms, among other things, regarding parallels. [14]

Poincaré believed that axioms in geometry cannot be synthetic a priori judgments, because it would mean that non-Euclidean geometry is unthinkable, which is not the case. But axioms cannot be experimental facts either, given the fact that lines and planes of elementary geometry are not physical objects and therefore cannot be experimented upon. [4] Indeed, Poincaré wrote that “*geometry is exact but no science founded upon experiments can be exact*” [4]. The sciences are subject to eternal revision, and geometry is not, which gives another objection to axioms being experimental facts: axioms in geometry cannot be reduced to data obtained from observation. [16]

Therefore, Poincaré regarded the axioms as conventions or as definitions in disguise:

“Si la géométrie était une science expérimentale, elle ne serait pas une science exacte. [...] Les axiomes géométriques ne sont donc ni des jugements synthétiques ni des faits expérimentaux. Ce sont des conventions. [...] En d’autres termes, les axiomes de la géométrie (je ne parle pas de ceux d’arithmétique) ne sont que des définitions déguisées.” [La Science et l’Hypothèse, page 66]

«If geometry were an experimental science, it would not be an exact science. [...]. The axioms of geometry therefore are neither synthetic a priori judgments nor experimental facts. They are conventions. [...] In other words, the axioms of geometry (I do not speak of those of arithmetic) are merely disguised definitions.” [Science and Hypothesis, page 66]

However, even though Poincaré said that “geometry is exact”, if geometrical axioms are not based on experimental science, we could argue that geometry is then a hypothetical and arbitrary science. Which would mean that its truth is questionable and cannot be guaranteed. [16]

5. The influence of Poincaré on the mathematics of the future

5.1 The conjecture of Poincaré

“As the recent excitement over the solution of the so-called Poincaré conjecture showed, Poincaré’s work in topology lives among mathematicians to this day as he does among philosophers of science, who still discuss his philosophy of conventionalism. It fed into the ideas of the Vienna circle⁷⁸, and it has kept his popular essays in print for a century.” [4]

Conjecture of Poincaré:

Every simply connected, closed 3-manifold is homeomorphic to the 3-sphere.

The conjecture of Poincaré is one of the seven famous ‘one million dollars conjectures’ of the Clay Mathematics Institute⁷⁹. A conjecture is an unproven mathematical or scientific theorem [Oxford Dictionaries]. Indeed, after Poincaré established his conjecture, it took years for mathematicians to be able to write a proof for it. It was finally solved in 2003, almost a century after it was proposed, by a Russian mathematician, Grigori Perelman⁸⁰. So far, the conjecture is the only problem of the Clay Institute that has been solved.

Let’s first define the words employed in the wording of the conjecture:

- A *simply connected space* is a space that has no ‘holes’. For example, a football is simply connected, but a donut is not.
- A *closed space* is a space that is finite and has no boundaries. A sphere is closed, but a plane is not because it is infinite.
- A *manifold* is a topological space that locally resembles Euclidean space near each point. Each point of an n-dimensional manifold has a neighbourhood that is homeomorphic to the Euclidean space of dimension n. Surfaces are two-dimensional manifolds. For instance, a standard sphere (a balloon) is a two-manifold. On each small portion of surface, only two coordinates are needed to identify the points (on the surface of the Earth, a point is identified with a latitude and a longitude). The topological property of a surface survives to all kinds of deformations applied to this surface. [30]

⁷⁸ The Vienna Circle was a group of philosophers and scientists drawn from the natural and social sciences, logic and mathematics, who met regularly from 1924 to 1936 at the University of Vienna.

⁷⁹ The Millenium Prize Problems are seven problems in mathematics that were stated by the Clay Mathematics Institute in 2000. The problems are Birch and Swinnerton-Dyer conjecture, Hodge conjecture, Navier-Stokes existence and smoothness, P versus NP problem, Poincaré conjecture, Riemann hypothesis, and Yang-Mills existence and mass gap.

⁸⁰ Grigori Perelman (1966-) is a Russian mathematician.

- If one space is *homeomorphic* to another, then we can continuously deform the one space into the other. For instance, a two-dimensional sphere and a football are homeomorphic but a two-dimensional sphere and a donut are not. [31]

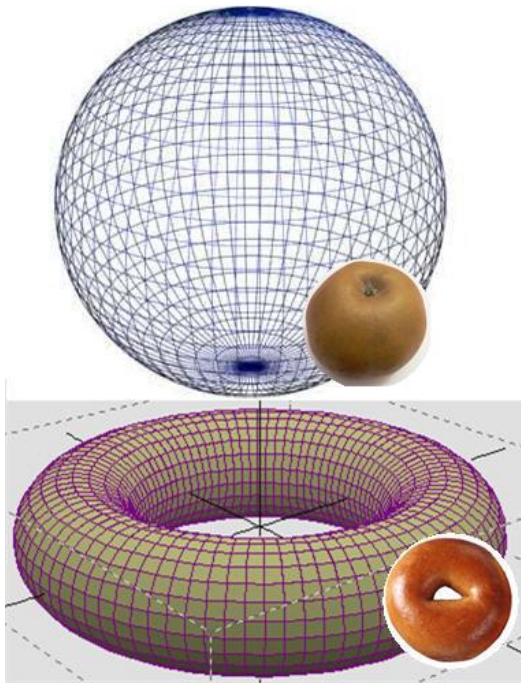


Figure 9 The sphere and the torus (donut) on the picture are not homeomorphic.

The primary tool used by Grigori Perelman to solve the conjecture of Poincaré is the Ricci-Hamilton flow, which was named after Gregorio Ricci-Curbastro⁸¹ and was first introduced by Richard Hamilton⁸² in 1981. In differential geometry, the Ricci flow is an intrinsic geometric flow. It is a process that deforms the metric of a Riemannian manifold in a way formally analogous to the diffusion of heat, smoothing out irregularities in the metric. [33] The geometric flow defined by Poincaré and proved by Perelman can be measured by entropy, using Ricci flow equations.

Topology has applications in many fields and Perelman's work introduced important techniques which can now be used on other problems. What Perelman did not know at the time, is that this concept of entropy can be used in social networks to calculate information diffusion. Indeed, the social networks tend to have similar geometric properties and to adhere to similar rules defined by Perelman. When taking into account degrees of freedom of a node of the social network and its association with geometric flow measured by entropy, it can define the virality of the given post in social network. [33]

Finally, the proven conjecture and Perelman's discovery could also have repercussions in the fabrication of chips or other electronical devices used for transportation or for research on the brain.

⁸¹ Gregorio-Ricci Curbastro (1853-1925) was an Italian mathematician. He is most famous as the inventor of the tensor calculus.

⁸² Richard Hamilton (1943-) is a professor of mathematics at Columbia University.

5.2 Poincaré and the Three Body Problem: establishing the stability of the solar system

5.2.1 Kepler's laws of planetary motion

The Three Body Problem is one of the most famous problems of mathematical physics. It starts with Kepler⁸³'s laws of planetary motion, describing at the beginning of the 17th century the motion of planets around the Sun. In this problem, we can notice that mathematics had a double purpose: solving the problem of trajectories, and solving the problem of ellipse.

The three laws of Kepler (1571-1630) are the following:

- First Law: Law of the Ellipses
- Second Law: Law of Equal Areas
- Third Law: Law of Harmonics

“Those laws are very beautiful and powerful, and they make us believe that there is really an order in the universe.” [34]

5.2.2 Newton, the Two Body Problem, and the involvement of other mathematicians

A few years after Kepler's death Newton⁸⁴ published a book called *Principia Mathematica*, introducing the fact that differential equations can predict the motion of objects. He explains that gravitational force exerted between two bodies is proportional to the product of the two masses. With one single law, Newton was able to express the three laws of Kepler. “Newton's laws permit the completely accurate solution of two bodies orbiting around their common centre of gravity: as Kepler had been the first to suggest, each body moves in an ellipse.” [4] He published this discovery in 1680, solved the Two Body Problem and rediscovered Kepler's ellipse with a very complex reasoning. However, Newton did not find a solution to the Three Body Problem (and no one did). In the Three Body Problem, “the trajectories are not ellipses anymore, and the simplicity of Kepler's system is gone.” [34] At this point, no law prohibits two planets from colliding into each other. This is the appearance of the famous and ancient problem of mathematical physics: the stability of the solar system. How can we predict the future events and movements of the planets? Newton's formula is very powerful because it is applicable to all bodies, irregardless of their size, weight, etc. But with his formula, Newton does not explain the small perturbations endured by the planets. [34]

⁸³ Johannes Kepler (1571-1630) was a German mathematician, astronomer and astrologer.

⁸⁴ Isaac Newton (1642-1726) was an English mathematician, astronomer and physicist.

The text published by Newton in 1680 created a controversy between several scientists, all of them trying to give their own opinion about the stability of the solar system; among which Newton, Leibnitz⁸⁵ or Voltaire⁸⁶. A hundred years later, mathematical techniques improved, and Laplace⁸⁷, Lagrange⁸⁸ and Gauss got involved in the controversy. They developed mathematical techniques to precisely evaluate the impact of perturbations. “Euler and Lagrange made mistakes, but Laplace obtained correct results.” [34] He proved that distances between the planets and the Sun stay unchanged on a length of time of about $1/\varepsilon^2$. This equals to durations of around a million years. “This is the beginning of the Laplacien determinism, built on the idea that if we know the state of the world at a t moment, then we can predict it at all the moments that will come.” [34]

5.2.3 Poincaré’s contribution to the Three Body Problem

Poincaré got involved in this debate because he replied to a competition organised by the Swedish King Oscar II in 1885, with the help of the Swedish mathematician Mittag-Leffler. Poincaré offered to work on the Three Body Problem and to study the stability of the solar system, but on an even greater time length: billions of years instead of millions. Up to this time, “no one had been able to show that the infinite series that expressed these solutions converged and so could be used to establish the behaviour of the solar system at all future times.” [4] Poincaré then sent an article proving that the solar system is stable on an extremely long length of time and won the price. He showed that there were reasons to believe that most orbits were stable. The article was published in *Acta Mathematica*, a magazine owned by Mittag-Leffler. But his assistant, the young mathematician Lars Phragmén⁸⁹, noticed a few mistakes. So Poincaré added some modifications and sent in his article again. However, he then realised that his whole work was incorrect! His mistake was due to a problem of intersection of curves that could intersect in dimension 3 but could not in dimension 2. He had written in his article that if two curves intersected, then they were inclusive. And if two curves are inclusive, then the system is stable. This intersection problem is way more complex than assumed by Poincaré. Figure 10 shows the kind of figure Poincaré was working with:

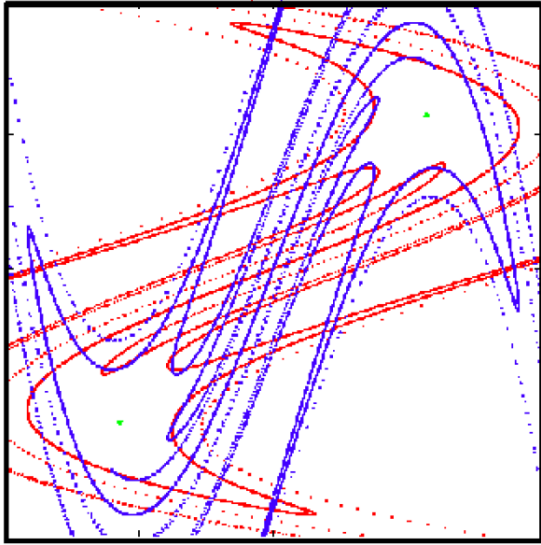
⁸⁵ Gottfried Wilhelm Leibniz (1646-1716) was a German polymath and philosopher.

⁸⁶ François-Marie Arouet (1694-1778), known by his *nom de plume* Voltaire, was a French writer, historian and philosopher.

⁸⁷ Pierre-Simon Laplace (1749-1827) was an influential French mathematician, physician and astronomer.

⁸⁸ Joseph-Louis Lagrange (1736-1813) was an Italian mathematician and astronomer.

⁸⁹ Lars Edvard Phragmén (1863-1937) was a Swedish mathematician.



"... One will be struck by the complexity of this figure, which I do not even try to trace. Nothing is more suitable to give us an idea of the complication of the problem of the three bodies and, in general, of all problems of dynamics in which there is no uniform integral. " Poincaré

Figure 10 Intersection of curves (traced on a computer)

5.2.4 The chaos theory

When Poincaré's mistake was found out, chaos theory was introduced for the first time, showing the unpredictability of determinist trajectories, due to the sensibility to initial conditions. "It is very complicated to predict the behaviour of the system." [34] We can relate this phenomenon to what we call the "butterfly effect". It is not possible to predict the future of a system with a determinist equation if we cannot exactly establish the initial condition. Yet, we will never be able to exactly express the initial condition, so we will never be able to predict what happens in the long run. This is completely opposed to the Laplacian determinism. (see 5.2.2)

Poincaré said *"You are asking me to predict the phenomena that will happen. If, by misfortune, I knew the laws of those phenomena, I would only manage to do so by inextricable calculations and I would have to give up on answering you; but, like I am lucky enough to ignore them, I will answer you immediately. And, what is most extraordinary, my answer will be correct."* [34]

The meteorologist Lorenz⁹⁰ also found the theory of chaos when trying to express with equations a simplified model of the atmosphere. He was not able to precisely describe the evolution of the system, because of the sensibility to initial conditions. Nobody talked about stability anymore, and everybody believed the world was ruled by chaos. Later on, in 1954, Kolmogorov⁹¹ demonstrated a new surprising

⁹⁰ Edward Lorenz (1917-2008) was an American mathematician, meteorologist, and a pioneer of chaos theory. He introduced the term butterfly effect.

⁹¹ Andrey Kolmogorov (1903-1987) was a Russian mathematician who made significant contributions to the mathematics of probability theory, topology, intuitionistic logic, turbulence, classical mechanics, algorithmic information theory and computational complexity.

theorem, suggesting that Poincaré had misinterpreted the solar system. It was the beginning of the KAM (Kolmogorov-Arnold-Moser) theory, using determinist equations and probability to handle the initial conditions. The theorem says that if we disrupt Kepler's system and chose a configuration at random, then there is a very high probability that the trajectory will stay stable. Although not entirely true, this revolutionary theorem changed mathematicians and physicians' opinions. According to Kolmogorov's theorem, to a Diophantine number equals a stable configuration. And to a non-Diophantine number equals a non-stable configuration.

Definition: An irrational number α is Diophantine if there exists a constant $C > 0$ and an exponent $r \geq 2$ such as for any rational p/q ($q > 0$), we have the inequality $|\alpha - p/q| > C/q^r$.

So Kolmogorov's theorem suggests that the trajectory and systems are stable, and therefore that Poincaré was wrong.

5.2.5 Computer science

However, in the 1980's, the development of computer science enabled new techniques to help answer this problem of stability of the solar system. Jacques Laskar⁹² and some others were able to numerically simulate the solar system on a length period of tens of millions of years. Laskar transformed Newton's equations in a simplified system with 150 000 terms, and proved that the system is almost not predictable after 10 million of years, and not predictable at all after 60 million of years. [34] This means that the stability of the solar system is not foreseeable, and supports Poincaré's point of view. However, it is possible to conduct simulations and to calculate probabilities of what might happen. Of course, the computer greatly helped Laskar in his work, and without the computer, such conclusions would not have been possible. Computers helped science progress at an extraordinary speed. Shannon⁹³, Von Neumann⁹⁴ or Alan Turing⁹⁵ are also great mathematicians that made a great use of computers and technology.

⁹² Jacques Laskar (1955-) is a French astronomer. He is a research director at the French National Centre for Scientific Research (CNRS), and a member of the Institute of Celestial Mechanics and Computing Ephemerides (IMCCE).

⁹³ Claude Elwood Shannon (1916-2001) was an American mathematician, electrical engineer and cryptographer known as « the father of information theory ».

⁹⁴ John Von Neumann (1903-1957) was a Hungarian-American mathematician, physicist, inventor and computer scientist.

⁹⁵ Alan Turing (1912-1954) was an English computer scientist, mathematician, and cryptanalyst. He was highly influential in the development of theoretical computer science.

5.3 Presence of Poincaré in the science of today

Poincaré had relatively few students working on his ideas, and furthermore, approximately two years after Poincaré's death, Europe plunged into a war in which many French mathematicians and scientists were killed. After the war, there were not many mathematicians available to spread and develop Poincaré's ideas. So Poincaré's ideas slowly started being rediscovered a few years after the war by mathematicians and the Bourbaki. Nicolas Bourbaki is the collective pseudonym under which a group of mainly French 20th-century mathematicians wrote a series of books beginning in 1935. Their work and mathematical publications led to the discovery of several concepts, and influenced modern branches of mathematics. Moreover, "Many mathematicians see Poincaré today not only as the exemplary creator of new branches of mathematics, but as the source of ongoing topics of research in several areas." [4] Indeed, the longevity of Poincaré's popular essays is remarkable. "So many of them are worth reading well over a hundred years after they were written, and they remain fresh because they address issues that will always come around in the education of mathematicians, scientists, and the general public." [4] He particularly made lasting contributions to Sophus Lie's theory of transformation groups and to algebraic geometry. "But his major contribution to mathematics is undoubtedly that of topology." [4]

Just after World War I, mathematicians Emile Borel⁹⁶ and George Birkhoff⁹⁷ decided to build a mathematics research institute, which was inaugurated in 1928 and named after Henri Poincaré. The institute's objective is to promote mathematical physics, and is still today a privileged place for scientists to meet, exchange, and diffuse knowledge. Since 2009, the head of the Institut Henri Poincaré is Cédric Villani⁹⁸.

Finally, during his career, Poincaré has expressed strong feelings about the freedom of science. In 1909, promoted doctor honoris causa at the 75th anniversary of the Université Libre de Bruxelles, he says: *"Freedom is for Science what air is for animal; deprived of this freedom, it dies from suffocation, like a bird deprived of oxygen. And this freedom must be without limits, because, if one wants to impose some, one gets a half-science only, and half-science is no more science, because it can be, it necessarily is, a false science. The thought must never be subordinated to any dogma, political party, passion, interest, preconceived idea, to anything indeed, except the facts themselves, because, for science, to be subordinated means to die."* The last sentence is reproduced on the walls of the main building of the University of Brussels. [5]

⁹⁶ Emile Borel (1871-1956) was a French mathematician and politician.

⁹⁷ George Birkhoff (1884-1944) was an American mathematician.

⁹⁸ Cédric Villani (1973-) is a French mathematician working primarily on partial differential equations, Riemannian geometry and mathematical physics. He was awarded the Fields medal in 2010.

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Henri Poincaré, *La Science et l'Hypothèse*. Paris: Ernest Flammarion, 1902.

An english version of the book, translated by George Bruce Halsted in May 2012, is also used. It was translated in the framework of the Project Gutenberg eBook of The Foundations of Science, and can be found on the following link: <http://www.gutenberg.org/files/39713/39713-h/39713-h.htm>.

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