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# Limit Laws, Homogenizable Structures and Their Connections

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**Abstract**
Ahlman, O. 2018. Limit Laws, Homogenizable Structures and Their Connections. (Gränsvärdeslagar, Homogeniserbara Strukturer och Deras Samband). *Uppsala Dissertations in Mathematics* 104. 43 pp. Uppsala: Department of Mathematics. ISBN 978-91-506-2672-8.

This thesis is in the field of mathematical logic and especially model theory. The thesis contain six papers where the common theme is the Rado graph R. Some of the interesting abstract properties of R are that it is simple, homogeneous (and thus countably categorical), has SU-rank 1 and trivial dependence. The Rado graph is possible to generate in a probabilistic way. If we let K be the set of all finite graphs then we obtain R as the structure which satisfy all properties which hold with assymptotic probability 1 in K. On the other hand, since the Rado graph is homogeneous, it is also possible to generate it as a Fraïssé-limit of its age.

Paper I studies the binary structures which are simple, countably categorical, with SU-rank 1 and trivial algebraic closure. The main theorem shows that these structures are all possible to generate using a similar probabilistic method which is used to generate the Rado graph. Paper II looks at the simple homogeneous structures in general and give certain technical results on the subsets of SU-rank 1.

Paper III considers the set K consisting of all colourable structures with a definable pregeometry and shows that there is a 0-1 law and almost surely a unique definable colouring. When generating the Rado graph we almost surely have only rigid structures in K. Paper IV studies what happens if the structures in K are only the non-rigid finite structures. We deduce that the limit structures essentially try to stay as rigid as possible, given the restriction, and that we in general get a limit law but not a 0-1 law.

Paper V looks at the Rado graph's close cousin the random t-partite graph and notices that this structure is not homogeneous but almost homogeneous. Rather we may just add a definable binary predicate, which hold for any two elemenets which are in the same part, in order to make it homogeneous. This property is called being homogenizable and in Paper V we do a general study of homogenizable structures. Paper VI conducts a special case study of the homogenizable graphs which are the closest to being homogeneous, providing an explicit classification of these graphs.

*Keywords:* Model theory, random structure, finite model theory, simple theory, homogeneous structure, countably categorical, 0-1 law

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*To Gro*

# List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I Ahlman, O.
**Simple structures axiomatized by almost sure theories**
*Annals of Pure and Applied Logic 167 (2016) 435-456.*

II Ahlman, O., Koponen, V.
**On sets with rank one in simple homogeneous structures**
*Fundamenta Mathematicae 228 (2015) 223-250.*

III Ahlman, O., Koponen, V.
**Random $l-$colourable structures with a pregeometry**
*Mathematical Logic Quarterly 63 (2017) 32-58.*

IV Ahlman, O., Koponen, V.
**Limit laws and automorphism groups of random nonrigid structures**
*Journal of Logic & Analysis 7:2 (2015) 1-53.*

V Ahlman, O.
**Homogenizable structures and model completeness**
*Archive for Mathematical Logic 55 (2016) 977-995.*

VI Ahlman, O.
**$>k-$homogeneous infinite graphs**
*Journal of Combinatorial Theory, Series B 128 (2018) 160-174.*

# Contents

# 1. Introduction

This thesis is written in the subject of mathematical logic and especially model theory. Model theory is the abstract study and construction of mathematical structures and their theories. A structure could be a graph, group, field, tree, vector space etc. In this thesis however we will almost exclusively consider structures with only relations in their vocabularies.

The **Rado graph** was first created by Willhelm Ackerman [1] in 1937 and named after Richard Rado [39] who further discovered its properties in 1964. In both articles the Rado graph is constructed by, on the natural numbers, adding an edge between any numbers $a < b$ such that the $a$:th number in the binary expansion of $b$ is 1. We will not explicitly study the Rado graph in this thesis. The Rado graph could however be seen as a common denominator between all of the articles through its abstract model theoretic properties and its many construction methods. We will cover properties such as homogeneity and supersimplicity, and use construction methods such as probabilistic limits, Fraïssé-limits and extension axioms. In order to conduct these studies we will use tools from logic, algebra, combinatorics and probability theory.

The six articles presented in this thesis can be briefly summarized by referencing the Rado graph. The first two articles concern the abstract model theoretic properties of the Rado graph and notices that the probabilistic construction used to create the Rado graph is also possible to use when creating other structures with the same model theoretic properties. The third and fourth article discuss the probabilistic method which can be used in order to create the Rado graph. This results both in new limit laws and infinite "random" structures which are similar to the Rado graph. The Rado graph is closely related to the random $t-$partite graph which is not homogeneous, but is homogenizable. The last two papers look at the concept of homogenizability and study this both as an abstract concept and in order to give a specific classification.

This thesis consists of four chapters followed by the six appended papers briefly described above. This introductory chapter contain definitions, theorems and history on the theory related to the articles in this thesis. Chapter 2 contain extended summaries of the appended papers including references to the examples, theorems and definitions which are mentioned in the introduction. Chapter 3 is a summary of the thesis, written in Swedish. This chapter is recommended for anyone (who speaks Swedish) who have not studied model theory, as it is accessibly written while still somewhat presenting the background and results of the thesis. Lastly Chapter 4 is acknowledgments.

## 1.1 Preliminaries

In this section we quickly present the notation and basic model theoretic notions which will be used throughout this thesis. It can be viewed as a sharp introduction for the mathematician who has not read any model theory, as a quick reminder for anyone who has worked a little bit with model theory, or an introduction to the notation for any researcher in the subject. More advanced concepts are defined in the other sections of this introduction. Anyone who wants a more complete introduction to the subject should study, for instance, Hodges book [24].

A **vocabulary** $V$ is a set of constant, function and relation symbols, where each function and relation symbol has a certain finite **arity**. In this thesis we will almost always consider a finite vocabulary which only contains relation symbols, such a vocabulary is called **finite relational**. A **language** $L$ is the set of all formulas which we can create using the symbols in a specific vocabulary. In this thesis we will only consider first order formulas. A **theory** is a set of sentences (i.e. closed formulas) from a specific language $L$. We say that a theory $T$ is **complete** if for each sentence $\varphi \in L$ either $\varphi \in T$ or $\neg\varphi \in T$.

Given a vocabulary $V$, a **structure** $\mathscr{M}$ (or a $V-$structure if we want to be specific) is a set $M$ together with an interpretation of each symbol in the vocabulary as an element, a function or a relation on $M$ respectively. The structures we use in this thesis will be denoted with calligraphic letters $\mathscr{A},\mathscr{B},\mathscr{C},\ldots$ with their universes being denoted by the corresponding roman letters $A,B,C,\ldots$. If $V' \subseteq V$ and $\mathscr{M}$ is a $V-$structure then the **reduct** of $\mathscr{M}$ to $V'$, written $\mathscr{M} \restriction V'$, is the $V'-$structure with universe $M$ where all symbols in $V'$ are interpreted like they are in $\mathscr{M}$. The **complete theory** of a structure $\mathscr{M}$, denoted $Th(\mathscr{M})$, is the set of all sentences which are true in $\mathscr{M}$. The abbreviation $[n] = \{1,\ldots,n\}$ is common practice, especially in a combinatorial context, and we will make good use for it here. We will often abuse notation on tuples writing $\bar{a} \in A$ when we mean $\bar{a} \in A^k$ for some $k \in \mathbb{Z}^+$.

An $n-$**type** of a theory $T$ is a set of formulas, who all have $n$ free variables, such that all formulas are satisfied by a tuple of elements in some model of $T$. A tuple which satisfies all formulas in a type is said to **realize** the type. Inside a structure $\mathscr{M}$ we may speak of the type of a tuple $\bar{a}$ over a set $B \subseteq M$, denoted $tp^{\mathscr{M}}(\bar{a}/B)$, by which we mean the set of all formulas $\varphi(\bar{x},\bar{b})$, where $\bar{b} \in B$ such that $\mathscr{M} \models \varphi(\bar{a},\bar{b})$. A type $p$ in a model $\mathscr{M}$ is **isolated**, by a satisfiable formula $\varphi$, if $\mathscr{M} \models \forall x(\varphi(\bar{x}) \to \psi(\bar{x}))$ for every $\psi(\bar{x}) \in p$.

For $V-$structures $\mathscr{M},\mathscr{N}$, an **embedding** $f : \mathscr{M} \to \mathscr{N}$ is an injective function $f : M \to N$ such that for each constant symbol $c \in V$, function symbol $g \in V$ and relation symbol $R \in V$, the following hold for any $a_1,\ldots,a_n \in M$.

- $f(c^{\mathscr{M}}) = c^{\mathscr{N}}$.
- $f(g^{\mathscr{M}}(a_1,\ldots,a_n)) = g^{\mathscr{N}}(f(a_1),\ldots,f(a_n))$.
- $R^{\mathscr{M}}(a_1,\ldots,a_n)$ if and only if $R^{\mathscr{N}}(f(a_1),\ldots,f(a_n))$.

An **isomorphism** is a bijective embedding, while an **automorphism** is an isomorphism whose range is the same structure as its domain. A **substructure** $\mathscr{N} \subseteq \mathscr{M}$ is a structure such that $N \subseteq M$ and the inclusion function is an embedding.

For a cardinal $\kappa$ a theory is $\kappa-$**categorical** if it has only a single model, up to isomorphism, of cardinality $\kappa$. The $\aleph_0-$categorical theories are especially nice due the following theorem.

**Theorem 1.1.1** (Engeler [12], Ryll-Nardzevski [40] and Svenonius [44] all independently)**.** *Let $T$ be a countable and complete theory with some infinite model. The following are equivalent.*

1. *$T$ is $\aleph_0-$categorical.*
2. *For each $n$ there are only finitely many $n-$types of $T$.*
3. *All types of $T$ are isolated.*

A graph $\mathscr{G}$ is a structure over the vocabulary $\{E\}$ with only a single binary relation such the interpretation of $E$ in $\mathscr{G}$, $E^{\mathscr{G}}$, is a symmetric, anti-reflexive relation. The relation $E$ is called an edge relation, while the elements in the universe of $\mathscr{G}$ are called vertices. Graphs have a special place in this thesis since the nicest non-trivial relational structures are the graphs. Because of this we will give some extra definitions just for the graphs. The complete graph on $n$ vertices, denoted $K_n$, is the graph with an edge between every pair of vertices. For a graph $\mathscr{G}$ the complement graph $\mathscr{G}^c$ is the graph with the same universe as $\mathscr{G}$ but for any $a,b \in G$ we have that $\mathscr{G} \models aE^{\mathscr{G}}b$ if and only if $\mathscr{G}^c \not\models aE^{\mathscr{G}^c}b$. For graphs $\mathscr{G}$ and $\mathscr{H}$ we define the disjoint union graph $\mathscr{G} \dot{\cup} \mathscr{H}$ as the graph with vertex set $G \dot{\cup} H$ and edge set $E^{\mathscr{G}} \dot{\cup} E^{\mathscr{H}}$. Note that when we defined substructures, if $\mathscr{N} \subseteq \mathscr{M}$ then all relations which hold on a tuple in $\mathscr{N}$ also hold for that tuple in $\mathscr{M}$. In graph theory this is often called induced subgraph, however in our model theoretic context we will refer to this just as a subgraph.

Ramsey theory is sometimes introduced as the fact that in big enough chaos there needs to exist small sections of order. More specifically when we talk about graphs, for any $m \in \mathbb{N}$, if we take a big enough graph then there exists a subgraph with $m$ vertices which is either the complete graph or the independent graph. In the infinite case we get the following theorem.

**Theorem 1.1.2.** *If $\mathscr{G}$ is an infinite graph then there exists an infinite subgraph $\mathscr{A} \subseteq \mathscr{G}$ such that $\mathscr{A}$ is either complete or independent.*

Theorems of a similar fashion exist in many different forms and, even though the concept is purely combinatorial, the methods of Ramsey theory are often used in model theory. In this thesis Ramsey theoretical concepts are used as part of proofs in articles III, IV and VI. For more information about Ramsey Theory see [19].

## 1.2 $0-1$ laws

For each $n \in \mathbb{Z}^+$, let $\mathbf{K}_n$ be some set of finite structures and associate a probability measure $\mu_n$ with each such set. Put $\mathbf{K} = (\mathbf{K}_n, \mu_n)_{n\in\mathbb{N}}$ to be the collection of these sets and their probability measures. We may extend the probability measures $\mu_n$ such that for any property $P$, not necessarily in a first order language, we define

$$\mu_n(P) = \mu_n(\{\mathscr{M} \in \mathbf{K}_n : \mathscr{M} \text{ satisfies } P\}).$$

Define $\mu(P) = \lim_{n\to\infty} \mu_n(P)$. We say that $\mathbf{K}$ has a **limit law** if for each first order sentence $\varphi$, in the specific language, $\mu(\varphi)$ converges. We say that $\mathbf{K}$ has a **$0-1$ law** if the limit $\mu(\varphi)$ always converges to 0 or 1. Define the **almost sure theory** $T_{\mathbf{K}}$ associated with $\mathbf{K}$ as the set of all first order sentences $\varphi$ such that $\mu(\varphi) = 1$, these sentences are referred to as **almost sure sentences**. It is a quick exercise, using the definitions, to show that $\mathbf{K}$ has a $0-1$ law if and only if $T_{\mathbf{K}}$ is a complete theory.

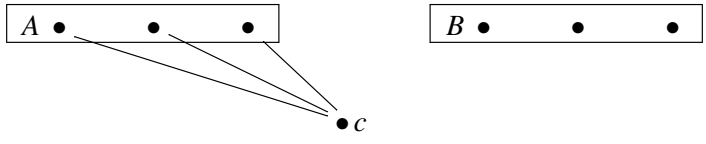
A couple of different probability measures will be used throughout this thesis. The most common probability measure, which we use unless we say anything else, is the **uniform** measure which, for each $\mathscr{M} \in \mathbf{K}_n$, puts $\mu_n(\mathscr{M}) = 1/|\mathbf{K}_n|$. We will often let $\mathbf{K}_n$ be the set of all structures with universe $[n]$ which satisfy some specific property. We refer to this by saying that the structures in $\mathbf{K}_n$ are **labeled**. This means that many structures in $\mathbf{K}_n$ will (except in trivial cases) be isomorphic to each other since if one renames the elements in the universe of a structure we get a different (yet isomorphic) structure. The other common case is to not allow for multiple structures with the same isomorphism type in $\mathbf{K}_n$. We then say that the structures in $\mathbf{K}_n$ are **unlabeled**. It is often easier to count labeled sets of structures than unlabeled, which is why many $0-1$ laws are first calculated on labeled sets and then transfered, through careful calculations, to the unlabeled case.

**Example 1.2.1.** For a fixed finite relational vocabulary $V$, let $\mathbf{K}_n$ be the set of all structures with universe $[n]$ and put $\mu_n$ to be the uniform probability measure on $\mathbf{K}_n$. Both Glebskii, Kogan, Liogon'kii, Talanov [18] and Fagin [14] independently proved that $\mathbf{K}$ has a $0-1$ law, however they used quite different methods to show this. The proof which Fagin used is important for the rest of this thesis and thus we will give a short sketch of it here.

The first thing which Fagin does is to define extension properties. These are formulas $\varphi_1, \ldots, \varphi_k, \ldots$ such that $\mathscr{M} \models \varphi_k$ if for any structure $\mathscr{A}$ of size $k$ and $\mathscr{A} \subseteq \mathscr{B}$ such that $|B| - |A| = 1$, if $\mathscr{A}_0 \subseteq \mathscr{M}$ and $\mathscr{A}_0 \cong \mathscr{A}$ then there is $\mathscr{B}_0 \subseteq \mathscr{M}$ such that $\mathscr{A}_0 \subseteq \mathscr{B}_0$ and $\mathscr{B}_0 \cong \mathscr{B}$. These extension properties are then proven to be almost surely true in $\mathbf{K}$ and thus they are in the almost sure theory $T_{\mathbf{K}}$. Fagin then show that $T_{\mathbf{K}}$ is countably categorical by building an isomorphism between any two countable models using the extension properties. Since $T_{\mathbf{K}}$ is countably categorical and does not have any finite models, as this is clearly

an almost sure property, the Łoś-Vaught test tells us that $T_{\mathbf{K}}$ is complete hence $\mathbf{K}$ has a $0-1$ law.

If we consider the previous example in the case of graphs, thus let $\mathbf{K}_n$ consist of all graphs with universe $[n]$, then the extension properties will state that for any sets $A, B$ such that $A \cap B = \emptyset$ and $|A \cup B| = k$ there is an element $c$ such that $c$ is adjacent to all elements in $A$, but adjacent to no elements in $B$.




The unique countable model for $T_{\mathbf{K}}$ will be isomorphic to the Rado graph. Because of this isomorphism the Rado graph is sometimes called "the random graph". In the same way "the random structure" often refers to the countable model of the almost sure theory $T_{\mathbf{K}}$ created in Example 1.2.1. This notation will not however be used further in this thesis as we have two other definitions of a structure being random in Paper I and Paper II.

**Example 1.2.2.** For a positive integer $l$ let $\mathbf{K}_n$ consist of all graphs $\mathscr{G}$ with universe $[n]$ such that the complete graph on $l+1$ vertices is not embeddable in $\mathscr{G}$. In 1976 Erdős, Kleitman, Rothschild [13] showed that almost surely the graphs in $\mathbf{K}$ are $l-$partite, thus the structures may be partitioned into $l$ parts such that no edges exist between elements in the same part. Note that this implies that if $\mathbf{C}_n$ is the set of all $l-$partite graphs with universe $[n]$ then $\mathbf{C}_n$ and $\mathbf{K}_n$ are almost surely the same. In 1987 Kolaitis, Prömel, Rothschild [25] use this result in order to show that $\mathbf{K}$ has a $0-1$ law. The proof of the $0-1$ law is done in the fashion of Fagin [14], which we sketched in Example 1.2.1. The extension properties are similar except that they only concern $l-$partite graphs. The proof is then conducted just like in 1.2.1, but in the current context, resulting again in a complete countably categorical almost sure theory.

For a fixed $l$, we define the **random $l-$partite graph** as the unique countable model of $T_{\mathbf{K}}$ when $\mathbf{K}_n$ consists of all $l-$partite graphs, as in the above example.

In general there is no reason to think that a certain set of structures should satisfy a $0-1$ law or even a limit law. In the following example we provide a couple of quick illustrations of some interesting instances of such sets of structures.

**Example 1.2.3.** If we let $\mathbf{K}_{2k}$ consist of only complete graphs on $2k$ vertices, while $\mathbf{K}_{2k+1}$ consists of only non-complete graphs on $2k+1$ vertices, then $\mathbf{K}$ will not have a limit law. The sentence $\forall x \forall y (x \neq y \rightarrow xEy)$ is true in each

structure in $\mathbf{K}_{2k}$ while false in each structure in $\mathbf{K}_{2k+1}$, thus the limit

$$\lim_{n\to\infty} \mu_n(\forall x \forall y(x \neq y \to xEy))$$

will not converge, as the probability will shift back and forth between 0 and 1 when going to infinity.

A more natural example of a set of structures without a limit law comes from an article by Compton, Henson, Shelah [7]. They prove that if $\mathbf{K}_n$ consists of all structures with universe $[n]$ over the language $\{\leq, R\}$, where $\leq$ is always interpreted as the linear order on $[n]$ and $R$ is a binary relation, then there is a sentence in the language whose asymptotic probability does not converge. In the same paper a non-limit law is also proved for $\mathbf{C}$ when $\mathbf{C}_n$ consist of all structures over the universe $[n]$ using the language consisting of a single binary function symbol. This is in sharp contrast with the results of Lynch [35] who prove that if $\mathbf{S}_n$ consists of all structures with universe $[n]$ over a vocabulary with a finite amount of unary function symbols then $\mathbf{S}$ have a limit law, but not a $0-1$ law.

Further examples of $0-1$ laws include Partial orders [6], Colored structures [26] and Sparse graphs [41]. For other expositions of limit and $0-1$ laws the reader may look at [10, 43, 46].

## 1.3 Homogeneous structures

**Definition 1.3.1.** Let $\mathscr{M}$ be a structure and $\mathscr{A} \subseteq \mathscr{M}$. We say that $\mathscr{M}$ is $\mathscr{A}-$**homogeneous** if for each embedding $f : \mathscr{A} \to \mathscr{M}$ there is an automorphism $g : \mathscr{M} \to \mathscr{M}$ such that $g(a) = f(a)$ for each $a \in A$. We say that $\mathscr{M}$ is **homogeneous** if $\mathscr{M}$ is $\mathscr{A}-$homogeneous for each finite $\mathscr{A} \subseteq \mathscr{M}$.

What we here call homogeneous is sometimes in the literature referred to as ultrahomogeneous [24] since the term homogeneous is used in other contexts in model theory. In this thesis we will however only use the above notion of homogeneous, thus we will not need to use the term ultrahomogeneous.

**Example 1.3.2.** The most trivial example of a homogeneous structure is just taking a trivial structure, having no relations. Trivially any embedding from a substructure is extendable to an automorphism. The rational numbers $\mathbb{Q}$ with the usual dense linear order relation is a homogeneous structure. This is a consequence of the denseness of the rationals making it possible to stretch and shrink the rational line without changing any properties. The Rado graph is also a homogeneous structure. In order to show this we can use the extension properties, which were also used to show the $0-1$ law in Example 1.2.1.

In order to show that the previously mentioned structures are homogeneous

we can use a so called back-and-forth argument to build finite partial isomorphisms in such a way that if we take the union of these partial isomorphisms we get an isomorphism of the whole structure.

**Definition 1.3.3.** Let **K** be a class of structures. The class **K** is **closed under isomorphism** if for each structure $\mathscr{M} \in \mathbf{K}$ and isomorphic structure $\mathscr{N}$, we have that $\mathscr{N} \in \mathbf{K}$. We say that **K** has the **hereditary property**, or in short just HP, if for each $\mathscr{A} \in \mathbf{K}$ and $\mathscr{B} \subseteq \mathscr{A}$ we have that $\mathscr{B} \in \mathbf{K}$. The class **K** satisfies the **joint embedding property**, or in short just JEP, if for each $\mathscr{A}', \mathscr{B}' \in \mathbf{K}$ there exists a structure $\mathscr{C}' \in \mathbf{K}$ such that both $\mathscr{A}'$ and $\mathscr{B}'$ are embeddable in $\mathscr{C}'$. Finally **K** has the **amalgamation property**, or in short just AP, if for each $\mathscr{A}, \mathscr{B}, \mathscr{C} \in \mathbf{K}$ and embeddings $f_0 : \mathscr{A} \to \mathscr{B}$ and $g_0 : \mathscr{A} \to \mathscr{C}$ there exists a structure $\mathscr{D} \in \mathbf{K}$ and embeddings $f_1 : \mathscr{B} \to \mathscr{D}$ and $g_1 : \mathscr{C} \to \mathscr{D}$ such that for each $a \in A$, $f_1(f_0(a)) = g_1(g_0(a))$.

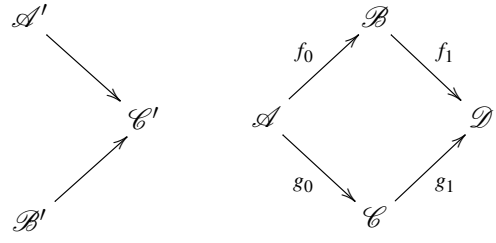



*Figure 1.1.* JEP and AP respectively.

Define the **age** of a structure

$$Age(\mathscr{M}) = \{\mathscr{A} : \mathscr{A} \text{ is finite and embeddable in } \mathscr{M}\}.$$

This is a class of structures which is not a set. We could easily make it into a set (which is countable in the case of a countable vocabulary) by only choosing the structures of size $n$ which have universe $[n]$. Defining the age so that it only becomes a class will however be convenient for some theorems such as the one below.

**Theorem 1.3.4** (Fraïssé [16])**.** *If $\mathscr{M}$ is an infinite homogeneous structure, then $Age(\mathscr{M})$ satisfies HP, JEP and AP.*

*If* **K** *is a class of finite structures closed under isomorphism satisfying HP, JEP and AP then there is a unique countable homogeneous structure $\mathscr{M}$ such that $Age(\mathscr{M}) = \mathbf{K}$.*

Due to this theorem one may say that a structure $\mathscr{M}$ is the **Fraïssé-limit** of a class of structures **K**, which means that $\mathscr{M}$ is the unique (up to isomorphism) countable homogeneous structure such that $Age(\mathscr{M}) = \mathbf{K}$.

We say that a structure $\mathscr{M}$ has **quantifier elimination** if for each formula

$\varphi(\bar{x})$ there is a quantifier free formula $\psi(\bar{x})$ such that $\mathscr{M} \models \forall \bar{x}(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}))$. This essentially means that the isomorphism type of a tuple, which in some sense is the most narrow quantifier free formula possible, determines the type of the tuple. The following connection between quantifier elimination and homogeneous structures is a consequence of this argument, where the countable categoricity is important to make all the types isolated by their atomic diagrams.

**Fact 1.3.5.** *Let $\mathscr{M}$ be a countably categorical structure. The structure $\mathscr{M}$ is homogeneous if and only if $\mathscr{M}$ has quantifier elimination.*

We have thus got three different characterization of a homogeneous structure: quantifier elimination, embedding extensions and the age satisfying $HP$, $JEP$ and $AP$. In some cases the homogeneous structures have been classified, but the general question what a homogeneous structure over a finite relational vocabulary looks like is still far from being solved. Even in the case of homogeneous 3−hypergraphs, there does not exist a known classification.

We will now present the classifications for countable (finite and infinite) graphs which are important as basic references in this discussion of homogeneous structures. The theorems are however also very important for Paper VI where the results are explicitly used.

**Theorem 1.3.6** (Gardiner [17] and independently Golfand and Klin [21])**.** *If $\mathscr{M}$ is a finite homogeneous graph, then $\mathscr{M}$ (or $\mathscr{M}^c$) is isomorphic to the 5−cycle, the $3 \times 3$−rook graph or a finite disjoint union of complete graphs of the same size.*

Note that the $3 \times 3$−rook graph is the graph which is created when you, on an empty $3 \times 3$ chess board, add an edge between any two squares which a rook may move between. This is isomorphic to the line graph of the complete bipartite graph with 3 elements in each part.

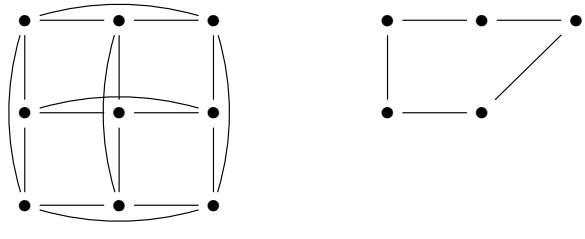


*Figure 1.2.* The $3 \times 3$−rook graph and the 5−cycle respectively.

**Theorem 1.3.7** (Lachlan and Woodrow [32])**.** *If $\mathscr{M}$ is a countably infinite homogeneous graph then $\mathscr{M}$ (or $\mathscr{M}^c$) is isomorphic to one of the following.*

- *The Rado graph.*

- *For an integer $n > 2$, the Fraïssé-limit of the class of finite graphs which does not embedd $K_n$.*
- *For $n \in \mathbb{Z}^+$, the infinite disjoint union of multiple $K_n$.*
- *A finite or infinite disjoint union of multiple $K_\infty$.*

Further classification theorems, which will not be relevant for this thesis to describe in detail, considering homogeneous structures include Cherlin's [5] classification of homogeneous digraphs and Lachlan's [33] classification of homogeneous tournaments. It is interesting to note that while there exists only $\aleph_0$ countable homogeneous graphs, there are $2^{\aleph_0}$ different isomorphism types of countable digraphs, which follows from a result from Henson [23].

In a structure which is not homogeneous there exist tuples who have different types yet induce the same local substructure. In a homogenizable structure we can, by just adding a finite amount of new relation symbols, distinguish the induced substructures of these types and thus make the structure homogeneous.

**Definition 1.3.8.** A $V-$structure $\mathscr{M}$ is called **homogenizable** if there exists a finite amount of $\emptyset-$definable relations $R'_1, \ldots, R'_n$ in $\mathscr{M}$ such that if we create a new vocabulary $V' = V \cup \{R_1, \ldots, R_n\}$ of relation symbols of corresponding arity and let $\mathscr{N}$ be the $V'-$structure such that $\mathscr{N} \restriction V = \mathscr{M}$ and $R_i^{\mathscr{N}} = R'_i$, then $\mathscr{N}$ is a homogeneous structure.

**Example 1.3.9.** The random bipartite graph $\mathscr{M}$ is not homogeneous. This structure was constructed in Example 1.2.2 as the unique countable model of the almost sure theory generated from the set of finite $K_3-$free graphs. One consequence of the extension properties, which were used to prove the $0-1$ law, is that for any two elements $a, b$ which belong to the same part there exists an element $c$ such that both $a$ and $b$ are adjacent to $c$, thus $c$ is in a different part than $a$ and $b$. Note however that this property can not hold for any elements $a', b'$ which are in different parts, since then the corresponding element $c'$ would be in either the part of $a'$ or $b'$ and thus there would be an edge inside a part, which is not allowed in a bipartite graph. Hence if we let $a', b'$ be elements which do not have an edge between them yet are in different parts and map $ab$ to $a'b'$, this embedding can not be extended to an automorphism.

Even though $\mathscr{M}$ is not homogeneous, we can use the extension properties in order to show that $\mathscr{M}$ is homogenizable. Define a new relation $P(x,y)$ by stating $\exists z(xEz \wedge yEz)$. From the above discussion it is clear that $\mathscr{M} \models P(c,d)$ if and only if $c$ and $d$ are elements in the same part. If we add $P$ as a new symbol in the vocabulary and intepret it in the above way, we can thus distinguish pairs of elements which come from the same part from pairs of elements from different parts. The new structure which has this extra relation will be homogeneous, which may be shown using a back-and-forth proof. A

similar, yet more technical, discussion is possible in order to show that for any $l \in \mathbb{Z}^+$ the random $l-$partite graph is homogenizable but not homogeneous.

One of the first to explicitly study homogenizable structures was Covington who, in her 1989 article [9], studied the class $\mathbf{K}$ consisting of all graphs which does not embed the 4 vertex path. Covington found that there is a unique model complete structure $\mathscr{M}$ such that $\mathscr{M}$ is homogenizable and $Age(\mathscr{M}) = \mathbf{K}$. In a later paper Covington [8] generalized her method in order to show that any class of structures satisfying the "Local failure of amalgamation property" generates a homogenizable structure. This property holds for the graphs $\mathbf{K}$ which does not embed a 4 vertex path, however it does not hold for the bipartite graphs described in Example 1.3.9.

In more recent years the subject has come alive again with results coming from Atserias and Toruńczyk [2] who found a necessary condition for a class of finite structures to generate a homogenizable structure and Hartman, Hubička and Nešetřil [22] who found that certain sets of structures with certain forbidden substructures all generate a homogenizable structure. For a review of homogeneous structures and its applications we refer to Macpherson's [36] article.

## 1.4 Simple theories and Pregeometries

One of the first to take on the quest to abstractly characterize models and their theories was Morley [37] who, in 1968, introduced the concept of a transcendental theory and Morley rank. These are abstract properties which can be used to classify theories and tell them apart on an abstract level. The field took a huge leap through Saharon Shelah who, among other things, published a book called Classification theory [42] (first printed in 1978) further developing the field and introducing new abstract properties. The concepts were often quite concrete, such as "There is a formula which defines a tree" or "There is a formula which defines a linear order". It turns out though that some of the nicest theories we can imagine, such as infinite sets, algebraically closed fields or sets with a finite number of equivalence relations, do not define any of these combinatorial structures, thus we get properties such as $NIP$, $NSOP$, $NTP$ which state that such combinatorial structures can not be created. It could seem that such negative information implies no information. We do however get strong abstract properties since a theory needs to be very restricted to not, in any way, define orders or trees.

In this part of the introduction we will take a quick look at some of the concepts from the abstract part of model theory. The reader who wants to see more details, consequences and examples the books [3, 45] are recommended or, for someone new to these concepts, the paper [20] which is written in an accessible format.

For a structure $\mathscr{M}$ and $A \subseteq M$ let $S_n^{\mathscr{M}}(A)$ be the set of all complete $n-$types over $A$ realized in $\mathscr{M}$. For a cardinal $\kappa$ we say that a theory $T$ is $\kappa-$**stable** if for any $\mathscr{M} \models T$ and $A \subseteq M$ with $|A| \leq \kappa$ we have that $|S_n^{\mathscr{M}}(A)| \leq \kappa$. A theory is **stable** if it is $\kappa-$stable for some cardinal $\kappa$. It is customary to write $\omega-$stable instead of $\aleph_0-$stable. It should be noted that in any structure $\mathscr{M}$, $|S_n^{\mathscr{M}}(A)| \geq \kappa$ for $|A| = \kappa$ since we can create the trivial type generated by $\bar{a} = \bar{x}$ which is clearly distinct for each $\bar{a} \in A$.

**Example 1.4.1.** Many of the most trivial structures are stable. Just taking an infinite set with no relations is $\omega-$stable since the only non-trivial type in $S_1^{\mathscr{M}}(A)$ is the type generated by $\{a \neq x : a \in A\}$. Some less trivial theories which are $\omega-$stable include algebraically closed fields and finitely/infinitely cross cutting equivalence relations.

Let $T$ be a theory and $\mathscr{M} \models T$. For $\bar{a}, A \subseteq M$ we say that a formula $\varphi(\bar{x}, \bar{a})$ **divides** over $A$ if there is a sequence $(\bar{a}_i)_{i \in \mathbb{N}}$ and a number $k \in \mathbb{Z}^+$ such that each subset of size $k$ of $\{\varphi(\bar{x}, \bar{a}_i) : i \in \mathbb{N}\}$ is inconsistent with $T$. A type $p$ **forks** over $A$ if there are formulas $\varphi_1(\bar{x}), \ldots, \varphi_n(\bar{x})$ such that $p$ implies $\varphi_1(\bar{x}) \vee \ldots \vee \varphi_n(\bar{x})$ and each $\varphi_i$ divides over $A$. If $A \subseteq B$, $p \in S_n^{\mathscr{M}}(B)$ and $q$ is the restriction of $p$ to only the formulas with parameters in $A$, then we say that $p$ is a **non-forking extension** of $q$ if $p$ does not fork over $A$. We write

$$A \underset{C}{\downarrow} B \text{ if } tp(A/C \cup B) \text{ is a non-forking extension of } tp(A/C).$$

The relation $\downarrow$ is called an independence relation and its negation, which indicate that the extension is forking, will be denoted $\not\downarrow$. We say that $T$ has **trivial independence** if for every $A, B, C_1, C_2$ if $A \underset{B}{\not\downarrow} (C_1 \cup C_2)$ then $A \underset{B}{\not\downarrow} C_1$ or $A \underset{B}{\not\downarrow} C_2$.

A complete theory $T$ is **simple** if for each model $\mathscr{M} \models T$, subset $B \subseteq M$ and type $p \in S_n^{\mathscr{M}}(B)$ there is $A \subseteq B$ such that $|A| \leq |T|$ and $p$ does not fork over $A$. The theory is **supersimple** if the set $A$ may always be chosen finite. The $SU-$rank of a type $p$ is defined in the following way, where $\alpha$ is an ordinal.

$$SU(p) \geq 0 \text{ if } p \text{ is consistent.}$$

$$SU(p) \geq \alpha + 1 \text{ if there is a forking extension } q \text{ of } p \text{ with } SU(q) \geq \alpha.$$

$$SU(p) \geq \alpha \text{ for a limit ordinal } \alpha \text{ if } SU(p) > \beta \text{ for each } \beta < \alpha.$$

Equality for $SU-$rank is defined by $SU(p) = \alpha$ if and only if $SU(p) \geq \alpha$ but $SU(p) \not\geq \alpha + 1$. Note that in general the $SU-$rank of a type can be any ordinal and sometimes the process can even go on forever, which is usually denoted with $SU(p) = \infty$. As an abbreviation for $SU(tp(a/B))$ we write $SU(a/B)$. We now present a few useful facts regarding the concepts just introduced.

**Fact 1.4.2.**

- *If a theory is stable then it is simple.*
- *In a simple theory, a type $p$ forks over a set $A$ if and only if $p$ divides over $A$.*
- *A theory is supersimple if and only if $SU(p) < \infty$ for all types $p$ (which are real).*

**Example 1.4.3.** The Rado graph is the standard example which is used to show something which is simple but not stable. This follows quickly using the extension properties. For any infinite set $A$ and disjoint $B, C \subseteq A$, let $p_{BC}$ be the type which consists of formulas which state that $x$ is adjacent to all elements in $B$ but adjacent to no elements in $C$. Using the extension properties and compactness we can show that $p_{BC}$ is consistent. For any disjoint $B, C$ such that $B \cup C = A$ it is clear that $p_{BC}$ is distinct, thus this method clearly creates $2^{|A|}$ different types. In Paper I we show (in a slightly more general setting) that the Rado graph is supersimple with SU-rank 1, a proof which very smoothly uses the extension properties and works directly with the definition of dividing.

In Example 1.2.2 we noted that the class **K** of all graphs not embedding the complete graph on 3 vertices has a $0-1$ law, where the countable structure which satisfy the almost sure theory is the random bipartite graph $\mathscr{M}$. It is a very similar proof (as in the Rado graph case) to show that $\mathscr{M}$ is simple with SU-rank 1, but $\mathscr{M}$ is not stable. We can also show that the class **K** satisfies the amalgamation, joint embedding and hereditary property, thus Theorem 1.3.4 implies that there exists a unique countable homogeneous structure $\mathscr{N}$ such that $Age(\mathscr{N}) = \mathbf{K}$. From Example 1.3.9 we know that $\mathscr{M}$ is not homogeneous (but homogenizable) and thus $\mathscr{M} \ncong \mathscr{N}$. Furthermore one can show that $\mathscr{N}$ is not simple. This distinction between $\mathscr{N}$ and $\mathscr{M}$ is a sharp contrast to the Rado graph which is both generated as the Fraïssé-limit from the class of all graphs and as the unique countable model of the almost sure theory coming from that class.

**Definition 1.4.4.** Let $A$ be a set and let $cl : \mathscr{P}(A) \to \mathscr{P}(A)$ be a function acting on the subsets of $A$. We say that $cl$ is a **closure operator** on $A$ if the following properties are satisfied for any $X, Y \subseteq A$.

**Reflexive** $X \subseteq cl(X)$.

**Monotonicity** $X \subseteq cl(Y)$ implies $cl(X) \subseteq cl(Y)$

**Finite Character** $cl(X) = \bigcup\{cl(X_0) : X_0 \subseteq X, |X_0| < \aleph_0\}$.

The pair $(A, cl)$ is called a **pregeometry** (or a matroid) if $cl$ is a closure operator on $A$ and the following property is also satisfied for any $a, b \in A$ and $X \subseteq A$.

**Exchange property** If $a \in cl(X \cup \{b\}) - cl(X)$ then $b \in cl(X \cup \{a\})$.

We say that a structure $\mathscr{M}$ has a definable pregeometry if there is a pregeometry $(M, cl)$ and there exists formulas $\theta_0(x_0), \theta_1(x_0, x_1), \ldots$ such that for any $a, b_1, \ldots, b_n \in M$ we have $a \in cl(b_1, \ldots, b_n)$ if and only if $\mathscr{M} \models \theta_n(a, b_1, \ldots, b_n)$.

**Remark 1.4.5.** The axioms for a closure operator are often chosen to include the statement $cl(X) = cl(cl(X))$. This is not necessary using the above chosen monotonicity axiom. If we let $X = cl(Y)$ and then apply the monotonicity axiom to $X \subseteq cl(Y)$ we get

$$cl(cl(Y)) = cl(X) \subseteq cl(Y).$$

On the other hand $cl(Y) \subseteq cl(Y)$, thus applying reflexivity we get $cl(Y) \subseteq cl(cl(Y))$. These two facts together imply that $cl(Y) = cl(cl(Y))$, which we got using only the above reflexivity and monotonicity. The reason that $cl(cl(X)) = cl(X)$ is not chosen as an axiom is that we then would have to add another monotonicity axiom in order to make the axiom schema equally strong, such as

$$X \subseteq Y \text{ implies } cl(X) \subseteq cl(Y).$$

Thus we would have 4 axioms instead of 3.

**Example 1.4.6.** The nicest pregeometry $(A, cl)$ is the **trivial pregeometry** which is defined by first arbitrarily choosing $cl(\emptyset)$, then put $cl(X) = X \cup cl(\emptyset)$ for any $X \subseteq A$. There are other versions for the "trivial pregeometry" in the literature such as $cl(X) = X$ or $cl(X) = \bigcup_{x \in X} cl(x)$. The concept of a trivial pregeometry, as defined above, will be used in Paper I.

A less trivial example can be created if we have a vector space $V$ and let $cl$ be the *linear span* operator. We can show that this is a pregeometry using standard linear algebra. This pregeometry is called a **vector space pregeometry**. The affine pregeometry and projective pregeometry are both pregeometries which are similar to the vector space pregeometry but with certain modifications. For more information on and examples of pregeometries see [38].

We say that a type is algebraic if it is only realized by a finite amount of elements. The **algebraic closure** in a structure $\mathscr{M}$ is defined on sets $X \subseteq M$, denoted $acl(X)$, as the set of all elements $a \in M$ such that the type $tp(a/X)$ is algebraic. In any structure the algebraic closure defines a closure operator. This means that we can talk about the algebraic closure being trivial, the monotonicity of the algebraic closure etc. just like we do for arbitrary closure operators and pregeometries. If a structure is simple with SU-rank 1 then the algebraic closure even defines a pregeometry. If the structure is also countably categorical then the algebraic closure, and thus also the pregeometry, is definable.

# 2. On the appended papers

## 2.1 Paper I

When studying $0-1$ laws such as Example 1.2.1, Example 1.2.2, the partial orders [6] and the colourable structures [26] a certain pattern seems to occur. The almost sure theories are all countably categorical, supersimple, have $SU-$rank 1 and trivial algebraic closure. These are also properties of the almost sure theories found in Paper IV. We can also note that the proofs of the $0-1$ laws which generate the almost sure theories are all done in a similar way as Fagins original proof, using extension properties, illustrated in Example 1.2.1. Paper I investigates why these three properties occur and shows that indeed the connection is not a coincidence. It is important to note that the above mentioned properties are not found in all almost sure theories as the Sparse graphs found by Shelah and Spencer [41] are not countably categorical, the theory is not even small, yet the almost sure theory is stable but not $\omega-$stable.

Remember from Section 1.2 that we denote the almost sure theory with respect to a set $\mathbf{K}$ as $T_{\mathbf{K}}$. We say that a vocabulary is binary if the arity of the relation symbols is at most 2.

**Theorem 2.1.1.** *If $T$ is countably categorical, simple with $SU-$rank 1 and has trivial algebraic closure over a finite binary relational vocabulary then there exists a set $\mathbf{K} = (\mathbf{K}_n, \mu_n)_{n\in\mathbb{N}}$ with a probability measure $\mu_n$ such that $T_{\mathbf{K}} = T$.*

This theorem even comes with an explicit construction showing what these structures look like and how we can generate them using finite structures. Since colourable structures such as the $l-$partite graphs have the above properties it is not that surprising that we may have a definable equivalence relation. In general however the equivalence relations do not need to follow the rules indicating that no edges exist inside any part, but rather we have $l$ parts and the relations inside each part and between parts are in some sense randomly placed, with the partition relation definable in the structure. Further more this means that all of these structures are homogenizable.

As a corollary we get a similar way to generate the stable and strongly minimal structures, since these are all special cases of the simple structures. For stable structures we have $l$ parts, however instead of placing relations randomly we have a unique choice between parts and inside each part.

In this paper we define a random structure as a structure whose complete

theory is equal to the almost sure theory generated from its set of finite substructures equipped with the uniform measure. In the proof of the above theorem the set $\mathbf{K}_n$ is created in a very specific way so that the theory $T$ gets generated as the almost sure theory. However in many of the examples of $0-1$ laws [6, 14, 18, 25] the set of structures considered is also the set of finite substructures of a model of the almost sure theory. Thus one might ask how the above result may extend to consider only random structures.

**Theorem 2.1.2.** *If $\mathscr{M}$ is binary, countable, $\aleph_0-$categorical, simple with $SU-$rank 1 and has trivial algebraic closure such that $acl(\emptyset)=\emptyset$, then $\mathscr{M}$ is a reduct of a binary random structure which is also $\aleph_0-$categorical, simple with $SU-$rank 1 and has trivial algebraic closure.*

Note that the extra condition $acl(\emptyset)=\emptyset$ is there because otherwise the structure which exist inside $acl(\emptyset)$ will probably disappear when generating the almost sure theory from the set of substructures. One way to solve this issue is if we would redefine a random structure as generated from the set of substructures where the structure of $acl(\emptyset)$ always is preserved. Another solution which works is to use another measure than the uniform. An instance of this was found by Elwes [11] who showed that if we use a preferential attachement process to get a probability measure, then the almost sure theory will be that of the Rado graph with a finite amount of vertices added which are either universal or isolated. These extra vertices will thus be spanning $acl(\emptyset)$ of this structure.

## 2.2 Paper II

In the ongoing task to understand the homogeneous structures we may add assumptions and restrictions from abstract model theory in order to get further tools to work with. The stable homogeneous structures are quite clearly understood from the work of Lachlan [34], however when generalizing to the simple structures not much work has been done. Paper II, which is coauthored with Vera Koponen, is the first in a sequence of papers [27, 28, 29, 30] where Koponen continues to study the binary simple homogeneous structures. One of the threads which which is followed to an end is a complete description of the binary simple homogeneous structures in [28].

In [29] Koponen shows that the binary simple homogeneous structures are all supersimple with finite $SU-$rank. If a structure is supersimple with finite rank then, for any element $a$, there is a finite set $A$ such that $SU(a/A)=1$. Thus if we understand the structure of the definable sets of $SU-$rank 1 we will have a quite good understanding of what the whole structure looks like.

Given a vocabulary $V$ with only binary relation symbols and a set $\Delta$ of binary atomic diagrams, let $R\Delta$ be the class of all finite $V-$structures $\mathscr{A}$ such

that any binary atomic diagram in $\mathscr{A}$ belong to $\Delta$. The class $R\Delta$ is clearly a class satisfying HP, JEP and AP, thus there is a unique homogeneous structure $\mathscr{M}$ such that $Age(\mathscr{M}) = R\Delta$. If we allow $\Delta$ to also contain unary relation symbols, such that all structures are "compatible" with each other, then a structure which is homogeneous with an age $R\Delta$ is called a **binary random structure**. Just like the Rado graph is both generated as the unique homogeneous structure from an amalgamation class and as the a unique structure satisfying an almost sure theory, so do also the binary random structures satisfy these two properties.

For sets $A, B \subseteq \mathscr{M}$, the **canonically embedded structure in** $A$ **over** $B$ is the structure with universe $A$ but for each distinct type $tp^{\mathscr{M}}(\bar{a}/B)$ add a relation symbol $R_{\bar{a}}$ which hold for exactly the tuples which satisfy the type. So the canonically embedded structure essentially has a relation for everything which is possible to express. The main result of Paper II states (in an even more general setting) that in a countable, binary, homogeneous, simple structure with trivial dependence any canonically embedded structure on a $SU-$rank 1 set is a reduct of a binary random structure. This means that when we are constructing structures satisfying the above properties, the binary random structures play a very important part.

## 2.3 Paper III

The four colour map theorem state that given any map one can colour the countries using only four colours such that no two adjacent countries get the same colour. We can abstract the concept of colourability to graph theory where we say that a graph is $l-$colourable if we can colour its vertices using $l$ different colours such that no two adjacent vertices get the same colour. In Example 1.2.2 we looked at graphs which are partitioned into $l$ different parts where no edge may exist inside any part, and noticed that if $\mathbf{K}_n$ consists of all such graphs with universe $[n]$ then we have a $0-1$ law. If we colour the elements which belong to the same part in an $l-$partite graph with the same colour we clearly get an $l-$coloured graph. We distinguish between coloured structures, structures with a unary relation for each colour, and colourable structures, structures where unary relations can be added (but does not exist) in order to make it coloured.

When we try to generalize the concept of colours to vocabularies containing relation symbols with arity 3 or higher, it is not clear exactly in which direction to go. If $R(a,b,c)$ holds we may either demand that all of $a, b$ and $c$ have different colours or that at least some pair of elements have different colours. This is what we call a **strong colouring** or **weak colouring** respectively. If a pregeometry is definable in the structure we can generalize this concept even more by adding the following extra assumptions.

- For $a, b$ such that $a \in cl(b) - cl(\emptyset)$ both $a$ and $b$ have the same colour.

- $\mathscr{M}$ is strongly (weakly) coloured if whenever $R(a_1,\ldots,a_n)$ holds for some relation symbol $R$ and $a_1,\ldots,a_n \in M$ then for all (there exists) $d,e \in cl(a_1,\ldots,a_n)$ which are independent we have that $d$ and $e$ have different colour.

Note in the above definition that we get the normal strong and weak coloured structures if we let $cl(X) = X$ for any set $X$. In [26] Koponen studies extension properties for sets of structures $\mathbf{K}_n$ where certain substructures are forbidden. In particular this setting includes the case where we have coloured and colourable structures. Adding the extra assumption of having an underlying pregeometry Koponen shows that if $\mathbf{K}_n$ consists of coloured structures with dimension $n$, then they have a $0-1$ law. This uses a very general theorem which may also be applied to the colourable case. In the colourable case Koponen also deduces certain structural properties, however this is only done in the case where the pregeometry is trivial.

In Paper III we expand the structural results which Koponen leaves out, creating the $0-1$ law in a concrete way and showing that the $l-$colourable structures almost surely have a uniformly definable colouring. More specifically we prove the following main theorem. Note that in favor for readability we state the assumptions in the Theorem rather vaguely. All details may however be found in the article.

**Theorem 2.3.1.** *For $l \in \mathbb{Z}^+$, let $\mathbf{K}_n$ be all labeled weakly $l-$colourable structures with a vector space, affine or projective pregeometry with dimension $n$ equipped with the dimension conditional measure $\delta_n$. The following then hold:*

- *There is a formula $\xi(x,y)$ such that almost surely in $\mathbf{K}$ if $a,b \in \mathscr{M} \in \mathbf{K}$ then $\mathscr{M} \models \xi(a,b)$ if and only if $a$ and $b$ are only colourable with the same colour.*
- *The structures in $\mathbf{K}$ are almost surely $l-$colourable in a unique way.*
- *The structures in $\mathbf{K}$ are almost surely not $(l-1)-$colourable.*
- *The almost sure theory $T_{\mathbf{K}}$ is countably categorical and axiomatized by $\forall\exists-$formulas.*

We assume above that we use a weak colouring, in the strong case however things are even better and we may relax the condition on the pregeometries to just having a certain property called polynomial $k-$saturation (which is already implied by the above pregeometries). The dimension conditional measure $\delta_n$ is a probability measure which give higher probability for structures which are easier to generate, where the generation process adds relations to small dimensions first and go up. This is different from the more common uniform measure which assigns all structures the same probability and we do not know whether the above theorem holds if we consider the uniform measure instead.

## 2.4 Paper IV

A year after Fagin published his proof [14] of the $0-1$ law for the set of structures $\mathbf{K}$, where $\mathbf{K}_n$ consists of all structures with universe $[n]$ over a finite relational vocabulary, Fagin continued to study $\mathbf{K}$ and discovered in [15] that almost surely the automorphism group of a structure in $\mathbf{K}$ is trivial. This result could then be used to show that if $\mathbf{C}_n$ consists of all structures of size $n$, but with only a single structure of each isomorphism type (the unlabeled case), then $\mathbf{C}$ has a $0-1$ law under the uniform measure. It also becomes clear that the set $\mathbf{D}_n$ consisting of all structures with universe $[n]$ and with non-trivial automorphism group will have asymptotic probability 0 when compared to the set $\mathbf{K}_n$. Note that the automorphism group of a structure can in general not be described using the first order formulas in the language, thus it is not clear from the $0-1$ law of $\mathbf{K}$ that any property of the automorphism group even converges.

Cameron [4] generalized Fagin's result in the graph case by studying the following question. Given a group $\mathscr{G}$, and letting each graph $\mathscr{M}$ in $\mathbf{C}_n$ have universe $[n]$ and $\mathscr{G} \leq Aut(\mathscr{M})$, what is the asymptotic probability that $Aut(\mathscr{M}) = \mathscr{G}$ for $\mathscr{M} \in \mathbf{C}$? Fagin's study shows that if $\mathscr{G}$ is the trivial group then this probability is 1, while Cameron proves that for any group this probability will exist and that it goes to 1 if and only if it is a direct product of symmetric groups.

Paper IV generalizes both Cameron's and Fagin's results to the case with sets of structures without trivial automorphism group over a finite relational vocabulary. The paper only considers sets of structures with the uniform measure, however the theorems stated below work in both the labeled and unlabeled setting. For notation in this paper we use $\mathbf{S}_n$ to denote the set of all structures with universe $[n]$ over a fixed relational vocabulary with at least one relation symbol of arity at least 2.

The first main theorem of Paper IV is the following which extend Cameron's results.

**Theorem 2.4.1.** *Let $\mathscr{G}, \mathscr{H}$ be finite groups. Then each of the following limits converge to a number in $\mathbb{Q}$ or goes to $\infty$.*

$$\lim_{n\to\infty} \frac{|\{\mathscr{M} \in \mathbf{S}_n : \mathscr{H} \leq Aut(\mathscr{M})\}|}{|\{\mathscr{M} \in \mathbf{S}_n : \mathscr{G} \leq Aut(\mathscr{M})\}|}, \qquad \lim_{n\to\infty} \frac{|\{\mathscr{M} \in \mathbf{S}_n : \mathscr{H} \cong Aut(\mathscr{M})\}|}{|\{\mathscr{M} \in \mathbf{S}_n : \mathscr{G} \cong Aut(\mathscr{M})\}|}$$

$$\lim_{n\to\infty} \frac{|\{\mathscr{M} \in \mathbf{S}_n : \mathscr{G} \cong Aut(\mathscr{M})\}|}{|\{\mathscr{M} \in \mathbf{S}_n : \mathscr{G} \leq Aut(\mathscr{M})\}|}$$

There are more possible fraction combinations than those listed in the above theorem, however through some easy algebraic manipulation we may deduce the others. Notice that we need to have infinity as a possibility for the limit since if we, for instance, choose $\mathscr{H}$ as the trivial group, but $\mathscr{G}$ as any non-trivial group then both of the first two limits will go to infinity, due to the

result by Fagin [15].

To write down the next theorem we will need to introduce some notation. For a finite structure $\mathscr{M}$ define the following concepts.

$$spt^*(\mathscr{M}) = |\{a \in M : f(a) \neq a \text{ for some } f \in Aut(\mathscr{M})\}|.$$

$$spt(\mathscr{M}) = \max_{f \in Aut(\mathscr{M})} |\{a \in M : f(a) \neq a\}|.$$

Thus $spt^*(\mathscr{M})$ is the total number of elements in $\mathscr{M}$ which are moved by at least one automorphism, while $spt(\mathscr{M})$ is the highest amount of elements moved by any automorphism.

**Theorem 2.4.2.**

- *For any finite group $G$, if $\mathbf{K}_n = \{\mathscr{M} \in \mathbf{S}_n : Aut(\mathscr{M}) \cong G\}$ or $\mathbf{K}_n = \{\mathscr{M} \in \mathbf{S}_n : Aut(\mathscr{M}) \geq G\}$ then $\mathbf{K}$ has a limit law.*
- *For any integer $m \geq 2$, if $\mathbf{K}_n = \{\mathscr{M} \in \mathbf{S}_n : spt^*(\mathscr{M}) \geq m\}$, $\mathbf{K}_n = \{\mathscr{M} \in \mathbf{S}_n : spt^*(\mathscr{M}) = m\}$ or $\mathbf{K}_n = \{\mathscr{M} \in \mathbf{S}_n : spt(\mathscr{M}) \geq m\}$ then $\mathbf{K}$ have a limit law.*
- *In all sets of structures previously considered in this theorem there is a finite set $A \subseteq \mathbb{Q}$ such that for any sentence $\varphi$ the asymptotic probability of $\varphi$ in $\mathbf{K}_n$ tends to a number in $A$.*

To prove these theorems we deduce the general structure which almost surely hold for structures in $\mathbf{K}_n$. We show that there are certain basic building blocks consisting of the structures where we fixate exactly what the structure of the support is, and then fixate how the automorphisms can move the support, call these sets of structures $\mathbf{S}_n(\mathscr{A},H)$. The set $\mathbf{S}(\mathscr{A},H)$ will then have a $0-1$ law and all of the above sets of structures (in the theorems) can be constructed as combinations of multiple sets similar to $\mathbf{S}(\mathscr{A},H)$, thus we get the limit laws (which in general are not $0-1$ laws) and limits with rational numbers. Further study of which automorphism groups are asymptotically found on structures depending on the support was done by Koponen [31].

## 2.5 Paper V

In the introduction we mentioned (Definition 1.3.8) that a structure is homogenizable if we can add a finite amount of new relation symbols to represent already definable relations in order to make the structure homogeneous. A homogeneous structure has restrictions on the automorphisms, quantifier elimination and certain properties of the age (Fact 1.3.5 and Theorem 1.3.4). The focus of Paper V is to study how these properties of the homogeneous structures generalize when we look at the homogenizable structures.

In the definition below we weaken the homogeneity property in order to

study homogenizable structures which are not homogeneous. It is shown in the paper that there are equivalent definitions which discuss quantifier elimination and the age in a similar way as the homogeneous structures do.

**Definition 2.5.1.** Let $\mathscr{M}$ be a homogenizable structure. The structure $\mathscr{M}$ is called

- **unavoidably homogenizable** if for some $k \in \mathbb{N}$ and any finite $\mathscr{A} \subseteq \mathscr{M}$ such that $|\mathscr{A}| > k$, $\mathscr{M}$ is $\mathscr{A}-$homogeneous.
- **uniformly homogenizable** if there is a finite structure $\mathscr{B} \subseteq \mathscr{M}$ such that for any finite structure $\mathscr{A} \subseteq \mathscr{M}$ with $\mathscr{B} \subseteq \mathscr{A}$, $\mathscr{M}$ is $\mathscr{A}-$homogeneous.
- **boundedly homogenizable** if for any finite $\mathscr{A} \subseteq \mathscr{M}$ there is a finite $\mathscr{B} \subseteq \mathscr{M}$ such that $\mathscr{A} \subseteq \mathscr{B}$ and $\mathscr{M}$ is $\mathscr{B}-$homogeneous.

The unavoidably homogenizable structures are essentially the most trivially homogenizable structures which are not homogeneous. We however show that the structures are not necessarily trivial, which we might guess since the homogeneous structures in general are not at all trivial. The unavoidably homogeneous graphs are studied and classified in Paper VI.

The uniformly homogeneous structures have a central place in understanding the homogenizable structures. They contain all structures homogenizable with algebraic formulas and any homogenizable structure can be made into a uniformly homogenizable structure by adding extra elements which witness the homogenizing formulas. This holds even if the homogenizable structure is not model-complete.

The paper provides a couple of examples showing that model-completeness is an important property of the homogenizable structures in order to keep them behaving nicely. If a structure is boundedly homogenizable it follows that the structure is model-complete. The question whether all model-complete homogenizable structures are boundedly homogenizable remains. In the case of homogenizable structures which are $\omega-$stable we prove that the answer is yes. Furthermore the paper studies specifically how these new definitions of homogenizable structures relate to being unary homogenizable, i.e. homogenizable using only unary relations, and find the following theorem.

**Theorem 2.5.2.** *If $\mathscr{M}$ is a countable infinite unary boundedly homogenizable structure with trivial algebraic closure such that $acl(\emptyset) = \emptyset$ then there are infinite uniformly homogenizable structures $\{\mathscr{N}_i\}_{i\in I}$ with only finitely many different isomorphism types such that*

$$\mathscr{M} = \bigcup_{i\in I} \mathscr{N}_i$$

## 2.6 Paper VI

While studying the homogenizable structures, one comes across the question "what is the most trivial example of a homogenizable structure which is not homogeneous?". There are essentially two ways for something to be "easy" to homogenize, either we have only a finite amount of elements who are in the new definable relations, called algebraically homogenizable, or we actually do not need the new relations after we have chosen a big enough structure, that is what we called unavoidably homogenizable in Paper V. In Paper VI we continue the study of the unavoidable homogenizable structures through a classification in the case of graphs.

Given a positive integer $k$, a structure $\mathscr{M}$ is called $k-$**homogeneous** if for each $\mathscr{A} \subseteq \mathscr{M}$ such that $|\mathscr{A}| = k$, $\mathscr{M}$ is $\mathscr{A}-$homogeneous. If $\mathscr{M}$ is $t-$homogeneous for each $t \geq k$ ($t < k$) then $\mathscr{M}$ is called $\geq k-$homogeneous ($<k-$homogeneous). Note that we could reformulate that $\mathscr{M}$ is homogeneous, from Definition 1.3.1, by saying that $\mathscr{M}$ is $k-$homogeneous for each $k \in \mathbb{N}$. Considering that Lachlan and Woodrow, see Theorem 1.3.7, classified all countable infinite homogeneous graphs, the next step is to look at graphs $\mathscr{G}$ which are $k-$homogeneous for all $k \in \mathbb{N}$ but a cofinite subset. This is equivalent with saying that $\mathscr{G}$ is $>k-$homogeneous for some $k \in \mathbb{N}$.

In order to study these graphs we will need to define a couple of new specific graphs which play important parts in the following constructions. Let $\mathscr{G}_t^c$ be the graph which consists of an infinite disjoint union of complete graphs on $t$ vertices, thus $\mathscr{G}_t$ is the complement graph of this graph. It is clear from Theorem 1.3.7 that $\mathscr{G}_t$ is a homogeneous graph.

**Lemma 2.6.1.** *Let $\mathscr{M}$ be a countable infinite graph. The graph $\mathscr{M}$ is $>k-$homogeneous but not $1-$homogeneous if and only if for some finite homogeneous graph $\mathscr{H}$ and $t \in \mathbb{Z}^+$ we have that $\mathscr{M}$, or $\mathscr{M}^c$, is isomorphic to $\mathscr{G}_t \dot{\cup} \mathscr{H}$.*

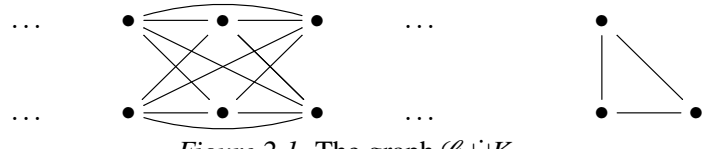


*Figure 2.1.* The graph $\mathscr{G}_2 \dot{\cup} K_3$.

The "only if" direction is straight forward to prove since if we take at least $t + 2|H| + 1$ vertices in $\mathscr{M}$ then we have found a connected component consisting of more than $|H|$ vertices, hence this component has to be a part of $\mathscr{G}_t$ while the other vertices are a part of $\mathscr{H}$. Since each component is homogeneous it follows that $\mathscr{M}$ is $>(t + 2|H|)-$homogeneous. The number $t + 2|H|$ is not the smallest number $k$ for which $\mathscr{M}$ is $>k-$homogeneous. However it is a number which trivially works for any choice of $t$ and $\mathscr{H}$. In order to describe the minimal $k$ one needs to conduct a case study depending on the choice of $\mathscr{H}$ and $t$. This is not done in Paper VI but should be a rather fun,

and not to hard, exercise.

Going further, for $t \geq 2$ we define the graph $\mathscr{H}_{t,1}$ as the disjoint union $\mathscr{G}_t \dot{\cup} \mathscr{G}_t$. For $t \geq 1$ define $\mathscr{H}_{t,2}$ as the graph $\mathscr{G}_t \dot{\cup} \mathscr{G}_t$ but where each $t-$tuple of independent vertices in one connected component gets completely connected to a unique $t-$tuple in the other connected component.

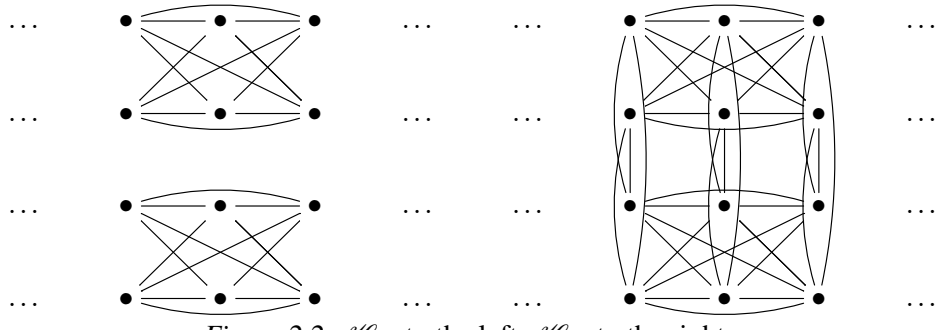


*Figure 2.2.* $\mathscr{H}_{2,1}$ to the left, $\mathscr{H}_{2,2}$ to the right.

Using these new graphs we may take care of the second case of the $>k-$homogeneous graphs, where we have $1-$homogenity but not $2-$homogenity.

**Lemma 2.6.2.** *Let $\mathscr{M}$ be a countable infinite graph. The graph $\mathscr{M}$ is $>k-$homogeneous, $1-$homogeneous but not $2-$homogeneous if and only if for some integer $t \geq 2$ $\mathscr{M}$, or $\mathscr{M}^c$, is isomorphic to $\mathscr{H}_{t,1}$, $\mathscr{H}_{t,2}$ or $\mathscr{H}_{1,2}$.*

To wrap up the paper we show that a $>k-$homogeneous infinite graph which is $1-$ and $2-$homogeneous is also homogeneous. The above results together form the main theorem of Paper VI which is a classification of the countable infinite $\geq k-$homogeneous graphs.

**Theorem 2.6.3.** *Let $\mathscr{M}$ be a countable infinite graph. The graph $\mathscr{M}$ is $>k-$homogeneous if and only if $\mathscr{M}$, or $\mathscr{M}^c$, is isomorphic to one of the following.*

- *A homogeneous graph.*
- *$\mathscr{G}_t \dot{\cup} \mathscr{H}$ for some positive integer $t$ and finite homogeneous graph $\mathscr{H}$.*
- *$\mathscr{H}_{1,2}$, $\mathscr{H}_{t,1}$ or $\mathscr{H}_{t,2}$ for some positive integer $t \geq 2$.*

The graph $\mathscr{G}_t \dot{\cup} \mathscr{H}$ in the above theorem is homogenizable by defining a unary relation which hold for all elements in the infinite component, while the graph $\mathscr{H}_{t,i}$ is homogenizable by the definable binary relation which states that two elements are in the same part. We may thus conclude that the $\geq k-$homogeneous graphs are homogenizable using only a single extra relation of arity at most 2.

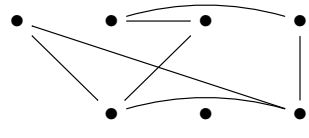
# 3. Sammanfattning på Svenska (Summary in Swedish)

Denna svenska sammanfattning är skriven på ett så populärvetenskapligt sätt som bara är möjligt för att kunna beskriva materialet i avhandlingen utan att samtidigt skriva en hel lärobok i matematik. Målet med sammanfattningen är att vem som helst[1] ska kunna läsa den för att få insikt i ungefär vad avhandlingen handlar om. Jag har inkluderat en hel del fotnoter för att förtydliga lite extra eller för att lägga till detaljer. För att få denna sammanfattning så lättläst som möjligt så är vissa definitioner och beskrivningar så pass vagt skrivna att de kan tolkas tvetydigt eller känns ofullständiga. Detta är ett medvetet val och jag uppmanar alla som vill ha formella tydliga definitioner att läsa den engelska texten som kom tidigare i denna avhandling, alternativt de bifogade artiklar vilken denna avhandling bygger på.

Vi börjar med en introduktion 3.1 där de centrala begreppen, som är bra att ha koll på i samtliga artiklar, diskuteras. Därefter, i Sektion 3.2, beskriver vi kort innehållet i artiklarna I och II. Det är tyvärr svårt att säga mycket om dessa artiklar utan att använda abstrakta tekniska termer, varpå denna sektion är ganska kort. I Sektion 3.3 beskriver vi innehållet i artiklarna III och IV, vilka båda studerar sannolikhetsgränsvärdeslagar. Den enklaste delen att läsa är kanske Sektion 3.4 där vi beskriver artiklarna V och VI. Dessa artiklar är de mest konkreta av de som finns med i avhandlingen, och speciellt artikel VI innehåller en mycket explicit klassifikation av en viss typ av graf.

## 3.1 Introduktion

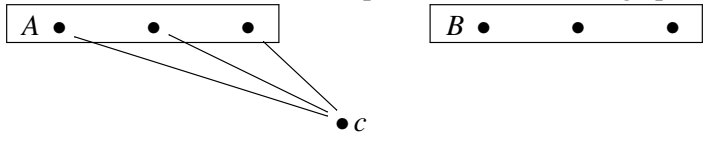
Detta är en avhandling i matematik, specifikt matematisk logik med inriktning på modellteori. Modellteori är studien av abstrakta matematiska modeller, deras egenskaper och teorier. Vi kommer nästan helt uteslutande använda oss av strukturer som har ett ändligt antal grundläggande relationer, men man kan i princip genom hela avhandlingen tänka sig att vi bara tittar på grafer dvs. en matematisk struktur som innehåller punkter/noder samt streck/kanter mellan dessa punkter.



[1] Personer som läst matematik vid universitetet kommer självklart ha en fördel, men förhoppningsvis kan alla åtminstone få en känsla för vad det handlar om.

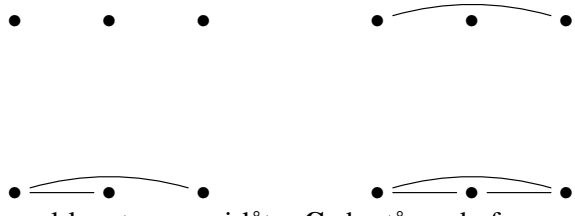
Den röda tråden i denna avhandling ges av den så kallade Radografen, som vi kommer beteckna med $\mathscr{R}$ genom denna sammanfattning. Radografen kan definieras på många olika sätt. Ett sätt är att vi tar ett (uppräkneligt) oändligt antal punkter och mellan varje par av två punkter så singlar vi en slant för att se om vi ska sätta en kant eller inte där. Genom denna process kommer vi med sannolikhet 1 att komma fram till Radografen[2]. En av de viktigaste egenskaperna hos Radografen är följande som vi kallar för en *förlängningsegenskap*.

- För varje två disjunkta ändliga mängder $A, B \subseteq \mathscr{R}$ så finns det en punkt $c \in \mathscr{R}$ så att $c$ har en kant till alla punkter i $A$ men till inga punkter i $B$.




Vi hade kunnat definiera Radografen med hjälp av förläningsegenskapen ovan. Det är till och med så att om vi tar någon annan graf $\mathscr{G}$ med ett (uppräkneligt) oändligt antal punkter som också uppfyller förlängningsegenskapen så kommer denna graf $\mathscr{G}$ vara isomorf[3] med Radografen.

Ett tredje sätt att skapa Radografen är också med hjälp av sannolikhetsteori. För varje naturligt tal $n$ låt $\mathbf{K}_n$ vara mängden av alla grafer där punkterna numrerats $1, \ldots, n$ . Om vi nu tittar på en egenskap $\varphi$ hos grafer, så kommer denna egenskap att vara sann i några av graferna i $\mathbf{K}_n$ och falsk i några. På detta sätt så får vi, för varje grafegenskap $\varphi$, en sannolikhet $\mathbf{P}_n(\varphi)$ för att denna egenskap gäller i en slumpmässig graf i $\mathbf{K}_n$.



Ett enkelt exempel kan tas om vi låter $\mathbf{C}_3$ bestå av de fyra graferna i bilden ovan. Då kommer sannolikheten att vi inte har någon kant vara $1/4$, medans sannolikheten att vi har minst två kanter vara $2/4$ och sannolikheten att vi har 4 kanter är 0. Om vi tar en specifik egenskap $\varphi$ så kan vi studera hur denna egenskaps sannolikhet förändras i olika $\mathbf{K}_n$, när $\mathbf{K}_n$ är som vi beskrev ovan. Om vi låter $n$ växa mot oändligheten (och alltså växer även grafernas storlek i $\mathbf{K}_n$) så kommer vi förhoppningsvis få en sannolikhet för $\varphi$ som stabiliserar sig och rör sig mot ett tal, detta tal kallas (om den existerar) den *asymptotiska sannolikheten* för $\varphi$. Ett enkelt exempel är att den asymptotiska sannolikheten för egenskapen "Det finns inga kanter" går mot 0, eftersom antalet grafer med

[2]Notera att sannolikhet 1 *inte* är samma sak som att det helt säkert kommer att hända. Detta underliga fenomen uppstår eftersom vi har ett oändligt antal punkter.
[3]Likadan.

minst en kant växer medans antalet grafer med ingen kant är exakt en (för varje val av antal noder). Om varje egenskap[4] har asymptotisk sannolikhet 0 eller 1 så säger vi att $\mathbf{K}_n$ har en $0-1$ lag. Låt $T_{\mathbf{K}}$ beteckna mängden av alla egenskaper som har sannolikhet 1. Fagin [14] och Glebskii, Kogan, Liogon'kii, Talanov [18] visade, oberoende av varandra, att om $\mathbf{K}_n$ består av alla grafer med storlek $n$ så har $\mathbf{K}_n$ en $0-1$ lag. Fagins bevis använde sig av förlängningsegenskaperna som han visade har asymptotisk sannolikhet 1 i $\mathbf{K}_n$. Eftersom förlängningsegenskaperna definierar Radografen så medför detta i sin tur att Radografen är den unika oändliga grafen som uppfyller alla egenskaper som har asymptotisk sannolikhet 1.

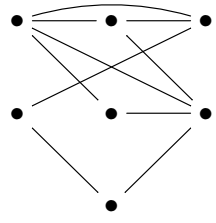


*Figure 3.1.* En 3-partit graf.

Ett annat viktigt exempel för oss är de $l-$partita slumpgraferna. Låt $l$ vara ett positivt heltal. En $l-$partit graf är en graf som kan delas upp i $l$ stycken delar så att inga kanter finns mellan två punkter som ligger i samma del. Låt $\mathbf{K}_n$ vara mängden av alla $l-$partita grafer där noderna numreras med $1,\ldots,n$. Kolaitis, Prömel och Rothschild [25] visade att $\mathbf{K}_n$ kommer att ha en $0-1$ lag på ett liknande sätt som Fagin visade $0-1$ lagen som beskrivs ovan, nämligen med hjälp av speciella förlängningsegenskaper. Även denna gång så kommer det att finnas en unik oändlig struktur som uppfyller alla egenskaper med asymptotisk sannolikhet 1. Dock så kommer denna oändliga struktur inte vara Radografen utan den så kallade $l-$partita slumpgrafen. Denna graf kan vi generera på liknande sätt som Radografen, nämligen genom att ta $l$ stycken delar med oändligt antal punkter i varje del, och därefter singla en slant för varje par av punkter i olika delar för att bestämma om en kant ska finnas eller ej.

Vi säger att en struktur är homogen om dess lokala egenskaper definierar dess globala egenskaper. Mer specifikt; om vi i en graf $\mathscr{G}$ hittar några punkter, med kanter utplacerade på ett visst sätt, och på ett annat ställe i grafen hittar andra punkter med precis samma konstellation av kanter, då ska dessa båda mängder med punkter uppfylla exakt samma egenskaper i hela grafen. Som exempel kan vi se att $5-$cykeln, och den kompletta $4-$grafen, i bilden nedan, är homogena. Däremot så är $6-$cykeln inte homogen eftersom paret med noder uppe till vänster och nere till höger inte har samma globala egenskaper som paret med noder uppe i mitten och nere till vänster, eftersom det ena paret har avstånd[5] 2 och det andra har avstånd 3 mellan sig, samtidigt som

[4] Om man ska vara petig så menar vi egentligen första ordningens sats från språket.

[5] Antal kanter man behöver gå över som minst för att ta sig från ena punkten till den andra.

båda paren med noder ser likadana ut, eftersom båda paren med noder saknar kant.

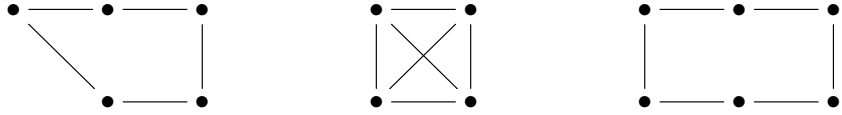


*Figure 3.2.* 5−cykeln, den kompletta 4−grafen och 6−cykeln respektive.

## 3.2 Artiklarna I och II

Radografen har några modellteoretiska egenskaper som är intressanta ur ett teoretiskt perspektiv. Den är enkel[6] har SU-rang 1, trivial pregeometri och är uppräkneligt kategorisk (läs den engelska introduktionen för definitioner). Dessa egenskaper innehas även av andra grafer (och strukturer) som skapas på samma slumpmässiga sätt så som Radografen och de $l-$partita slumpgraferna. En fråga som man därför kan ställa sig är hur dessa egenskaper hänger ihop med den genereringsprocess som skapade både Radografen och de $l-$partita slumpgraferna. Vi begränsar oss ytterligare genom att titta på så kallade binära strukturer, det vill säga grafer men med ett fixerat antal olikfärgade kanter som man kan ha mellan punkterna.

I artikel I visar vi att Fagins [14] bevismetod med förlängningsegenskaper, som används för att bevisa ett flertal $0-1$ lagar, medför att den oändliga struktur som skapas innehar alla egenskaper som beskrevs ovan. Dessutom så visar vi att samtliga strukturer som uppfyller egenskaperna ovan kan genereras på just detta sätt.

Artikel II skrevs tillsammans med Vera Koponen och är den första artikeln i en serie som Koponen [27, 28, 29, 30] fortsatte skriva där de binära enkla homogena strukturerna klassificeras. Det vi kommer fram till i denna artikel är framförallt tekniska beskrivningar av de delmängder av strukturerna som har SU-rang 1, vilket i senare artiklar används för att kunna klassificera enkla homogena strukturer.

## 3.3 Artiklarna III och IV

I dessa två artiklar studerar vi en av genereringsprocesserna som användes för Radografen. Det vill säga vi har en mängd $\mathbf{K}_n$ med strukturer[7] och sen kollar vi på hur sannolikheten för olika egenskaper förändras när storleken på strukturerna ökar.

[6]Detta har inget att göra med algebrans "enkel grupp" eller liknande begrepp och det betyder definitivt inte att strukturen är enkel att förstå eller beskriva.

[7]Man kan tänka på grafer, men detta blir i dessa artiklar lite trivialt i vissa aspekter.

I artikel III studerar vi $\mathbf{K}_n$ som består av färgbara strukturer med en underliggande pregeometri. En färglagd graf är i princip samma sak som en färglagd karta. Den ska vara färglagd så att två punkter som ligger bredvid varandra aldrig har samma färg. Notera att de $l-$partita graferna som vi tidigare tittade på är $l-$färgbara. Detta kan man se genom att man färglägger alla noder som ligger i en viss del med samma färg.

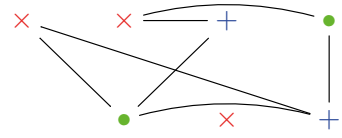


*Figure 3.3.* En graf färglagd[8] i tre färger

Problemet som dyker upp är hur vi generaliserar detta till strukturer, och inte bara grafer. Specifikt, om kanterna kan innehålla tre eller fler punkter (och inte bara två som i grafer) vilka regler ska vi då ha för färgläggning? Svaret blir att vi ger två olika regler och får då starkt och svagt färglagda strukturer där vi antingen vill att alla punkter som ligger på en kant ska ha olika färg, respektive att minst två punkter som ligger på en kant har olika färg. En *$l-$färgbar* struktur är en struktur som inte är färglagd än, men som kan färgläggas med högst $l$ stycken färger. I artikel III visar vi att om $\mathbf{K}_n$ består av alla $l-$färgbara strukturer med en pregeometri[9], så kommer vi asymptotiskt ha ett unikt sätt att färglägga strukturerna i $\mathbf{K}_n$ och detta måste göras med exakt $l$ färger. Vi visar dessutom att samtliga egenskaper kommer att ha asymptotisk sannolikhet 0 eller 1, vi har alltså en $0-1$ lag.

En graf kallas rigid om det inte går att byta plats på några av punkterna (och medföljande kanter) och få tillbaka samma graf.

*Figure 3.4.* En rigid graf till vänster och en icke-rigid graf till höger.

Fagin [15] visade att om $\mathbf{K}_n$ består av alla strukturer av storlek $n$ så kommer den asymptotiska andelen rigida grafer att närma sig 1 när $n$ växer. Detta betyder alltså att strukturer där vi kan byta plats på punkterna är ganska ovanliga. Om vi tittar på vilka egenskaper som har asymptotisk sannolikhet 0 eller 1 i $\mathbf{K}_n$ så kommer graferna som är icke-rigida alltså inte kunna påverka resultatet eftersom de är för få. I artikel IV så studerar vi därför vad som händer med dessa strukturer. Mer specifikt så begränsar vi $\mathbf{K}_n$ så att den endast får bestå av strukturer som inte är rigida på olika sätt och i olika grad, bland annat genom

[8] För att tillåta läsande i svart-vitt så använder vi symboler som visar om två punkter har samma färg.

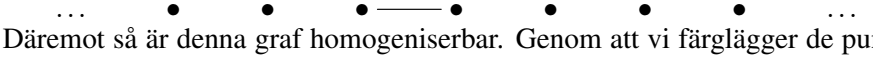
[9] Detta är ytterligare en begränsning som gör färgläggandet lite krångligare.

att ha en viss automorfigrupp[10]. Det vi hittar är att för dessa mängder av strukturer så har vi generellt sett inte asymptotisk sannolikhet 0 eller 1 för samtliga egenskaper, men vi har en sannolikhet som åtminstone närmar sig ett tal, och inte bara oscillerar när vi går mot oändligheten[11].

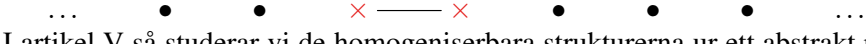
## 3.4 Artiklarna V och VI

I introduktionen ovan definierar vi de $l-$partita slumpgraferna och begreppet homogenitet. Den $l-$partita slumpgrafen är inte homogen eftersom om vi väljer två punkter som ligger i samma del (och därför har ingen kant mellan sig) och jämför dessa med punkter som ligger i olika delar och inte har någon kant mellan sig så kommer det ena paret ha avstånd 2 och det andra ha avstånd 3 mellan sig. Alltså är de två paren med punkter globalt olika vilket medför att grafen inte är homogen. Man kan dock fixa denna miss i homogeniteten med ett litet trick. Om vi lägger till en ny röd kant mellan varje par av noder som ligger i samma del, så kommer vi inte längre lokalt stöta på problemet som jag nyss beskrev eftersom att par av noder som ligger i olika delar har ingen röd kant medans alla par av noder i samma del numera har en röd kant. I fallet med $l-$partita grafer så är detta det enda problemet vi har, och alltså kommer denna nya struktur som vi skapar genom att lägga till en extra röd kant vara homogen.

En homogeniserbar graf är en graf som kan göras homogen genom att vi lägger till ett ändligt antal extra färgade kanter[12] eller färglägger punkterna. Som vi beskrev ovan så är de $l-$partita graferna homogeniserbara. Nedan syns grafen som består av ett oändligt antal punkter där exakt två punkter har en kant mellan sig, kalla denna graf $\mathscr{G}$ för referens till senare. Denna graf är inte homogen eftersom alla punkter ser, när man bara tittar på dem en i taget, lika ut men när man tittar på dem i hela grafen ser man att några punkter är annorlunda eftersom de har en granne[13].



Däremot så är denna graf homogeniserbar. Genom att vi färglägger de punkter som har en granne så kommer vi kunna upptäcka skillnaden mellan de med granne och de utan granne även när vi bara ser en punkt i taget. Detta är det enda problemet som finns i grafen, vilket gör att den nu är homogen.



I artikel V så studerar vi de homogeniserbara strukturerna ur ett abstrakt perspektiv och klassificerar dem i förhållande till hur nära de är att vara homogena. Vi studerar sedan egenskaperna hos dessa klasser och visar hur

[10] Sätt att flytta runt punkterna.

[11] För sådana exempel se 1.2.3 i den engelska delen.

[12] Rättare sagt så kan dessa extra "kanter" vara relationer av valfri ställighet.

[13] En granne till en punkt a är en annan punkt b så att det finns en kant mellan a och b.

klasserna förhåller sig till varandra.

I artikel VI studerar vi i större noggrannhet klassen med de enklaste homogeniserbara strukturerna. Den trevligaste av dessa grafer är grafen, som vi kallar $\mathscr{H}$, med ett oändligt antal noder som alla (utom en) har exakt en kant och dessa kanter går alla till samma nod.

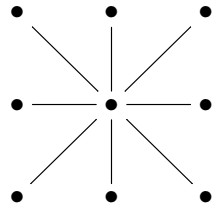


*Figure 3.5.* Grafen[14] $\mathscr{H}$

Denna graf har två typer av punkter, den unika punkten som alla har en kant till (kalla denna mittenpunkten) samt resten av punkterna. Om vi dock bara tittar på en punkt i taget så vet vi inte vilken punkt vi har, och alltså är grafen inte homogen. Grafen är tydligt homogeniserbar genom att färglägga mittenpunkten.

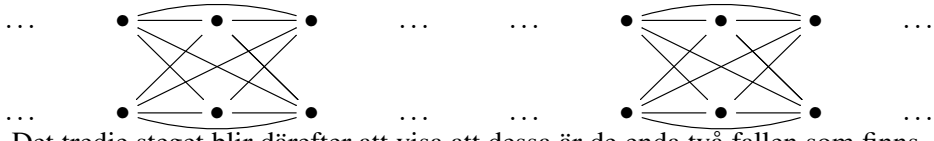
Om vi tittar på två punkter i $\mathscr{H}$ som inte har någon kant mellan sig så vet vi att dessa punkter inte kan vara mittenpunkten, vi har alltså koll på vilka punkter dessa är globalt. Tittar vi däremot på två punkter som har en kant mellan sig, men vi ser inget mer, så vet vi inte vilken av dessa två punkter som är mittenpunkten. Om vi dock väljer en till punkt, så vi har tre punkter, så måste två av dessa inte ha en kant mellan sig och alltså är den återstående punkten mittenpunkten. Om vi nu lägger till fler punkter så kommer vi alltid ha koll på vilken av dessa som är mittenpunkten och även veta vilka som inte är det. Slutsatsen är att efter vi valt minst 3 punkter i $\mathscr{H}$ så kommer vi ha helt koll på exakt vilka globala egenskaper som finns hos de valda punkterna. Detta kallar vi att $\mathscr{H}$ är $\geq 3-$homogen. Alltså den är homogen, men bara om vi tittar på minst 3 punkter. Generellt sett så kan vi definiera $\geq k-$homogenitet som är homogenitet när vi valt minst $k$ punkter. Notera att den grafen $\mathscr{H}$ som vi beskrev på förra sidan inte är $\geq k-$homogen för något tal $k$, eftersom om vi väljer ett stort antal punkter, men så att inga av punkterna har en kant så kommer vi inte veta om någon, eller vilken, av våra valda punkter som har en kant till en annan punkt globalt.

Huvudsyftet med artikel VI är att klassificerar de $\geq k-$homogena graferna[15]. Detta gör vi i tre steg. Det första steget är att visa att om det finns flera typer av punkter, så som i grafen $\mathscr{H}$ ovan, då kommer de ungefär se ut som $\mathscr{H}$, men kanske med lite fler punkter som sitter ihop på ett symmetriskt sätt. Det andra steget är att vi visar att om vi har en $\geq k-$homogen graf där alla punkter ser

[14] Här borde vi ha oändligt antal noder som är runtomkring (alla fick dock inte plats i bilden), men ändå bara exakt en nod i mitten.

[15] I denna artikel menas verkligen grafer, gentemot resterande artiklar där vi egentligen vill prata om godtyckliga strukturer med ändligt många (möjligtvis binära) relationer.

likadana ut men det finns par av punkter (som båda har en kant eller ej) som ser olika ut globalt, då kommer de att se ut ungefär som grafen nedan.



Det tredje steget blir därefter att visa att dessa är de enda två fallen som finns för $\geq k-$homogena grafer. Bevisen består till stor del av att man använder Ramseyteori.

# 4. Acknowledgments

As of writing this thesis I have been studying for 22 years. During my elementary school I never really saw any challenge or had any huge interest, I never did anything more than I had to and never saw any point in doing so. When I came to high school I got an interest for mathematics which carried me to the university to take a bachelor and master in mathematics. Even though not all courses on the university were easy they still had boundaries with specific things to learn, read and think about with no reward or encouragement for exploring outside the box. Starting my PhD-studies I finally got to let go of these boundaries and actually see no limits in what to think about and how to explore it. This may be both the most fascinating and the hardest part which I experienced when writing this thesis. I was very lucky to have the patient, encouraging and inspiring Vera Koponen as advisor. Throughout these years she has been able to guide me and support me in my research without putting up boundaries.

A 5 year PhD education is not without monetary costs and I would like to thank the department of mathematics for funding me through the whole process. I would also like to thank the Knut and Alice Wallenberg foundation, Liljewalch's, SveFUM and the G.S. Magnuson's foundation for their travel grants which enabled me to go on many inspiring conferences.

For 9 years I have worked at the department of mathematics. First as a teaching assistant, then as a doctoral student with full course responsibilities. During this time I have met a lot of colleagues who have been interesting, inspiring, friendly and annoying. To sum it up, the years here have been truly both developing and fun. Among the teachers I have been working with, I have met Jörgen and Inger the most and they have greatly influenced me with their incredible teaching skills. Although they have very different styles of teaching they have influenced me to develop my own thinking and philosophy of teaching. The administrative staff of the department have made the whole PhD-journey very smooth. Special thanks goes to Inga-Lena, who probably knows everything, and Fredrik, who always found a solution to the problems which teaching generated.

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Uppsala University, 2018

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