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Localised Radial Basis Function Methods for Partial Differential Equations

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Abstract

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Radial basis function methods exhibit several very attractive properties such as a high order convergence of the approximated solution and flexibility to the domain geometry. However the method in its classical formulation becomes impractical for problems with relatively large numbers of degrees of freedom due to the ill-conditioning and dense structure of coefficient matrix. To overcome the latter issue we employ a localisation technique, namely a partition of unity method, while the former issue was previously addressed by several authors and was of less concern in this thesis.

In this thesis we develop radial basis function partition of unity methods for partial differential equations arising in financial mathematics and glaciology. In the applications of financial mathematics we focus on pricing multi-asset equity and credit derivatives whose models involve several stochastic factors. We demonstrate that localised radial basis function methods are very effective and well-suited for financial applications thanks to the high order approximation properties that allow for the reduction of storage and computational requirements, which is crucial in multi-dimensional problems to cope with the curse of dimensionality. In the glaciology application we in the first place make use of the meshfree nature of the methods and their flexibility with respect to the irregular geometries of ice sheets and glaciers. Also, we exploit the fact that radial basis function methods are stated in strong form, which is advantageous for approximating velocity fields of non-Newtonian viscous liquids such as ice, since it allows to avoid a full coefficient matrix reassembly within the nonlinear iteration.

In addition to the applied problems we develop a least squares radial basis function partition of unity method that is robust with respect to the node layout. The method allows for scaling to problem sizes of a few hundred thousand nodes without encountering the issue of large condition numbers of the coefficient matrix. This property is enabled by the possibility to control the coefficient matrix condition number by the rate of oversampling and the mode of refinement.

Keywords: Radial basis function, Partition of unity, Computational finance, Option pricing, Credit default swap, Glaciology, Fluid dynamics, Non-Newtonian flow, Anisotropic RBF

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Dedicated to Inna and Daniel

List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I V. Shcherbakov and E. Larsson. Radial basis function partition of unity methods for pricing vanilla basket options. *Comput. Math. Appl.*, 71(1):185–200, 2016.

II L. von Sydow, L. J. Höök, E. Larsson, E. Lindström, S. Milovanović, J. Persson, V. Shcherbakov, Y. Shpolyanskiy, S. Sirén, J. Toivanen, J. Waldén, M. Wiktorsson, J. Levesley, J. Li, C. W. Oosterlee, M. J. Ruijter, A. Toropov, Y. Zhao. BENCHOP—The BENCHmarking project in option pricing. *Int. J. Comput. Math.*, 92(12):2361–2379, 2015.

III V. Shcherbakov. Radial basis function partition of unity operator splitting method for pricing multi-asset American options. *BIT*, 56(4):1401–1423, 2016.

IV S. Milovanović and V. Shcherbakov. Pricing derivatives under multiple stochastic factors by localized radial basis function methods. Available as *arXiv:1711.09852*, 2017.

V A. Itkin, V. Shcherbakov, and A. Veygman. Influence of jump-at-default in IR and FX on Quanto CDS prices. Available as *arXiv:1711.07133*, 2017.

VI J. Ahlkrona and V. Shcherbakov. A meshfree approach to non-Newtonian free surface ice flow: Application to the Haut Glacier d'Arolla. *J. Comput. Phys.*, 330:633–649, 2017.

VII G. Cheng and V. Shcherbakov. Anisotropic radial basis function methods for continental size ice sheet simulations. Available as *arXiv:1711.09947*, 2017.

VIII E. Larsson, V. Shcherbakov, and A. Heryudono. A least squares radial basis function partition of unity method for solving PDEs. *SIAM J. Sci. Comput.*, 39(6):2538–2563, 2017.

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Related Work

Although not explicitly discussed in the comprehensive summary, the following papers are related to the contents of this thesis

1. E. Larsson, S. Milovanović, V. Shcherbakov, L. von Sydow, et al. BENCHOP—The BENCHmarking project in Option Pricing: Basket options. Manuscript in preparation, 2017.

2. E. Larsson, S. Milovanović, V. Shcherbakov, L. von Sydow, et al. BENCHOP—The BENCHmarking project in Option Pricing: Stochastic and local volatility. Manuscript in preparation, 2017.[1]

[1] These two papers are parts of a collaborative project with a large number of participants. Please note that the authors named here form the executive group of the project and the final list of authors will be longer.

Contents

1. Introduction

In this thesis we develop radial basis function (RBF) methods for partial differential equations (PDEs) and apply them to complex problems that have no analytical solution but require accurate approximations. We consider problems in financial mathematics and glaciology which are both relevant for everyday life and difficult enough to demonstrate the advantage of RBF methods over other well established numerical methods such as finite differences and finite elements.

The first use of RBFs was mentioned by Hardy for cartography, geodesy and digital terrain models in order to reduce errors in data interpolation [51]. Later it was adapted for partial differential equations by Kansa [63, 64] and then became intensively researched in late 1990s and early 2000s [32, 36, 88, 99, 106]. Nowadays, RBF methods are used both for interpolation [20, 30, 31] and for solving PDEs [9, 39, 65]. The most attractive properties of RBF methods are their fast convergence and meshfree nature, which makes them flexible and easy to apply to problems stated in domains with complex geometries.

In spite of great theoretical convergence properties, the global RBF method becomes impractical even for relatively small problems (with a few thousands of degrees of freedom) because (i) the discretisation results in a system of equations with dense coefficient matrix, which is computationally expensive to solve; (ii) the coefficient matrix of the linear system becomes highly ill-conditioned even for relatively large values of the shape parameter. Potential remedies are to use (1) a localisation technique to sparsify the matrix; (2) a stabilisation algorithm that removes or, at least, weakens the dependence on the shape parameter.

The localisation techniques, which are currently in the most extensive use, are the radial basis function partition of unity method (RBF-PUM) [19, 21, 68, 87, 91], the radial basis function generated finite difference method (RBF-FD) [8, 41, 97, 98, 103, 104], and the compactly supported radial basis function method [100, 107]. The advantage of all these local RBF methods is that they result in a much sparser linear system of equations, which improves the computational efficiency, while maintaining high accuracy and flexibility [2, 79, 91]. Moreover, localised methods are suitable for parallel simulations [22, 96, 107].

However, if large problems are solved and high accuracy is required, then RBF methods need to be stabilised to remove the dependency on the shape parameter and decrease the condition number of the coefficient matrix. Several techniques were developed for this purpose, such as the Contour-Padé

method [45, 103, 105], the RBF-QR method [40, 42, 44, 69], and the RBF-GA method [43]. A drawback of such techniques is that the stabilisation comes at a certain cost, and if only a moderately accurate approximation is required, the use of a stable method may be unjustified, since the time to obtain a stable basis may severely dominate the solving time. Nevertheless, we attempt to overcome this issue in Paper VIII where by the use of a least squares method we can design an approach that is able to reduce the total computational time associated with obtaining a stable RBF-QR basis within a partition of unity framework by reusing computations.

Apart from developing RBF-PUM, in this thesis we study properties and benefits of the method in applications to computational finance and glaciology. The problems in finance are typically stated in domains with simple geometries, however, require high efficiency to swiftly determine prices of financial contracts and calibrate models. Therefore, RBF methods are well suited to these applications since they provide high accuracy with relatively low numbers of discretisation points. This property becomes extremely important when contracts on several assets or under multi-factor models are priced since formulations of these problems result in high-dimensional PDEs, and the high accuracy of the methods helps to use fewer discretisation points and, consequently, lower the storage requirements and computational complexity. RBF methods were applied to problems in option pricing in [39, 53, 85] and their advantages over both standard and state-of-the-art methods were demonstrated in [91, 93]. Also, RBF methods were used for pricing quanto credit derivatives [62].

In contrast to the problems in finance, the problems in ice sheet modelling and glaciology are stated in very complex domains. Therefore, the meshfree flexibility of RBF methods is of the highest relevance here. Due to massive sizes it is difficult to obtain high resolution on continental ice sheets, for example, a resolution of 10–20 kilometres is considered to be sufficient, but more importantly, even with such a resolution the problem sizes reach a few millions of computational nodes, which makes the problems very challenging to solve. In turn, RBF methods allow to reduce the number of nodes while maintaining a similarly high accuracy, hence, reduce the computational effort thanks to the high order approximation properties. It is also worth mentioning that RBF methods are formulated in strong form, which makes them more suitable for nonlinear problems than finite element methods (FEM) because RBF methods do not require a full matrix reassembly in every nonlinear iteration. It was shown in [1] that the cost of matrix reassembly in FEM may seriously dominate the simulation time. Thus, the advantage of RBF approximation for finding the velocity field of a glacier was first demonstrated in [2], and later the method was further developed for synthetic ice sheets of continental size using anisotropic basis functions in [24].

The remainder of this thesis is structured as follows. In Chapter 2 we introduce the basics of radial basis functions as well as discuss our recent fundings

and developments of the method based on results from Paper VIII. In Chapter 3 we formulate the problems of computational finance and discuss the advantages of RBF methods for financial applications observed in Papers I–V. In Chapter 4 we introduce the mathematical model for simulation of the ice sheet and glacier velocity field and demonstrate the usefulness of RBF methods for glaciological applications based on findings from Papers VI–VII.

2. Radial Basis Function Methods

2.1 The Global RBF Method

Consider an elliptic linear boundary value problem (BVP):

$$\begin{cases} \mathscr{L}u(\underline{x}) = f(\underline{x}), & \underline{x} \in \Omega, \\ \mathscr{B}u(\underline{x}) = g(\underline{x}), & \underline{x} \in \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega \in \mathbb{R}^d$ is a domain and $\partial\Omega$ is its boundary, $\mathscr{L}$ is the interior differential operator, $\mathscr{B}$ is the boundary differential operator, and f, g are smooth forcing functions. In order to approximate the solution we define two sets of nodes $X = \{\underline{x}_i\}_{i=1}^n$, at which the basis functions are centred, and $Y = \{\underline{y}_i\}_{i=1}^m$, at which the RBF approximation is evaluated. If the evaluation node set is equal to the set of centre nodes, then we refer to this case as pure collocation; otherwise, if there are more evaluation points than centre nodes it is a least squares approach. An RBF approximation $\tilde{u}(\underline{x})$ to the solution $u(\underline{x})$ on the domain Ω has the form of a weighted sum

$$\tilde{u}(\underline{x}) = \sum_{j=1}^{n} \lambda_j \phi(\varepsilon, \|\underline{x} - \underline{x}_j\|), \tag{2.2}$$

where $\phi(\varepsilon, \|\underline{x} - \underline{x}_j\|)$ is a basis function centred at x_j and λ_j are unknown coefficients to be determined. Typical choices of smooth basis functions can be found in Table 2.1. It is important to notice that the basis functions depend on the so-called shape parameter ε that determines the width of the basis functions and plays a crucial role in the approximation properties of the selected finite-dimensional basis. Therefore, it has to be chosen with special care. The dependence on the value of the shape parameter for some basis functions is presented in Figure 2.1.

Table 2.1. *Commonly used radial basis functions.*

RBF	$\phi(\varepsilon, r)$
Gaussian (GA)	$\exp(-\varepsilon^2 r^2)$
Multiquadric (MQ)	$\sqrt{1+\varepsilon^2 r^2}$
Inverse Multiquadric (IMQ)	$1/\sqrt{1+\varepsilon^2 r^2}$
Inverse Quadratic (IQ)	$1/(1+\varepsilon^2 r^2)$

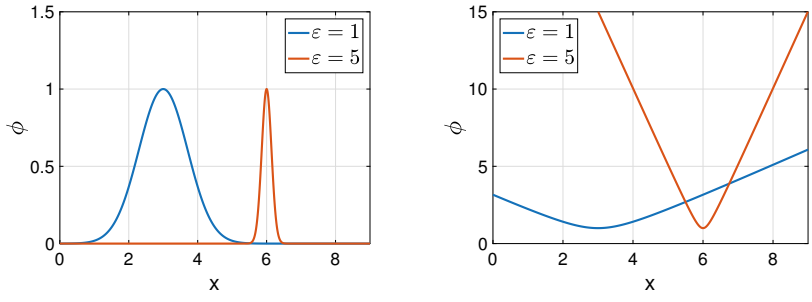

Figure 2.1. Commonly used RBFs for different values of the shape parameter ε. *Left:* Gaussian functions. *Right:* Multiquadric functions.

To simplify the forthcoming derivations we introduce the following notation: a function evaluated at the node points, $u(X)$, denotes a column vector $[u(\underline{x}_1),\ldots,u(\underline{x}_n)]^T$; a function of two arguments with centres at X and evaluated at Y, $\phi(Y,X)$, denotes an $(m\times n)$ matrix with elements $\phi(\varepsilon,||\underline{y}_i-\underline{x}_j||)$, $i=1,\ldots,m$, $j=1,\ldots,n$; a function evaluated at a point and a node set $\phi(\underline{x},X)$ is a row vector, while $\phi(Y,\underline{x})$ is a column vector. Thus, we can now rewrite (2.2) in a matrix-vector form

$$\tilde{u}(\underline{x}) = \phi(\underline{x},X)\Lambda, \tag{2.3}$$

where $\Lambda = [\lambda_1,\ldots,\lambda_n]^T$. We can relate the coefficients Λ and the function values evaluated at the node set X via the interpolation condition

$$\tilde{u}(X) = \phi(X,X)\Lambda. \tag{2.4}$$

It has been shown [32, 67, 89] that for smooth RBFs the magnitude of the coefficients Λ becomes unbounded as $\varepsilon\to 0$, while the values $\tilde{u}(X)$ remain well-behaved. Therefore, we prefer to express the problem in terms of the nodal values $\tilde{u}(X)$. For the basis functions presented in Table 2.1, the interpolation matrix $\phi(X,X)$ is nonsingular for distinct node points [77]. Hence, we can write

$$\Lambda = \phi(X,X)^{-1}\tilde{u}(X), \tag{2.5}$$

which provides us with the possibility to reformulate (2.3) as

$$\tilde{u}(\underline{x}) = \phi(\underline{x},X)\phi(X,X)^{-1}\tilde{u}(X). \tag{2.6}$$

The formulation (2.6) in its form is well-conditioned, albeit the matrix $\phi(X,X)^{-1}$ can be highly ill-conditioned, and some stabilisation techniques may be needed to maintain stability for small values of the shape parameter. In Paper VIII of this thesis we use the RBF-QR approach [40, 42, 44, 69] that allows for stable computation and evaluation of differentiation matrices for any small value of ε. Also, other stabilisation methods, such as the Contour-Padé method [45, 103, 105] or RBF-GA [43], can be applied. However, stabilisation by all the above

mentioned techniques comes at a certain cost. Therefore, it might be unnecessary to employ a stabilisation technique if the desired accuracy can be attained without it. On the other hand, if a high resolution is required, then such a stabilisation approach is the only way to overcome ill-conditioning and achieve the desired resolution. Moreover, RBF-QR or another stable method is vital for convergence in an RBF partition of unity method [68].

Applying a linear differential operator to the RBF approximation (2.6) leads to

$$\mathscr{L}\tilde{u}(\underline{x}) = \mathscr{L}\phi(\underline{x},X)\phi(X,X)^{-1}\tilde{u}(X). \tag{2.7}$$

Collocation of (2.7) on the set of evaluation points Y results in

$$\mathscr{L}\tilde{u}(Y) = L(Y,X)\tilde{u}(X), \tag{2.8}$$

where $L(Y,X)$ is a differentiation matrix defined as

$$L(Y,X) = \mathscr{L}\phi(Y,X)\phi(X,X)^{-1}. \tag{2.9}$$

Thus, we can discretise the BVP (2.1) and define a linear system of equations to solve for the function values $\tilde{u}(X)$. We enforce the boundary conditions at boundary points Y^b and the PDE at internal points Y^i, where $Y = Y^i \bigcup Y^b$. The resulting linear system has size $(m \times n)$ and takes the form

$$\begin{cases} L(Y^i,X)\tilde{u}(X) = f(Y^i), \\ B(Y^b,X)\tilde{u}(X) = g(Y^b), \end{cases} \tag{2.10}$$

where $B(Y,X)$ is the discrete boundary operator $\mathscr{B}$ that can be constructed in a similar fashion as $L(Y,X)$.

2.2 The Global RBF Method for Nonlinear Problems

In the same manner solutions of nonlinear BVPs can be approximated by the global RBF method. Consider a nonlinear BVP

$$\mathscr{P}[\underline{x}, u(\underline{x}), \mathscr{D}u(\underline{x})] = 0 \quad \Longleftrightarrow \quad \begin{cases} \mathscr{P}_1 = 0, & \underline{x} \in \Omega, \\ \mathscr{P}_2 = 0, & \underline{x} \in \partial\Omega, \end{cases} \tag{2.11}$$

where $\mathscr{P}_1$ is the interior nonlinear operator, $\mathscr{P}_2$ is the boundary nonlinear operator, and $\mathscr{D}$ is a shorthand notation for differential operators, such as ∂_x, ∇, Δ. Applying the nonlinear operator (2.11) to the RBF approximation (2.6), we obtain a nonlinear system of equations

$$\mathscr{P}[\underline{x}, \tilde{u}(\underline{x}), \mathscr{D}\tilde{u}(\underline{x})] = \mathscr{P}\left[\underline{x}, \phi(\underline{x},X)\phi(X,X)^{-1}\tilde{u}(X), \mathscr{D}\phi(\underline{x},X)\phi(X,X)^{-1}\tilde{u}(X)\right] = 0. \tag{2.12}$$

Collocation of (2.12) on the set of evaluation nodes leads to a nonlinear system of equations

$$P(Y,X) := \mathscr{P}[Y, \tilde{u}(Y), \mathscr{D}\tilde{u}(Y)] = 0. \tag{2.13}$$

A root of the nonlinear system (2.13) can be sought by a nonlinear solver, such as Newton's method [37, 75, 90], fixed point iteration method [2, 24], or trust-region methods [9, 25], that iteratively solves a linearised problem. Hence, the problem eventually reduces down to a linear system similar to (2.10). Thus, for simplicity of notation we consider only linear operators in the future description.

2.3 Time-Dependent Problems

Solutions of time-dependent parabolic and hyperbolic initial value problems (IVP) can also be approximated by RBF methods. There are different ways to approach the time derivative: to employ a finite difference scheme for its discretisation [2, 39, 87, 91], or to view the time dimension as an additional space dimension and apply an RBF method to a $(d+1)$-dimensional problem [33, 52]. However, in this thesis we adhere to the former approach, which is more commonly used. Without loss of generality, consider a parabolic IVP of the following form (the case of hyperbolic problems is identical)

$$\frac{\partial u}{\partial t}(t,\underline{x}) + \mathscr{L}u(t,\underline{x}) = 0, \quad \underline{x} \in \Omega,\ t \in (0,T], \tag{2.14}$$

$$u(0,\underline{x}) = f(\underline{x}). \tag{2.15}$$

We can apply the method of lines, leaving the time dimension be continuous, and obtain a system of ordinary differential equations (ODEs), which then can be solved by, e.g., finite different methods. Assuming that $\lambda_i = \lambda_i(t)$ we repeat steps (2.2)–(2.9) and arrive at a semi-discrete system of equations

$$\frac{\partial \tilde{u}}{\partial t}(t,Y) + L(Y,X)\tilde{u}(t,X) = 0, \tag{2.16}$$

subject to the initial condition

$$\tilde{u}(0,Y) = f(Y). \tag{2.17}$$

2.4 The RBF Partition of Unity Method

Despite some valuable properties, such as high accuracy and exponential convergence for smooth problems [23, 63, 74, 86], the above approach has a major drawback—the coefficient matrix $L(Y,X)$ is dense, which makes the solving procedure expensive. In order to overcome this issue localised RBF methods

such as radial basis function partition of unity methods (RBF-PUM) [19, 21, 68, 87, 91, 101] and radial basis function generated finite difference methods (RBF-FD) [8, 32, 97, 98, 103] were introduced. Also there is an opportunity to use compactly supported basis functions [101, 107].

In this thesis we follow and further develop the partition of unity approach that allows for a significant matrix sparsification. We introduce the method for and investigate its approximation properties by solving problems of pricing multi-asset vanilla and exotic equity derivatives as well as pricing derivatives whose models involve several stochastic factors. In addition, we design an RBF-PUM framework for modelling ice sheet and glacier velocity fields, which in principle can be extended to a more general problem of finding the velocities of a steady flow. Apart from applied problems we put an effort into the core method development to improve the method approximation characteristics. We propose a least squares approach instead of collocation and prove it to be beneficial for the approximation quality and for the method robustness.

The idea of partition of unity was introduced by Babuška and Melenk [5] for finite elements and was later adopted for RBF methods [21, 38, 87, 91, 101]. The main principle is to subdivide the computational domain into several overlapping subdomains, construct a local RBF approximation in each subdomain and then blend them together by partition of unity weight functions.

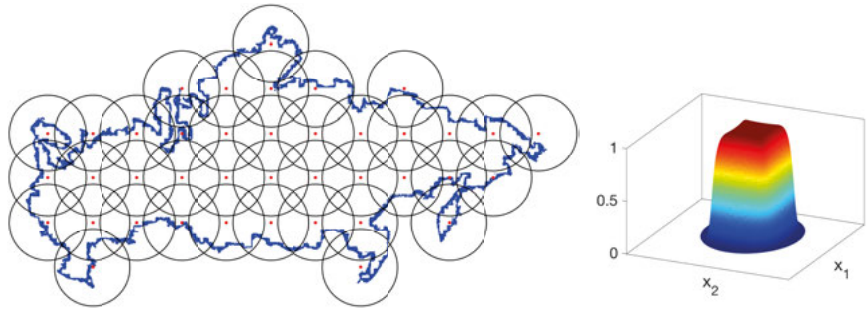

Figure 2.2. Left: A partitioning of a domain performed with patches of circular shape. *Right:* A partition of unity weight function w_i that is compactly supported on the patch Ω_i.

We construct the partition of unity weights $w_j(\underline{x})$ as functions locally supported and subordinated to an open cover $\{\Omega_j\}_{j=1}^N$ of the computational domain Ω

$$\Omega \subseteq \bigcup_{j=1}^{N} \Omega_j, \tag{2.18}$$

and the functions combine into unity at every point $\underline{x} \in \Omega$

$$\sum_{j=1}^{N} w(\underline{x}) = 1, \quad \forall \underline{x} \in \Omega. \tag{2.19}$$

Throughout this thesis we select d-dimensional spheres as patches in $\mathbb{R}^d$ for the domain partitioning (see Figure 2.2), except for Paper VII where we use ellipsoids. The weight function can be constructed by Shepard's method [92] from compactly supported generating functions $\varphi_j(\underline{x})$

$$w_j(\underline{x}) = \frac{\varphi_j(\underline{x})}{\sum_{i=1}^{N} \varphi_i(\underline{x})}, \quad j = 1, \ldots, N. \tag{2.20}$$

The generating functions must be smooth enough to fulfil the PDE smoothness requirements of the solution, for example, Wendland functions [99] are $C^2(\mathbb{R}^d)$ in up to three spatial dimensions

$$\varphi(r) = (4r+1)(1-r)_+^4. \tag{2.21}$$

In order to map the generating function to the patch Ω_j with centre $\underline{c}_j$ and radius ρ_j we shift and scale the function

$$\varphi_j(\underline{x}) = \varphi_j\left(\frac{||\underline{x} - \underline{c}_j||}{\rho_j}\right), \quad \forall \underline{x} \in \Omega. \tag{2.22}$$

Then the global RBF partition of unity solution $\tilde{u}(\underline{x})$ can be found as a weighted sum of the local RBF interpolants $\tilde{u}_j(\underline{x})$ constructed in each patch Ω_j and partition of unity weights $w_j(\underline{x})$

$$\tilde{u}(\underline{x}) = \sum_{j=1}^{N} w_j(\underline{x}) \tilde{u}_j(\underline{x}), \tag{2.23}$$

where

$$\tilde{u}_j(\underline{x}) = \sum_{i=1}^{n_j} \lambda_i^j \phi(\varepsilon, ||\underline{x} - \underline{x}_i^j||) \tag{2.24}$$

is a local RBF approximation constructed using n_j basis functions centred at local nodes X_j, and λ_i^j are the coefficient to be determined. Similarly to (2.3) we shorten the notation and write (2.24) as

$$\tilde{u}_j(\underline{x}) = \phi(\underline{x}, X_j)\Lambda_j, \tag{2.25}$$

where $\Lambda_j = (\lambda_1^j, \ldots, \lambda_{n_j}^j)$. Also following (2.5) and (2.6) we can get an expression for $\tilde{u}_j(\underline{x})$ that excludes the coefficients Λ_j

$$\tilde{u}_j(\underline{x}) = \phi(\underline{x}, X_j)\phi(X_j, X_j)^{-1}\tilde{u}_j(X_j). \tag{2.26}$$

Thus, applying a linear differential operator $\mathscr{L}$ to (2.23) we obtain

$$\begin{aligned} \mathscr{L}\tilde{u}(\underline{x}) &= \sum_{j=1}^{N} \mathscr{L}\left(w_j(\underline{x})\tilde{u}_j(\underline{x})\right) \\ &= \sum_{j=1}^{N} \mathscr{L}\left(w_j(\underline{x})\phi(\underline{x}, X_j)\right)\phi(X_j, X_j)^{-1}\tilde{u}_j(X_j). \end{aligned} \tag{2.27}$$

For example, if we are to apply the Laplace operator to (2.23) we will get

$$\Delta\tilde{u}(\underline{x}) = \sum_{j=1}^{N} \Big(\Delta w_j(\underline{x})\phi(\underline{x},X_j) + 2\nabla w_j(\underline{x}) \cdot \nabla\phi(\underline{x},X_j) + w_j(\underline{x})\Delta\phi(\underline{x},X_j) \Big) \phi(X_j,X_j)^{-1}\tilde{u}_j(X_j), \tag{2.28}$$

where the scalar product should be applied to the gradient vectors. To express (2.28) in terms of differentiation matrices we need to put the partition of unity weight functions into a proper matrix form

$$W_j^{\mathscr{L}}(Y) = \mathrm{diag}\big(\mathscr{L}w_j(Y)\big), \tag{2.29}$$

where Y is an arbitrary set of evaluation nodes. Then we can write the Laplacian as

$$\Delta\tilde{u}(Y) = \sum_{j=1}^{N} \Big(W_j^{\Delta}(Y)D^I(Y,X_j) + 2W_j^{\nabla}(Y) \cdot D^{\nabla}(Y,X_j) + W_j^I(Y)D^{\Delta}(Y,X) \Big) D^I(X_j,X_j)^{-1}\tilde{u}_j(X_j), \tag{2.30}$$

where $D^{\mathscr{L}}$ denotes the corresponding differentiation matrix

$$D^{\mathscr{L}}(Y_j,X_j) = \mathscr{L}\phi(Y_j,X_j)\phi(X_j,X_j)^{-1}. \tag{2.31}$$

2.5 Error Estimates

In order to asses the convergence properties of the method we need to estimate the difference between the true solution and its RBF approximation $||\tilde{u}-u||_{L_2(\Omega)}$. To do so we start by introducing the RBF-PUM interpolant

$$\mathscr{I} = \sum_{j=1}^{N} w_j \mathscr{I}(u_j), \tag{2.32}$$

where $\mathscr{I}(u_j)$ is the local RBF interpolant defined in (2.24) satisfying the interpolation condition $\mathscr{I}(u_j)(X_j) = u(X_j)$. We also define the interpolation error and its derivatives as

$$\mathscr{E}_{\mathscr{L}} = \mathscr{L}(\mathscr{I}(u) - u). \tag{2.33}$$

Estimates of the interpolation error (2.33), when Gaussian basis function are used, were studied in [68, 87] with respect to two refinement techniques: (i) refine the partitioning while preserving the number of nodes in patches, i.e., decreasing the patch size H_j; and (ii) refine node sets in patches while preserving the total number of patches in the partitioning, i.e., decreasing the internodal distance h_j. For the former case the estimate is

$$||\mathscr{E}_{\mathscr{L}}||_{L_\infty(\Omega)} \leq K \max_{1\leq j\leq N} C_j^A H_j^{q(n_j)+1-\frac{d}{2}-\alpha} ||u||_{\mathscr{N}(\Omega_j)}, \tag{2.34}$$

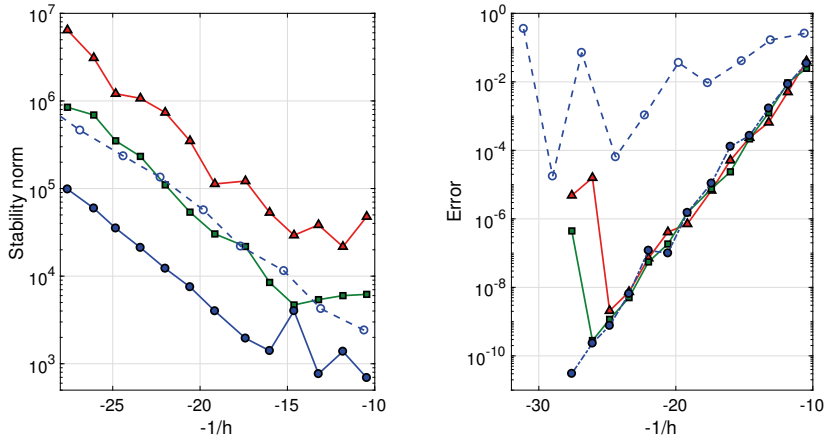

Figure 2.3. Left: The numerically estimated stability norm. *Right:* The corresponding error as a function of H for $n = 28$ ($\triangle$), $n = 55$ ($\square$), and $n = 91$ ($\bigcirc$) for the least squares (solid lines, solid markers) and collocation (dashed lines, open markers). Note that only selected data points, those that are optimal for collocation, have markers.

where the constants C_j^A depend on the problem dimension d, the chosen weight functions, the number of local nodes n_j, and the order of the differential operator α. The function $q(n_j)$ corresponds to the polynomial degree q supported by the local number of points n_j. That is, if $n_{q,d}$ denotes the dimension of the polynomial space of degree q in d dimensions and the number of local points satisfies $n_{q,d} \leq n_j \leq n_{q+1,d}$, then $q(n_j) = q$. The norm $||u||_{\mathcal{N}(\Omega_j)}$ is the native norm in the space generated by the basis functions [38]. Error estimate (2.34) indicates algebraic convergence in patch size H_j. For the latter case the error can be estimated as

$$||\mathscr{E}_{\mathscr{L}}||_{L_\infty(\Omega)} \leq KC^E \max_{1\leq j\leq N} e^{\gamma \log(h_j)/\sqrt{h_j}} ||u||_{\mathcal{N}(\Omega_j)}, \tag{2.35}$$

where the constant C^E and the rate γ both depend on the problem dimension d and the order of differential operator α, and C^E additionally depends on the chosen weight function. Error estimate (2.35) indicates exponential convergence with respect to the internodal distance h_j. Similar estimates can be constructed for other types of basis functions, for example for inverse multiquadrics [86].

The interpolation error estimates (2.34), and (2.35) will not essentially differ whether the pure collocation or least squares approach is used. Hence, the convergence rates will be the same for both approaches since the interpolation error drives the convergence. However, as we show in Paper VIII the solution in the least squares sense is numerically more robust due to higher flexibility in the formation of the set of evaluation points [71]. Thus, we need to look at

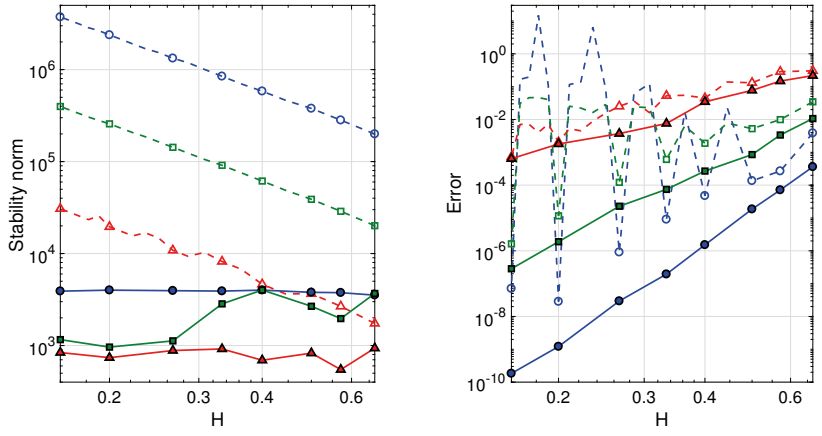

Figure 2.4. Left: The numerically estimated matrix norm $||L(\cdot,X)L^+||_{L_\infty(\Omega)}$. *Right:* The corresponding error as a function of $-1/h$ for oversampling $\beta = 1.1$ ($\triangle$), $\beta = 1.2$ ($\square$), and $\beta = 1.5$ ($\bigcirc$) for the least squares (solid lines, solid markers) and collocation (dashed lines, open markers).

the full error estimate

$$||\tilde{u}-u||_{L_2(\Omega)} \leq C_P||\mathscr{E}_{\mathscr{L}}||_{L_\infty(\Omega)} + C_P||L(\cdot,X)L^+||_{L_\infty(\Omega)}\left(C_M\delta_M + ||\mathscr{E}_{\mathscr{L}}||_{L_\infty(\Omega)}\right), \tag{2.36}$$

where L stands for $L(Y,X)$, $L^+ = (L^TL)^{-1}L^T$ is a pseudo inverse such that $L^+L = I$, and δ_M is the machine precision. We see that the approximation error estimate is proportional to the interpolation error estimate, which makes sense, but it also depends on the multiplier $||L(\cdot,X)L^+||_{L_\infty(\Omega)}$ that is similar to a condition number for the PDE approximation. The value of the matrix norm correlates with the problem size and problem parameters and is important for robustness of the methods. However, it can be controlled by oversampling in the least squares setup, which makes this approach behave more predictably. We can see in the right panel of Figure 2.3 how the error profiles differ for the collocation and least squares approaches for similar number of points per patch when approximating a solution to Poisson's problem. Although, for some selected partitioning sizes the error values for the two approaches may be close, the results for the collocation case look much more uncertain. Moreover, we can notice that the value of the matrix norm $||L(\cdot,X)L^+||_{L_\infty(\Omega)}$ increases algebraically under refinement for the collocation approach (see Figure 2.3, left panel). This makes the collocation approach difficult to apply for large problems. In contrast, the matrix norm stays constant for the least squares approach, which allows for scaling to large problem sizes in terms of the number of patches used. We can also observe an error reduction when decreasing the internodal distance while keeping the number of partitions fixed (see Figure 2.4, right panel). The left panel of Figure 2.4 shows that the matrix norm

increases exponentially for both approaches but can be lowered by oversampling in the least squares case. It is difficult to suggest a unique strategy for implementing refinements in the least squares RBF-PUM, but based on these results we could propose selecting a sufficient number of points per patch according to an acceptable condition number and then refining the partition size.

3. Applications in Computational Finance

3.1 Pricing Equity Derivatives

The most common equity derivative is an option. A European option is a financial contract that gives the buyer the right, but not the obligation, to buy or sell an underlying asset at a specific strike price on a specified date. Nowadays, the range of traded options is wide, and some contracts can even be customised and exist only in a single copy. However, only a small fraction of all option contracts is traded on exchanges, while the bulk of contracts is traded over-the-counter. This means that the value of most of the options cannot be determined through the market trading. That is, some other valuation methods need to be employed.

Black and Scholes [12] and Merton [76] were the first to suggest a strictly mathematical approach for the valuation of options. Given the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, the Black and Scholes market consists of a risk-free bank account B_t and a risky asset X_t, which follow the dynamics

$$\mathrm{d}B_t = rB_t\mathrm{d}t, \tag{3.1}$$

$$\mathrm{d}X_t = \mu X_t\mathrm{d}t + \sigma X_t\mathrm{d}W_t, \tag{3.2}$$

where r is the risk-free interest rate, μ is the return on the asset, σ is the volatility, and W_t is a Weiner process. The price of an option that at time of maturity T pays out $\Phi(X_T)$ can be found as a discounted expected cashflow

$$u(X,t) = e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}[\Phi(X_T)]. \tag{3.3}$$

Note that in order to avoid arbitrage opportunities the expectation is taken under the risk neutral measure $\mathbb{Q}$ that is equivalent to $\mathbb{P}$ and under which the discounted asset price is a martingale [11]. Formulation (3.3) is equivalent to a partial differential equation by the Feynman–Kac theorem [11]

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 x^2\frac{\partial^2 u}{\partial x^2} + rx\frac{\partial u}{\partial x} - ru = 0, \tag{3.4}$$

$$u(x,T) = \Phi(x). \tag{3.5}$$

Equation (3.4) is a standard parabolic PDE and can be solved by all well-known numerical methods, such as finite differences, finite elements or radial basis functions. For some types of payoff functions $\Phi(x)$, analytical solutions can be found. However, if we assume stochastic volatility, stochastic interest

rate, or if the option contract is written on several assets, then the problem will grow in dimensionality and its complexity will increase dramatically. Analytical solutions for such problems exist very seldom and typically for very specific stochastic models and payoff functions. If an option has the American property, i.e., can be exercised at any time before maturity, the valuation problem turns into a free boundary problem that has no analytical solution, and the price of such a contract can only be found approximately.

The standard methods that are used for pricing option contracts in the absence of analytical or semi-analytical solutions are of three kinds: (i) stochastic such as Monte Carlo methods [48, 49, 73] and binomial and trinomial trees [29, 56], which deal with formulations similar to (3.1)–(3.2); (ii) deterministic methods such as finite difference methods [61, 94, 102], more rarely finite elements [3, 34] and finite volumes [46], which deal with formulations similar to (3.4)–(3.5); and (iii) Fourier transform based methods such as Carr–Madan [18], COS [35], which deal with formulations similar to (3.3). The third category is the most efficient [93], but requires the existence of either the density function or the characteristic function of the underlying process distribution, which in some cases does not exist, for example, when the volatility is determined by a parametric function [93]. Then the methods from categories (i) and (ii) can be useful. RBF methods belong to the second category and are gaining in popularity and becoming more and more actively researched in applications to problems in finance because they are able to cope with problems with moderately high dimensions efficiently [7, 54, 83].

Practically, the deterministic methods work for problems with up to four dimensions and often employ splitting schemes to split a higher-dimensional problem into a set of one-dimensional problems [55, 60]. However, this procedure is rather complex in presence of nonzero correlations between the stochastic processes. Therefore, for problems with dimensionality greater than three, Monte Carlo methods were typically the only choice. In this thesis we develop and demonstrate that RBF methods can be an alternative to Monte Carlo methods for moderately high-dimensional problems.

In Paper I we develop an RBF partition of unity method for pricing basket European and American options [91] and demonstrate its greater approximation power compared with a standard finite difference method. RBF-PUM requires three times less nodes per dimension than the finite difference method to achieve the tolerance 10^{-4} due to a higher order convergence (see Figure 3.1). Firstly, this leads to a shorter computational time. Secondly, this is crucial for a transition to higher-dimensional problems since RBF-PUM consumes less memory to store the data. The combination of the two properties can mitigate the curse of dimensionality to some extent.

In Paper I we also improve the nonlinear penalty approach for valuing American options proposed by Nielsen [80] to account for dividend paying stocks and derive an estimate for the error introduced by the penalty.

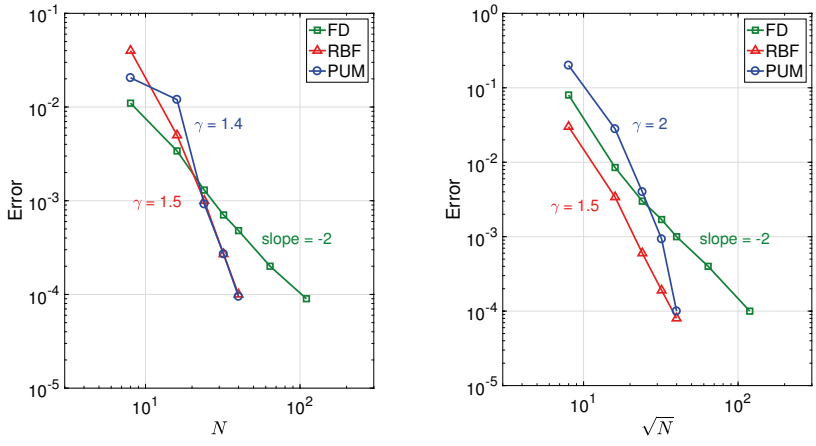

Figure 3.1. Left: Convergence in l_∞-norm for a European option on one underlying asset. *Right:* Convergence in l_∞-norm for a European basket option on two underlying assets. γ denotes the rate of convergence as in inequality (2.35).

Table 3.1. *Computational time to compute a solution that has a relative error* $< 10^{-4}$. *The dash "−" denotes absence of results for the method.*

Method	European	American	Up-and-out
FD	1.8e−02	7.6e−02	2.5e−02
FD-NU	9.2e−03	5.8e−02	1.6e−02
FD-AD	9.7e−03	4.3e−02	9.6e−03
RBF	6.2e−02	4.6e+00	1.4e−01
RBF-FD	2.9e−01	1.3e+00	2.8e−01
RBF-PUM	2.8e−02	3.6e+00	5.4e−02
RBF-LSML	4.2e−02	–	3.0e−02
RBF-AD	7.9e−01	1.7e+01	2.4e+01
RBF-MLT	1.6e+01	–	2.4e+02

However, it turns out that the penalty approach is less efficient than another commonly used technique to handle the free boundary, namely, the operator splitting method [59]. This was shown in Paper II, where several methods were tested on a set of benchmark problems with respect to the execution time to achieve the relative error tolerance level 10^{-4}, and RBF-PUM with a penalty function was outperformed by all other methods that employed the operator splitting method (see Table 3.1) [93]. We exclude the possibility of failure of RBF-PUM itself for the American option pricing problem because the method performed well for the rest of the benchmark tests. We determined that the issue was in the selection of a suitable technique for treating the free boundary as well as in the selection of a proper time integration scheme. In Paper II we used an implicit-explicit scheme with the nonlinear penalty term being ex-

plicit, which led to a severe restriction on the time step size and, consequently, to a long run time. In Paper III we improve RBF-PUM for the American option pricing problem by combining it with the operator splitting and, for the sake of science, implement three versions of the penalty method: fully explicit (PEX), implicit-explicit as was used in Paper II (PIMEX), and fully implicit with nonlinear Newton iterations (PIM). We also test them within the framework of Paper II. Surprisingly, the most efficient out of the three penalty versions was PIM. This was possible thanks to the fast convergence of the Newton iteration. However, the combination of RBF-PUM and the operator splitting method, indeed, resulted in a very efficient solver that outperformed all realisations of the penalty method (see Table 3.2)[1].

Table 3.2. *Computational time to compute a solution that has a relative error* $< 10^{-4}$.

Method	American
PEX	1.5e+00
PIMEX	5.0e−01
PIM	3.8e−02
OS	2.5e−02

In general, the results of Paper II verify RBF-PUM to be a method well suited for various types of contracts with exotic payoffs and implied volatility functions and as the most appropriate deterministic solver from category (ii) for the listed higher-dimensional problems. In Table 3.3 we report computational time for the node based methods to achieve a relative error lower than 10^{-4} for problems with dimensionality higher than one as was specified by the experimental framework in Paper II. We observe that among the methods RBF-PUM is the most favourable. Moreover, the results of the forthcoming study [70] demonstrate that RBF-PUM is able to approximate a solution of a five-asset index option with the accuracy of 10^{-6} in about four seconds on an ordinary laptop.

Also, we notice that RBF-PUM performs well for the model with two stochastic factors, namely the Heston model. Therefore based on this result, we extend the method formulation for option pricing under models with several stochastic factors in Paper IV [78]. There we assume stochastic nature of the volatility and interest rate, which indeed corresponds to the reality and to the data observed on the market. It is a well known fact that the volatility and interest rate are stochastic factors, but they are still often assumed to be constant in order to avoid raising the problem dimensionality since every stochastic factor gives a rise to an additional dimension. However, RBF-PUM is well suited for that type of problems since we can effectively exploit the high order approximation properties of RBFs. Furthermore, in Paper V we develop

[1]Note, that a different machine was used here, therefore the run time values in Table 3.2 do not align with the corresponding values in Table 3.1.

Table 3.3. *Computational time to compute a solution that has a relative error* $< 10^{-4}$. *The dash "−" denotes absence of results for the method.*

Method	Heston	Spread
FD	−	−
FD-NU	4.3e+00	7.4e+01
FD-AD	−	4.7e+01
RBF	1.9e+01	7.7e+01
RBF-FD	−	2.2e+03
RBF-PUM	4.3e+00	1.3e+01

an RBF-PUM approach for valuing credit derivatives such as credit default swaps under a model with four stochastic factors that experience jumps.

3.2 Valuing Credit Default Swaps

A credit default swap (CDS) is a financial agreement that the seller of the CDS will compensate the buyer in the event of a default. The CDS contract is set up in such a way that the seller receives periodic payments (coupons) from the buyer until a default event occurs and the seller is obliged to pay out the protection upon the default. Entering such contracts helps the investor to hedge risks and achieve the desired risk exposure. Credit default swaps are the most common form of credit derivatives and may involve municipal bonds, emerging market bonds, mortgage-backed securities or corporate bonds. However, in this thesis we focus on CDS contracts that involve sovereign obligations.

In order to determine the fair value of the contract we need to understand what payments are being made by each contracting party. The expected cashflow received by the seller consists of the periodic payments, which in general are paid quarterly and stop being paid out after a default event [14, 72]

$$L_c = \mathbb{E}_t \left[\sum_{i=1}^{m} cNB(t,t_i)\Delta t \, \mathbb{1}_{\{\tau \geq t_i\}} \right], \tag{3.6}$$

and the amount that is accrued from the nearest past payment date until the time of the default event τ

$$L_a = \mathbb{E}_t \left[cNB(t,\tau)(\tau - t_{\beta(\tau)}) \mathbb{1}_{\{t < t_{\beta(\tau)} \leq \tau < T\}} \right], \tag{3.7}$$

where c is the CDS coupon, N is the notional amount, $(t_0, t_1, \ldots, t_m)$ are the predefined payment dates, $B(t,t_i)$ is the stochastic discount factor from date t_i to date t, t is the date when the contract comes into force, Δt is the payment time interval, T is the maturity date of the contract, such that $m\Delta t = T - t$, $t_{\beta(\tau)}$ is the payment date preceding the default event. The sum of the coupon

and accrued amounts is called the *premium leg*. The opposite expected protection cashflow, also known as the *protection leg*, is

$$L_p = \mathbb{E}_t \left[(1-R) NB(t,\tau) \mathbb{1}_{\{t<\tau\leq T\}} \right], \tag{3.8}$$

where R is the recovery rate, which is unknown beforehand, and is determined at, or right after, the default, e.g., in court. In modern mathematical finance theory it is customary to consider the recovery rate to be stochastic, see e.g., [27] and references therein. However, we assume that the recovery rate is constant. Thus, we define the CDS par spread s as the coupon which equalises these two legs and makes the CDS contract fair at time t. Hence, s solves the following equation

$$\sum_{i=1}^{m} \mathbb{E}_t \left[sNB(t,t_i)\Delta t \mathbb{1}_{\{\tau\geq t_i\}} \right] + \mathbb{E}_t \left[sNB(t,\tau)(\tau - t_{\beta(\tau)}) \mathbb{1}_{\{t<\tau<T\}} \right] = \mathbb{E}_t \left[(1-R) NB(t,\tau) \mathbb{1}_{\{t<\tau\leq T\}} \right]. \tag{3.9}$$

The CDS spread is often viewed as an indicator of the riskiness of investment into the reference entity's bonds. A large CDS spread value means a high cost of protection against default, implying a greater risk of that to occur. In contrast, low CDS spread values demonstrate good credit worthiness of the reference entity.

In this thesis we consider quanto CDS contracts, which have a special feature that the payments are set in a different currency to that of the reference entity. A typical example would be a CDS which has its reference as a dollar denominated bond for which the premium of the swap is payable in euros. These contracts are widely used to hedge holdings in bonds or bank loans that are denominated in a foreign currency (other than the investor's home currency).

As far as the value of the quanto CDS is concerned, it is assumed that their market quotes are available in both domestic and foreign currencies, which we denote as CDS_d and CDS_f. Typically the US dollar is viewed as the domestic currency since the prices and market factors are calibrated under the USD measure. In this thesis we consider euro as the foreign currency. Also, we assume that the exchange rate between the foreign and domestic currencies is 1 to Z_t i.e., loosely speaking, 1 EUR is worth Z_t USD.

In case of default, the foreign currency tends to depreciate, which results in a discrepancy between CDS_d and CDS_f. For instance, from the historical data on five-year Portuguese CDS quoted in USD and EUR (see Figure 3.2) we observe that the difference can reach as high values as 145 bps (basis points, 1 bp = 0.01%). Recent work [4] presents the term structure of spreads, defined as the difference between the USD and EUR denominated CDS spreads, for six Eurozone countries: Germany, Belgium, France, Ireland, Italy, and Portugal, and for maturities 3, 5, 7, 10, and 15 years relative to the 1 year quanto spread. This difference could reach 30 bps at the time horizon of 15 years

(France, Ireland). The results presented in [15] indicate a significant discrepancy across domestic and foreign CDS quotes for Italy. A USD CDS spread quote of 440 bps could translate into a EUR quote of 350 bps in the middle of the European debt crisis in the first week of May 2012.

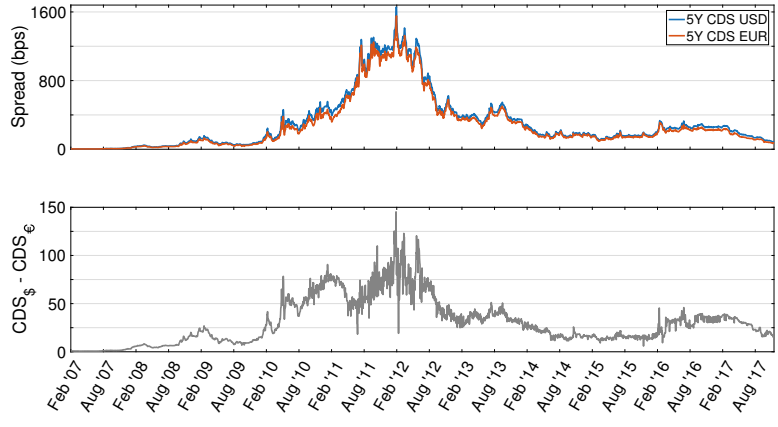

Figure 3.2. Historical data on five-year Portuguese sovereign CDS quoted in US dollars and euro and the difference between the rates.

Quanto effects drew a lot of attention on the modelling side. Various aspects of the problem were under investigation including the relationship between sovereign credit and currency risks, the pricing of sovereign CDS, the impact of contagion on credit risk, see survey in [4] and references therein. But we primarily focus on pricing quanto CDS, and determining and testing an appropriate framework that provides a reasonable explanation of these effects from a mathematical finance point of view.

Since the underlying security of a CDS contract is a bond, we need to model the bond prices, but before going into modelling details we need to introduce a probability space $(\Omega, \mathscr{F}, \mathbb{Q})$ satisfying the standard hypothesis, namely:

1. $(\mathscr{F}_t, t \geq 0)$ is a filtration under which the dynamics of the risk factors are adapted and under which the default time of the reference entity is a stopping-time;
2. The risk neutral probability measure $\mathbb{Q}$ corresponds to the domestic currency money market;
3. By $\mathbb{E}_t[\cdot]$ we denote the expectation conditioned on the information received by time t, i.e. $\mathbb{E}[\cdot|\mathscr{F}_t]$.

Further we consider two money markets: B_t associated with the domestic currency (USD), and $\hat{B}_t$ associated with the foreign currency (EUR), where $t \geq 0$ is the calendar time. We assume that dynamics of the two money market

accounts are given by

$$\mathrm{d}B_t = R_t B_t \mathrm{d}t, \quad B_0 = 1, \tag{3.10}$$

$$\mathrm{d}\hat{B}_t = \hat{R}_t \hat{B}_t \mathrm{d}t, \quad \hat{B}_0 = 1, \tag{3.11}$$

where the stochastic interest rates $R_t, \hat{R}_t$ follow the Cox-Ingersoll-Ross (CIR) process [28]

$$\mathrm{d}R_t = a(b - R_t)\mathrm{d}t + \sigma_r \sqrt{R_t}\mathrm{d}W_t^{(1)}, \quad R_0 = r, \tag{3.12}$$

$$\mathrm{d}\hat{R}_t = \hat{a}(\hat{b} - \hat{R}_t)\mathrm{d}t + \sigma_{\hat{r}} \sqrt{\hat{R}_t}\mathrm{d}W_t^{(2)}, \quad \hat{R}_0 = \hat{r}, \tag{3.13}$$

where $a, \hat{a}$ are the mean-reversion rates, $b, \hat{b}$ are the mean-reversion levels, $\sigma_r, \sigma_{\hat{r}}$ are the volatilities, and $W_t^{(1)}, W_t^{(2)}$ are Brownian motions. Without loss of generality we assume $a, \hat{a}, b, \hat{b}, \sigma_r, \sigma_{\hat{r}}$ to be constant.

We assume that the exchange rate Z_t of the two currencies is stochastic with dynamics driven by the following stochastic differential equation

$$\mathrm{d}Z_t = \mu_z Z_t \mathrm{d}t + \sigma_z Z_t \mathrm{d}W_t^{(3)}, \quad Z_0 = z, \tag{3.14}$$

where μ_z, σ_z are the corresponding drift and volatility, and $W_t^{(3)}$ is another Brownian motion.

As the underlying bond is subject to a potential default we need to account for the credit risk. We define the hazard rate λ_t to be a stochastic process given by

$$\lambda_t = e^{Y_t}, \quad t \geq 0, \tag{3.15}$$

with Y_t following the Ornstein–Uhlenbeck process

$$\mathrm{d}Y_t = \kappa(\theta - Y_t)\mathrm{d}t + \sigma_y \mathrm{d}W_t^{(4)}, \quad Y_0 = y, \tag{3.16}$$

where κ is the corresponding mean-reversion rate, θ is the mean-reversion level, σ_y is the volatility, and $W_t^{(4)}$ is a Brownian motion.

We assume that all Brownian motions $W_t^{(i)}$, $i \in [1,4]$, are dependent, and this dependence can be specified through the constant instantaneous correlation ρ between each pair of the Brownian motions, i.e., $\langle \mathrm{d}W_t^{(i)}, \mathrm{d}W_t^{(j)} \rangle = \rho_{ij}\mathrm{d}t$.

Finally, we define the default process $(D_t,\, t \geq 0)$ as

$$D_t = \mathbb{1}_{\{\tau < t\}}, \tag{3.17}$$

where τ is the default time of the reference entity.

Brigo et al. [15] could to some extent explain the discrepancy between the USD and EUR denominated CDS by introducing a jump-at-default in the FX rate. However, we believe that a default also impacts the foreign interest rate and therefore a jump in the interest rate should be taken into account, since

the default lowers the creditability and dramatically increases the cost of borrowing. Indeed, looking at the historical data of the Russian crisis of 1998, we can observe a very pronounced jump in the interest rate. The short interest rate grew from 20% in April 1998 to 120% in August 1998 [84].

To incorporate jumps into the dynamics of the FX rate in (3.14), we assume the jump to be proportional to the current rate

$$\mathrm{d}Z_t = \gamma_z Z_{t^-}\mathrm{d}M_t, \tag{3.18}$$

where $\gamma_z \in [-1,\infty)$ is a devaluation/revaluation parameter and t^- indicates the time just before a default occurs.

The hazard process Γ_t of a random time τ with respect to a reference filtration is defined through the equality $e^{-\Gamma_t} = 1 - \mathbb{Q}\{\tau \le t|\mathscr{F}_t\}$. It is well known that if the hazard process Γ_t of τ is absolutely continuous, i.e.,

$$\Gamma_t = \int_0^t (1-D_s)\lambda_s\,\mathrm{d}s, \tag{3.19}$$

and increasing, then the process $M_t = D_t - \Gamma_t$ is a martingale (which is called the compensated martingale of the default process D_t) under the full filtration $\mathscr{F}_t \vee \mathscr{H}_t$ with $\mathscr{H}_t$ being the filtration generated by the default process. So, M_t is a martingale under $\mathbb{Q}$, [10].

It can be shown that under the risk-neutral measure associated with the domestic currency, the drift μ_z is, ([15])

$$\mu_z = R_t - \hat{R}_t. \tag{3.20}$$

Therefore, with allowance for (3.14), (3.18) we obtain

$$\mathrm{d}Z_t = (R_t - \hat{R}_t)Z_t\mathrm{d}t + \sigma_z Z_t \mathrm{d}W_t^{(3)} + \gamma_z Z_t \mathrm{d}M_t. \tag{3.21}$$

Thus, Z_t is a martingale under the $\mathbb{Q}$-measure with respect to $\mathscr{F}_t \vee \mathscr{H}_t$ as it should be, since it is a tradable asset. Certainly, we are more interested in the negative values of γ_z because revaluation of the foreign currency in case of its sovereign default is extremely unlikely.

Similarly, we add jump-at-default to the stochastic process for the foreign interest rate $\hat{R}_t$

$$\mathrm{d}\hat{R}_t = \gamma_{\hat{r}}\hat{R}_{t^-}\mathrm{d}D_t, \tag{3.22}$$

thus (3.13) transforms to

$$\mathrm{d}\hat{R}_t = \hat{a}(\hat{b} - \hat{R}_t)\mathrm{d}t + \sigma_{\hat{r}}\sqrt{\hat{R}_t}\mathrm{d}W_t^{(2)} + \gamma_{\hat{r}}R_t\mathrm{d}D_t, \tag{3.23}$$

where $\gamma_{\hat{r}} \in [-1,\infty)$ is the parameter that determines the post-default cost of borrowing. We are interested in positive values of $\gamma_{\hat{r}}$ as the interest rate most likely will grow after a default has occurred. Note that $\hat{R}_t$ is not tradable, and so is not a martingale under the $\mathbb{Q}$-measure.

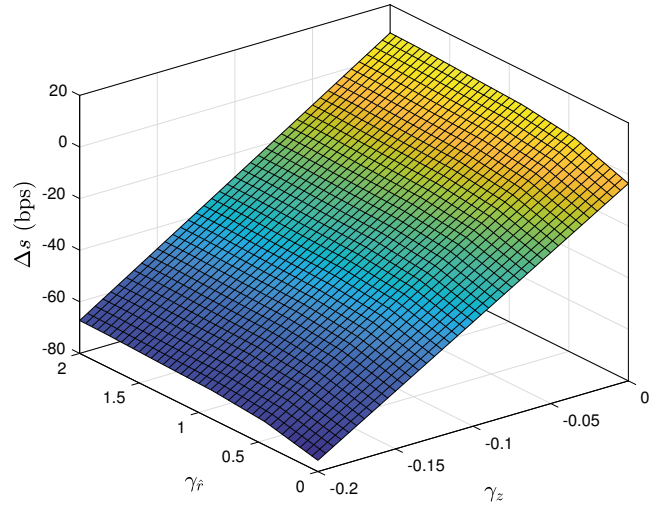

Figure 3.3. The difference between the foreign and domestic CDS quotes as a function of the jump amplitude in the foreign exchange and interest rates.

Thereby, we can derive a set of pricing four-dimensional PDEs, however, the derivations are a bit lengthy, therefore, we refer the reader to Paper V and omit the details here. We use RBF-PUM to approximate the solutions of the set of PDEs, and the numerical solutions is presented in Figure 3.3. We plot not the CDS value itself but the difference between the spread of the CDS denominated in US dollar and its counterpart denominated in the foreign currency as a function of the jump sizes in the FX and interest rates.

In the absence of jumps the domestic and foreign spreads are quoted with a difference in 3 bps, which would be close to the normal situation if no currency and interest rate depreciation occurred. In fact, this was the case until recently when quanto effects were not taken into account. For example, Greek CDS with payments in dollars and in euros were traded with 1 bp difference in 2006 [95]. However, nowadays the quanto effects are regarded very seriously by practitioners and the market values are adjusted accordingly. Our model demonstrates that the largest portion of discrepancies can be explained by the jumps-at-default in the FX and foreign interest rates. The FX rate is, of course, the dominant impact factor since the devaluation of the foreign currency has an immediate effect on the amount of protection being paid out when converted into the US dollars. Nevertheless, the impact of the jump in the foreign interest rate in our setup is responsible for about 15 bps difference between the quotes in dollars and foreign currency.

4. Applications in Ice Sheet Modelling

Accurate modelling of ice sheets and glaciers is becoming of an increasing importance since melting ice is one of the main contributors to the sea level rise [26]. Glacial ice slowly moves and deforms under its own weight, creeping down a valley or spreading over a continent. In this context, glacial ice can be viewed as a non-Newtonian, incompressible, very viscous fluid. The flow can be modelled by a set of nonlinear Stokes equations

$$-\nabla p + \nabla \cdot \left(\eta(\mathbf{v})(\nabla \mathbf{v} + (\nabla \mathbf{v})^T)\right) + \rho \mathbf{g} = \mathbf{0}, \tag{4.1}$$

$$\nabla \cdot \mathbf{v} = 0, \tag{4.2}$$

where p is the pressure, $\mathbf{v} = (v_x, v_y, v_z)$ is the velocity field, ρ is the density, $\rho\mathbf{g}$ is the force of gravity, and η the nonlinear viscosity.

The physics of the ice is determined by the bedrock topography $z = b(x,y)$ and the ice surface position $z = h(t,x,y)$. The bedrock is often assumed to be immobile while the surface changes over time and its dynamics are defined by

$$\partial_t h + v_x|_{z=h}\partial_x h + v_y|_{z=h}\partial_y h = v_z + a_s, \tag{4.3}$$

where a_s is the net accumulation/ablation at the ice surface, which depends on precipitation and surface air temperature. Note, the ice-atmosphere interface is open and constitutes a free boundary.

Numerical solution of the full-Stokes system demands an extensive computational effort. In spite of the fact that the full-Stokes is regarded as the most accurate model, the glaciologists are still not completely aware of the exact physical processes taking place in ice. Therefore, one can argue that the full-Stokes equations may be unnecessary since their modelling capacity is dispersed by processes which the model does not account for. Hence, there are approximations to the Stokes model of different accuracy. The least accurate but computationally inexpensive are the Shallow Ice Approximation (SIA) [57] and Shallow Shelf Approximation (SSA) [16]. They are often used for simulating dynamics of inner parts of ice sheets since the velocity is low there and SIA and SSA give a relatively good approximation. However, for more dynamic parts of ice sheets more accurate approximations have to be used, such as the First Order Stokes model, also known as the Blatter–Pattyn model [13, 81]. It gives a second order approximation with respect to the ratio thickness/length of an ice sheet. The Blatter–Pattyn model is sufficiently good for capturing the dynamics of ice sheet edges but still less accurate than the

full-Stokes for simulating the grounding line where ice comes in contact with water [66].

Some models, such as the Ice Sheet Coupled Approximation Levels (ISCAL) [1] or the Ice Sheet System Model (ISSM) produced by NASA [66], combine several approaches and substitute the full-Stokes equations by SIA or the Blatter–Pattyn model in regions where this is appropriate, hence, significantly reducing simulation time. Most of the current models are built upon libraries that implement finite element methods. For example, ISCAL uses Elmer/Ice [47] and ISSM uses PETSc [6].

In this thesis we make a first attempt to develop an RBF solver for simulating ice sheet dynamics. We employ the Blatter–Pattyn model for modelling the velocity field. As mentioned above, this model is just an approximation to the full-Stokes equations but recognised as sufficiently accurate [66], while being much less computationally demanding. It exploits the fact that the ratio between ice thickness and length is small, and under such an assumption the derivatives $\partial_x v_z$ and $\partial_x v_z$ can be disregarded, and $\partial_x\big(\eta(\partial_z v_x + \partial_x v_z)\big)$ and $\partial_y\big(\eta(\partial_z v_y + \partial_y v_z)\big)$ are considered negligible in comparison to $\partial_z(2\eta\partial_z v_z)$. Thus, the system dimensionality can be reduced and the horizontal velocity components (v_x, v_y) can be obtain from the following system of equations

$$\partial_x\big(\eta(2\partial_x v_x + \partial_y v_y)\big) + \frac{1}{2}\partial_y\big(\eta(\partial_x v_y + \partial_y v_x)\big) + \frac{1}{2}\partial_z\left(\eta\partial_z v_x\right) = \rho g \partial_x h, \quad (4.4)$$

$$\frac{1}{2}\partial_x\big(\eta(\partial_x v_y + \partial_y v_x)\big) + \partial_y\big(\eta(2\partial_y v_y + \partial_x v_x)\big) + \frac{1}{2}\partial_z\left(\eta\partial_z v_y\right) = \rho g \partial_y h. \quad (4.5)$$

Once the horizontal velocity is found, the vertical velocity v_z is given by the mass conservation equation (4.2) and the pressure is given by

$$p = -2\eta(\partial_x v_x + \partial_y v_y) + \rho g(h - z). \quad (4.6)$$

In Paper VI we suggest a numerical approach to this problem based on an RBF method and the use of immobile nonuniform background node sets [2] to model the velocity field of the Haut glacier d'Arolla [82]. This allows for swift and efficient solution of the nonlinear problem and effective tracking of the free ice surface. We demonstrate that, in contrast to finite element methods, our approach does not require full matrix reassembly in every nonlinear iteration, and therefore allows for avoiding the most computationally demanding procedure, which in FEM may take up to 70–90% of the total run time [1].

In Table 4.1 we present the error and execution time values for several numbers of degrees of freedom. As the reference solution we use a finite element solution on a fine mesh with 12167 nodes. We observe that RBF-PUM is about an order of magnitude faster then FEM, which makes it attractive for application in problems of finding velocity fields of large ice sheets such as Greenland and Antarctica [24], where due to large domain sizes and, hence, large computational problems, highly efficient performance is vital. Moreover, an important property of the RBF method is conservation of the total volume

Table 4.1. *Error and CPU time for several numbers of degrees of freedom for FEM, the global RBF method, and RBF-PUM.*

FEM			RBF			RBF-PUM		
Error	N	Time	Error	N	Time	Error	N	Time
0.0033	482	11.4055	0.0045	524	1.2621	0.0045	500	0.3563
0.0024	563	15.1375	0.0028	711	1.6162	0.0027	669	0.4959
0.0022	805	20.9487	0.0023	855	2.4665	0.0025	814	0.8257
0.0018	972	26.8749	0.0016	1101	4.6602	0.0015	1002	2.0894
0.0010	1251	36.4145	0.0010	1505	10.2750	0.0013	1348	3.7643

of the system. In Figure 4.1 among others we plot the relative change in the total volume over a two-year transient simulation. The difference in volume diminishes when decreasing the internodal distance.

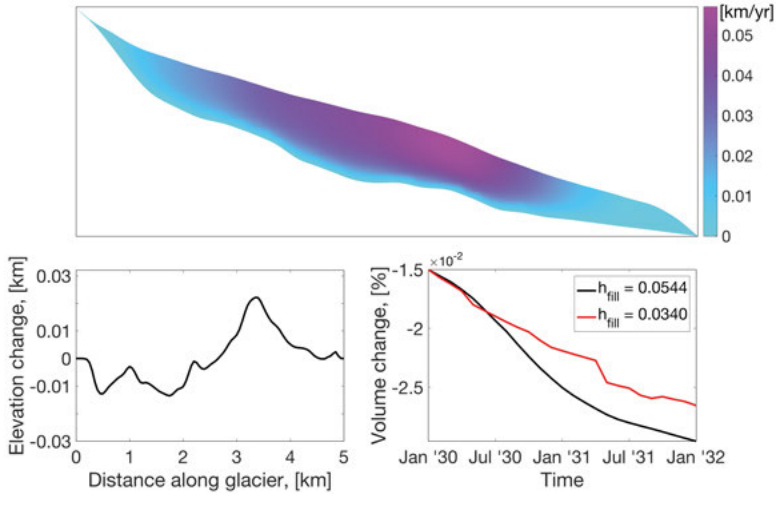

Figure 4.1. Top: The horizontal velocity (kilometres per year) of the d'Arolla Glacier after evolving the surface for two years. *Bottom:* The surface elevation change after a transient simulation (*left*). The relative volume change over the two-year period with respect to different values of the internodal distance (*right*).

4.1 Anisotropic Radial Basis Functions

The standard RBF-PUM with isotropic (i.e., direction independent) basis functions are well suited for problems that are set on domains with reasonably low aspect ratios such as glaciers, for example, the Haut glacier d'Arolla. However, if we are to apply the method for simulating velocity fields of real

size ice sheets, the method must be capable of dealing with extremely large length/thickness ratios, since a typical length of an ice sheet is much larger than its typical height. The ratio may reach $500:1$. Unfortunately, isotropic RBF methods cease to give a reasonable approximation under such circumstances. Therefore, in Paper VII we modify the formulation to better match the domain properties [24]. We do it by adapting anisotropic basis functions, which are not uniform in all directions under the standard Euclidean norm, but are uniform under the norm $||\cdot||_a$, which defines the distance between two points $x, y \in \Omega \subset \mathbb{R}^2$ as

$$||x-y||_a = \sqrt{(x_1-y_1)^2 + a^2(x_2-y_2)^2}, \tag{4.7}$$

where a is the aspect ratio of the domain discretisation with resolutions h_x and h_z in the horizontal and vertical directions, respectively,

$$a = \frac{h_x}{h_z}. \tag{4.8}$$

Figure 4.2 displays an anisotropic Gaussian basis function in the Euclidean and $||\cdot||$-norm.

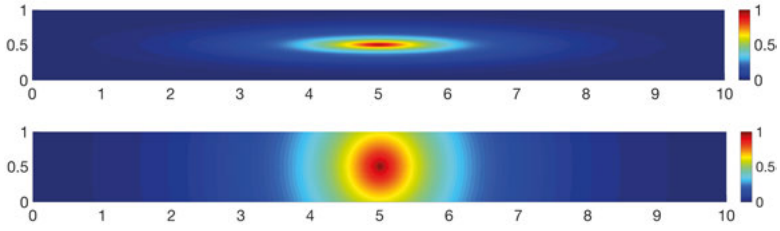

Figure 4.2. Top: An anisotropic Gaussian RBF. *Bottom:* An isotropic Gaussian RBF. The aspect ratio of the domain discretisation $a = 10$.

In order to develop the anisotropic approach we select a synthetic ice sheet with the so-called Bueler profile [17, 50] that is inspired by the celebrated benchmark EISMINT [58]. The synthetic ice sheet resembles an ice sheet of continental size with aspect ratio $428:1$. We use the Blatter–Pattyn model to simulate the dynamics of ice masses. For that problem we define a as in (4.8) and scale the basis functions accordingly. Besides scaling the basis functions we also scale the patches as was suggested in [87]. That is, the patches have a circular form in the $||\cdot||_a$-norm but are elliptic in the standard Euclidean norm.

To compute the velocity fields of the ice sheet we use both structured Cartesian nodes and unstructured quasi-random Halton nodes. Using the Halton nodes we demonstrate the meshfree property of the RBF method that can be very valuable for simulating velocity fields of real ice sheets. In Figure 4.3 we show the horizontal and vertical velocities obtained by the anisotropic

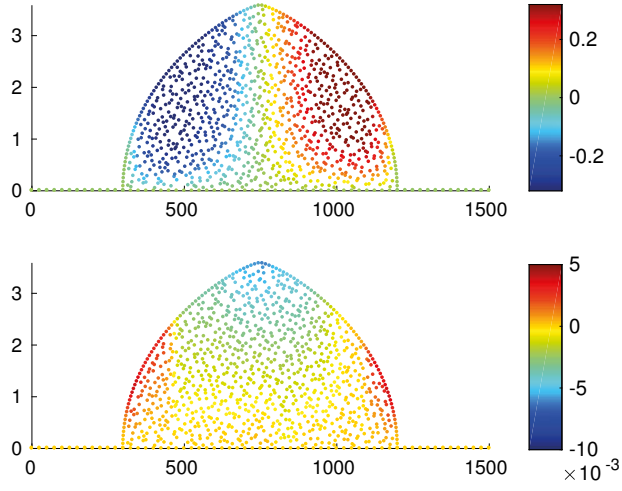

Figure 4.3. Top: The horizontal velocity (kilometres per year) of the ice cap computed on the Halton nodes by anisotropic RBF-PUM. *Bottom:* The respective vertical velocity (kilometres per year).

RBF-PUM on a Halton set of nodes. The velocities are well recovered and the reader is referred to [58] to verify that the presented solution behaves as expected.

5. Summary of Papers

5.1 Paper I

V. Shcherbakov and E. Larsson. Radial basis function partition of unity methods for pricing vanilla basket options. *Comput. Math. Appl.*, 71(1):185–200, 2016.

We derive a penalty approach for dividend paying American basket call options and find estimates of the error introduced by the penalty approach. We develop a radial basis function partition of unity method for pricing multi-asset options, and numerically prove that the method converges rapidly for both European and American style options. We numerically investigate the method efficiency and show its advantages over global radial basis function methods and standard finite difference methods. Additionally, we analyse the domain partitioning for the partition of unity approach and study various partitioning setups with respect to l_∞-error and computational time.

Contribution

The author of this thesis wrote the manuscript and performed the numerical experiments. The stability analysis for the penalty approach was done in close collaboration with the second author. The ideas were also developed in close collaboration with the second author.

5.2 Paper II

L. von Sydow, L. J. Höök, E. Larsson, E. Lindström, S. Milovanović, J. Persson, V. Shcherbakov, Y. Shpolyanskiy, S. Sirén, J. Toivanen, J. Waldén, M. Wiktorsson, J. Levesley, J. Li, C. W. Oosterlee, M. J. Ruijter, A. Toropov, Y. Zhao. BENCHOP—The BENCHmarking project in option pricing. *Int. J. Comput. Math.*, 92(12):2361–2379, 2015.

This project was a large joint effort to produce benchmark tests in option pricing. We designed six benchmark problems with several subproblems in each and tested different computational methods to solve them. All methods were compared in terms of time-to-error, and all CPU times to achieve relative error tolerance 10^{-4} were reported in tables. RBF-PUM was shown to be competitive among the methods that rely on node discretisation of the domain. It was the most efficient method in this group for problems with dimensionality

greater than one. However, there was a clear drop of efficiency for American style options, which was explained by the use of the nonlinear penalty term to deal with the free boundary. We had a semi-implicit time integration scheme with explicit penalty to avoid nonlinear iterations. It resulted in a constraint on the time step size, which turned out to be quite severe. This issue was resolved in Paper III where we switched to the operator splitting approach that was used by other node-based methods in the benchmarking experiments.

Contribution

The author of this paper was responsible for development and implementation of the RBF-PUM method for the benchmark problems. The ideas regarding the paper structure and what problems were relevant for the tests were discussed in close collaboration between all authors.

5.3 Paper III

V. Shcherbakov. Radial basis function partition of unity operator splitting method for pricing multi-asset American options. *BIT*, 56(4):1401–1423, 2016.

In this paper we improve the approach for pricing American style options. Instead of the penalty method for the free boundary we use an operator splitting method. However, we also experiment with different implementations of the penalty approach. We test fully explicit, semi-implicit, and fully implicit schemes, with nonlinear Newton's iterations for the latter version. It turns out that the fully implicit nonlinear scheme is the most efficient time integration scheme for the penalty method and Newton's method converges in a few iterations. However, all penalty method implementations are outperformed by the operator splitting method. Importantly, this advantage increases with the dimensionality. The operator splitting is the only technique that allows for solving problems in dimension higher than two in reasonable time within our experimental framework. We also show that the combination of RBF-PUM and the operator splitting method is very efficient for calculating the Δ and Γ Greeks for American options since we can get the solution on the entire span of the underlying asset spot values.

Contribution

The author of this thesis is the sole author of this paper.

5.4 Paper IV

S. Milovanović and V. Shcherbakov. Pricing derivatives under multiple stochastic factors by localized radial basis function methods. Available as *arXiv:1711.09852*, 2017.

In this paper we introduce two localised RBF approaches, namely, RBF-PUM and RBF-FD, for pricing derivatives under models with several stochastic factors. It is well known that asset volatilities and economy interest rates are stochastic, and therefore, much be modelled accordingly. However, every additional stochastic factor gives a rise to an extra dimension in the formulation. Thus, valuing derivatives under multi-factor models results leads to complex multi-dimensional PDE problems, which have to be solved numerically. In this paper we demonstrate how the RBF methods can be used to attack these problems and give a detailed guideline on how to implement the methods. Additionally, the paper contains a comparison of RBF-PUM and RBF-FD in terms of convergence rate and computational efficiency.

Contribution

The manuscript was written and the numerical experiments were performed in close collaboration between the authors.

5.5 Paper V

A. Itkin, V. Shcherbakov, and A. Veygman. Influence of jump-at-default in IR and FX on Quanto CDS prices. Available as *arXiv:1711.07133*, 2017.

In this paper we focus on modelling aspects of quanto credit default swap values. We develop a model that can explain discrepancies in values of CDS denominated in different currencies. This is achieved by incorporating jumps-at-default into the exchange and foreign interest rates. We observe that our model is capable of explaining the largest portion of the difference between the domestic and foreign CDS values. The greatest impact factor is the jump in the exchange rate since the risk of devaluation of the foreign currency makes the investors willing to pay less for the protection denominated in the foreign currency. However, the jump in the foreign interest rate is also responsible for around 15 bps basis between the CDS quotes in the domestic and foreign currencies. We use RBF-PUM to approximate the solution of a set of four-dimensional PDEs numerically.

Contribution

The author of this thesis participated in the model development, performed the numerical experiments, and contributed to writing the manuscript. The ideas for the paper were developed in close collaboration between all authors.

5.6 Paper VI

J. Ahlkrona and V. Shcherbakov. A meshfree approach to non-Newtonian free surface ice flow: Application to the Haut Glacier d'Arolla. *J. Comput. Phys.*, 330:633–649, 2017.

In this paper we introduce a radial basis function method for solving for the velocity field of ice flow. The meshfree nature of the method allows for easy discretisation of the computational domain, that typically has complex geometry, and for efficient handling of the free surface. As a test case we select the Haut glacier d'Arolla, which is a part of a well-known benchmark ISMIP-HOM. We compare the RBF method with the standard finite element method with linear hat functions. The RBF method significantly outperforms the finite element method in terms of CPU time for a given number of discretisation nodes even though the convergence rates are similar. The convergence of the RBF method is limited by the non-Lipschitz continuous domain geometry. The method benefits from avoiding the full matrix reassembly within the nonlinear solver in every iteration, which turns out to be the most expensive procedure. Additionally, we implement an RBF partition of unity method that reduces the computational effort even more thanks to sparsification of the system of equations.

Contribution

The manuscript was written and the numerical experiments were performed in close collaboration between the authors.

5.7 Paper VII

G. Cheng and V. Shcherbakov. Anisotropic radial basis function methods for continental size ice sheet simulations. Available as *arXiv:1711.09947*, 2017.

We extend the approach from Paper VI to simulate the dynamics of continental size ice sheets. In order to overcome the numerical issues associated with the large aspect ratios of the computational domains we modify the basis functions to match the domain geometry. We achieve this by using anisotropic basis functions that are scaled accordingly. We implement an anisotropic RBF-PUM and apply it for finding the velocity field of a large ice cap that is defined by the Bueler profile that resembles a continental size ice sheet. By the anisotropic RBF-PUM we can recover the velocity with an error of just a few meters on a moderate size node sets.

Contribution

The manuscript was written and the numerical experiments were performed in close collaboration between the authors.

5.8 Paper VIII

E. Larsson, V. Shcherbakov, and A. Heryudono. A least squares radial basis function partition of unity method for solving PDEs. *SIAM J. Sci. Comput.*, 39(6):2538–2563, 2017.

We advance RBF-PUM by switching the approach from pure collocation to least squares. Moreover, we use a single template of centre nodes for all partitions, that yields an opportunity to discard some computations, such as computations of a stable basis for the RBF-QR method for each patch. This is a crucial advantage since the computation of an RBF-QR basis is computationally very demanding and otherwise has to be performed for every partition. Furthermore, the method becomes more suitable for complex geometries due to the possibility to have centre nodes outside the computational domain. We experimentally show that the least squares based approach is more robust than pure collocation and allows for stable computations for problems with large numbers of degrees of freedom. We derive analytical error estimates for both collocation based RBF-PUM and least squares based RBF-PUM. Also, we show how the numerical stability can be controlled by oversampling for the latter method.

Contribution

The author of this thesis participated in the method development together with the first author and performed numerical experiments for the two-dimensional case. The ideas for the paper were developed in close collaboration between all authors.

6. Summary in Swedish

I denna avhandling utvecklar vi radiella basfunktionsmetoder (RBF) för partiella differentialekvationer (PDE) och tillämpar dem på komplexa problem som inte har någon analytisk lösning men kräver noggranna approximationer. Vi betraktar problem i finansiell matematik och glaciologi som är relevanta för vardagslivet och svåra nog för att visa fördelarna med RBF-metoder jämfört med andra väletablerade numeriska metoder, såsom finita differensmetoder och finita elementmetoder.

Den första användningen av RBF-metoder beskrevs av Hardy för kartografi, geodesi och digitala terrängmodeller för att minska fel i datainterpolering [51]. Senare anpassades de för partiella differentialekvationer av Kansa [63, 64] och blev sedan intensivt utforskade i slutet av 1990-talet och början av 2000-talet [32, 36, 88, 99, 106]. Nu för tiden används RBF-metoder både för interpolering [20, 30, 31] och för att lösa PDE:er [9, 39, 65]. De mest attraktiva egenskaperna hos RBF-metoder är deras snabba konvergens och nätfrihet, vilka gör dem flexibla och lätta att tillämpa på problem definerade i domäner med komplex geometri.

Trots goda teoretiska konvergensegenskaper blir den globala RBF-metoden opraktisk även för relativt små problem (med några tusen frihetsgrader) eftersom (i) diskretiseringen resulterar i ett ekvationssystem med full koefficientmatris, vilket är beräkningsmässigt dyrt att lösa; (ii) koefficientmatrisen för det linjära systemet blir mycket illa-konditionerad även för stora värden på formparametern. Potentiella lösningar är att använda (a) en lokaliseringsteknik för att sparsifiera matrisen; (b) en stabiliseringsalgoritm som avlägsnar eller minskar beroendet på formparametern.

Lokaliseringsteknikerna, som för närvarande är populärast, är approximation med RBF kombinerad med enhetsuppdelning (RBF-PUM) [19, 21, 68, 87, 91], radiella basfunktionsgenererade finita differenser (RBF-FD) [8, 41, 97, 98, 103, 104] och mindre vanliga, men fortfarande förekommande i litteraturen, RBF med kompakt stöd [100, 107]. Fördelen med alla dessa lokala RBF-metoder är att de resulterar i ett mycket glesare linjärt ekvationsystem, vilket förbättrar beräkningseffektiviteten samtidigt som hög noggrannhet och flexibilitet upprätthålls [2, 79, 91]. Dessutom är lokaliserade metoder lämpliga för parallella simuleringar [22, 96, 107].

Dock, om stora problem löses och hög noggrannhet krävs, måste RBF-metoderna stabiliseras för att ta bort beroendet på formparametern och minska konditionstalet hos koefficientmatrisen. Flera tekniker har utvecklats för detta ändamål, såsom Contour-Padé-metoden [45, 103, 105], RBF-QR-metoden [40,

42, 44, 69] och RBF-GA-metoden [43]. En nackdel med sådana tekniker är att stabiliseringen har en viss kostnad och om endast en måttligt noggrann approximation erfordras, kan användningen av en stabil metod vara omotiverad, eftersom tiden för att beräkna en stabil bas kan dominera lösningstiden. I Artikel VIII minskar vi istället kostnaden för den stabila basen med hjälp av en minstakvadratmetod som tillåter att samma bas används för varje partition i en enhetsuppdelningsmetod.

Förutom att utveckla RBF-PUM, studerar vi egenskaper och fördelar hos metoden i tillämpningar inom finansiell matematik och glaciologi. Problemen i finans defineras vanligtvis i domäner med enkel geometri, men kräver hög effektivitet för att snabbt kunna bestämma priserna hos finansiella kontrakt och kalibrera modeller. Därför är RBF-metoder väl lämpade för dessa applikationer eftersom de ger hög noggrannhet med relativt få diskretiseringspunkter. Den här egenskapen blir oerhört viktig när kontrakt på flera aktier eller multifaktormodeller prissätts eftersom formuleringar av dessa problem resulterar i högdimensionella PDE:er och metodens höga noggrannhet hjälper till att använda färre diskretiseringspunkter och därmed minska lagringsbehovet. RBF metoder tillämpades på problem i optionsprissättning i [39, 53, 85] och deras fördelar över både standard och state-of-the-art metoder demonstrerades i [91, 93]. Dessutom har RBF-metoder använts för att prissätta quanto kreditderivat [62].

I motsats till problemen inom finans defineras problemen i ismodellering och glaciologi på mycket komplexa domäner. Därför är flexibiliteten hos RBF-metoder av högsta relevans här. På grund av massiva storlekar är det svårt att få hög upplösning på kontinentala istäcken, en upplösning på 10–20 kilometer anses till exempel vara tillräcklig, men ännu viktigare, även med en sådan upplösning når problemstorlekarna några miljoner beräkningspunkter, vilket gör problemen mycket utmanande att lösa. RBF-metoder tillåter att minska antalet noder, samtidigt som en lika hög noggrannhet upprätthålls, vilket reducerar beräkningsansträngningen tack vare egenskaperna med hög order approximation. Det är också värt att nämna att RBF-metoder är formulerade på stark form, vilket gör dem mer lämpade för olinjära problem än finita elementmetoder (FEM), eftersom RBF metoder inte kräver fullständig matrisassemblering vid varje olinjär iteration. Det visades i [1] att kostnaden för matrisåteruppbyggnad i FEM kan dominera simuleringstiden. Således demonstrerades fördelen med RBF-approximation för att hitta hastighetsfältet hos en glaciär i [2] och senare utvecklades denna ytterligare för ett modellproblem med en inlandsis av kontinentalstorlek genom användning av anisotropa basfunktioner [24].

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First of all, I would like to thank my supervisor Elisabeth Larsson for providing guidance and knowledge whenever I needed it. Elisabeth, needless to say, you were the most influential person in my Ph.D. journey. I am sincerely grateful to you for these five years of productive collaboration. I also had the great luck to have Lina von Sydow as my second supervisor. Lina, you are the best manager I have ever met. Although we did not work much together, I always felt your support. You organised such a phenomenal project as BENCHOP and it was my pleasure to be a part of it.

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I am grateful to Andrey Itkin for organising my visit to New York University and to Alex Veygman for making me better understand the banking industry. Andrey and Alex, I was happy to work with you and I am honoured to have you as my friends. I would also like to acknowledge the Department of Finance and Risk Engineering of New York University and personally Peter Carr for hosting me at the department. This was a fantastic experience.

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