Abstract. Random graphs is a well-studied field of probability theory, and have proven very useful in a range of applications. However, most random graphs are static in the sense that the network structure does not change over time; they also tend to consist of single-type objects. This puts restrictions on possible applications. In this thesis we extend two standard models to a dynamic and multi-type setting, respectively.

In the first paper we study a dynamic version of the famous Erdős-Rényi graph. The graph changes dynamically over time, but still has the static Erdős-Rényi graph as its stationary distribution. In studying the dynamic graph we present two results. The first one concerns the time to stationarity, and the second one the time to reach a certain number of edges.

In the second paper we introduce and study an extension of the preferential attachment model. The standard preferential attachment model is already dynamic, but its vertices are only allowed to be of one type. We introduce a multi-type analogue of the preferential attachment model and study its asymptotic degree distributions as well as its asymptotic composition.
List of Papers

The thesis consists of two papers:


Notes on collaboration. Paper I is based on the ideas of P. Trapman. The results on strong stationary times in the first part of the paper were derived by S. Rosengren, and the second part of the paper was done jointly. The writing of the manuscript was done mainly by S. Rosengren.
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1. Introduction

Graphs are mathematical objects used to represent some kind of network. They consist of vertices, meant to represent e.g. individuals; together with edges between vertices, meant to represent some kind of relation. Mathematically, a graph $G$ is a collection of vertices $V$ together with a collection of edges $E \subset V \times V$, i.e. $G = (V, E)$. Random graphs arise when the construction of the graph involves randomness.

Random graphs are interesting mathematical objects in their own right, and they are often the suitable choice when modeling networks that are inherently random, e.g. when the relationships between vertices can be regarded as random. However, maybe the most useful application of random graphs is to let them serve as models for complex networks, which are not necessarily random. A complex network is a network with a complicated structure, e.g. the social structures of Facebook, LinkedIn, or just a large group of people. These networks are difficult to study. For example, if you want to know the size of the largest connected component you would need full access to the network, and even then the problem can be computationally infeasible — the time complexity of general search methods (e.g. breadth first search) is of order $O(|V| + |E|)$, which in turn can be of order $O(n^2)$. One technique of dealing with complexity is to replace it with randomness, e.g. replace the complex network with a random graph. For instance, instead of knowing the full network we may have access to local properties, e.g. the average number of friends of a uniformly chosen individual. If we can find a random graph model with the same local properties, we could model the complex network as an outcome of that random graph, which in turn can be analyzed. This often allows us to make useful predictions about the complex network. For extensive treatments of the area of random graphs, see e.g. [4], [7], [10].

Next we give a short introduction to two standard random graph models — the Erdős-Rényi graph and the preferential attachment model, both of which will be extended in Papers I and II.

1.1. Erdős-Rényi Graph. The Erdős-Rényi graph model(s) was first introduced in [5] and [6], and is a well-studied model of random graphs. It is defined as either: (i) consisting of $n$ vertices and $m$ edges, where the edges are assigned uniformly to the $\binom{n}{2}$ vertex pairs — this graph model is denoted $G(n, m)$; or (ii) consisting of $n$ vertices where edges are assigned independently between vertex pairs with probability $p$ — this graph model is denoted $G(n, p)$, see [3] for more details and many properties of the model. In what follows we shall focus mainly on the $G(n, p)$-model. It is the simplest of random graph models, but still has an interesting behavior as $n$ grows large. Perhaps the most famous property of the model is that it exhibits a phase transition. Namely, for the number of vertices in the largest component — denoted $|C_1(n, p)|$ — we have the following classic result.

**Theorem 1.** [10, Thm. 4.4, 4.5, and 4.8] Let $p_n = \frac{\lambda}{n}$ where $\lambda > 0$.

(i) If $\lambda < 1$ then

$$\frac{|C_1(n, p_n)|}{\log(n)} \xrightarrow{p} \frac{1}{\lambda - 1 - \log(\lambda)} \text{ as } n \to \infty.$$ 

(ii) If $\lambda > 1$ then

$$\frac{|C_1(n, p_n)|}{n} \xrightarrow{p} \xi \text{ as } n \to \infty$$

where $\xi$ is the survival probability of a branching process with offspring distribution $X \sim Po(\lambda)$.

The critical case when $\lambda = 1$ is also of mathematical interest, in which the size of the largest component is of order $n^{2/3}$, see e.g. Theorem 5.1 of [10].
In Paper I ([9]) we shall extend the Erdős-Rényi graph to a dynamic graph and see that, among other things, it too exhibits a phase transition.

Although interesting and worth studying, the model is often not suitable for real-world applications. One reason for this is that (with \( p_n = \lambda/n \)) the limiting degree distribution of a vertex is Poisson(\( \lambda \)). This is a distribution with exponentially decaying tail — the fraction of vertices with degree \( k \) decays exponentially in \( k \). However, in many real networks the degree tail decays more slowly. Often, the fraction of vertices with degree \( k \) decays as an inverse power of \( k \) — that is, the degrees follow a power law distribution, see [10, Section 1]. This type of degree distribution is something that is generated by the preferential attachment model.

1.2. Preferential Attachment. The preferential attachment model is a well-studied model of random network growth which was first introduced in [2]. In preferential attachment models, traditionally, new vertices arrive according to some process (often at integer times \( \{1, 2, \ldots\} \)) and upon arrival attach to an existing vertex with probability proportional to that vertex’s degree. Therefore, vertices that have a high degree are more likely to attract new vertices — this is also called the rich-get-richer effect, and the model can be suitable to model networks exhibiting this phenomenon.

Preferential attachment is particularly interesting for modeling empirical networks since it, as mentioned above, gives rise to power law degree distributions — something often observed in empirical networks.

The preferential attachment model is already dynamic, but the vertices are only allowed to be of one single type. In Paper II ([8]) we extend the preferential attachment model by allowing vertices to be of different types. This does not change the inherent properties that the preferential attachment dynamic gives rise to (e.g. the multi-type model still has power law degree distributions) — but gives rise to a more flexible modeling framework.

2. Summary of Papers

2.1. Paper I. In this paper we study a dynamic Erdős-Rényi graph model (first introduced in [1]), in which, independently for each vertex pair, edges appear and disappear according to a Markov on-off process. For \( \alpha, \beta > 0 \) and \( n \) a positive integer, the dynamic Erdős-Rényi graph is a stochastic process evolving according to the following dynamics.

(i) The number of vertices is fixed at \( n \).
(ii) Independently for each vertex pair, if no edge is present an edge is added after an \( \text{Exp}(\beta n^{-1}) \)-distributed time; if an edge is present, the edge is removed after an \( \text{Exp}(\alpha) \)-distributed time.

Our first result concerns the time it takes for the graph to reach stationarity. We show that, conditioned on that the graph starts in a given state, the (fastest) time to stationary is distributed as the maximum of \( \binom{n}{2} \) independent exponentially distributed random variables with rate \( \alpha + \frac{\beta}{n-1} \). We also show that among all strong stationary times this is the fastest one.

The main result concerns the time it takes until the dynamic graph contains \( i = \lceil cn \rceil \) edges, where \( c \) is a positive constant. For large \( n \) the expected time to go from 0 to \( i = \lceil cn \rceil \) exhibits three different behaviors depending on the value of \( c \). For \( c < \frac{n}{2\alpha} \) the graph reaches \( i \) edges after roughly a constant time; for \( c = \frac{n}{2\alpha} \) the graph reaches \( i \) edges after an logarithmic time (in \( n \)); while for \( c > \frac{n}{2\alpha} \) the graph reaches \( i \) edges after an exponentially (in \( n \)) large time.

2.2. Paper II. In this paper we introduce a multi-type preferential attachment model, and study it using general multi-type branching processes. We derive a framework for studying the model where a type \( i \) vertex generates new type \( j \) vertices with rate \( w_{ij}(n_1, n_2, \ldots, n_p) \) where \( n_k \) is the number of type \( k \) vertices previously generated by the type \( i \) vertex, and \( w_{ij} \) is a
function from \( N^p \) to \( \mathbb{R} \). We derive general results concerning asymptotic degree distributions and asymptotic compositions (total proportion of vertices with a specific type). We then apply the theory to models with more specific attachment rates.

In the case with linear preferential attachment — where type \( i \) vertices generate new type \( j \) vertices with rate \( w_{ij}(n_1, n_2, \ldots, n_p) = \gamma_{ij}(n_1+n_2+\cdots+n_p)+\beta_{ij} \), where \( \gamma_{ij} \) and \( \beta_{ij} \) are positive constants — the main result concerns the asymptotic degree distribution. We show that under mild regularity conditions on the parameters \( \{\gamma_{ij}\},\{\beta_{ij}\} \) the asymptotic degree distribution of a vertex is a power law distribution. Roughly speaking, the proportion of type \( i \) vertices with \( k \) children (of any type) satisfies

\[
p_i(k) \approx C \cdot k^{-(1+\alpha_\gamma+\cdots+\alpha_{\gamma p})} \quad \text{if} \quad \gamma_{i1} + \cdots + \gamma_{ip} > 0,
\]

where \( \alpha \) is the Malthusian parameter of the multi-type branching process associated with the graph, and \( C \) is a constant.

References