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Clustering and increased settling speed of oblate particles at finite Reynolds number

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We study the settling of rigid oblates in quiescent fluid using interface-resolved Direct Numerical Simulations. In particular, an immersed boundary method is used to account for the dispersed solid phase together with lubrication correction and collision models to account for short-range particle-particle interactions. We consider semi-dilute suspensions of oblate particles with aspect ratio \( AR = 1/3 \) and solid volume fractions \( \phi = 0.5\% - 10\% \). The solid-to-fluid density ratio \( R = 1.5 \) and the Galileo number (i.e. the ratio between buoyancy and viscous forces) based on the diameter of a sphere with equivalent volume \( Ga = 60 \). With this choice of parameters, an isolated oblate falls vertically with a steady wake with its broad side perpendicular to the gravity direction. At this \( Ga \), the mean settling speed of spheres is a decreasing function of the volume \( \phi \) and is always smaller than the terminal velocity of the isolated particle, \( V_t \). On the contrary, we show here that the mean settling speed of oblate particles increases with \( \phi \) in dilute conditions and is 33\% larger than \( V_t \). At higher concentrations, the mean settling speed decreases becoming smaller than the terminal velocity \( V_t \) between \( \phi = 5\% \) and 10\%. The increase of the mean settling speed is due to the formation of particle clusters that for \( \phi = 0.5\% - 1\% \) appear as columnar-like structures. From the pair-distribution function we observe that it is most probable to find particle-pairs almost vertically aligned. However, the pair-distribution function is non-negligible all around the reference particle indicating that there is a substantial amount of clustering at radial distances between 2 and 6c (with \( c \) the polar radius of the oblate). Above \( \phi = 5\% \), the hindrance becomes the dominant effect, and the mean settling speed decreases below \( V_t \). As the particle concentration increases, the mean particle orientation changes and the mean pitch angle (the angle between the particle axis of symmetry and gravity) increases from 23° to 47°.

Key words:

1. Introduction

There is a wide range of environmental processes and industrial applications that involve suspensions of particles settling under gravity. Among these we recall the pollutant transport in underground water, soot particle dispersion, fluidized beds and the settling of micro-organisms such as plankton, rain droplets and snow. Often, these applications involve a large number of particles settling in quiescent fluids and despite the large number of studies on the topic, the understanding of this complex

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phenomenon is still far from clear. Sedimentation depends indeed on a wide range of parameters. Particles may differ in density, shape, size and stiffness, and real suspensions are hardly monodispersed. In the present work, we focus on the effects due to particle shape. In particular, we consider suspensions of buoyant oblate particles of fixed aspect ratio \(AR = 1/3\), and show how particle orientation leads to different dynamics and microstructures in comparison to the ideal case of spherical particles.

If we limit our attention to the case of an isolated rigid sphere, it is known that the settling speed depends on the solid-to-fluid density ratio, \(R\), and the Galileo number \(Ga\), namely the ratio between buoyancy and viscous forces acting on the particle. Even in this two-parameter space a variety of particle path regimes are encountered, involving vertical, oblique, time-periodic oscillating, zig-zagging, helical and chaotic motion as shown numerically and experimentally by Jenny et al. (2004); Horowitz & Williamson (2010). The a-priori estimation of the particle terminal falling velocity, \(V_t\) is also non-trivial. With the assumption of Stokes flow in an unbounded quiescent fluid, it is found that the Stokes terminal velocity \(V_s\) is function of the sphere radius, \(a\), the density ratio, \(R\), the magnitude of the gravitational acceleration \(g\) and the kinematic viscosity of the fluid \(\nu\) (Guazzelli & Morris 2011). However, when the Reynolds number of the settling particle \((Re_t = V_t d/\nu)\) becomes finite, the fore-aft symmetry of the fluid flow around the particle is broken leading to the generation of a rear wake. As previously mentioned, the terminal speed depends on \(R\) and more pronouncedly on \(Ga\), but no theoretical formula that relates these quantities exists. Up to date, only formulae that make use of empirical relations for the drag coefficient of an isolated sphere, \(C_D\), are available (Schiller & Naumann 1935; Clift et al. 2005; Yin & Koch 2007). However, these formulae relate only the terminal Reynolds number \(Re_t\) to the Galileo number \(Ga\), neglecting the dependence on the density ratio \(R\).

When particle suspensions are considered, the scenario is further complicated by hydrodynamic and particle-particle interactions. A most relevant effect occur when a sphere is entrained in the wake of another particle of comparable size settling at finite \(Re_t\), as the particle behind will accelerate towards the leading particle. The particles will hence touch and finally the rear particle will tumble laterally. This phenomenon is denoted as drafting-kissing-tumbling of a particle-pair (Fortes et al. 1987); during the draft phase the rear particle reaches speeds larger than the terminal velocity \(V_t\). The extent of the increase of the rear particle speed with respect to \(V_t\) depends on \(Ga\).

Generally speaking, the mean settling speed of a suspension of particles, \(\langle V_z \rangle\), is also a function of the solid volume fraction \(\phi\). For very dilute suspensions under the assumption of Stokes flow, Hasimoto (1959) and later Sangani & Acrivos (1982) obtained expressions for the drag force exerted by the fluid on three different cubic arrays of settling spheres. A different expression was instead found by Batchelor (1972) who used a different approach based on conditional probability arguments. All these formulae relate the mean settling speed directly to the solid volume fraction \(\phi\) but are unable to properly predict \(\langle V_z \rangle\) for semi-dilute and dense suspensions. For such suspensions, the empirical formula proposed by Richardson & Zaki (1954) is probably the most used. This was obtained from experimental results in creeping flow conditions and relates the mean settling speed normalized by the Stokes terminal velocity to the solid volume fraction \(\phi\), via a power-law. More specifically, the mean settling speed of the suspension \(\langle V_z \rangle\) is a decreasing function of \(\phi\) and is always smaller than \(V_s\). This formula has been shown to be accurate also for concentrated suspensions and for low Reynolds numbers \(Re_t\). A wide number of more recent studies have been devoted to the improvement of this empirical formula to account for larger \(Re_t\). Among these we recall the experimental studies by Garside & Al-Dibouni (1977); Di Felice (1999) and the numerical study by Yin & Koch (2007). These authors
showed that the power-law exponent is a non-linear function of the Reynolds number $Re_t$, and that a correction coefficient should be introduced.

The mean settling speed $\langle V_z \rangle$ decreases with $\phi$ due to the hindrance effect (Climent & Maxey 2003; Guazzelli & Morris 2011). In a batch sedimentation system, the fixed bottom of the container forces the fluid to move in the opposite direction such that the flux of the particle-fluid mixture remains zero. Hindrance becomes more pronounced as the volume fraction $\phi$ increases, leading a monotonic decrease of $\langle V_z \rangle$ with respect to $V_t$. At large $Re_t$, however, the suspension behavior is further complicated by the particle-particle hydrodynamic interactions. In our previous work, we have studied semi-dilute suspensions ($\phi = 0.5\% - 1\%$) of spheres with density ratio $R = 1.02$ and $Ga = 145$ (Fornari et al. 2016a) and have found that drafting-kissing-tumbling events are indeed frequent, with the involved particles reaching speeds more than twice the mean $\langle V_z \rangle$. It was estimated that without these intermittent events the mean settling speed, $\langle V_z \rangle$, would be smaller by about 3%.

For suspensions of spheres with $\phi = 0.5\%, R = 1.5$ and a larger value $Ga = 178$, Uhlmann & Doychev (2014) found that particle clusters form. These clusters settle faster than $V_t$ and as a result, the mean settling speed $\langle V_z \rangle$ increases by 12% with respect to the terminal speed of an isolated particle, $V_t$. The formation of clusters is related to the steady oblique motion observed for isolated spheres with $R = 1.5$ and $Ga = 178$. Indeed, at a lower $Ga = 121$, for which an isolated sphere exhibits a steady vertical motion, no clustering is observed. An increased mean settling speed at large $Ga$ was also observed by Zaidi et al. (2014); Fornari et al. (2016b). Recently, these results were also confirmed experimentally by Huisman et al. (2016) who also observed the formation of a columnar structure of spheres at high $Ga$.

In the past few years, numerical investigations were also devoted to the study of the sedimentation of suspensions of finite-size spheres in stratified environments (Doostmohammadi & Ardekani 2015), in homogeneous isotropic (Chouippe & Uhlmann 2015; Fornari et al. 2016a,b) and shear turbulence (Tanaka & Teramoto 2015). The case of finite-size spherical bubbles rising in vertical turbulent channel flow has been considered by Santarelli & Fröhlich (2015, 2016).

When considering non-spherical particles, the sedimentation process is further complicated as the particle orientation plays a role in the dynamics. Feng et al. (1994) performed two-dimensional numerical simulations of settling elliptic particles to show that in stable conditions an elliptic particle always falls with its long axis perpendicular to gravity. Three-dimensional oblates settling in steady motion at low $Re_t$ also display the symmetry axis in the gravity direction. However, increasing $R$ or $Ga$ the system becomes unstable and disc-like particles are observed to oscillate horizontally. As explained by Magnaudet & Mougin (2007); Ern et al. (2012), the path instability of spheroidal particles is closely related to their wake instability. Indeed, the release of vortices in the wake of a spheroidal particle is modified as soon as the angle between the particle symmetry axis and the velocity direction is changed. Therefore, the ensuing wake instability is also strongly related to the particle aspect ratio $AR$. A complete parametric study on disc-shaped cylinders and oblates with different aspect and density ratios falling under gravity was performed by Chrust (2012).

Recently we extended the immersed boundary method (IBM) of Breugem (2012) to account for ellipsoidal particles (Ardekani et al. 2016). We have shown that above a threshold $Ga$, oblate particles perform a zigzagging motion whereas prolate particles rotate around the vertical axis with their broad side facing the falling direction. The threshold $Ga$ is shown to decrease as the aspect ratio departs from 1. Particle-pair interactions were also studied. It has been found that the drafting-kissing-tumbling is modified
with respect to the case of settling spheres. In particular, for two oblate particles with \( AR = 1/3 \) and \( Ga = 80 \), the tumbling part is suppressed and the particles fall together with a mean speed that is substantially larger than \( V_t \). Also, spheroidal particles are attracted in the wake of a leading particle from larger lateral distances than in the case of spheres. The absence of the tumbling phase was also found experimentally for pairs of falling disks with \( AR = 1/6 \) by Brosse & Ern (2011).

Much less is known about the sedimentation of suspensions of spheroidal particles. A pioneering study is that by Fonseca & Herrmann (2005), who studied numerically the settling of suspensions of oblate ellipsoids with \( AR \simeq 0.27 \) and \( R = 4 \) at the relatively low Reynolds numbers, \( Re_t = 0.04 \) and 7. Volume fractions up to \( \phi \sim 0.2 \) were considered. At the smallest \( Re_t \), these authors found a local maxima of the mean settling speed \( \langle V_z \rangle \) at \( \phi = 0.05 \), that is however smaller than the settling speed of an isolated oblate, \( V_t \). On the other hand, at \( Re_t = 7 \) it was found that \( \langle V_z \rangle \simeq 1.1V_t \) for \( \phi \sim 1\% \).

In the present study, we investigate the sedimentation of semi-dilute suspensions of oblate particles at finite Reynolds number \( Re_t \). In particular, we consider particles with aspect ratio \( AR = 1/3 \), density ratio \( R = 1.5 \) and Galileo number (based on the diameter of a sphere with an equivalent volume) \( Ga = 60 \). With this choice of parameters, a single oblate falls steadily with its broad side perpendicular to gravity and with a terminal Reynolds number of approximately 40. Four solid volume fractions of \( \phi = 0.5\%, 1\%, 5\% \) and 10\% are studied. We find that differently from spheres of equal \( Ga \), the mean settling speed of the suspension, \( \langle V_z \rangle \), first increases with \( \phi \), and is therefore larger than the terminal velocity of a single particle, \( V_t \). The mean settling speed decreases for \( \phi > 0.5\% \) and becomes smaller than \( V_t \) between \( \phi = 5\% \) and 10\%. In this range of \( \phi \), a power-law fit similar to that by Richardson & Zaki (1954) is proposed. We then show that the enhancement of \( \langle V_z \rangle/V_t \) at low \( \phi \) is related to the formation of a columnar structure of particles. Within this structure, intense particle clustering is observed. For \( \phi = 0.5\%−1\% \), the particle pair-distribution function is found to be high in the range \( r \in [2c, 6c] \) and between \( \psi \simeq 2^\circ−8^\circ \), with maximum values at \( r = 2.02c \) and \( \psi = 17^\circ−10^\circ \) (with \( c \) the polar radius of the oblate and \( \psi \) the polar angle with respect to the direction of gravity). Hence, particles are almost vertically piled-up at low \( \phi \) as also shown by the order parameter. At higher \( \phi \) the amount of clustering is reduced. We also show that the mean particle orientation (computed as the cosine of the angle between the particle symmetry axis and gravity) decreases with \( \phi \). A power-law fit in terms of \( \phi \) is also proposed. The particle mean pitch angle with respect to the horizontal plane increases with \( \phi \), from \( 22.8^\circ (\phi = 0.5\%) \) to \( 47^\circ (\phi = 10\%) \). It should be noted that for an isolated oblate with \( Ga = 60 \) and \( R = 1.5 \) the pitch angle is \( 0^\circ \). Finally, we calculate joint probability functions of settling speeds \( V_z \) and orientation \( |O_z| \). By means of conditioned averages we show that particles settling with larger speeds than the mean, \( \langle V_z \rangle/V_t \), settle on average with higher pitch angles.

2. Set-up and Methodology

The sedimentation of semi-dilute suspensions of oblate particles is considered in a computational domain with periodic boundary conditions in the \( x \), \( y \) and \( z \) directions for both the fluid and the particles, with gravity acting in the positive \( z \) direction. Oblates with aspect ratio \( AR = 1/3 \) are considered. We name \( b \) and \( c \) the equatorial and polar radii of the ellipsoid. The computational box has size \( 20d \times 20d \times 160d \), being \( d \) the diameter of a sphere with the same volume as the ellipsoidal particle. Four solid volume fractions are investigated, \( \phi = 0.5\%, 1\%, 5\% \) and 10\%. These correspond to 611, 1222, 6111 and 12222 particles. Oblates are initially randomly distributed in the computational
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Figure 1. Definition of the equatorial and polar radii, \( b \) and \( c \), and of the pitch angle \( |\zeta| \).

domain with zero angular and translational velocity, and with their orientation vector \([O_x, O_y, O_z] = [0, 0, 1]\). Hence, their broad side is perpendicular to gravity and the pitch angle (defined between their symmetry axis and gravity) is 0°. Note that this angle is equal to the angle between the plane defined by the equatorial radius \( b \) and the (horizontal) \( xy \) plane, as shown in figure 1. Therefore we name it as pitch angle (i.e. when the pitch angle is larger than zero, the spheroid is inclined with respect to the horizontal plane). For comparisons, the case of an isolated oblate is also simulated as reference.

We consider non-Brownian rigid oblate particles slightly heavier than the suspending fluid with density ratio \( R = 1.5 \) and Galileo number (based on the diameter \( d \) of the equivalent sphere)

\[
Ga = \sqrt{(R - 1) g d^3 \nu} = 60
\]  

As previously mentioned, this non-dimensional number quantifies the importance of the gravitational forces acting on the particle with respect to viscous forces. At this \( Ga \) isolated spheres and oblates settle vertically with steady wakes. The Reynolds number based on the terminal falling speed of a single oblate is found to be \( Re_t = 38.7 \).

The simulations have been performed using the version of the immersed boundary method developed by Breugem (2012) and modified by Ardekani et al. (2016) to account for ellipsoidal particles. With this approach, the coupling between the fluid and solid phases is fully modelled. The flow is evolved according to the incompressible Navier-Stokes equations, whereas the particle motion is governed by the Newton-Euler Lagrangian equations for the particle centroid linear and angular velocities

\[
\rho_p V_p \frac{d\mathbf{u}_p}{dt} = \oint_{\partial V_p} \mathbf{\tau} \cdot \mathbf{n} dS + (\rho_p - \rho_f) V_p \mathbf{g} 
\]

\[
\frac{dI_p \omega_p}{dt} = \oint_{\partial V_p} \mathbf{r} \times \mathbf{\tau} \cdot \mathbf{n} dS
\]

where \( \rho_p, V_p \) and \( I_p \) are the particle density, volume and moment of inertia; \( \mathbf{g} \) is the gravitational acceleration; \( \mathbf{\tau} = -p \mathbf{I} + 2\mu \mathbf{E} \) is the fluid stress, with \( \mathbf{E} = \left( \nabla \mathbf{u}_f + \nabla \mathbf{u}_f^T \right) / 2 \) the deformation tensor; \( \mathbf{r} \) is the distance vector from the center of the particle while \( \mathbf{n} \) is the unity vector normal to the particle surface \( \partial V_p \). Dirichlet boundary conditions for the fluid phase are enforced on the particle surfaces as \( \mathbf{u}_f |_{\partial V_p} = \mathbf{u}_p + \omega_p \times \mathbf{r} \).

Using the immersed boundary method, the boundary condition at the moving fluid/solid interfaces is indirectly imposed by an additional force on the right-hand side of the
Navier-Stokes equations. It is hence possible to discretize the computational domain with a fixed staggered mesh on which the fluid phase is evolved using a second-order finite-difference scheme together with a set of Lagrangian points, uniformly distributed on the surface of the particle to represent the interface. Time integration is performed by a third-order Runge-Kutta scheme combined with pressure correction at each sub-step. When the distance between two particles becomes smaller than twice the mesh size, a lubrication model is used to correctly reproduce the interaction between the particles. In particular, the closest points on the surfaces of two ellipsoids are found. From these, the Gaussian radii of curvature are calculated, and these correspond to the radii of the best fitting spheres tangent to the given surface points. The lubrication model based on Jeffrey (1982) asymptotic solution for spheres of different size is then employed. Additionally, the soft-sphere model is used to account for normal and tangential collisions between the ellipsoids (Costa et al. 2015). As for lubrication, collision forces are calculated for the best fitting spheres at the points of contact and are later transferred to the spheroids centres. More details and validations for the specific immersed boundary method used for ellipsoids can be found in Ardekani et al. (2016). Other validations specific to the immersed boundary method for spherical particles are found in Breugem (2012); Lambert et al. (2013); Picano et al. (2015); Fornari et al. (2016a).

A cubic mesh with approximately eight points per particle polar radius (\(c \sim 24\) points per equatorial radius \(b\)) is used for the results presented, which corresponds to \(640 \times 640 \times 5120\) grid points in the computational domain and 3220 Lagrangian points on the surface of each particle. Note finally that zero total volume flux is imposed in the simulations.

For \(\phi = 0.5\% - 1\%\), simulations were run for 182 particle relaxation times defined using the equivalent diameter \(d = \tau_p/Rd^2/(18\nu)\). Defining as reference time the time it takes for an isolated oblate to fall over a distance equal to its polar radius, \(c/V_t\), the simulation time corresponds to \(2430\ c/V_t\). For denser cases, the statistically steady-state condition is reached earlier and simulations were run for \(72\ \tau_p = 962\ c/V_t\ (\phi = 5\%)\) and \(42\ \tau_p = 561\ c/V_t\ (\phi = 10\%)\). Statistics are collected after \(90\ \tau_p = 1202\ c/V_t\) for \(\phi = 0.5\% - 1\%\), \(24\ \tau_p = 320\ c/V_t\) for \(\phi = 5\%\), and \(6\ \tau_p = 80\ c/V_t\) for \(\phi = 10\%\).

### 3. Results

#### 3.1. Settling speed and suspension microstructure

The most striking result of our study is that semi-dilute suspensions of oblate particles with \(Ga \gg 1\), settle on average substantially faster than isolated particles, about 33% faster at \(\phi = 0.5\%\). As the Galileo number \(Ga\) increases, the effects of particle inertia and hydrodynamic interactions due to particle wakes become progressively more important overcoming the hindrance effect described above (Yin & Koch 2007; Guazzelli & Morris 2011). For spherical particles at moderate \(Ga\), the hindrance leads to a power-law decay of the mean settling speed \(\langle V_z\rangle\) with the volume fraction \(\phi\). Hence, the mean settling speed \(V_z\) is smaller than the terminal falling speed \(V_t\) of an isolated sphere for all volume fractions \(\phi\). At low and moderate terminal Reynolds numbers, \(Re_t = V_t d/\nu \leq 20\), the hindrance effect is well-described by the modified Richardson & Zaki empirical formula (Richardson & Zaki 1954)

\[
\frac{\langle V_z\rangle}{V_t} = \kappa (1 - \phi)^n
\]

(3.1)

where \(n\) is an exponent that depends on \(Re_t\) (Garside & Al-Dibouni 1977)

\[
\frac{5.1 - n}{n - 2.7} = 0.1 Re_t^{0.9}
\]

(3.2)
Figure 2. Mean settling speed $\langle V_z \rangle / V_t$ as function of the solid volume fraction $\phi$. The dash-dotted line is a fit of $\langle V_z \rangle / V_t$ in the range $\phi \in [0.5; 10\%]$. The empirical fit proposed by Richardson and Zaki for settling spheres is also shown (dashed line). In the inset, the mean settling speed is normalized by the settling speed of an isolated spherical particle of same Galileo number $Ga$, based on the equivalent diameter. 

while $\kappa$ is a correction coefficient for finite $Re_t$ that has been found to be in the range $0.8 - 0.92$ (Di Felice 1999; Yin & Koch 2007). Note that we have checked that we can get consistent values of $\kappa$ with our code. In particular, we have also performed simulations of suspensions of spheres with $Ga \sim 9$ settling under gravity, and found that $\kappa \sim 0.91$.

From the simulation of the isolated settling oblate we find that the terminal Reynolds number is $Re_t = V t d_{eq} / \nu = 38.7$ (being $d_{eq}$ the equivalent diameter of a sphere with the same volume). At these $R$, $Ga$ and $Re_t$ the corresponding wake behind both spheres and oblates is steady and vertical (Bouchet et al. 2006; Ardekani et al. 2016). For the case of oblique wakes past isolated spheres, Uhlmann & Doychev (2014) have shown an increase of the mean settling speed (these authors considered spheres with $R = 1.5$ and $Ga = 178$). The mean settling speed of the suspension ($\phi = 0.5\%$) increases above $V_t$ by about 12%. This increase of $\langle V_z \rangle$ is due to the formation of particle clusters.

The results for the mean settling speed of the oblate suspension, $\langle V_z \rangle$, normalized by $V_t$, are shown in figure 2. The expected velocity predicted via the empirical fit (3.1) using $\kappa = 1$ and $n = 4.17$ (obtained from equation 3.2 and $Re_t = 38.7$) is also shown for comparison. In contrast to what expected for spheres, we find that the mean settling speed $\langle V_z \rangle$ is larger than $V_t$ for volume fractions approximatively lower than 7%. For $\phi = 0.5\%$ and 1%, $\langle V_z \rangle \sim 1.3 V_t$, and for $\phi = 5\%$, $\langle V_z \rangle = 1.08 V_t$. Hence, as $\phi$ increases there is an initial increase of $\langle V_z \rangle / V_t$. However, the hindrance effect becomes progressively stronger above $\phi = 0.5\%$, reducing $\langle V_z \rangle / V_t$; this observable becomes lower than 1 for the largest volume fraction considered here. We fitted our data in the range $\phi \in [0.5\% - 10\%]$ using equation (3.1) to find $\langle V_z \rangle / V_t = 1.353(1 - \phi)^{4.227}$. Clearly, this relation is valid only for this range of volume fractions and possibly for larger $\phi$. We leave as future work the study of more dilute cases to understand how $\langle V_z \rangle / V_t$ initially increases with $\phi$.

In the inset of figure 2 we report the same data normalized by the terminal velocity of an isolated spherical particle with $Ga = 60$. We see that oblates settle at a substantially slower rate than spheres for all $\phi$. For $\phi = 0.5$, $1\%$, $\langle V_z \rangle \sim 0.36 V_{t,sphere}$. 

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Figure 3. Instantaneous snapshots of oblate particles falling under gravity: a) front view; b) top view; c) zoom on a cluster of 3 piled up particles.

To understand the enhancement of $\langle V_z \rangle / V_t$ at moderate $\phi$, we first show in figure 3a) an instantaneous snapshot of the settling suspension at $\phi = 0.5\%$. It can be seen that most particles are located on the right half of the snapshot (i.e. for $x \geq 0.5L_x$) where they seem to form a columnar structure. As we will soon show, particles within this columnar structure fall on average faster than the more isolated particles and than the
Clustering and increased settling speed of oblate particles

Figure 4. (a) Probability density function, *p.d.f.*, of the settling speed for all *φ*. (b) Probability density functions of the settling speed calculated for particles contained in the volumes defined by *x_c* ∈ [0; 0.5] *L* (dashed line) and *x_c* ∈ [0.5; 1] *L* (solid line), for *φ* = 0.5%.

whole suspension. This peculiar particle distribution can also be observed in figure 3b), displaying a the top view of the same instantaneous configuration. These observations confirm the importance of particle-particle hydrodynamic interactions at finite inertia. Note that the smaller region delimited by a box in figure 3a) highlights particles that fall approximately 3 – 3.5 times faster than an isolated particle.

As mentioned above, Ardekani et al. (2016) studied the settling of two isolated oblate particles at a similar *Ga*. While two spheres would undergo the so-called *drafting-kissing-tumbling* phenomenon (Fortes et al. 1987), Ardekani et al. (2016) showed that the rear oblate accelerates in the wake of the front particle until it approaches it and almost perfectly sticks to it. The tumbling stage is therefore suppressed and the particles fall in contact. The particle-pair falls with a speed that is 1.5 times the terminal velocity of an isolated oblate. In the same study, it was also found that the maximum radius of the collision (or entrainment area) for oblates of *AR* = 1/3 is approximately 4 times larger than that of spheres of equal *Ga* for several different vertical separations. These results obtained for particle-pairs are reflected in and determine the suspension behavior. From the close-up in figure 3c), we note that in a suspension with *φ* = 0.5% more than 3 particles can pile up during drafting-kissing-tumbling events. These particle clusters generate strong wakes that, in turn, lead to the formation of the columnar-like structure. This columnar structure is observed also for larger *φ*. Note that the (*x, y*) location of the structure is purely random. Similar columnar structures were observed also for spherical particles by Uhlmann & Doychev (2014); Huisman et al. (2016), although for much larger *Ga* = 178, while these were not observed for spheres at *Ga* = 120.

Next, we display in figure 4a) the probability density function, *p.d.f.*, of the settling speed *V_z/VT* for all *φ* under investigation. The moments of the *p.d.f.s* are reported in table 1. For *φ* = 0.5% and 1% the distributions are similar and (positively) skewed towards larger speeds than the mean value. As *φ* increases, the skewness of the *p.d.f.s* (*S_{V_z}*) decreases becoming negligible for the denser case, indicating that the dynamics is mostly governed by excluded volume effects, rather than by pair-interactions and clustering formation. On the contrary, the standard deviation of the *p.d.f.s* increases with *φ* up to *φ* = 5%. A slightly smaller *σ_{V_z}*, is found instead for *φ* = 10%. The flatness, *F_{V_z}*, is always around 3. It is interesting to observe that as *φ* increases, the *p.d.f.s* tend progressively towards a normal distribution indicating that the settling dynamics,
Initially governed mostly by particle interaction through wakes, becomes progressively dominated by the hindrance effect.

As can be seen from the p.d.f.s, the probability of having particles rising increases with $\phi$. This is due to the imposition of the zero total volume flux condition. This ensures that $\langle W_z \rangle = (1 - \phi) \langle U_z \rangle + \phi \langle V_z \rangle = 0$, where $W_z$ and $U_z$ are the bulk and fluid velocities (Guazzelli & Morris 2011). As said, particle clusters settle substantially faster than the whole suspension and to satisfy the condition $\langle W_z \rangle = 0$, strong upward local fluid streams are generated in their surroundings. When these updrafts encounter slowly settling particles, they drag them in the opposite direction with respect to gravity. Notice that we have observed this effect also for dense suspensions of spheres with $Ga \sim 9$, for which we found results in agreement with the corrected Richardson-Zaki fit and with Yin & Koch (2007).

In figure 4b) we show the p.d.f.s of settling speeds for particles whose centers are located within $x \in [0; 0.5) L_x$ or within $x \in [0.5; 1] L_x$ (i.e. the computational domain is divided in two parts denoted as left and right). Particles located within $x \in [0.5; 1] L_x$ (i.e. where the columnar structure is found, right side) settle with a mean velocity larger than that of the suspension ($\langle V_{r,z} \rangle / V_t = 1.45$). A smaller mean settling speed is found instead in the left half ($\langle V_{l,z} \rangle / V_t = 1.1$). Concerning the distribution standard deviation, this is also slightly larger in the right half, $x_c \in [0.5; 1] L_x$ ($\sigma_r = 0.48$ and $\sigma_l = 0.45$). On the other hand, it is interesting to note that the skewness is larger for the slower particles, located in the region $x_c \in [0; 0.5) L_x$ ($S_l = 0.75$ and $S_r = 0.44$). This is because on the left half, there are less particles that less frequently undergo intense drafting-kissing-tumbling interactions. These interactions lead to the large skewness, while the mean value is similar to $V_t$ being the particles more isolated. Since the drafting-kissing-tumbling events are more intermittent in the left half, also the flatness is larger than for the velocities of the particles forming the fast falling column ($F_t = 4$ versus $F_r = 3$).

We now turn to the discussion of the microstructure of the whole suspension. To this aim we calculate the pair-distribution function $P(r)$, the conditional probability of finding a particle at $r$, given one at the origin. Following Kulkarni & Morris (2008), this is defined as

$$P(r) = P(r, \theta, \psi) = \frac{H(r, \theta, \psi)}{n \Delta V}$$

where $\theta$ is the polar angle (measured from the positive $x$ axis), $\psi$ is the azimuthal angle (measured from the positive $z$ axis), $n$ is the average particle number density, $\Delta V$ is the total number of sampling points, $\Delta V = r^2 \Delta \sin \psi \Delta \psi \Delta \theta$ is the volume of the sampling bin, and $H(r, \theta, \psi)$ is the histogram of particle-pairs. More specifically, the pair space is discretized in $(r, \theta, \psi)$ and at each sampling time we obtain $N_p (N_p - 1) / 2$ pair separation vectors $r$ from the simulated particle configurations. Each vector $r$ is put

---

<table>
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<tr>
<th>$\phi$ (%)</th>
<th>$\langle V_z \rangle$</th>
<th>$\sigma_{V_z}$</th>
<th>$S_{V_z}$</th>
<th>$F_{V_z}$</th>
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<td>3.06</td>
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</tbody>
</table>

Table 1. Moments of the p.d.f.s of settling speed for all $\phi$: mean value, $\langle V_z \rangle$; standard deviation, $\sigma_{V_z}$; skewness, $S_{V_z}$; flatness, $F_{V_z}$. The first two value are normalized by the terminal velocity $V_t$. 

---
Figure 5. Pair-distribution function $P(r, \theta)$ for all $\phi$ under investigation with $\phi$ increasing from left to right and top to bottom. The insets in panels (c) and (d), corresponding to $\phi = 5\%$ and $10\%$, show $P(r, \theta)$ with the same contour levels as in (a) and (b). Panel (e) shows the radial distribution function $g(r)$ for all $\phi$, while panel (f) shows the order parameter $\langle P_2 \rangle(r)$.

For our cases, $P(r)$ is axisymmetric about the direction of gravity. Therefore we first report $P(r)$ as function of the center-to-center distance $r$ (normalized by $c$), and the azimuthal angle $\psi$, averaging over $\theta$. The pair-distribution function $P(r)$ is shown in figure 5a),b),c),d) for the four volume fractions under investigation. For simplicity, we denote the axis at $\psi = 90^\circ$ as $x/c$. The results are substantially different from what observed for spheres at comparable and smaller $Ga$ (Yin & Koch 2007; Uhlmann & Doychev 2014) for which no evident clustering is observed. For $\phi = 0.5\%$ we see that the
pair-distribution function \( P(r) \) is large all around the reference particle. On average, each particle is surrounded by other particles, with a higher probability in the region between \( \theta \sim 5^\circ \) and \( \theta \sim 80^\circ \). From this figure it is also clear that particles preferentially cluster on top of each other, with their broad sides almost perpendicular to the vertical direction, and with an inclination that increases with \( x/c \), see also figure 3c). The maximum of \( P(r) \) is located at \( r = 2.02c \) and \( \psi \sim 17^\circ \). Notice that horizontal clusters of sticking particles falling with their broad sides perfectly parallel to gravity do not occur. Indeed we see that \( P(r) = 0 \) for \( \psi = 90^\circ \), below \( x \sim 4c \). For \( \phi = 1\% \) the results are similar although \( P(r) \) is lower over the entire \( (r, \theta, \psi) \) space. The probability of finding a second particle is again large all around the reference particle, with the highest values between \( \theta \sim 2^\circ \) and \( \theta \sim 70^\circ \). Particles are hence more vertically aligned within clusters, with a maximum \( P(r) \) located at \( r = 2.02c \) and \( \psi \sim 10^\circ \) (i.e. with the symmetry axis almost parallel to gravity). The maximum value is just slightly smaller than that found for \( \phi = 0.5\% \). The region of high \( P(r) \) is found to translate towards \( \psi = 0^\circ \).

For \( \phi = 5\% \) and \( 10\% \), \( P(r) \) decreases substantially for all values of \( (r, \theta, \psi) \). The maximum of \( P(r) \) is now found between \( \psi = 0^\circ \) and \( 2^\circ \). Note also that for these cases \( P(r) \sim 1 \) for \( r \gtrsim 6c \), indicating that the random (Poissonian) distribution of particles (i.e. an uncorrelated statistical particle distribution) is already reached above this radial distance. Conversely, for the cases with \( \phi = 0.5\% \) and \( 1\% \) we see that \( P(r) \sim 1 \) only in a small region between \([5; 9]c \times [30^\circ; 70^\circ]\). The correlation of the particle distribution at large \( r/c \) for low volume fractions is indicative of the presence of the above-mentioned columnar structures.

A measure of the suspension microstructure that can be more easily quantified in a plot is the average of \( P(r, \theta, \psi) \) over all possible orientations. This is known as the radial distribution function \( g(r) \) and it is shown in figure 5e) for all volume fractions. We see that by averaging over all \( \theta \) and \( \psi \), the maximum of the radial distribution function appears at a radial distance of approximately \( 4c \). The peak is almost halved as \( \phi \) is doubled from \( 0.5\% \) to \( 1\% \). For these cases, the decorrelation of the particle distribution, \( g(r) \sim 1 \), occurs for \( r > 20c \). This is also approximately the radius of the columnar structure identified in figure 3. The extent of clustering is sharply reduced for \( \phi \gtrsim 5\% \). Indeed, the maxima of \( P(r) \) are between \( 2 \) and \( 1.5 \), and the uncorrelated value (i.e. 1) is quickly reached, \( r/c \sim 6 \).

Finally, we consider in figure 5f) the order parameter. This is used to quantify the preferential orientation of particle pairs and it is a function of the radial separation \( r \) (Yin & Koch 2007). This is defined as the angular average of the second Legendre polynomial

\[
\langle P_2(r) \rangle = \frac{\int_0^\pi P(r, \psi)P_2(\cos \psi) \sin \psi d\psi}{\int_0^\pi P(r, \psi) \sin \psi d\psi}
\]

(3.4)

where \( P_2(\cos \psi) = (3\cos^2 \psi - 1)/2 \). The order parameter \( \langle P_2 \rangle \) is 1 for vertically aligned pairs at a separation \( r \), \( -1/2 \) for horizontally aligned pairs and 0 for isotropic configurations. We see that for \( r \sim 2c \), the order parameter \( \langle P_2 \rangle \) is between 0.6 and 0.8 for all \( \phi \). This confirms the observation that when the separation distance \( r \) is of the order of \( 2c \), most particles are almost perfectly piled up. Above \( r \sim 3.5c \), \( \langle P_2 \rangle \) becomes negative for \( \phi = 0.5\% \) and \( 1\% \), with a minimum value of approximately \(-0.2 \sim 0.15\) around \( r \sim 4.5c \). Hence, at these radial distance particle pairs tend to be more horizontally aligned, with a finite inclination or pitch angle between their axis of symmetry and the direction of gravity plane (as we will show later). For the higher volume fractions, \( \langle P_2 \rangle \) is only slightly negative, confirming the disappearance of strong clustering in these
cases. The configuration becomes more isotropic after \( r = 8c \) for the more dilute cases, and after \( r = 6c \) for the denser cases.

3.2. Particle dynamics

In the previous section we have analyzed the p.d.f.s of particle settling speeds for all \( \phi \). We now study the horizontal component of the translational velocity (in the direction perpendicular to gravity), as well as the rotational velocities.

The probability density function of the horizontal component of the particle velocity in the plane perpendicular to gravity is reported in figure 6a). For the sake of simplicity, we define this component of the velocity as \( V_z \) (note that at this stage of the simulation, most particles are found within the columnar structure and hence, the horizontal velocity is symmetric around the direction of gravity). The p.d.f.s of \( V_z/V_l \) are similar to normal distributions centered around \( \langle V_z \rangle \sim 0 \), with skewness \( S \sim 0 \) and flatness \( F \) slightly larger than 3, for all \( \phi \). On the other hand, the standard deviation \( \sigma_{V_z} \) increases until \( \phi = 5\% \). For the more dilute cases, the fact that both \( \langle V_z \rangle \sim 0 \) and \( S \sim 0 \) indicate that there is a constant inflow/outflow of particles to/from the columnar structure.

The standard deviation \( \sigma_{V_z} \) as function of \( \phi \) is shown in figure 6b), together with \( \sigma_{V_x} \). It is interesting that both \( \sigma_{V_x} \) and \( \sigma_{V_z} \) decrease after \( \phi = 5\% \). This is probably an excluded volume effect as at this high \( \phi \), particles are more uniformly distributed and settle as a bulk. Next, we display the ratio \( \sigma_{V_z}/\sigma_{V_x} \) (i.e. the anisotropy of the velocity fluctuations), see figure 6c). Considering that for 1 particle \( \sigma_{V_x}/\sigma_{V_z} = 0 \), we see a sharp increase of the ratio \( \sigma_{V_z}/\sigma_{V_x} \) up to 1\%. Above \( \phi = 1\% \), the dynamics is controlled by excluded volume and hindrance effects, and the increase of the anisotropy with \( \phi \) becomes approximately linear.

Finally, we examine the particle angular velocities around the directions parallel (\( z \)) and perpendicular (\( x \)) to gravity, see the p.d.f.s of in figures 6d) and e). The angular velocities are normalized by \( V_l/c \). First of all, we observe that both p.d.f.s are centred around \( \langle \omega_z \rangle = 0 \) and \( \langle \omega_x \rangle = 0 \). Concerning the p.d.f.s of \( \omega_z \) we see that the standard deviation \( \sigma_{\omega_z} \) increases substantially with \( \phi \) (for \( \phi = 10\% \) \( \sigma_{\omega_z} \) is almost 3 times that found for \( \phi = 0.5\% \)), the skewness \( S \) is \( \sim 0 \), while the flatness \( F \) is larger than 3 and decreases from 10 (\( \phi = 0.5\% \)) to 5.8 (\( \phi = 10\% \)). In the \( x \) direction, the standard deviation \( \sigma_{\omega_x} \) also increases substantially with \( \phi \) (\( \sigma_{\omega_x}(\phi = 0.5\%) \) is 42\% of \( \sigma_{\omega_x}(\phi = 10\%) \)), \( S \) is again approximately 0 and the flatness \( F \) decreases from 16 to 6. The comparison between \( \sigma_{\omega_z} \) and \( \sigma_{\omega_x} \) is shown in figure 6f). Differently from the translational velocities, we observe that the fluctuations of angular velocities are larger in the direction perpendicular to gravity and that these increase more rapidly with the volume fraction \( \phi \). We see indeed in figure 3 that instantaneously many particles are inclined with respect to horizontal planes. These particles may be undergoing rotations around axes perpendicular to \( g \), while settling with an average pitch angle.

An isolate oblate with \( Ga = 60 \) falls with its broad side perpendicular to the direction of gravity. The orientation vector, defined by the direction of the particle symmetry axis, is hence \( [O_x, O_y, O_z]^T = [0, 0, 1] \). This means that the pitch angle between the axis of symmetry and gravity is 0°. Due to hydrodynamic and particle–particle interactions in suspensions the mean particle orientation changes. The p.d.f. of \( |O_z| \) is shown for all \( \phi \) in figure 7a). We first observe that the probability of having particles with \( |O_z| \simeq 0 \) (i.e. with the symmetry axis aligned with gravity) increases significantly with \( \phi \). For \( \phi = 10\% \) the probability of having \( |O_z| \simeq 0 \) is 1 order of magnitude larger than for \( \phi = 0.5\% \). On the contrary, the mean value of \( |O_z| \) decreases with \( \phi \). The mean values \( \langle |O_z| \rangle \) are shown in figure 7b). For \( \phi = 0.5\% \), \( \langle |O_z| \rangle = 0.922 \): on average particles are inclined by 22.8° with respect to the horizontal plane. Increasing the volume fraction we find \( \langle |O_z| \rangle = 0.895 \).
Figure 6. (a) Particle speed in the direction perpendicular to gravity $V_x/V_t$ for all $\phi$. (b) Standard deviation of the particle velocities parallel, $\sigma_{V_z}$, and perpendicular to gravity, $\sigma_{V_x}$ as function of $\phi$. (c) Anisotropy of the velocity fluctuations, $\sigma_{V_x}/\sigma_{V_z}$ for all cases. (d) and (e), probability density functions, p.d.f. of the particle angular velocities in the directions parallel, $\omega_z$, and perpendicular to gravity, $\omega_x$. These are normalized by $V_t/c$ ($c$ is the polar radius of the oblates). (f) Standard deviation of the particle angular velocities in the directions parallel and perpendicular to gravity.

(corresponding to an angle of $26.4^\circ$, $0.765$ ($40^\circ$) and $0.679$ ($47^\circ$) for $\phi = 1\%$, $5\%$, $10\%$. The increase of the mean pitch angle with $\phi$ is an interesting effect and indicates that particles change their orientational configuration to better sample the available volume. An instantaneous snapshot of the settling particles for the case with $\phi = 10\%$ is shown in figure 8. We see indeed that particles exhibit all possible orientations between $|O_z| = 0$
Clustering and increased settling speed of oblate particles

Figure 7. (a) Probability density function of the absolute value of the particle orientation as it falls under gravity, \(|O_z|\). (b) Mean value of the orientation \(\langle|O_z|\rangle\) as function of \(\phi\). The solid line represents an attempt to fit the data with an exponent that is itself a function of the volume fraction \(\phi\).

Figure 8. Instantaneous snapshot of the suspension with \(\phi = 10\%\). For the sake of clarity, 25\% of the particles are shown in the first quarter of the computational domain.

and 1. Note also that some particles clusters can still be observed regardless of the high volume fraction.

Since the change in \(|O_z|\) is an excluded volume effect, we believe that it should be described by a function that depends directly on the volume fraction \(\phi\) and on the remaining parameters, \(Ga, R\) and \(AR\), only via some coefficients. Therefore we also report
in figure 7b) the function

\[ f(\phi) = (1 - \phi)^{\phi/(1-\phi)} \]

that is shown to fit our data sufficiently well. This observation can have implications for the modelling of settling suspensions.

It must be noted that if a single oblate is constrained to fall with a finite pitch angle \( (O_z < 1) \), it will reach a terminal velocity larger than that for \( O_z = 1 \). We hence decided to perform an additional simulation of an isolated oblate settling with the mean pitch of the \( \phi = 0.5\% \) case \( (O_z = 0.921 \text{ or } 22.8^\circ) \) to see how the terminal velocity \( V_t \) of an inclined particle compares to \( \langle V_z \rangle \). We find that \( V_z/V_t = 0.41 \) and \( V_z/V_t = 1.03 \). The increase of the falling speed is limited, significantly lower than that of the suspension, showing again the importance of particle-pair interactions. The drift speed is instead large and about 1.6\( V_z \) \( (\phi = 0.5\%) \). We therefore believe that the drift speed of inclined particles within the suspension plays a role in the formation of the columnar structure.

To conclude this section we report in figures 9a),b),c),d) the joint probability density functions \( J \) of particle settling speed \( V_z/V_t \) and orientation \( |O_z| \) for \( \phi = 0.5\%, 1\%, 5\% \) and 10\%. We also show the mean values of \( V_z/V_t \) and \( |O_z| \) that we found for the suspension, indicated by dashed lines. The solid blue and red lines in the plots represent the mean \( V_z/V_t \) and \( |O_z| \) obtained by conditioned averages of \( J(V_z/V_t, |O_z|) \)

\[ \langle V_z/V_t \mid |O_z| \rangle = \int_{-\infty}^{\infty} V_z J(V_z/V_t \mid |O_z|) \, dV_z, \quad (3.6) \]

\[ \langle |O_z| \mid V_z/V_t \rangle = \int_{0}^{1} |O_z| J(|O_z| \mid V_z/V_t) \, d|O_z|. \quad (3.7) \]

For \( \phi = 0.5\% \) and 1\%, we find that \( \langle V_z/V_t \mid |O_z| \rangle \) increases from about 1 at \( |O_z| = 1 \) to an almost asymptotic value of 1.5. Hence, particles settling with an inclination \( \geq \cos^{-1}(\langle |O_z| \rangle) \) fall on average with speeds that are 12\% larger than \( \langle V_z \rangle \). In particular, we find \( \langle V_z/V_t \mid |O_z| = \langle |O_z| \rangle \rangle = 1.49 \) and 1.42 for \( \phi = 0.5\%, 1\% \) (red circles). Concerning \( \langle |O_z| \mid V_z/V_t \rangle \) we see that it decreases almost linearly with \( V_z/V_t \). Particles with larger \( V_z/V_t \) fall on average with smaller \( \langle |O_z| \rangle \) (i.e. more inclined with respect to the horizontal plane). The values found at speeds equal to \( \langle V_z/V_t \rangle \) are 1\% larger than \( \langle |O_z| \rangle \) (green squares).

Similar observations apply also to the remaining cases, those at higher \( \phi \). As expected, for all \( |O_z| \), \( \langle V_z/V_t \mid |O_z| \rangle \) decreases with \( \phi \). Specifically, \( \langle V_z/V_t \mid |O_z| = \langle |O_z| \rangle \rangle = 1.12 \) and 0.90 for \( \phi = 5\% \text{ and } 10\% \), decreases by 4\% less than \( \langle V_z \rangle/V_t \). For \( \langle |O_z| \mid V_z/V_t \rangle \) we again observe values that are 1\% larger than \( \langle |O_z| \rangle \).

Summarizing, from the joint \( p.d.f.s \) of particle settling speeds and falling orientation, we find that on average particles settling with higher velocities tend to fall with their axis more inclined with respect to the direction of gravity. Hence, as particles interact through their wakes, eventually forming clusters, they tend to increase their pitch angle.

### 3.3. Fluid phase velocity statistics

Finally, we look at the statistics of the fluid-phase velocity. Figures 10(a),(b) report the instantaneous particle concentration averaged over the settling direction, whereas figures 10(c),(d) show the instantaneous vertical component of the fluid velocity, also averaged along the \( z \)-direction. For these statistics we choose a single timestep towards the end of the simulation \( (t \sim 182\tau_p \text{ and } 42\tau_p \text{ for } \phi = 1\% \text{ and } 10\%) \). For \( \phi = 1\% \), we observe that most particles accumulate around the centre of the computational domain; see figure 10(a). Within this columnar structure, particles settle substantially faster than
Clustering and increased settling speed of oblate particles

\[ \frac{V_z}{V_t} \]

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\[ \frac{V_z}{V_t} \]

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\[ \frac{V_z}{V_t} \]

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<td>0.5 0.6 0.7 0.8 0.9</td>
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\[ \frac{V_z}{V_t} \]

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</table>

Figure 9. Joint probability density functions of the settling speed \( V_z/V_t \), and the absolute value of the orientation \( |O_z| \), for increasing volume fraction \( \phi \) in panels (a),(b),(c) and (d). The dashed lines correspond to the mean \( V_z/V_t \) and \( |O_z| \). The red circles correspond to the mean \( V_z/V_t \) conditioned to \( |O_z| = |\langle O_z \rangle| \) of the suspension. The green squares correspond to the mean \( |O_z| \) conditioned to \( V_z = \langle V_z \rangle \) of the suspension. The conditioned averaged values for \( \langle V_z/V_t \mid |O_z| \rangle \) and \( \langle |O_z| \mid V_z/V_t \rangle \) are showed by the blue and red curves.

\( V_t \), and due to the no-slip boundary condition, the fluid surrounding the particles is forced to move in the same direction (positive \( z \)-direction). Indeed, in figure 10(c) we see that the fluid speed \( U_z \) is strongly positive in the same regions, with maxima at the locations of higher concentration. The highest fluid speed in the settling direction is almost of the order of the terminal velocity \( (\max(U_z) \approx 0.6V_t) \). Note that due to the zero volume flux condition, in the locations depleted of particles the fluid moves in the direction opposite to gravity with non-negligible speeds \( (\min(U_z) \approx -0.4) \). This contributes to the hindrance effect. Similar results are found for \( \phi = 0.5\% \).

In figures 10(b),(d) we show instead the mean particle concentration and mean fluid speed for \( \phi = 10\% \). The volume fraction is relatively high and particles are almost uniformly distributed in the domain. The locations of high positive fluid speed cannot be easily related to the positions of high concentration. However, we still observe that where the particle concentration is lower, the mean fluid speed is large and negative (i.e., rising fluid). So, for \( \phi = 10\% \) the maximum positive \( U_z \) is reduced \( (\approx 0.3V_t) \), while the rising fluid becomes faster \( (\min(U_z) \approx -0.7) \) leading to an increased hindrance effect.

Last, we show in figures 11(a),(b) the standard deviation of the vertical and horizontal components of the fluid velocity, \( \sigma_{U_z} \) and \( \sigma_{U_x} \), and the velocity fluctuations anisotropy, \( \sigma_{U_x}/\sigma_{U_z} \), for all \( \phi \). We see that both \( \sigma_{U_z} \) and \( \sigma_{U_x} \) increase with \( \phi \). However, for \( \sigma_{U_z} \) an approximately constant value of 0.25\( V_t \) is reached after \( \phi = 5\% \). The standard deviations
Figure 10. Instantaneous particle concentration averaged over the settling (z-) direction for \( \phi = 1\% \) (a) and 10\% (b). Instantaneous vertical component of the fluid velocity averaged over the settling direction for \( \phi = 1\% \) (c) and 10\% (d). These statistics are calculated at a single timestep towards the end of the simulation (\( t \sim 182\tau_p \) and \( \sim 42\tau_p \) for \( \phi = 1\% \) and 10\%).

Figure 11. (a) Standard deviation of the vertical and horizontal components of the fluid velocity, \( \sigma_{U_z} \) and \( \sigma_{U_x} \), for all \( \phi \). In the inset, the standard deviation of the particles velocities are also shown. (b) Anisotropy of fluid velocity fluctuations, \( \sigma_{U_x}/\sigma_{U_z} \), together with those of the solid phase, for all \( \phi \).

of the fluid velocities are substantially smaller than those of the particles, (especially in the horizontal direction), for the smaller volume fractions; see the inset of figure 11(a). The difference is progressively reduced as \( \phi \) increases. For example, for \( \phi = 0.5\% \) we find that \( \sigma_{U_z} = 0.17\sigma_{V_z} \), while for \( \phi = 10\% \), \( \sigma_{U_z} = 0.42\sigma_{V_z} \). This indicates that at high
volume fractions, the dynamics of both phases is governed mostly by excluded volume effects. Regarding the anisotropy of fluid velocity fluctuations, $\sigma_{U_z}/\sigma_{U_x}$, we see from figure 11(b) that it also increases with $\phi$. As for the solid phase, the increase with $\phi$ is almost linear.

4. Final remarks

We have studied the sedimentation of suspensions of oblate particles in quiescent fluid at finite $Re_l$. We chose the aspect ratio $AR = 1/3$, density ratio $R = 1.5$, Galilean number $Ga = 60$ (based on the diameter of a sphere with equal volume), and four volume fractions $\phi = 0.5\%, 1\%, 5\%$ and $10\%$. The single particle case was also simulated and at this combination of $R$ and $Ga$, the particle settles with its broad side perpendicular to the direction of gravity with a straight, steady wake. The orientation vector of the isolated oblate is hence $[O_x, O_y, O_z] = [0, 0, 1]$ (i.e. the pitch angle with respect to the plane perpendicular to gravity is $0^\circ$).

The average settling speed of a suspension changes with the particle volume fraction as the results of the competition of two different physical mechanisms: i) the hindrance effect, which is more pronounced in higher volume fractions and tends to reduce the average settling speed; and ii) the hydrodynamics of particle pair interactions, e.g. drafting-kissing-tumbling, which tends to increase the average speed and form piles of particles. We report that, unlike the case of spherical particles of equal $Ga$, the mean settling speed of the oblate particles suspension, $\langle V_z \rangle/V_i$, increases with $\phi$ in dilute conditions. For $\phi = 0.5\%-1\%$; the mean settling speed is about $30\%$ larger than the terminal velocity of an isolated oblate. Note that for suspensions of spheres it has been shown that $\langle V_z \rangle/V_i$ is always a decreasing function of $\phi$ for $Ga \sim 60$ (Richardson & Zaki 1954; Yin & Koch 2007; Uhlmann & Doychev 2014). The mean settling speed becomes smaller than the terminal velocity $V_i$ only for $\phi > 5\%$. This implies that at lower volume fractions the hindrance effect is overcome by hydrodynamic and particle-particle interactions and this leads to an increase of $\langle V_z \rangle/V_i$. Indeed, we have shown that in dilute conditions most particles are arranged at steady-state in a columnar-like structure with a radius of about $20c$, where $c$ is the oblate polar radius. Within this structure, particle clusters settle with velocities up to 4 times the mean, $\langle V_z \rangle/V_i$. Therefore, the probability density functions, p.d.f.s, of $V_z/V_i$ display a clear positive skewness, $S_{V_z} \sim 0.4$ (i.e. many particles fall with speeds larger than the mean value). While $\langle V_z \rangle/V_i$ is reduced increasing the volume fraction, the velocity standard deviation $\sigma_{V_z}$ increases up to $\phi \sim 5\%$, the skewness $S_{V_z}$ tends to 0 and the flatness $F_{V_z}$ is always approximately 3. Additionally, within the columnar-like structure, the fluid is strongly dragged by the particles (due to the no-slip condition) and in the settling direction it reaches speeds almost of the order of $V_i$ ($\max(U_z) \sim 0.6 V_i$).

To study the suspension microstructure, the pair- and radial distribution functions are calculated. While no clustering is observed for spheres of same $Ga$, the pair-distribution function $P(r)$ is found to be large all around a reference particle and especially between $\psi \sim 0^\circ - 80^\circ$ and $r/c \sim 2 - 5$. The highest probability of finding a neighbour particle is at $r/c = 2.02$, $\psi \simeq 17^\circ$ for $\phi = 0.5\%$, $r/c = 2.02$, $\psi \simeq 10^\circ$ for $\phi = 1\%$, and around $r/c = 2.02$, $\psi \simeq 0^\circ - 2^\circ$ for the highest volume fractions investigated. Hence, on average particles are almost piled up. The extent of clustering sharply decreases for $\phi > 1\%$ as it is shown by the pair-distribution function $P(r)$ and its angular average, $g(r)$ (the radial distribution function). The radial distribution function is found to be maximum around $r/c \simeq 4$ (i.e. regardless of the orientation, the highest probability of finding a particle-pair is located at a separation distance of $4c$). In the cases with lowest volume fractions, the particle distribution is found to be uncorrelated at distances larger than
20c, a distance of the order of the radius of the columnar structure. For the denser cases, the decorrelation of the structure occurs at shorter separations, at distances of the order of 6c. Using the definition of order parameter \( \langle P_2 \rangle (r) \) we have shown that for distances of about 2c particles are almost perfectly vertically aligned. Above \( r/c > 3 \) particles are almost horizontally aligned (with a finite inclination with respect to the horizontal plane). The suspension structure becomes more isotropic for distances \( r/c \geq 6 \) in the denser cases and \( r/c \geq 8 \) in the more dilute cases.

We have also considered the particle lateral velocity as well as the angular velocities. The mean particle lateral speed \( \langle V_x \rangle / V_t \) is approximately 0, while its standard deviation \( \sigma_{V_x} \) increases with the volume fraction up to \( \phi = 5\% \) and is smaller than \( \sigma_{V_z} \). The anisotropy of the particle velocity fluctuations \( \sigma_{V_x} / \sigma_{V_z} \) increases abruptly until \( \phi = 0.5\% \) and then approximately linearly with \( \phi \). From the p.d.f.s of angular velocities, we find that particles rotate more around the directions perpendicular to gravity. It is found that particles settle on average inclined with respect to the horizontal plane (the particle orientation with respect to the axis of symmetry, \( |O_z| \), is less than 1, where 1 corresponds to a pitch angle of 0°). The mean pitch angle increases with \( \phi \) from 22.8° to 47° (i.e. \( |O_z| \) decreases with \( \phi \)). A power-law fit that depends solely on the solid volume fraction \( \phi \) and one coefficient is proposed. Computing the terminal speed of an isolated oblate settling with the mean pitch of the suspension with \( \phi = 0.5\% \) (22.8°) we find that the lateral velocity \( V_x/V_t = 0.41 \) and the vertical only \( V_z/V_t = 1.03 \), significantly lower than that of the suspension, showing the importance of particle-pair interactions.

Finally, we have calculated the joint probability functions of particle settling speed and orientation. We used conditioned averages to show that particles settling with speeds larger than the mean \( \langle V_z \rangle / V_t \), have on average lower mean orientations (higher pitch angles).

With this study we have began to look at the effects of particle shape in sedimenting suspensions of inertial particles. In the future, it will be interesting to consider oblate and prolate particles of different aspect ratios and Galileo numbers.

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Clustering and increased settling speed of oblate particles


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