Envelopes of holomorphy for Bounded Holomorphic Functions

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DOCTORAL DISSERTATION

by

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Abstract

Some problems concerning holomorphic continuation of the class of bounded holomorphic functions from bounded domains in $\mathbb{C}^n$ that are domains of holomorphy are solved. A bounded domain of holomorphy $\Omega$ in $\mathbb{C}^2$ with nonschlicht $H^\infty$-envelope of holomorphy is constructed and it is shown that there is a point in $\Omega$ for which Gleason's Problem for $H^\infty(\Omega)$ cannot be solved. Furthermore a proof of the existence of a bounded domain of holomorphy in $\mathbb{C}^2$ for which the volume of the $H^\infty$-envelope of holomorphy is infinite is given. The idea of the proof is to put a family of so-called "Sibony domains" into the unit bidisk by a packing procedure and patch them together by thin neighbourhoods of suitably chosen curves.

If $H^\infty(\Omega)$ is the Banach algebra of bounded holomorphic functions on a bounded domain $\Omega$ in $\mathbb{C}^n$ and if $p$ is a point in $\Omega$, then the following problem is known as Gleason's Problem for $H^\infty(\Omega)$:

Is the maximal ideal in $H^\infty(\Omega)$ consisting of functions vanishing at $p$ generated by $(z_1 - p_1), \ldots, (z_n - p_n)$?

A sufficient condition for solving Gleason's Problem for $H^\infty(\Omega)$ for all points in $\Omega$ is given. In particular, this condition is fulfilled by a convex domain $\Omega$ with $\text{Lip}_{1+\epsilon}$-boundary ($0 < \epsilon < 1$) and thus generalizes a theorem of S.L.Leibenson. It is also proved that Gleason's Problem can be solved for all points in certain unions of two polydisks in $\mathbb{C}^2$. One of the ideas in the methods of proof is integration along specific polygonal lines.

Certain properties of some open sets defined by global plurisubharmonic functions in $\mathbb{C}^n$ are studied. More precisely, the sets $D_u = \{z \in \mathbb{C}^n : u(z) < 0\}$ and
$E_h = \{(z, w) \in \mathbb{C}^n \times \mathbb{C} : h(z, w) < 1\}$ are considered where $u$ is a plurisubharmonic function of minimal growth and $h \neq 0$ is a non-negative homogeneous plurisubharmonic function. (That is, the functions $u$ and $h$ belong to the classes $L(\mathbb{C}^n)$ and $H_+ (\mathbb{C}^n \times \mathbb{C})$ respectively.) It is examined how the fact that $E_h$ and the connected components of $D_u$ are $H^\infty$-domains of holomorphy is related to the structure of the set of discontinuity points of the global defining functions and to polynomial convexity. Moreover it is studied whether these notions are preserved under a certain bijective mapping from $L(\mathbb{C}^n)$ to $H_+ (\mathbb{C}^n \times \mathbb{C})$. Two counterexamples are given which show that polynomial convexity is not preserved under this bijection. It is also proved, for example, that if $D_u$ is bounded and if the set of discontinuity points of $u$ is pluripolar then $D_u$ is of type $H^\infty$.

A survey paper on general properties of envelopes of holomorphy is included. In particular, the paper treats aspects of the theory for the bounded holomorphic functions. The results for the bounded holomorphic functions are compared with the corresponding ones for the holomorphic functions.

**Key words:** holomorphic function, bounded holomorphic function, domain of holomorphy, envelope of holomorphy, Gleason’s problem, convex set, plurisubharmonic function, pluripolar set, polynomially convex set

**1991 Mathematics Subject Classification:** 32A17, 32D10, 32E25.

Papers summarized in this dissertation:

1. U. Backlund, *$H^\infty$- envelopes of holomorphy*.
5. U. Backlund, *Comparison of certain $H^\infty$- domains of holomorphy*.

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Summary

This thesis consists of the following five papers:

1. U. Backlund, *$H^\infty$ - envelopes of holomorphy*.


5. U. Backlund, *Comparison of certain $H^\infty$-domains of holomorphy*.

In the papers [2], [3] and [5] in this thesis we solve some problems concerning holomorphic continuation of the class of bounded holomorphic functions. Given a bounded domain $\Omega$ in $\mathbb{C}^n$ that is a domain of holomorphy (i.e. a maximal domain of existence for a holomorphic function), one could ask the question whether all bounded holomorphic functions on $\Omega$ still can be holomorphically continued to a strictly larger domain (not necessarily lying in $\mathbb{C}^n$). A punctured disk in the complex plane is the simplest example of such a domain. The first example of such a domain with the additional property that the interior of the closure of the domain is equal to the domain itself was constructed by N. Sibony [S]. His example is in two dimensions and cannot occur in one dimension. In ([2], Theorem 3.1) we construct a bounded domain of holomorphy $\Omega$ in $\mathbb{C}^2$ with nonschlicht $H^\infty$-envelope of holomorphy. That is, the maximal domain of definition for the bounded holomorphic functions on $\Omega$ is a Riemann domain (a covering space) spread over $\mathbb{C}^2$ which cannot be biholomorphically embedded into $\mathbb{C}^2$. Furthermore we show that the constructed domain $\Omega$ contains a point for which Gleason’s Problem for $H^\infty(\Omega)$ cannot be solved ([2], Proposition 4.1).
In paper [3] we give a proof of the existence of a bounded domain of holomorphy in $\mathbb{C}^2$ for which the volume of the $H^\infty$-envelope of holomorphy is infinite ([3], Theorem 1). The idea of the proof is to put a family of domains of the same kind as the above-mentioned example of N. Sibony into the unit bidisk by a packing procedure and patch them together by thin neighbourhoods of suitably chosen curves.

In paper [4] we consider Gleason's Problem for $H^\infty(\Omega)$. If $H^\infty(\Omega)$ is the Banach algebra of bounded holomorphic functions on a bounded domain $\Omega$ in $\mathbb{C}^n$ and if $p$ is a point in $\Omega$, then the following problem is known as Gleason's Problem for $H^\infty(\Omega)$:

Is the maximal ideal in $H^\infty(\Omega)$ consisting of functions vanishing at $p$ generated by $(z_1 - p_1), \ldots, (z_n - p_n)$?

We give a sufficient condition for solving Gleason's Problem for $H^\infty(\Omega)$ for all points in $\Omega$ ([4], Theorem 1). In particular ([4], Corollary 1), this condition is fulfilled by a convex domain $\Omega$ with $\text{Lip}_{1+\epsilon}$-boundary ($0 < \epsilon < 1$) and thus generalizes a theorem of S.L. Leibenson. It is also proved that Gleason's Problem can be solved for all points in certain unions of two polydisks in $\mathbb{C}^2$ ([4], Proposition 1). One of the ideas in the methods of proof is integration along specific polygonal lines.

In paper [5] we study certain properties of some open sets defined by global plurisubharmonic functions in $\mathbb{C}^n$. More precisely, we consider the sets

$$D_u = \{z \in \mathbb{C}^n : u(z) < 0\}$$

$$E_h = \{(z, w) \in \mathbb{C}^n \times \mathbb{C} : h(z, w) < 1\}$$

where $u$ is a plurisubharmonic function of minimal growth and $h \not\equiv 0$ is a non-negative homogeneous plurisubharmonic function. That is, the functions $u$ and $h$ belong to the classes $L(\mathbb{C}^n)$ and $H_+(\mathbb{C}^n \times \mathbb{C})$ respectively. We examine how the fact that $E_h$ and the connected components of $D_u$ are $H^\infty$-domains of holomorphy is related to the structure of the set of discontinuity points of the global defining functions and to polynomial convexity. We also examine whether these notions are preserved under a
certain bijection between \( L(\mathbb{C}^n) \) and \( H_+(\mathbb{C}^n \times \mathbb{C}) \):

\[
S : L(\mathbb{C}^N) \to H_+(\mathbb{C}^N \times \mathbb{C}) \quad \text{defined by}
\]

\[
(Su)(z, w) := |w| \exp u(z/w) \quad , \quad (z, w) \in \mathbb{C}^n \times (\mathbb{C} \setminus \{0\})
\]

\[
:= \limsup_{(z, \zeta) \to (z, 0)} |\zeta| \exp u(z/\zeta) \quad , \quad (z, w) \in \mathbb{C}^n \times \{0\}
\]

and

\[
T : H_+(\mathbb{C}^n \times \mathbb{C}) \to L(\mathbb{C}^n) \quad \text{defined by}
\]

\[
(Th)(z) := \log h(z, 1) \quad , \quad z \in \mathbb{C}^N.
\]

In particular, we prove that polynomial convexity is not preserved by any of the mappings \( S \) and \( T \). The counterexamples are given in ([5], Theorem 11) and ([5], Theorem 12). It is also proved, for example, that if \( D_u \) is bounded and if the set of discontinuity points of \( u \) is pluripolar then \( D_u \) is of type \( H^\infty \) ([5], Theorem 9).

For more background on the questions that are discussed in this thesis we refer the reader to the included survey paper [1] on general properties of envelopes of holomorphy. In particular, paper [1] treats aspects of the theory for the bounded holomorphic functions. The results for the bounded holomorphic functions are compared with the corresponding ones for the holomorphic functions.

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Reference
