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Abstract

In this paper we deal with the classical problem of random cover times. We investigate the distribution of the time it takes for a Poisson process of cylinders to cover a set $A \subset \mathbb{R}^d$. This Poisson process of cylinders is invariant under rotations, reflections and translations, and in addition we add a time component so that cylinders are "raining from the sky" at unit rate. Our main results concerns the asymptotic of this cover time as the set A grows. If the set A is discrete and well separated, we show convergence of the cover time to a Gumbel distribution. If instead A has positive box dimension (and satisfies a weak additional assumption), we find the correct rate of convergence.

1 Introduction

There are many variants of coverage problems that has been studied in the probabilistic literature. One of the first papers on this subject was by Dvoretzky ([4]) and dealt with the problem of covering the circle by using a sequence of sets placed randomly around the circle. The related problem of covering $\mathbb{R}^d$ was then later studied by Shepp in ([11]) for $d = 1$, Biermé and Estrade ([2]) for general d and also by Broman, Jonasson and Tykesson ([3]) for general d. In common to all of these papers is that there were no time-component involved. Instead, they all consider (infinite) measures μ on the set of compact subsets of $\mathbb{R}^d$. Then, they study a Poisson process using μ as the intensity measure, and ask whether $\mathbb{R}^d$ will be completely covered. Of course, this will depend on the particular choice of μ and this dependence is what they investigate.

A variant of this covering problem is to do the following. Start with some bounded set $A \subset \mathbb{R}^d$, and throw down other (possibly random) sets B_i in random locations, and proceed until A is covered. It is then natural to ask about the distribution of the number of sets needed in order to cover A. Alternatively, if the sets are dropped at unit rate, one can ask about the distribution of the cover time, i.e. the time it takes until A is covered. An example would be to let A be the unit square in $\mathbb{R}^2$, and letting B_i all be squares of side length ϵ with their centres uniformly distributed in A. Another example would be to let the side length of the sets B_i be random. Such problems was studied by for instance Siegel and Holst ([12]) and Janson [7] on the circle, while a much more general result was later obtained by Janson in [8]. In particular, all of these papers studied asymptotics of the cover time as the set A increases.

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More recently, Belius [1] studied the problem of covering a bounded set $A \subset \mathbb{Z}^d$ by what is known as random interlacements. This is basically a Poisson process on the trajectories of bi-infinite random walks in $\mathbb{Z}^d$ and was introduced by Sznitman in [13]. The major difference between the paper by Belius and the others mentioned above, is that the interlacement trajectories are infinite objects, whereas in the classical setting, the corresponding sets are finite. The use of infinite objects introduces a number of new challenges as the cover levels of separated sets are no longer independent.

The aim of this paper is to combine the classical problem of covering sets $A \subset \mathbb{R}^d$ of non-zero dimension (rather than say a subset of $\mathbb{Z}^d$), with the use of bi-infinite objects to cover the set. In order to explain our main result, we shall here give informal descriptions of the mathematical quantities and tools needed. Precise definitions and explanations will be given in Section 2. For the models to make sense, we will always assume that the dimension d is at least 2, and this will be assumed throughout the paper without any further comment.

We will use a Poisson process Ψ where an element $(L, s) \in \Psi$ consists of a bi-infinite line $L \subset \mathbb{R}^d$ and a "time-stamp" $s \in \mathbb{R}^+$. The set of lines with time-stamp smaller than t will then be a Poisson process on the set of bi-infinite lines in $\mathbb{R}^d$, and this process will be invariant under rotations, reflections and translations (see Section 2). For such a line L we consider the corresponding cylinder $\mathfrak{c}(L)$ with base radius 1, i.e.

$$\mathfrak{c}(L) := \{x \in \mathbb{R}^d : d(x, L) \leq 1\} = L + B(0, 1).$$

Then we ask whether

$$A \subset \bigcup_{(L,s)\in\Psi: s\leq t} \mathfrak{c}(L)?$$

That is, we ask whether A has been covered by the cylinders $\mathfrak{c}(L)$ which have dropped before time t, and we let the *cover time* of A be

$$\mathcal{T}(A) := \inf \left\{ t > 0 : A \subset \bigcup_{(L,s)\in\Psi: s\leq t} \mathfrak{c}(L) \right\}.$$

For any $0 < \rho < 1$, let A^ρ be a maximal set of points $x \in A$ such that the distance between any two such points is at least ρ. We will then let $|A^\rho|$ denote the cardinality of the set A^ρ. There are in general many choices of such a set A^ρ, but we will assume that one is picked according to some predetermined rule. While we do not explicitly state which rule we use (as it will not be important), we stress that given $A \subset \mathbb{R}^d$, this rule determines A^ρ uniquely.

If $A \subset \mathbb{R}^d$ is such that for some $0 < \dim_B(A) \leq d$ we have that

$$0 < \liminf_{\rho\to 0} \rho^{\dim_B(A)} |A^\rho| \leq \limsup_{\rho\to 0} \rho^{\dim_B(A)} |A^\rho| < \infty \tag{1}$$

then it follows that the so-called box dimension (see Section 2) of A is $\dim_B(A)$. Here and in the future, $|\cdot|$ denotes cardinality, while $nA := \{x \in \mathbb{R}^d : x/n \in A\}$. We can now state the main theorem of the paper.

Theorem 1.1. *For any set A satisfying* (1)*, we have that the sequence*

$$(\mathcal{T}(nA) - \dim_B(A)(\log n + \log\log n))_{n\geq 1}$$

is tight.

Remarks: If A satisfies the stronger assumption that for some $0 < c_A < \infty$,

$$\lim_{\rho \to 0} \rho^{\dim_B(A)} |A^\rho| = c_A, \tag{2}$$

it is reasonable (in light of similar results such as the ones in [8]), to expect that

$$\mathcal{T}(nA) - \dim_B(A)(\log n + \log \log n) + C$$

converges to a Gumbel distribution for a suitable choice of constant C. Our main result does not quite achieve this. However, it does show that if $\mathcal{T}(nA) - f(n)$ converges to a non-trivial random variable, then the function $f(n)$ must in fact take the form $\dim_B(A)(\log n + \log \log n) + O(1)$. Furthermore, as will be shown in the proof of Theorem 1.1, the quantity $-\log c_A - \dim_B(A) \log \dim_B(A)$ comes up naturally in the calculations, and it is not far-fetched to believe that convergence takes place with this choice of constant C.

It might not be clear whether a "typical" set should satisfy condition (2) or not. While we will not discuss this question in detail (as it is more a statement that belongs to fractal geometry), we will show in Proposition 4.2 that $A = [0,1]^d$ does satisfy this condition. It will be clear from this example that property (1) in fact holds for many sets, for instance any bounded set that contains a box.

In Section 7 we will prove an auxiliary result (see Theorem 2.1) which can be applied to the cases when the upper and lower box dimensions (see Section 2) do not agree or if

$$\lim_{\rho \to 0} \rho^{\dim_B(A)} |A^\rho| \in \{0, \infty\}.$$

In order to prove Theorem 1.1, we will study two quantities that are closely related to $\mathcal{T}(A)$. Firstly, the *discrete cover time*

$$T_d^\rho(A) := \inf \left\{ t > 0 : A^\rho \subset \bigcup_{(L,s) \in \Psi : s \leq t} \mathfrak{c}(L) \right\}. \tag{3}$$

The name is derived from the fact that $T_d^\rho(A)$ is the time when the discrete subset A^ρ of A is covered. Recall that A^ρ is uniquely determined by some rule, and so there is no ambiguity in the definition of $T_d^\rho(A)$. We will always assume that $\mathcal{T}(A)$ and $T_d^\rho(A)$ are generated by the same cylinder process, thus defining their joint distributions. It is then easy to prove that (see Proposition 7.1) for any $\rho > 0$,

$$T_d^\rho(A) \leq \lim_{\delta \to 0} T_d^\delta(A) = \mathcal{T}(A), \tag{4}$$

where the inequality and equality should be interpreted in a pointwise sense (see Section 2). We see that the cover time $\mathcal{T}(A)$ can be estimated from below by using $T_d^\rho(A)$, and that in order to get a good estimate, one should take ρ as small as possible. However, by taking ρ too small, the estimates that we will obtain for $T_d^\rho(A)$ will become useless. Therefore, it will be important to pick ρ in an optimal way. It turns out that this will correspond to letting ρ be of order $(\log |A^1|)^{-1}$, see further Section 7.

Secondly, we define the *well cover time*

$$T_w^\rho(A) := \inf\{t > 0 : \forall x \in A^\rho \ \exists (L,s) \in \Psi \text{ such that } B(x,\rho) \subset \mathfrak{c}(L) \text{ and } s \le t\}. \quad (5)$$

We see that A is well covered if every ball $B(x,\rho)$ with centres in A^ρ has been covered by a single cylinder $\mathfrak{c}(L)$. This implies that $\mathcal{T}(A) \le T_w^\rho(A)$ for every $\rho > 0$, so that the well cover time can be used for estimating $\mathcal{T}(A)$ from above. Again, $T_w^\rho(A)$ is generated by using the same cylinder process used to generate $\mathcal{T}(A)$ and $T_d^\rho(A)$. We mention that a relation between $T_w^\rho(A)$ and $T_d^\rho(A)$ can be obtained through scaling, see the remark after Theorem 6.4 where this is discussed in more detail.

The discrete cover time $T_d^\rho(A)$ is interesting in its own right. The reason for this is that it describes the cover time of A when A is a ρ-separated set, which simply means $d(x,y) \ge \rho$ for every $x,y \in A$ such that $x \ne y$. In particular, we then have that $\dim_B(A) = 0$, and so this does not fall under the conditions of Theorem 1.1. Our secondary result is the following.

Theorem 1.2. *Consider any* $0 < \rho < \min\left(\frac{d-1}{6d}, \frac{C_d}{5}\right)$*, where* C_d *is defined in* (22). *Consider further a sequence of bounded sets* $(A_n)_{n=1}^\infty$ *such that* $|A_n^\rho| \to \infty$ *as* $n \to \infty$. *Then,* $T_d^\rho(A_n) - \log|A_n^\rho|$ *converges in distribution to the Gumbel distribution as* $n \to \infty$.

The following example shows that there is a fundamental difference between the case where the sets $(A_n)_{n\ge1}$ consists of finitely many points, and the case where the sets $(A_n)_{n\ge1}$ are "large". The example also illustrates that the condition $\rho < \min\left(\frac{d-1}{6d}, \frac{\tilde{C}_d}{5}\right)$ is not really a limitation.

Example 1: Let $A_n = [0,n-1]^d \cap \mathbb{Z}^d$ so that $|A_n^1| = n^d$ for every n. Observe also that $A_n^1 = A_n^\rho$ for every $\rho < 1$. Theorem 1.2 then implies that

$$\mathcal{T}(A_n) - d\log n = T_d^1(A_n) - \log\left|A_n^1\right| = T_d^\rho(A_n) - \log|A_n^\rho|$$

converges in distribution to a Gumbel distribution. Note that the scaling $-d\log n$ is different from the scaling $-d\log n - d\log\log n$ which according to Theorem 1.1 would be the correct choice if $A_n = [0,n]^d$ instead of $[0,n-1]^d \cap \mathbb{Z}^d$. We point out that the results of [1] are similar to when $A_n = [0,n-1]^d \cap \mathbb{Z}^d$, while the results of [8] are similar to when $A_n = [0,n]^d$.

A result similar to Theorem 1.2 but for $T_w^\rho(A)$ can readily be obtained. However, since this does not have the clear interpretation that Theorem 1.2 does, we do not include it in the paper.

The structure of the rest of the paper is as follows: In Section 3 we present some preliminary results concerning the Poisson cylinder model. In Section 4 we prove some fundamental properties of the set A^ρ. In Section 5 we prove Theorem 5.5 which will be our main tool for analyzing $T_w^\rho(A)$. In Section 6 we prove the corresponding result for $T_d^\rho(A)$, namely Theorem 6.4 which will then be used to provide the proof of Theorem 1.2. Section 6 will mainly contain sketches as most of it is very similar to what is done in Section 5. The overall proof-strategy of Sections 5 and 6 will be similar to the strategy of [1]. However, there will also be many differences stemming from the fact that we are working in $\mathbb{R}^d$ as opposed to the discrete space $\mathbb{Z}^d$ (see also the example above). Finally, in Section 7 it is all put together to prove Theorem 1.1.

2 Model and definitions

In the subsections below we will firstly discuss the Poisson cylinder model, secondly the set A^ρ and the related cover times $T_d^\rho(A), T_w^\rho(A)$, and thirdly provide some basic background on box dimensions.

2.1 The Poisson cylinder model

Our first step is to define the Poisson *line* model. To that end, let $G(d,1)$ be the set of bi-infinite lines in $\mathbb{R}^d$ that pass through the origin o, and let $A(d,1)$ be the set of bi-infinite lines in $\mathbb{R}^d$. Furthermore, let

$$\mathcal{L}_K := \{L \in A(d,1) : L \cap K \neq \emptyset\},$$

be the set of lines that intersect K where $K \subset \mathbb{R}^d$ is a compact set. For convenience, we let $\mathcal{L}_{A,B}$ denote the set $\mathcal{L}_A \cap \mathcal{L}_B$, i.e. the set of lines that intersects both sets $A, B \subset \mathbb{R}^d$. Furthermore, $B(x,\rho)$ will denote the closed (d-dimensional) ball of radius ρ centered at x.

On the space $G(d,1)$ there is a unique Haar measure $\nu_{d,1}$, normalized so that $\nu_{d,1}(G(d,1)) = 1$, and on $A(d,1)$, there is a unique (up to constants) measure which is invariant under rotations, reflections and translations. We will let $\mu_{d,1}$ denote this latter measure, normalized so that $\mu_{d,1}\left(\mathcal{L}_{B_d(o,1)}\right) = 1$ (see for instance [10] Chapter 13). For any subspace $H \subset \mathbb{R}^d$, and set $D \subset \mathbb{R}^d$, we let $\Pi_H(D)$ denote the projection of D onto H. Here, we will consider $\Pi_H(D)$ to be a subset of $\mathbb{R}^d$ (and not just a subset of H). Furthermore, we will let κ_d denote the volume of the unit ball $B(o,1)$ in $\mathbb{R}^d$, and λ_d denote Lebesgue measure on $\mathbb{R}^d$ so that $\kappa_d = \lambda_d(B(o,1))$.

For any $L \in G(d,1)$ we will let $L^\perp$ be the $(d-1)$-dimensional hyperplane orthogonal to L. The following representation ([10] Theorem 13.2.12) of the measure $\mu_{d,1}$ will be useful for us. For any $K \subset \mathbb{R}^d$ we have that

$$\mu_{d,1}(\mathcal{L}_K) = \frac{1}{\kappa_{d-1}} \int_{G(d,1)} \int_{L^\perp} \mathbb{1}(L + y \in \mathcal{L}_K) \lambda_{d-1}(dy) \nu_{d,1}(dL), \tag{6}$$

where $\mathbb{1}$ denotes an indicator function. Informally, for a fixed line L, the inner integral integrates over all lines parallel to L that intersects K. Then, the outer integral integrates over all possible choices of L.

Our next step is to consider the following space of point measures on $A(d,1)$,

$$\Omega = \{\omega = \sum_{i=1}^{\infty} \delta_{L_i} : L_i \in A(d,1) \text{ and } \omega(\mathcal{L}_A) < \infty \text{ for all compact } A \subset \mathbb{R}^d\},$$

where δ_L denotes point measure at L. By standard abuse of notation, we will confuse the random measure $\omega \in \Omega$ with its support supp(ω), which is really a subset of $A(d,1)$.

We define Ψ to be a Poisson point process on $A(d,1) \times \mathbb{R}^+$ with intensity measure $\mu_{d,1} \times \lambda_1^+$ where λ_1^+ denotes Lebesgue measure on $\mathbb{R}^+$. We think of an element $(L,s) \in \Psi$ as a line in $\mathbb{R}^d$ accompanied with a time-stamp s. We then let $\omega_t = \Pi_{A(d,1)}\{(L,s) \in A(d,1) \times \mathbb{R}^+ : s \leq t\}$, where $\Pi_{A(d,1)}$ denotes projection onto the space $A(d,1)$. Thus, ω_t is the collection of lines which have been placed before or at time t, and $\omega_t \in \Omega$. Of course, by our definition, ω_t is in fact a Poisson process on $A(d,1)$ with intensity measure $t\mu_{d,1}$. Similarly, we let $\omega_{t_1,t_2} = \Pi_{A(d,1)}\{(L,s) \in A(d,1) \times \mathbb{R}^+ : t_1 < s \leq t_2\}$, so that ω_{t_1,t_2} is the set of lines placed between times t_1 and t_2. The intensity measure of ω_{t_1,t_2} is therefore $(t_2 - t_1)\mu_{d,1}$. Obviously we have that $\omega_{t_1} \cup \omega_{t_1,t_2} = \omega_{t_2}$, and we will let $(\omega_t)_{t\geq 0}$ denote the corresponding process.

We will sometimes slightly abuse notation by writing $\mathfrak{c}(L) \in \omega_t$ instead of $L \in \omega_t$, and we will often think of (and refer to) ω_t as a collection of cylinders instead of lines.

2.2 The set A^ρ and the cover times $T_d^\rho(A)$ and $T_w^\rho(A)$

For any $\rho > 0$, a set of points $\{x_1, \ldots, x_N\}$ such that $d(x_i, x_j) \geq \rho$ whenever $i \neq j$ will be called ρ-separated. Given a bounded set $A \subset \mathbb{R}^d$ and $\rho \in (0,1)$ let

$$N_\rho(A) = \max\{n : \exists x_1, \ldots, x_n \in A \text{ such that } d(x_i, x_j) \geq \rho \text{ for all } i \neq j\}.$$

Then, let A^ρ be one of these sets chosen by some predetermined rule. We see that A^ρ is a ρ-separated subset of A containing the maximal number of points $N_\rho(A)$. For future reference, we observe that since $A^{\rho/n}$ is a ρ/n-separated set in A, it follows that $n(A^{\rho/n})$ is in fact a ρ-separated set in nA. It follows that $|A^{\rho/n}| \leq |(nA)^\rho|$, and since the reverse inequlity can be established in the same way, we conclude that

$$|A^{\rho/n}| = |(nA)^\rho|. \tag{7}$$

Recall the definitions of the discrete cover time $T_d^\rho(A)$ and the well cover time $T_w^\rho(A)$ (i.e (3) and (5)). We will need the following related concept. A ball $B(x,\rho)$ is *singularly covered* by time t if there exists $L \in \omega_t$ such that $B(x,\rho) \subset \mathfrak{c}(L)$. Alternatively, the point x is *ρ-singularly covered* by time t if the corresponding ball $B(x,\rho)$ is singularly covered. The quantity

$$T_s^\rho(x) := \inf\{t > 0 : \exists L \in \omega_t \text{ such that } B(x,\rho) \subset \mathfrak{c}(L)\}$$

is then the time at which the point x is ρ-singularly covered. It follows that the set A is ρ-well-covered if the balls $B(x_i,\rho)$ are singularly covered for all $x_i \in A^\rho$.

In this paper, we are interested in the asymptotic behaviour of $\mathcal{T}(A)$ as the size of set A increases. Clearly we have that

$$T_d^\rho(A, (\omega_t)_{t\geq 0}) \leq \mathcal{T}(A, (\omega_t)_{t\geq 0}) \leq T_w^\rho(A, (\omega_t)_{t\geq 0}), \tag{8}$$

which is the formal way of describing the coupling discussed in the Introduction. It is also in this sense that the inequality and equality of (4) should be understood. Therefore, by studying the distributions of $T_w^\rho(A)$ and $T_d^\rho(A)$, we can find bounds on $\mathcal{T}(A)$.

Note that $\mathcal{T}(A) \leq T_w^\rho(A)$ for *any* choice of $\rho > 0$. Similar to what we discussed in the Introduction concerning $T_d^\rho(A)$, we see that the smaller we take ρ, the less we "waste" by covering the point x ρ-singularly (instead of x just being covered). However, as before, by taking ρ very small, our methods will yield progressively worse bounds (see Theorem 5.5).

The overall strategy to deal with $T_d^\rho(A)$ and $T_w^\rho(A)$ can be explained as follows. Informally, if the cover time is τ, then immediately before this time (i.e. at time $\tau(1-\epsilon)$), the set which is uncovered will with high probability consist of small islands which are well separated. Indeed, this is the case as Proposition 5.3 shows. In addition, Proposition 5.3

tells us that the number of such islands will be highly concentrated around its expected value. This can then be combined with Proposition 5.1 which tells us that the cover times of these well separated islands behave almost independently. These two results are then combined into Theorem 5.5, which provides the final estimates on the well cover time $T_w^\rho(A)$. By a similar approach, we can then prove the corresponding statement for $T_d^\rho(A)$, namely Theorem 6.4. However, since the proof of this theorem is almost the same as the one for Theorem 5.5, we will only give a short walk-through of the minor changes necessary rather than providing a complete proof. Finally, Theorems 5.5 and 6.4 will be used to prove Theorem 1.1.

2.3 Box dimensions

In this subsection we shall review some basic properties of Box dimensions, sometimes referred to as Minkowski dimensions (see Falconer [5] Chapter 3).

Usually when defining box dimensions, one starts by counting the minimum number of boxes $N(A,\epsilon)$ of side length $\epsilon > 0$ needed to cover a set A, and then define the quantity

$$\overline{\dim}_B(A) := \limsup_{\epsilon \to 0} \frac{\log N(A,\epsilon)}{-\log \epsilon},$$

called the *upper box dimension* of the set A. It is not hard to prove that (see [5] p.41) we must then also have that

$$\overline{\dim}_B(A) = \limsup_{\rho \to 0} \frac{\log |A^\rho|}{-\log \rho}.$$

Similarly, by defining

$$\underline{\dim}_B(A) := \liminf_{\epsilon \to 0} \frac{\log N(A,\epsilon)}{-\log \epsilon},$$

to be the *lower box dimension* of the set A, one finds that

$$\underline{\dim}_B(A) = \liminf_{\rho \to 0} \frac{\log |A^\rho|}{-\log \rho}.$$

If these coincides, then

$$\dim_B(A) = \overline{\dim}_B(A) = \underline{\dim}_B(A)$$

is simply called the Box dimension of A.

The two quantities in (1) is related to the upper and lower Minkowski content (see [6] Sections 3.2.37-3.2.44). It is beyond the scope of this paper to investigate this relationship in detail. However, as mentioned already in the remarks after the statement of Theorem 1.1, it will be the case that (2) (and so in turn (1)) is satisfied for many sets. See in particular Proposition 4.2.

We can now present Theorem 2.1 mentioned in the remarks after Theorem 1.1.

Theorem 2.1. *For any $\underline{\alpha} < \underline{\dim}_B(A)$, we have that for every $z \in \mathbb{R}$,*

$$\lim_{n \to \infty} \mathbb{P}(\mathcal{T}(nA) - \underline{\alpha} \log n \leq z) = 0.$$

Furthermore, for any $\overline{\alpha} > \overline{\dim}_B(A)$, we have that for every $z \in \mathbb{R}$,

$$\lim_{n \to \infty} \mathbb{P}(\mathcal{T}(nA) - \overline{\alpha} \log n \leq z) = 1.$$

Remark: One can consider the cover times along a subsequence $(n_k)_{k\geq 1}$ such that

$$\lim_{k\to\infty} \frac{|A^{1/n_k}|}{-\log n_k} = \overline{\dim}_B(A).$$

If in addition,

$$0 < \liminf_{k\to\infty} n_k^{-\overline{\dim}_B(A)} |A^{1/n_k}| \leq \limsup_{k\to\infty} n_k^{-\overline{\dim}_B(A)} |A^{1/n_k}| < \infty,$$

then, one will obtain a result resembling Theorem 1.1 but along this subsequence, and with $\overline{\dim}_B(A)$ in place of $\dim_B(A)$.

We end this section with a comment on notation. We shall frequently use c to denote a constant (depending only on d) which may change from line to line. In contrast, numbered constants c_k will be fixed.

3 Preliminary results concerning the Poisson cylinder model

The quantity $(1-\rho)^{d-1}$ will appear in many places, so we will let $\gamma(\rho) := (1-\rho)^{d-1}$. The purpose of this section is to establish a number of preliminary results concerning the measure $\mu_{d,1}$. These results will then be used throughout the rest of this paper.

Lemma 3.1. *Let $0 < \rho < 1$ and $K \subset \mathbb{R}^d$ be a compact set. We have that*

a) for any $c > 0$, $\mu_{d,1}(\mathcal{L}_{cK}) = c^{d-1}\mu_{d,1}(\mathcal{L}_K)$,

b) $\mu_{d,1}\left(\mathcal{L}_{B(x,1-\rho)}\right) = \gamma(\rho)$,

c) $\mu_{d,1}\left(\mathcal{L}_{B(x,1-\rho)} \cup \mathcal{L}_{B(y,1-\rho)}\right) = \gamma(\rho)\mu_{d,1}\left(\mathcal{L}_{B(x',1)} \cup \mathcal{L}_{B(y',1)}\right)$,

d) $\mu_{d,1}\left(\mathcal{L}_{B(x,1-\rho),B(y,1-\rho)}\right) = \gamma(\rho)\mu_{d,1}\left(\mathcal{L}_{B(x',1),B(y',1)}\right)$,

where (x', y') is any pair of points such that $d(x,y) = (1-\rho)\, d(x',y')$.

Proof. For part a) we start by noting that for any fixed $L \in G(d,1)$ the set of $y \in L^\perp$ such that $L + y \in \mathcal{L}_K$ is precisely $\Pi_{L^\perp}(K)$. Recalling that λ_d denotes d-dimensional Lebesgue measure, we see that

$$\int_{L^\perp} \mathbb{1}(L + y \in \mathcal{L}_K)\lambda_{d-1}(dy) = \lambda_{d-1}\left(\Pi_{L^\perp}(K)\right).$$

Thus, by equation (6),

$$\begin{aligned}\mu_{d,1}(\mathcal{L}_{cK}) &= \frac{1}{\kappa_{d-1}} \int_{G(d,1)} \lambda_{d-1}\left(\Pi_{L^\perp}(cK)\right)\nu_{d,1}(dL) \\ &= \frac{c^{d-1}}{\kappa_{d-1}} \int_{G(d,1)} \lambda_{d-1}\left(\Pi_{L^\perp}(K)\right)\nu_{d,1}(dL) = c^{d-1}\mu_{d,1}(\mathcal{L}_K),\end{aligned}$$

as required.

For part b) we simply note that

$$\begin{aligned}\mu_{d,1}\left(\mathcal{L}_{B(x,1-\rho)}\right) &= \mu_{d,1}\left(\mathcal{L}_{B(o,1-\rho)}\right) \\ &= \mu_{d,1}\left(\mathcal{L}_{(1-\rho)B(o,1)}\right) = (1-\rho)^{d-1}\mu_{d,1}\left(\mathcal{L}_{B(o,1)}\right) = \gamma(\rho),\end{aligned}$$

where we use translation invariance, part a) and the normalization of $\mu_{d,1}$.

For parts c) and d) we can assume without loss of generality that $x = o$. Then we have that

$$\begin{aligned}&\mu_{d,1}\left(\mathcal{L}_{B(o,1-\rho)} \cup \mathcal{L}_{B(y,1-\rho)}\right) = \mu_{d,1}\left(\mathcal{L}_{B(o,1-\rho)\cup B(y,1-\rho)}\right) \\ &\quad= \mu_{d,1}\left(\mathcal{L}_{(1-\rho)\left(B(o,1)\cup B\left(\frac{y}{1-\rho},1\right)\right)}\right) = (1-\rho)^{d-1}\mu_{d,1}\left(\mathcal{L}_{B(o,1)\cup B\left(\frac{y}{1-\rho},1\right)}\right),\end{aligned}$$

and part c) is proved.

Finally, to prove part d), note that,

$$\begin{aligned}&\mu_{d,1}\left(\mathcal{L}_{B(o,1-\rho),B(y,1-\rho)}\right) \\ &\quad= \mu_{d,1}\left(\mathcal{L}_{B(o,1-\rho)}\right) + \mu_{d,1}\left(\mathcal{L}_{B(y,1-\rho)}\right) - \mu_{d,1}\left(\mathcal{L}_{B(o,1-\rho)} \cup \mathcal{L}_{B(y,1-\rho)}\right)\end{aligned}$$

and use part a), part c) and translation invariance to get

$$\begin{aligned}&\mu_{d,1}\left(\mathcal{L}_{B(o,1-\rho)}\right) + \mu_{d,1}\left(\mathcal{L}_{B(y,1-\rho)}\right) - \mu_{d,1}\left(\mathcal{L}_{B(o,1-\rho)} \cup \mathcal{L}_{B(y,1-\rho)}\right) \\ &\quad= (1-\rho)^{d-1}\left(\mu_{d,1}\left(\mathcal{L}_{B(o,1)}\right) + \mu_{d,1}\left(\mathcal{L}_{B\left(\frac{y}{1-\rho},1\right)}\right) - \mu_{d,1}\left(\mathcal{L}_{B(o,1)} \cup \mathcal{L}_{B\left(\frac{y}{1-\rho},1\right)}\right)\right) \\ &\quad= \gamma(\rho)\mu_{d,1}\left(\mathcal{L}_{B(o,1),B\left(\frac{y}{1-\rho},1\right)}\right).\end{aligned}$$

□

The next result allows us to estimate the measure of the set of cylinders intersecting two distant balls. This lemma was first published as Lemma 3.1 of [14]. Here, we present a sketch for sake of completeness.

Lemma 3.2. *Let $x_1, x_2 \in \mathbb{R}^d$ and let $r = d(x_1, x_2)$. There exists constants c_1 and c_2 depending only on d such that*

$$\frac{c_1}{r^{d-1}} \leq \mu_{d,1}\left(\mathcal{L}_{B(x_1,1),B(x_2,1)}\right) \leq \frac{c_2}{r^{d-1}},$$

for every $r \geq 4$.

Sketch of proof By translation invariance of $\mu_{d,1}$, we can, without loss of generality assume that $x_1 = o$ so that x_2 is located on the surface of $B(o, r)$. Furthermore, we need order $r^{(d-1)}$ balls of radius 1 to cover the surface of $B(o, r)$. By symmetry, a random line passing through $B(o, 1)$ will hit any fixed ball in the cover with equal probability. Thus, the probability that it will hit $B(x_2, 1)$ must be of order $r^{-(d-1)}$. □

A direct application of Lemmas 3.1 and 3.2 yields our next result.

Lemma 3.3. *There are constants c_1 and c_2 depending only on d such that for any $\rho > 0$, and $x_1, x_2 \in \mathbb{R}^d$ with $r := d(x_1, x_2) \geq 4(1-\rho)$, we have that*

$$c_1 \frac{\gamma(\rho)^2}{r^{d-1}} \leq \mu_{d,1}\left(\mathcal{L}_{B(x_1,1-\rho),B(x_2,1-\rho)}\right) \leq c_2 \frac{\gamma(\rho)^2}{r^{d-1}}.$$

Proof. By part d) of Lemma 3.1 and using Lemma 3.2 we get that

$$\begin{aligned}\mu_{d,1}\left(\mathcal{L}_{B(x_1,1-\rho),B(x_2,1-\rho)}\right) &= (1-\rho)^{d-1}\mu_{d,1}\left(\mathcal{L}_{B(x_1',1),B(x_2',1)}\right)\\ &\leq (1-\rho)^{d-1}\frac{c_2}{(r/(1-\rho))^{d-1}} = c_2\frac{\gamma(\rho)^2}{r^{d-1}}\end{aligned}$$

as required. The lower bound follows in the same way. □

The next lemma states that, despite the long-range correlation nature of the Poisson cylinder process, events occurring in two distant sets are almost independent. The key point of the proof is to note that the events become independent after we conditioned on the event that no lines intersect both sets.

Lemma 3.4. *Let $K_1, K_2 \subset \mathbb{R}^d$ be disjoint sets and let E_1 and E_2 be events depending only on ω_t in $\mathcal{L}_{K_1}$ and $\mathcal{L}_{K_2}$ respectively. Then:*

$$|\mathbb{P}(E_1 \cap E_2) - \mathbb{P}(E_1)\mathbb{P}(E_2)| \leq 4\mathbb{P}(\omega_t(\mathcal{L}_{K_1,K_2}) \neq 0).$$

Proof. Note that

$$\begin{aligned}&\mathbb{P}(E_1 \cap E_2) \\ &\quad= \mathbb{P}(E_1|\omega_t(\mathcal{L}_{K_1,K_2}) = 0)\mathbb{P}(E_2|\omega_t(\mathcal{L}_{K_1,K_2}) = 0)\mathbb{P}(\omega_t(\mathcal{L}_{K_1,K_2}) = 0)\\ &\quad\quad+\mathbb{P}(E_1 \cap E_2|\omega_t(\mathcal{L}_{K_1,K_2} \neq 0)\mathbb{P}(\omega_t(\mathcal{L}_{K_1,K_2}) \neq 0),\end{aligned} \tag{9}$$

since the events E_1 and E_2 are conditionally independent on $\omega_t(\mathcal{L}_{K_1,K_2}) = 0$. Furthermore, writing

$$\begin{aligned}\mathbb{P}(E_i) = \mathbb{P}(E_i|\omega_t(\mathcal{L}_{K_1,K_2}) = 0)\mathbb{P}(\omega_t(\mathcal{L}_{K_1,K_2}) = 0)\\ +\mathbb{P}(E_i|\omega_t(\mathcal{L}_{K_1,K_2}) \neq 0)\mathbb{P}(\omega_t(\mathcal{L}_{K_1,K_2}) \neq 0)\end{aligned}$$

for $i = 1, 2$ and using (9), a straightforward calculation gives us that

$$|\mathbb{P}(E_1 \cap E_2) - \mathbb{P}(E_1)\mathbb{P}(E_2)| \leq 4\mathbb{P}(\omega_t(\mathcal{L}_{K_1,K_2}) \neq 0),$$

as desired. □

As explained in Section 2, one step to prove the main theorems is to show that the cover times of distant and small sets $K_1, \ldots, K_{n+1}$ are almost independent. If we consider one of these sets K_j, the next lemma gives us bounds on the probability that there exists a line passing through K_j and any one of the other sets K_i, $i \neq j$.

Lemma 3.5. *Let $0 < \rho < 1$ and $\{K_i\}_{i=1}^{n+1}$ be a family of sets such that for every i, $K_i \subset B(x_i, 1-\rho)$ for some $x_i \in \mathbb{R}^d$. Assume also that $d(x_i, x_j) = r_{ij} \geq 4(1-\rho)$ for every $i \neq j$ and let $r = \min_{i\neq j} r_{ij}$. Then, for all $1 \leq j \leq n+1$ we have*

$$\mathbb{P}\left(\omega_t\left(\mathcal{L}_{\bigcup_{i=1}^{n+1} K_i \setminus K_j, K_j}\right) \neq 0\right) \leq ntc_2\frac{\gamma(\rho)^2}{r^{d-1}},$$

where c_2 is the same as in Lemma 3.3.

Proof. Fix j and let $B = \bigcup_{i=1}^{n+1} B(x_i, 1-\rho)$. By Lemma 3.3 and a simple union bound we have that

$$\mu_{d,1}\left(\mathcal{L}_{(B\setminus B(x_i,1-\rho)),B(x_i,1-\rho)}\right) \leq nc_2 \frac{(1-\rho)^{2(d-1)}}{r^{d-1}}.$$

Therefore we get that

$$\mathbb{P}\left(\omega_t\left(\mathcal{L}_{\bigcup_{i=1}^{n+1} K_i\setminus K_j, K_j}\right) \neq 0\right) \leq \mathbb{P}\left(\omega_t\left(\mathcal{L}_{B\setminus B(x_j,1-\rho),B(x_j,1-\rho)}\right) \neq 0\right)$$
$$= 1 - \exp\left(-t\mu_{d,1}\left(\mathcal{L}_{B\setminus B(x_j,1-\rho),B(x_j,1-\rho)}\right)\right) \leq 1 - \exp\left(-ntc_2\frac{(1-\rho)^{2(d-1)}}{r^{d-1}}\right).$$

The result follows by using that $e^{-x} \geq 1 - x$ for all $x \in \mathbb{R}$. □

In Lemma 3.2 we considered $\mu_{d,1}(\mathcal{L}_{B(o,1)} \cap \mathcal{L}_{B(re_1,1)})$ for large values of r. Next, we need to consider this for small values of r. In contrast to Lemma 3.2, no such result exists in the literature, and so we provide a full proof.

Proposition 3.6. *For any $d \geq 2$, if $r \leq 2\sqrt{1-4^{-1/(d-2)}}$ we have that*

$$\mu_{d,1}\left(\mathcal{L}_{B(o,1)} \cap \mathcal{L}_{B(re_1,1)}\right) \leq 1 - \frac{r}{12}.$$

Remark: For $d = 2$ we interpret the upper bound on r as being equal to 2.

Proof. From (6) it follows as in the proof of Lemma 3.1 that

$$\mu_{d,1}\left(\mathcal{L}_{B(o,1)} \cap \mathcal{L}_{B(re_1,1)}\right) = \frac{1}{\kappa_{d-1}} \int_{G(d,1)} \lambda_{d-1}\left(\Pi_{L^\perp}(B(o,1)) \cap \Pi_{L^\perp}(B(re_1,1))\right) \nu_{d,1}(dL). \tag{10}$$

If $L \in G(d,1)$ is written as $L = \{s(l_1, ..., l_d) : s \in \mathbb{R}\}$ where $l_1^2 + ... + l_d^2 = 1$, then the projection matrix $\Pi_{L^\perp}$ has elements $(\Pi_{L^\perp})_{ii} = 1 - l_i^2$ and $(\Pi_{L^\perp})_{ij} = -l_i l_j$ for $i \neq j$. Let $p_r = p_r(L) = \Pi_{L^\perp}((r, 0, \dots, 0))$ so that p_r is the projection of the center of $B(re_1, 1)$. Of course, the projection of $B(o, 1)$ onto $L^\perp$ is then a $(d-1)$-dimensional ball of radius 1 centred at o. Straightforward calculations yield that

$$p_r = r(1 - l_1^2, -l_1 l_2, \dots, -l_1 l_d),$$

so that

$$|p_r|^2 = r^2((1-l_1^2)^2 + l_1^2 l_2^2 + \cdots + l_1^2 l_d^2) = r^2(1 - 2l_1^2 + l_1^2(l_1^2 + \cdots + l_d^2)) = r^2(1 - l_1^2).$$

Of course, $\Pi_{L^\perp}(B(o,1))$ and $\Pi_{L^\perp}(B(re_1,1))$ intersects whenever $|p_r| \leq 2$, or equivalently whenever $r^2(1 - l_1^2) \leq 4$. Since we are assuming that $r \leq 2$, this is always satisfied. Furthermore, the ($(d-1)$-dimensional) volume of the lens-shaped area $\Pi_{L^\perp}(B(o,1)) \cap \Pi_{L^\perp}(B(re_1,1))$ is then the sum of the volumes of two spherical caps of height $h = 1 - |p_r|/2$. The volume of one such spherical cap is (see [9]) given by

$$\frac{1}{2}\kappa_{d-1} J_{2h-h^2}\left(\frac{d}{2}, \frac{1}{2}\right).$$

As above, κ_{d-1} is the volume of the unit ball in $\mathbb{R}^{d-1}$ while J_{2h-h^2} denotes a regularized incomplete beta function. We note that $2h - h^2 = 1 - |p_r|^2/4 = 1 - r^2(1 - l_1^2)/4$, so that (10) becomes

$$\mu_{d,1}\left(\mathcal{L}_{B(o,1)} \cap \mathcal{L}_{B(re_1,1)}\right) = \int_{G(d,1)} J_{1-r^2(1-l_1^2)/4}\left(\frac{d}{2}, \frac{1}{2}\right) \nu_{d,1}(dL). \tag{11}$$

Furthermore,

$$J_{1-r^2(1-l_1^2)/4}\left(\frac{d}{2}, \frac{1}{2}\right) = \frac{\int_0^{1-r^2(1-l_1^2)/4} t^{\frac{d}{2}-1}(1-t)^{\frac{1}{2}-1}dt}{\int_0^1 t^{\frac{d}{2}-1}(1-t)^{\frac{1}{2}-1}dt}.$$

Let $D_d := \int_0^1 t^{\frac{d}{2}-1}(1-t)^{-\frac{1}{2}}dt$ so that

$$J_{1-r^2(1-l_1^2)/4}\left(\frac{d}{2}, \frac{1}{2}\right) = 1 - \frac{1}{D_d}\int_{1-r^2(1-l_1^2)/4}^1 t^{\frac{d}{2}-1}(1-t)^{-\frac{1}{2}}dt. \tag{12}$$

Furthermore, we trivially have that (since $d \geq 2$,)

$$D_d = \int_0^1 t^{\frac{d}{2}-1}(1-t)^{-\frac{1}{2}}dt \leq \int_0^1 (1-t)^{-1/2}dt = 2. \tag{13}$$

We proceed to bound the integral on the right hand side of (12) from below. We have that

$$\begin{aligned}
&\int_{1-r^2(1-l_1^2)/4}^1 t^{\frac{d}{2}-1}(1-t)^{-\frac{1}{2}}dt \\
&\quad \geq (1-r^2(1-l_1^2)/4)^{\frac{d}{2}-1}\int_{1-r^2(1-l_1^2)/4}^1 (1-t)^{-\frac{1}{2}}dt \\
&\quad = (1-r^2(1-l_1^2)/4)^{\frac{d}{2}-1} 2\sqrt{r^2(1-l_1^2)/4} \\
&\quad = (1-r^2(1-l_1^2)/4)^{\frac{d}{2}-1} r\sqrt{1-l_1^2},
\end{aligned}$$

so that by (11), (12) and (13) we have that

$$\begin{aligned}
&\mu_{d,1}\left(\mathcal{L}_{B(o,1)} \cap \mathcal{L}_{B(re_1,1)}\right) \\
&\leq \int_{G(d,1)} 1 - \frac{1}{2}\left((1-r^2(1-l_1^2)/4)^{\frac{d}{2}-1} r\sqrt{1-l_1^2}\right)\nu_{d,1}(dL) \\
&= 1 - \frac{r}{2}\int_{G(d,1)} (1-r^2(1-l_1^2)/4)^{\frac{d}{2}-1}\sqrt{1-l_1^2}\nu_{d,1}(dL),
\end{aligned} \tag{14}$$

which uses that $\nu_{d,1}(G(d,1)) = 1$ (see Section 2).

In order to estimate the right hand side of (14), consider the sets $G_k := \{L \in G(d,1) : l_k^2 > 1/2\}$. Since $l_1^2 + \cdots + l_d^2 = 1$, we clearly have that $G_k \cap G_m = \emptyset$ if $k \neq m$. Therefore

$$1 = \nu_{d,1}(G(d,1)) \geq \nu_{d,1}(\cup_{k=1}^d G_k) = d\nu_{d,1}(G_1),$$

so that $\nu_{d,1}(G_1) \leq 1/d$. We get that

$$\int_{G(d,1)} \mathbb{1}(|l_1| \leq 1/\sqrt{2}) \nu_{d,1}(dL) = 1 - \nu_{d,1}(G_1) \geq \frac{d-1}{d}. \tag{15}$$

We will split (14) into two cases. First, consider $d = 2$, so that

$$\begin{aligned}
&\mu_{2,1}(\mathcal{L}_{B(o,1)} \cap \mathcal{L}_{B(re_1,1)}) \\
&\quad \leq 1 - \frac{r}{2} \int_{G(2,1)} \sqrt{1 - l_1^2} \nu_{2,1}(dL) \\
&\quad \leq 1 - \frac{r}{2} \int_{G(2,1)} \sqrt{1 - l_1^2} \mathbb{1}(|l_1| \leq 1/\sqrt{2}) \nu_{2,1}(dL) \\
&\quad \leq 1 - \frac{r}{2\sqrt{2}} \int_{G(2,1)} \mathbb{1}(|l_1| \leq 1/\sqrt{2}) \nu_{2,1}(dL) \leq 1 - \frac{r}{4\sqrt{2}},
\end{aligned}$$

where we use (15) in the last inequality.

Second, consider any $d \geq 3$. Since $r \leq 2\sqrt{1 - 4^{-1/(d-2)}}$ it follows that

$$1 - \frac{r^2}{4} \geq 2^{-2/(d-2)}.$$

We therefore get that

$$(1 - r^2(1 - l_1^2)/4)^{\frac{d}{2}-1} \geq (1 - r^2/4)^{\frac{d-2}{2}} \geq \frac{1}{2}.$$

Hence, by (14) and the above, we conclude that

$$\begin{aligned}
&\mu_{d,1}\left(\mathcal{L}_{B(o,1)} \cap \mathcal{L}_{B(re_1,1)}\right) \\
&\quad \leq 1 - \frac{r}{2} \int_{G(d,1)} (1 - r^2(1 - l_1^2)/4)^{\frac{d}{2}-1} \sqrt{1 - l_1^2} \nu_{d,1}(dL) \\
&\quad \leq 1 - \frac{r}{4} \int_{G(d,1)} \sqrt{1 - l_1^2} \nu_{d,1}(dL) \\
&\quad \leq 1 - \frac{r}{8} \int_{G(d,1)} \mathbb{1}(|l_1| \leq 1/\sqrt{2}) \nu_{d,1}(dL) \\
&\quad \leq 1 - \frac{r}{8}\frac{d-1}{d} \leq 1 - \frac{r}{12},
\end{aligned}$$

where we use (15) in the penultimate inequality. □

Define

$$\begin{aligned}
\beta(\rho, k) &:= \mu_{d,1}\left(\mathcal{L}_{B(o,1) \cup B\left(\frac{2^k \rho}{1-\rho} e_1, 1\right)}\right) \\
&= \gamma^{-1}(\rho) \mu_{d,1}\left(\mathcal{L}_{B(o,1-\rho)} \cup \mathcal{L}_{B\left(2^k \rho e_1, 1-\rho\right)}\right),
\end{aligned} \tag{16}$$

where the equality comes from part c) of Lemma 3.1. By part b) of Lemma 3.1 we have that $\gamma(\rho) = \mu_{d,1}\left(\mathcal{L}_{B(x,1-\rho)}\right)$ so we can think of $\beta(\rho, k)$ as being the ratio between the measure of the set of lines that intersects the union of two balls of radii $1 - \rho$ at distance $2^k \rho$, and the measure of the set of lines that intersect a ball of radius $1 - \rho$. The next lemma gives both upper and lower bounds for $\beta(\rho, k)$, that will be used in the proofs of Lemma 5.2. The proof is an application of the previous proposition.

Lemma 3.7. *We have that* $1 + \frac{2^k}{12}\rho < \beta(\rho, k) < 2$*, for* $0 < \rho < 2/3$ *and all* k *such that* $2^k \frac{\rho}{1-\rho} \leq 2\sqrt{1 - 4^{-1/(d-2)}}$.

Proof. Since $B(o, 1-\rho) \subset B(o, 1)$, we have $\beta(\rho, k) < 2\mu_{d,1}\left(\mathcal{L}_{B(o,1)}\right) = 2$. For the lower bound, note that,

$$\begin{aligned}
\beta(\rho, k) &= \mu_{d,1}\left(\mathcal{L}_{B(o,1)} \cup \mathcal{L}_{B\left(\frac{2^k\rho}{1-\rho}e_1, 1\right)}\right) \\
&= \mu_{d,1}\left(\mathcal{L}_{B(o,1)}\right) + \mu_{d,1}\left(\mathcal{L}_{B\left(\frac{2^k\rho}{1-\rho}e_1, 1\right)}\right) - \mu_{d,1}\left(\mathcal{L}_{B(o,1)} \cap \mathcal{L}_{B\left(\frac{2^k\rho}{1-\rho}e_1, 1\right)}\right) \\
&\geq 2 - \left(1 - 2^k \frac{\rho}{12(1-\rho)}\right) \geq 1 + \frac{2^k}{12}\rho,
\end{aligned}$$

by using Proposition 3.6 with $r = 2^k\rho/(1-\rho)$ and that $1 - \rho < 1$. □

4 Basic properties of the set A^ρ

In the end (see Theorems 5.5 and 6.4), many estimates will be of the form $|A^\rho|^{-\rho}$. In order for this to translate into something more tangible, we will need to be able to relate $|A^\rho|$ and $|A^\delta|$ for various values of δ, and especially to $\delta = 1$ which will play a central role. This is the purpose of Lemma 4.1.

Lemma 4.1. *Let* $0 < \rho < 1$.

a) For any A *bounded we have that*

$$A \subseteq \bigcup_{x \in A^\rho} B(x, \rho).$$

b) For any A *bounded and all* $\rho < \delta \leq 1$ *we have that* $\left|A^\delta\right| \leq |A^\rho|$.

c) Let $A_1 \cap A_2 = \emptyset$ *and* $B = A_1 \cup A_2$*. Then we have that* $|B^\rho| \leq |A_1^\rho| + |A_2^\rho|$.

d) For any set A *we have that*

$$|A^1| \leq |A^\rho| \leq 6^d \rho^{-d} |A^1|.$$

Proof. For part *a*) we start by assuming that $A \not\subseteq \bigcup_{x \in A^\rho} B(x, \rho)$ and take $y \in A \setminus \bigcup_{x \in A^\rho} B(x, \rho)$. Then, $y \in A$ and $d(x, y) \geq \rho$ for all $x \in A^\rho$. Therefore the set $\{x_1, \ldots, x_{N_\rho(A)}, y\} \subset A$ is ρ-separated, contradicting the maximality of $N_\rho(A)$.

For part *b*), let $\{x_1, \ldots, x_{N_\delta(A)}\} \subset A$ be a δ-separated set. Clearly,

$$\{x_1, \ldots, x_{N_\delta(A)}\}$$

is also a ρ-separated set and therefore $\left|A^\delta\right| = N_\delta(A) \leq |A^\rho|$.

For part *c*), assume that $|B^\rho| > |A_1^\rho| + |A_2^\rho|$. Then we get that

$$|A_1^\rho| + |A_2^\rho| < |B^\rho \cap A_1| + |B^\rho \cap A_2|,$$

and so without loss of generality we may assume that $|A_1^\rho| < |B^\rho \cap A_1|$. However, $B^\rho \cap A_1$ is a ρ-separated set in A_1, so this would contradict the maximality of A_1^ρ.

For part *d)* the lower bound is proved by noting that $\rho < 1$ and using part *b)* of this Lemma. For the upper bound, note that

$$\bigcup_{x \in B(o,1)^\rho} B(x, \rho/3) \subset B(o, 1+\rho)$$

and $B(x_i, \rho/3) \cap B(x_j, \rho/3) = \emptyset$ for all distinct $x_i, x_j \in B(o,1)^\rho$, since $d(x_i, x_j) \geq \rho$ by the definition of $B(o,1)^\rho$. Then

$$|B(o,1)^\rho| \, \lambda_d(B(x_1, \rho/3)) \leq \lambda_d(B(o, 1+\rho)) = \kappa_d(1+\rho)^d \leq \kappa_d 2^d,$$

where we used $\rho < 1$ in the last inequality. Since $\lambda_d(B(x_1, \rho/3)) = \kappa_d(\rho/3)^d$ we have that

$$|B(o,1)^\rho| \leq 6^d \rho^{-d}. \tag{17}$$

Let $x_1, \ldots x_{|A^1|}$ be an enumeration of the points in the set A^1. Set $D_1 = B(x_1, 1)$ and then recursively, let $D_j = B(x_j, 1) \setminus \cup_{i=1}^{j-1} B(x_i, 1)$. Obviously, $D_j \subset B(x_j, 1)$, $D_i \cap D_j = \emptyset$ and $A \subset \cup_{i=1}^{|A^1|} D_i$. Furthermore, note that

$$\begin{aligned}|A^\rho| &\leq |(\cup_{x \in A^1} B(x,1))^\rho| \\ &= \left|\left(\cup_{i=1}^{|A^1|} D_i\right)^\rho\right| \leq \sum_{i=1}^{|A^1|} |D_i^\rho| \leq \sum_{i=1}^{|A^1|} |B(x_i,1)^\rho| = \left|A^1\right| |B(o,1)^\rho|,\end{aligned}$$

where we use part *c)* in the last two inequalities. Equation (17) then implies the required upper bound. □

We now give an easy proposition showing that $A = [0,1]^d$ satisfies condition (2), and as a consequence many sets will satisfy (1). Presumably, results such as this are well known, even though we could not find a reference for this exact statement. Thus, the purpose of this proposition is to show that the conditions of Theorem 1.1 will indeed be satisfied for many sets. The purpose is not to provide an original statement.

Proposition 4.2. *Let $A = [0,1]^d$. Then,*

$$0 < \lim_{\rho \to 0} \rho^d |A^\rho| < \infty,$$

and in particular the limit exists. It follows that any bounded set A such that $[x, x+\delta]^d \subset A$ for some $x \in \mathbb{R}^d$ and $\delta > 0$, must satisfy (1).

Proof. Let $c \in \mathbb{R}^+$ and note that $A = [0,1]^d$ is contained in the union of $(\lceil (c\rho)^{-1} \rceil)^d$ translates of the set $A_{c\rho} = [0, c\rho]^d$. As in (7) we have that $|A_{c\rho}^\rho| = |A^{\rho/(c\rho)}| = |A^{1/c}|$. It follows from Lemma 4.1 part *c)*, that

$$|A^\rho| \leq (\lceil (c\rho)^{-1} \rceil)^d |A_{c\rho}^\rho| \leq ((c\rho)^{-1} + 1)^d |A^{1/c}|. \tag{18}$$

Therefore,

$$\limsup_{\rho \to 0} \rho^d |A^\rho| \leq \limsup_{\rho \to 0} (c^{-1} + \rho)^d |A^{1/c}| = c^{-d} |A^{1/c}|,$$

and so

$$\limsup_{\rho\to 0} \rho^d |A^\rho| \le \liminf_{c\to\infty} c^{-d}|A^{1/c}| = \liminf_{\rho\to 0} \rho^d|A^\rho|.$$

This proves that the limit exists.

We then observe that (18) immediately implies that the limit is finite. Furthermore, we must have that

$$|A^\rho| \ge |A \cap (\rho\mathbb{Z}^d)| \ge (\rho^{-1}-1)^d,$$

and so $\lim_{\rho\to 0} \rho^d|A^\rho| \ge 1$.

We now turn to the second statement. Let $B = [x, x+\delta]^d \subset A$ so that

$$\liminf_{\rho\to 0} \rho^d|A^\rho| \ge \liminf_{\rho\to 0} \rho^d|B^\rho| > 0,$$

by the first statement of the proposition. Similarly, by letting $D = [y, y+\Delta]^d \supset A$ we find that

$$\liminf_{\rho\to 0} \rho^d|A^\rho| \le \liminf_{\rho\to 0} \rho^d|D^\rho| < \infty.$$

□

The next result is of a more technical nature. As explained in Section 2, if τ is the (ρ-well) cover time of A, then the set which is uncovered at time $\tau(1-\epsilon)$ consists (with high probability) of small and distant sets. Indeed, this is what Proposition 5.3 proves. The proof of this proposition is based on a second moment argument, which involves a sum over pairs of points in A^ρ (see Lemma 5.2). In order to get a good bound for this sum, we will have to get estimates on the maximal number of points in A^ρ within a certain distance of a fixed point $x \in A^\rho$. This is what we do in Lemma 4.3. The proof explores Lemma 4.1 and straightforward packing arguments. Recall that c represent a constant, depending only on the dimension d, and that it may change from line to line.

Lemma 4.3. *Let $N(\rho, r) = N(\rho, r, A) = |A^\rho \cap (B(o, r+1) \setminus B(o, r))|$. There is a constant $c < \infty$ depending on d only such that for any $\rho < 1$ and any A,*

a) $N(\rho, r, A) \le c\rho^{-d}r^{d-1}$, for every $r \ge 1$;

b) $|A^\rho \cap B(o, r)| \le c\rho^{-d}r^d$ for $\rho < r$.

c) For any $y \in A^\rho$ we have that $\sum_{x\in A^\rho\setminus B(y,1)} d(y, x)^{1-d} \le c\rho^{-d}|A^\rho|^{1/d}$.

Proof. a) Let $x_1, \ldots, x_{N(\rho,r)}$ denote the points in $A^\rho \cap (B(o, r+1) \setminus B(o, r))$. For any $i = 1, \ldots, N(\rho, r)$ we have that $B(x_i, \rho/3) \subset (B(o, r+2) \setminus B(o, r-1))$. Also, for any $j \ne i$ we have that $B(x_i, \rho/3) \cap B(x_j, \rho/3) = \emptyset$, since $d(x_i, x_j) \ge \rho$ by the definition of A^ρ. Then

$$\begin{aligned} &N(\rho, r)\lambda_d(B(x_1, \rho/3)) \\ &\qquad \le \lambda_d(B(o, r+2) \setminus B(o, r-1)) = \kappa_d\left((r+2)^d - (r-1)^d\right) \le cr^{d-1}. \end{aligned}$$

Since $\lambda_d(B(x_1, \rho/3)) = \kappa_d(\rho/3)^d$ we have that

$$N(\rho, r) \le c\rho^{-d}r^{d-1}$$

and the proof is complete.

b) Let $\{x_1, \ldots, x_N\} = A^\rho \cap B(o, r)$. Then $B(x_i, \rho) \subset B(o, r+\rho)$ for all $i = 1, \ldots, N$. As in the previous case, for any $j \neq i$ we have that $B(x_i, \rho/3) \cap B(x_j, \rho/3) = \emptyset$, since $d(x_i, x_j) \geq \rho$ by the definition of A^ρ. Then

$$N(\rho, r)\lambda_d(B(x_1, \rho/3)) \leq \lambda_d(B(o, r+\rho)) = \kappa_d(r+\rho)^d \leq cr^d,$$

where we used $\rho < r$ in the last inequality. Since $\lambda_d(B(x_1, \rho/3)) = \kappa_d(\rho/3)^d$ we have that

$$N(\rho, r) \leq c\rho^{-d}r^d$$

and the proof is complete.

c) We assume without loss of generality that $y = o$. Start by ordering the points $\{x_1, \ldots, x_N\} \subset A^\rho \setminus \{o\}$ such that $d(o, x_i) \leq d(o, x_j)$ for all $i < j$. Then, let $N_k = |B(o, k)^\rho|$ and observe that for any $i > N_k$ we must have that $d(o, x_i) > k$. Then, note that by Lemma 4.1 part c), and part a) of the current lemma we get that for $k \geq 2$,

$$\begin{aligned} N_k - N_{k-1} &= |B(o, k)^\rho| - |B(o, k-1)^\rho| \\ &\leq |\left(B(o, k) \setminus B(o, k-1)\right)^\rho| \leq c\rho^{-d}k^{d-1} \leq c2^{d-1}\rho^{-d}(k-1)^{d-1}. \end{aligned} \tag{19}$$

We now define $K := \max\{k : N_k < |A^\rho|\}$ and consider the box $S_K = [-K/\sqrt{d}, K/\sqrt{d}] \subset B(o, K)$. We then have that $\left|S_K^\rho\right| \leq |B(o, K)^\rho| = N_K$. Consider then the set $\rho\mathbb{Z}^d \cap S_K$ where $\rho\mathbb{Z}^d$ is the d-dimensional hypercubic lattice whose vertices are at distance ρ. It is easy to see that $\rho\mathbb{Z}^d \cap S_K$ is a ρ-separated set in S_K and so we have that $\left|S_K^\rho\right| \geq \left|\rho\mathbb{Z}^d \cap S_K\right| \geq c\rho^{-d}K^d$, by the maximality of $\left|S_K^\rho\right|$. Therefore,

$$c\rho^{-d}K^d \leq |S_K^\rho| \leq N_K < |A^\rho|,$$

and therefore, $K \leq c\rho|A^\rho|^{1/d}$.

By part b), $N_1 \leq c\rho^{-d}$ and so by letting $N_0 = 0$ and using Equation (19) we get that,

$$\begin{aligned} \sum_{x \in A^\rho \setminus B(o,1)} d(o, x)^{1-d} &\leq \sum_{i=1}^{|A^\rho|} \max(d(o, x_i), 1)^{1-d} \\ &= \sum_{k=1}^{K+1} \sum_{i=N_{k-1}+1}^{\min(N_k, N)} \max(d(o, x_i), 1)^{1-d} \\ &\leq N_1 + \sum_{k=2}^{K+1} (N_k - N_{k-1})(k-1)^{1-d} \\ &\leq N_1 + \sum_{k=2}^{K+1} c\rho^{-d}(k-1)^{d-1}(k-1)^{1-d} \\ &\leq c\rho^{-d} + Kc\rho^{-d} \leq c\rho^{-d} + c\rho^{1-d}|A^\rho|^{1/d} \leq c\rho^{-d}|A^\rho|^{1/d} \end{aligned} \tag{20}$$

where we use that $\max(d(o, x_i), 1) \geq 1$ for $1 \leq i \leq N_1$ and the observation above that $d(o, x_i) \geq k - 1$ for $N_{k-1} + 1 \leq i \leq N_k$.

□

5 The ρ-well cover time $T_w^\rho(A)$

The purpose of this section is to prove Theorem 5.5 which is our main result concerning the well cover time. As mentioned in the Introduction, the strategy is similar to that of [1] although there are also many differences due to us working in $\mathbb{R}^d$ rather than $\mathbb{Z}^d$.

Recall the definition of $T_s^\rho(x)$ from Section 2. We extend the meaning of the notation T_s^ρ as follows. For any set $\{x_i\}_{i=1}^n$, let

$$T_s^\rho(\{x_i\}_{i=1}^n) := \max_{i=1,\dots,n}\{T_s^\rho(x_i)\}.$$

We have that $T_w^\rho(A) = T_s^\rho(A^\rho)$, while in general $T_s^\rho(\{x_i\}_{i=1}^n)$ is not the same as $T_w^\rho(\cup_{i=1}^n B(x_i,\rho))$ (unless of course $(\cup_{i=1}^n B(x_i,\rho))^\rho = \{x_i\}_{i=1}^n$). Our next proposition will deal with estimating $T_s^\rho(\{x_i\}_{i=1}^n)$ for well separated sets $\{x_i\}_{i=1}^n$. The proof uses Lemmas 3.4 and 3.5.

Proposition 5.1. *Let $m > 0$ and let $\{x_i\}_{i=1}^n$ be a set. Let $d(x_i,x_j) = r_{ij}$ and $r = \min r_{ij}$. Suppose $r \geq \max((n/2)^{\frac{m+2}{d-1}}, 4(1-\rho))$. Then, for $t \geq 0$ we have:*

$$|\mathbb{P}(T_s^\rho(\{x_i\}_{i=1}^n) \leq t) - \mathbb{P}(T_s^\rho(o) \leq t)^n| \leq tc\gamma(\rho)^2(n/2)^{-m}.$$

Proof. Let $\mathcal{X} = \{x_i\}_{i=1}^n$, $\mathcal{X}_i = \mathcal{X} \setminus \{x_k\}_{k=1}^i$ and $B_i = \bigcup_{x\in\mathcal{X}_i} B(x,1-\rho)$. Then

$$\begin{aligned}&|\mathbb{P}(T_s^\rho(\mathcal{X}) \leq t) - \mathbb{P}(T_s^\rho(o) \leq t)^n|\\ &\quad\leq |\mathbb{P}(T_s^\rho(\mathcal{X}) \leq t) - \mathbb{P}(T_s^\rho(\mathcal{X}_1) \leq t)\,\mathbb{P}(T_s^\rho(x_1) \leq t)|\\ &\qquad+\mathbb{P}(T_s^\rho(o) \leq t)\left|\mathbb{P}(T_s^\rho(\mathcal{X}_1) \leq t) - \mathbb{P}(T_s^\rho(o) \leq t)^{n-1}\right|,\end{aligned}$$

by translation invariance. Using Lemma 3.4, we get

$$|\mathbb{P}(T_s^\rho(\mathcal{X}) \leq t) - \mathbb{P}(T_s^\rho(\mathcal{X}_1) \leq t)\,\mathbb{P}(T_s^\rho(x_1) \leq t)| \leq 4\mathbb{P}\left(\omega_t\left(\mathcal{L}_{B_1,B(x_1,1-\rho)}\right) \neq 0\right),$$

and thus

$$\begin{aligned}&|\mathbb{P}(T_s^\rho(\mathcal{X}) \leq t) - \mathbb{P}(T_s^\rho(o) \leq t)^n|\\ &\quad\leq 4\mathbb{P}\left(\omega_t\left(\mathcal{L}_{B_1,B(x_1,1-\rho)}\right) \neq 0\right) + \left|\mathbb{P}(T_s^\rho(\mathcal{X}_1) \leq t) - \mathbb{P}(T_s^\rho(o) \leq t)^{n-1}\right|.\end{aligned}$$

Repeating the same steps $n-1$ more times, we get:

$$\begin{aligned}&|\mathbb{P}(T_s^\rho(\mathcal{X}) \leq t) - \mathbb{P}(T_s^\rho(o) \leq t)^n|\\ &\quad\leq 4\left(\mathbb{P}\left(\omega_t\left(\mathcal{L}_{B_1,B(x_1,1-\rho)}\right) \neq 0\right) + \dots + \mathbb{P}\left(\omega_t(\mathcal{L}_{B(x_n,1-\rho),B(x_{n-1},1-\rho)}) \neq 0\right)\right).\end{aligned}$$

Applying Lemma 3.5 (which requires $r \geq 4(1-\rho)$) to the above gives

$$\begin{aligned}&|\mathbb{P}(T_s^\rho(\mathcal{X}) \leq t) - \mathbb{P}(T_s^\rho(o) \leq t)^n|\\ &\quad\leq c\gamma(\rho)^2\left((n-1)\frac{t}{r^{d-1}} + (n-2)\frac{t}{r^{d-1}} + \dots + \frac{t}{r^{d-1}}\right)\\ &\quad= c\gamma(\rho)^2\frac{n(n-1)}{2}\frac{t}{r^{d-1}}.\end{aligned}$$

Therefore,

$$|\mathbb{P}(T_s^\rho(\mathcal{X}) \leq t) - \mathbb{P}(T_s^\rho(o) \leq t)^n| \leq n^2 c\gamma(\rho)^2 \frac{t}{r^{d-1}}.$$

Since $r \geq (n/2)^{\frac{m+2}{d-1}}$, we have that $r^{-(d-1)} \leq (n/2)^{-(m+2)}$ and thus

$$|\mathbb{P}(T_s^\rho(\mathcal{X}) \leq t) - \mathbb{P}(T_s^\rho(o) \leq t)^n| \leq n^2 c\gamma(\rho)^2 \frac{t}{r^{d-1}} \leq tc\gamma(\rho)^2 (n/2)^{-m}.$$

□

Let

$$A_\epsilon^\rho := \{x \in A^\rho : T_s^\rho(B(x,\rho)) > (1-\epsilon)\gamma^{-1}(\rho) \log |A^\rho|\} \tag{21}$$

so that A_ϵ^ρ is the subset of points $x \in A^\rho$ such that the ball $B(x,\rho)$ is not singularly covered at time $(1-\epsilon)\gamma^{-1}(\rho)\log|A^\rho|$.

The statements in the rest of the paper will all require some assumptions on A and ρ. For convenience we state them here (numbered (A1) through (A5)) so that we can refer to them later. These assumptions might look somewhat construed, but as we shall see later, they are in fact carefully chosen in order to allow us to optimize our choice of ρ. For convenience, we will let

$$C_d := \sqrt{1 - 4^{-1/(d-2)}} \tag{22}$$

where we interpret $C_2 = 1$. We also note that C_d should not be confused with the constant C_ϵ appearing in Theorem 1.1. Furthermore, let

$$\tilde{C}_d := \frac{C_d}{12(1+C_d)}. \tag{23}$$

Then, our assumptions are

$$0 < \rho < \min\left(\frac{d-1}{6d}, \frac{C_d}{5}\right) \tag{A1}$$

$$|A^\rho|^{1/(2d)} > 4 \tag{A2}$$

$$\rho \log |A^\rho| \leq |A^\rho|^{\rho/200} \tag{A3}$$

$$|A^\rho|^{\rho/200} \geq 2 \tag{A4}$$

$$\rho^{-d}(\log|A^\rho|)^d \leq |A^\rho|^{\tilde{C}_d/2} \tag{A5}$$

Next, we will provide bounds on two sums (over pairs of points in A^ρ) of the probabilities of two distinct points being in the set A_ϵ^ρ. This lemma is a preparation for the second moment argument of Proposition 5.3 (mentioned earlier in Section 4). The strategy of the proof of Lemma 5.2 is to split the sum into three parts. The first part is a sum over all pairs of points that are within distance C_d of each other, the second sums over pairs within the intermediate region between C_d and $\log|A^\rho|$, while the last sums over distant points. In order to obtain the desired result, we then use properties of μ_{d-1}

(Lemma 3.1) , A^ρ (Lemma 4.3) and $\beta(\rho)$ (Lemma 3.7) together with Assumptions A1 and A4 for the first sum. For the second sum we again use Lemma 3.7 and in addition we use Assumptions A1 and A5. For the final sum, we need Lemmas 3.1, 3.2, 3.3 and 4.3 and Assumptions A4 and A5.

We note that Assumptions A2 and A3 will be used later.

Lemma 5.2. *Let A, ρ satisfy Assumptions A1, A4 and A5, and let $\epsilon > 0$ be such that $\rho/1000 < \epsilon < \rho/36$. We then have that*

a) $\sum_{\substack{x,y\in A^\rho \\ x\neq y}} \mathbb{P}\left(x, y \in A^\rho_\epsilon\right) < c\,|A^\rho|^{-\epsilon} + |A^\rho|^{2\epsilon}$

b) $\sum_{\substack{x,y\in A^\rho \\ 0<d(x,y)<b|A^\rho|^{1/2d}}} \mathbb{P}\left(x, y \in A^\rho_\epsilon\right) < cb^d\,|A^\rho|^{-\epsilon}$,

for every $b \geq 1$, and where c is a constant depending only on the dimension d.

Proof. We will prove both statements by considering the sum

$$I = \sum_{\substack{x,y\in A^\rho \\ 0<d(x,y)\leq a}} \mathbb{P}\left(x, y \in A^\rho_\epsilon\right)$$

and choosing appropriate values for a later. Let us split I into three parts:

$$I_1 = \sum_{\substack{x,y\in A^\rho \\ \rho\leq d(x,y)<C_d}} \mathbb{P}\left(x, y \in A^\rho_\epsilon\right), \tag{24}$$

$$I_2 = \sum_{\substack{x,y\in A^\rho \\ C_d\leq d(x,y)<\log|A^\rho|}} \mathbb{P}\left(x, y \in A^\rho_\epsilon\right), \tag{25}$$

and

$$I_3 = \sum_{\substack{x,y\in A^\rho \\ \log|A^\rho|\leq d(x,y)\leq a}} \mathbb{P}\left(x, y \in A^\rho_\epsilon\right). \tag{26}$$

We start by noting that

$$\begin{aligned} &\mathbb{P}\left(x, y \in A^\rho_\epsilon\right) \qquad (27)\\ &= \mathbb{P}\left(\{T^\rho_s(x) > (1-\epsilon)\gamma^{-1}(\rho)\log|A^\rho|\} \cap \{T^\rho_s(y) > (1-\epsilon)\gamma^{-1}(\rho)\log|A^\rho|\}\right) \\ &= \mathbb{P}\left(\omega_{(1-\epsilon)\gamma^{-1}(\rho)\log|A^\rho|}\left(\mathcal{L}_{B(x,1-\rho)} \cup \mathcal{L}_{B(y,1-\rho)}\right) = 0\right) \\ &= \exp\left(-(1-\epsilon)\gamma^{-1}(\rho)\log|A^\rho|\mu_{d,1}\left(\mathcal{L}_{B(x,1-\rho)} \cup \mathcal{L}_{B(y,1-\rho)}\right)\right). \end{aligned}$$

We are now ready to address (24). Let $K = \lfloor \frac{\log C_d - \log\rho}{\log 2} \rfloor$, where $\lfloor\cdot\rfloor$ denotes the integer part. Note that

$$2^{K+1}\rho \leq 2^{\frac{\log C_d-\log\rho}{\log 2}+1}\rho \leq 2C_d \text{ and that } 2^{K+1}\rho \geq 2^{\frac{\log C_d-\log\rho}{\log 2}}\rho = C_d.$$

Note also that $K \geq 1$ since $\rho \leq C_d/2$ by Assumption A1. Thus, (24) becomes

$$I_1 = \sum_{\substack{x,y \in A^\rho \\ \rho \leq d(x,y) < C_d}} \mathbb{P}\left(x, y \in A^\rho_\epsilon\right) \leq \sum_{k=0}^{K} \sum_{\substack{x,y \in A^\rho \\ 2^k \rho \leq d(x,y) < 2^{k+1}\rho}} \mathbb{P}\left(x, y \in A^\rho_\epsilon\right). \tag{28}$$

For $2^k\rho \leq d(x,y) < 2^{k+1}\rho$, we have that

$$\begin{aligned}
&\mu_{d,1}\left(\mathcal{L}_{B(x,1-\rho)} \cup \mathcal{L}_{B(y,1-\rho)}\right) = \mu_{d,1}\left(\mathcal{L}_{B(o,1-\rho)} \cup \mathcal{L}_{B(d(x,y)e_1,1-\rho)}\right) \\
&\quad = \mu_{d,1}\left(\mathcal{L}_{B(o,1-\rho)}\right) + \mu_{d,1}\left(\mathcal{L}_{B(d(x,y)e_1,1-\rho)}\right) - \mu_{d,1}\left(\mathcal{L}_{B(o,1-\rho)} \cap \mathcal{L}_{B(d(x,y)e_1,1-\rho)}\right) \\
&\quad \geq \mu_{d,1}\left(\mathcal{L}_{B(o,1-\rho)}\right) + \mu_{d,1}\left(\mathcal{L}_{B(2^k\rho e_1,1-\rho)}\right) - \mu_{d,1}\left(\mathcal{L}_{B(o,1-\rho)} \cap \mathcal{L}_{B(2^k\rho e_1,1-\rho)}\right) \\
&\quad = \mu_{d,1}\left(\mathcal{L}_{B(o,1-\rho)} \cup \mathcal{L}_{B(2^k\rho e_1,1-\rho)}\right)
\end{aligned}$$

where the inequality follows since any line $L \in \mathcal{L}_{B(o,1-\rho)} \cap \mathcal{L}_{B(d(x,y)e_1,1-\rho)}$ must also belong to $\mathcal{L}_{B(o,1-\rho)} \cap \mathcal{L}_{B(2^k\rho e_1,1-\rho)}$. Thus

$$\begin{aligned}
&\exp\left(-(1-\epsilon)\gamma^{-1}(\rho)\log|A^\rho|\mu_{d,1}\left(\mathcal{L}_{B(x,1-\rho)} \cup \mathcal{L}_{B(y,1-\rho)}\right)\right) \\
&\quad \leq \exp\left(-(1-\epsilon)\gamma^{-1}(\rho)\log|A^\rho|\mu_{d,1}\left(\mathcal{L}_{B(o,1-\rho)} \cup \mathcal{L}_{B(2^k\rho e_1,1-\rho)}\right)\right) \\
&\quad = \exp\left(-(1-\epsilon)\gamma^{-1}(\rho)\log|A^\rho|\gamma(\rho)\mu_{d,1}\left(\mathcal{L}_{B(o,1)} \cup \mathcal{L}_{B\left(\frac{2^k\rho}{1-\rho}e_1,1\right)}\right)\right) \\
&\quad = \exp\left(-(1-\epsilon)\log|A^\rho|\,\beta(\rho,k)\right),
\end{aligned}$$

where the first equality comes from Lemma 3.1 part c).

Therefore, (27), (28) and the above gives us that

$$\begin{aligned}
I_1 &\leq \sum_{k=0}^{K} \sum_{\substack{x,y \in A^\rho \\ 2^k\rho \leq d(x,y) < 2^{k+1}\rho}} \exp\left(-(1-\epsilon)\log|A^\rho|\,\beta(\rho,k)\right) \\
&\leq \sum_{k=0}^{K} |A^\rho|^{1-(1-\epsilon)\left(1+\frac{2^k}{12}\rho\right)} c(2^{k+1}\rho)^d \rho^{-d},
\end{aligned}$$

where we used Lemma 4.3 part b) and Lemma 3.7. In order to motivate the use of Lemma 3.7 we note that

$$\frac{2^K\rho}{1-\rho} \leq \frac{2^{\frac{\log C_d - \log\rho}{\log 2}}\rho}{1-\rho} \leq \frac{C_d}{1-\rho} \leq \frac{C_d}{1-1/6} \leq 2C_d$$

since $\rho < 1/6$ by Assumption A1. Furthermore, for any k we claim that

$$1-(1-\epsilon)\left(1+\frac{2^k}{12}\rho\right) = -\frac{2^k}{12}\rho + \epsilon + \frac{2^k}{12}\epsilon\rho \leq -2^k\epsilon. \tag{29}$$

Indeed, (29) is equivalent to

$$\epsilon \leq \rho\frac{\frac{1}{12}2^k}{1+\frac{1}{12}2^k\rho+2^k} = \rho\frac{\frac{1}{12}}{2^{-k}+\frac{1}{12}\rho+1}.$$

Since the right hand side is larger than $\rho/36$ it follows that (29) holds by our assumption on ϵ. Continuing we get that

$$\begin{aligned} I_1 &\le c \sum_{k=0}^{K} |A^\rho|^{-2^k \epsilon} 2^{d(k+1)} \le c\,|A^\rho|^{-\epsilon} \sum_{k=0}^{\infty} |A^\rho|^{-(2^k-1)\epsilon} 2^{d(k+1)} \\ &\le c\,|A^\rho|^{-\epsilon} \sum_{k=0}^{\infty} (|A^\rho|^\rho)^{-(2^k-1)/1000} 2^{d(k+1)} \\ &\le c\,|A^\rho|^{-\epsilon} \sum_{k=0}^{\infty} 2^{-(2^k-1)/1000} 2^{d(k+1)} \le c\,|A^\rho|^{-\epsilon}, \end{aligned}$$

by our assumption that $\epsilon > \rho/1000$ and Assumption A4, since $|A^\rho|^\rho \ge |A^\rho|^{\rho/200} \ge 2$. Therefore,

$$I_1 \le c\,|A^\rho|^{-\epsilon}. \tag{30}$$

Consider now Equation (25). Equation (27) implies that

$$\begin{aligned} I_2 &= \sum_{\substack{x,y\in A^\rho \\ C_d \le d(x,y) < \log|A^\rho|}} \mathbb{P}\left(x, y \in A^\rho_\epsilon\right) \\ &= \sum_{\substack{x,y\in A^\rho \\ C_d \le d(x,y) < \log|A^\rho|}} \exp\left(-(1-\epsilon)\gamma^{-1}(\rho)\log|A^\rho|\,\mu_{d,1}\left(\mathcal{L}_{B(o,1-\rho)} \cup \mathcal{L}_{B(d(x,y)e_1,1-\rho)}\right)\right) \\ &= \sum_{\substack{x,y\in A^\rho \\ C_d \le d(x,y) < \log|A^\rho|}} \exp\left(-(1-\epsilon)\log|A^\rho|\,\mu_{d,1}\left(\mathcal{L}_{B(o,1)} \cup \mathcal{L}_{B\left(\frac{d(x,y)}{1-\rho}e_1,1\right)}\right)\right) \\ &\le \sum_{\substack{x,y\in A^\rho \\ C_d \le d(x,y) < \log|A^\rho|}} \exp\left(-(1-\epsilon)\log|A^\rho|\,\mu_{d,1}\left(\mathcal{L}_{B(o,1)} \cup \mathcal{L}_{B(C_d e_1,1)}\right)\right), \end{aligned}$$

since $d(x,y) \ge C_d$ and $\rho \ge 0$. Now, Lemma 3.7 applied to $\rho = C_d/(1+C_d)$ (so that $\rho/(1-\rho) = C_d$) and $k = 0$, implies that $\mu_{d,1}\left(\mathcal{L}_{B(o,1)} \cup \mathcal{L}_{B(C_d e_1,1)}\right) \ge 1 + \frac{C_d}{12(1+C_d)} = 1 + \tilde{C}_d$, which gives

$$\begin{aligned} I_2 &\le \sum_{\substack{x,y\in A^\rho \\ C_d \le d(x,y) < \log|A^\rho|}} \exp\left(-(1-\epsilon)\log|A^\rho|\,\mu_{d,1}\left(\mathcal{L}_{B(o,1)} \cup \mathcal{L}_{B(C_d e_1,1)}\right)\right) \\ &\le c\,|A^\rho|\,\rho^{-d}(\log|A^\rho|)^d \exp\left(-(1-\epsilon)\left(1+\tilde{C}_d\right)\log|A^\rho|\right) \\ &= c\rho^{-d}(\log|A^\rho|)^d\,|A^\rho|^{\epsilon - \tilde{C}_d + \epsilon\tilde{C}_d} \le c\,|A^\rho|^{\epsilon - \tilde{C}_d/2 + \epsilon\tilde{C}_d} \end{aligned}$$

where we used Lemma 4.3 part *b*) in the second inequality and Assumption A5 in the last inequality. Furthermore, we claim that $\epsilon - \tilde{C}_d/2 + \epsilon\tilde{C}_d \le -\epsilon$. Indeed, this follows since

$$\frac{\tilde{C}_d}{2(2+\tilde{C}_d)} = \frac{C_d}{2(24(1+C_d)+C_d)} \ge \frac{C_d}{148} \ge \frac{\rho}{36} \ge \epsilon$$

since $C_d \le 2$ and since $\rho \le C_d/5$ by Assumption A1. We conclude that

$$I_2 \le c\,|A^\rho|^{-\epsilon}. \tag{31}$$

We now turn to Equation (26) and the sum I_3. By Lemma 3.1 part c)

$$\begin{aligned}\mu_{d,1}\left(\mathcal{L}_{B(x,1-\rho)} \cup \mathcal{L}_{B(y,1-\rho)}\right) &= \gamma(\rho)\mu_{d,1}\left(\mathcal{L}_{B(x',1)} \cup \mathcal{L}_{B(y',1)}\right) \\ &= \gamma(\rho)\left(2\mu_{d,1}\left(\mathcal{L}_{B(x',1)}\right) - \mu_{d,1}\left(\mathcal{L}_{B(x',1)} \cap \mathcal{L}_{B(y',1)}\right)\right),\end{aligned}$$

for any x', y' such that $d(x,y) = (1-\rho)\, d(x',y')$. Since by Assumption A4 we have that $d(x',y') \geq d(x,y) \geq \log|A^\rho| \geq \rho \log|A^\rho| \geq 200 \log 2 \geq 4$, we can apply Lemma 3.2 to see that

$$\begin{aligned}&\gamma(\rho)\left(2\mu_{d,1}\left(\mathcal{L}_{B(x',1)}\right) - \mu_{d,1}\left(\mathcal{L}_{B(x',1),B(y',1)}\right)\right) \\ &\qquad \geq \gamma(\rho)\left(2 - \frac{c_2}{d(x',y')^{d-1}}\right) = \gamma(\rho)\left(2 - \frac{c_2\gamma(\rho)}{d(x,y)^{d-1}}\right).\end{aligned}$$

Then, we see from (26) that

$$\begin{aligned}I_3 &\leq \sum_{\substack{x,y \in A^\rho \\ \log|A^\rho| \leq d(x,y) \leq a}} \exp\left(-(1-\epsilon)\log|A^\rho|\left(2 - \frac{c_2\gamma(\rho)}{d(x,y)^{d-1}}\right)\right) \\ &= |A^\rho|^{-2(1-\epsilon)} \sum_{\substack{x,y \in A^\rho \\ \log|A^\rho| \leq d(x,y) \leq a}} \exp\left((1-\epsilon)\log|A^\rho|\, c_2\gamma(\rho)\, d(x,y)^{1-d}\right).\end{aligned}$$

Since $\log|A^\rho| \leq d(x,y)$, we have that

$$(1-\epsilon)\log|A^\rho|\, c_2\gamma(\rho)\, d(x,y)^{1-d} \leq (1-\epsilon)c_2\gamma(\rho)(\log|A^\rho|)^{2-d} \leq c_2.$$

It is easy to prove that for any $x \leq c_2$ we must have that $e^x \leq 1 + e^{c_2}x$, and so we get that

$$\begin{aligned}&|A^\rho|^{-2(1-\epsilon)} \sum_{\substack{x,y \in A^\rho \\ \log|A^\rho| \leq d(x,y) \leq a}} \exp\left((1-\epsilon)\log|A^\rho|\, c_2\gamma(\rho)\, d(x,y)^{1-d}\right) \qquad (32)\\ &\leq |A^\rho|^{-2(1-\epsilon)} \sum_{\substack{x,y \in A^\rho \\ \log|A^\rho| \leq d(x,y) \leq a}} \left(1 + e^{c_2}(1-\epsilon)\log|A^\rho|\,\gamma(\rho)\, d(x,y)^{1-d}\right) \\ &\leq \min\left(|A^\rho|^{2\epsilon}, c\rho^{-d}|A^\rho|^{2\epsilon-1} a^d\right) \\ &\quad + c(1-\epsilon)\gamma(\rho)|A^\rho|^{-2(1-\epsilon)}\log|A^\rho| \sum_{\substack{x,y \in A^\rho \\ \log|A^\rho| \leq d(x,y) \leq a}} d(x,y)^{1-d}\end{aligned}$$

where the minimum comes from summing 1 and using part b) of Lemma 4.3 to bound the number of elements in the sum.

Next, we provide a bound to the last term on the right hand side of (32). To that end, note that

$$\begin{aligned}\sum_{\substack{x,y \in A^\rho \\ \log|A^\rho| \leq d(x,y) \leq a}} d(x,y)^{1-d} &\leq \sum_{x \in A^\rho} \sum_{y \in A^\rho \setminus B(x,1)} d(x,y)^{1-d} \\ &\leq \sum_{x \in A^\rho} c\rho^{-d}|A^\rho|^{1/d} \leq c\rho^{-d}|A^\rho|^{1+1/d},\end{aligned}$$

where we used part c) of Lemma 4.3 in the second inequality. Furthermore, this and Assumption A5 implies that

$$\begin{aligned} c(1-\epsilon)\gamma(\rho)\left|A^{\rho}\right|^{-2(1-\epsilon)} \log\left|A^{\rho}\right| \sum_{\substack{x,y\in A^{\rho} \\ \log|A^{\rho}|\leq d(x,y)\leq a}} d\left(x,y\right)^{1-d} \qquad (33) \\ \leq c(1-\epsilon)\rho^{-d}\gamma(\rho)\log\left|A^{\rho}\right|\left|A^{\rho}\right|^{-2(1-\epsilon)+1+1/d} \\ \leq c\rho^{-d}(\log\left|A^{\rho}\right|)^{d}\left|A^{\rho}\right|^{-2(1-\epsilon)+1+1/d} \\ \leq c\left|A^{\rho}\right|^{-2(1-\epsilon)+1+1/d+\tilde{C}_d/2} \leq c\left|A^{\rho}\right|^{-\epsilon}. \end{aligned}$$

where we in the last inequality use that $-2(1-\epsilon)+1+1/d+\tilde{C}_d/2 \leq -\epsilon$. Indeed, this follows since

$$1-\frac{1}{d}-\frac{\tilde{C}_d}{2} = 1-\frac{1}{d}-\frac{C_d}{24(1+C_d)} \geq 1-\frac{1}{d}-\frac{2}{24} \geq \rho \geq 3\epsilon,$$

since $\rho < 1/6$ by Assumption A1 and the fact that $\epsilon < \rho/36$.

Combining (33) and (32) we see that

$$I_3 < \min\left(\left|A^{\rho}\right|^{2\epsilon}, c\rho^{-d}\left|A^{\rho}\right|^{2\epsilon-1}a^{d}\right) + c\left|A^{\rho}\right|^{-\epsilon}. \qquad (34)$$

By summing the contributions from I_1, I_2 and I_3 ((30), (31) and (34)), we then conclude that

$$I < \min\left(\left|A^{\rho}\right|^{2\epsilon}, c\rho^{-d}\left|A^{\rho}\right|^{2\epsilon-1}a^{d}\right) + c\left|A^{\rho}\right|^{-\epsilon}.$$

Taking the limit $a \to \infty$, we obtain

$$I < \left|A^{\rho}\right|^{2\epsilon} + c\left|A^{\rho}\right|^{-\epsilon},$$

and the first statement is proved.

Furthermore, taking $a = b\left|A^{\rho}\right|^{1/2d}$, we have that

$$\begin{aligned} c\rho^{-d}\left|A^{\rho}\right|^{2\epsilon-1}a^{d} = c\rho^{-d}\left|A^{\rho}\right|^{2\epsilon-1/2}b^{d} = cb^{d}|A^{\rho}|^{-\epsilon}\left(\rho^{-d}|A^{\rho}|^{3\epsilon-1/2}\right) \\ \leq cb^{d}|A^{\rho}|^{-\epsilon}\left(|A^{\rho}|^{3\epsilon-1/2+\tilde{C}_d/2}\right) \leq cb^{d}|A^{\rho}|^{-\epsilon}, \end{aligned}$$

where we used Assumption A5 and that $|A^{\rho}|^{3\epsilon-1/2+\tilde{C}_d/2} \leq 1$ again by our assumption that $\epsilon < \rho/36$ and Assumption A1. Thus

$$I < cb^{d}\left|A^{\rho}\right|^{-\epsilon}$$

and the proof is complete. □

The next proposition is another crucial step towards the proof of Theorem 5.5. Consider first

$$\begin{aligned} G_{A,\rho,\epsilon} = \{K \subset A^{\rho} : \left|\left|K\right| - \left|A^{\rho}\right|^{\epsilon}\right| \leq \left|A^{\rho}\right|^{2\epsilon/3} \text{ and} \qquad (35) \\ d\left(x,y\right) \geq \left|A^{\rho}\right|^{1/2d} \text{ for all distinct } x,y \in K\}, \end{aligned}$$

so that $G_{A,\rho,\epsilon}$ is a collection of subsets of A^ρ that are well separated and close in cardinality to $|A^\rho|^\epsilon$. Also note that the condition $||K| - |A^\rho|^\epsilon| \leq |A^\rho|^{2\epsilon/3}$ implies that $G_{A,\rho,\epsilon}$ consists only of non-empty sets. What we will prove in Proposition 5.3 is that just before A is ρ-well covered, the set that is not yet singularly covered consists (with high probability) of a small number of distant points, i.e. the set belongs to $G_{A,\rho,\epsilon}$. As mentioned, the proof is based on a second moment argument, where we use Lemma 5.2 to obtain the bounds.

Proposition 5.3. *Let ρ, A satisfy Assumptions A1, A4 and A5. Then, for $\rho/1000 < \epsilon < \rho/36$, we have that*

$$\mathbb{P}\left(A^\rho_\epsilon \notin G_{A,\rho,\epsilon}\right) \leq c\,|A^\rho|^{-\epsilon/3}.$$

Remark: As will transpire, we will in fact only use the above proposition for two values of ϵ, namely $\epsilon = \rho/50$ and $\epsilon = \rho/200$.

Proof. We will get the result by proving that

$$\mathbb{P}\left(\exists x, y \in A^\rho_\epsilon : 0 < d\left(x, y\right) < |A^\rho|^{1/2d}\right) \leq c\,|A^\rho|^{-\epsilon}, \tag{36}$$

and

$$\mathbb{P}\left(||A^\rho_\epsilon| - |A^\rho|^\epsilon| > |A^\rho|^{2\epsilon/3}\right) \leq c\,|A^\rho|^{-\epsilon/3}. \tag{37}$$

For (36) we have:

$$\begin{aligned}
&\mathbb{P}\left(\exists x, y \in A^\rho_\epsilon : 0 < d\left(x, y\right) < |A^\rho|^{1/2d}\right) \\
&= \mathbb{P}\left(\bigcup_{\substack{x,y\in A^\rho \\ 0<d(x,y)<|A^\rho|^{1/2d}}} \{x, y \in A^\rho_\epsilon\}\right) \\
&\leq \sum_{\substack{x,y\in A^\rho \\ 0<d(x,y)<|A^\rho|^{1/2d}}} \mathbb{P}\left(x, y \in A^\rho_\epsilon\right) \leq c\,|A^\rho|^{-\epsilon},
\end{aligned}$$

where the last inequality come from part b) of Lemma 5.2 with $b = 1$, and (36) is done.

For (37), observe that by part b) of Lemma 3.1 we have that

$$\begin{aligned}
&\mathbb{P}\left(T^\rho_s(x) > t\right) \\
&= 1 - \mathbb{P}\left(\exists L \in \omega_t : B(x, \rho) \subset \mathfrak{c}\left(L\right)\right) = 1 - \mathbb{P}\left(\omega_t(\mathcal{L}_{B(x,1-\rho)}) \neq 0\right) \\
&= \mathbb{P}\left(\omega_t(\mathcal{L}_{B(x,1-\rho)}) = 0\right) = \exp\left(-t\mu_{d,1}\left(\mathcal{L}_{B(x,1-\rho)}\right)\right) = \exp\left(-t\gamma(\rho)\right).
\end{aligned} \tag{38}$$

Using this, we get that

$$\begin{aligned}
\mathbb{E}(|A^\rho_\epsilon|) &= \sum_{x\in A^\rho} \mathbb{P}(T^\rho_s(x) > (1-\epsilon)\gamma^{-1}(\rho)\log|A^\rho|) \\
&= |A^\rho|\exp\left(-(1-\epsilon)\gamma^{-1}(\rho)\log|A^\rho|\gamma(\rho)\right) = |A^\rho|^\epsilon.
\end{aligned}$$

By Chebyshev's inequality,

$$\mathbb{P}\left(||A^\rho_\epsilon| - |A^\rho|^\epsilon| > |A^\rho|^{2\epsilon/3}\right) \leq \frac{\mathbb{E}\left(|A^\rho_\epsilon|^2\right) - |A^\rho|^{2\epsilon}}{|A^\rho|^{4\epsilon/3}}. \tag{39}$$

Now

$$\mathbb{E}\left(|A^\rho_\epsilon|^2\right) = \sum_{x,y\in A^\rho} \mathbb{P}\left(x,y \in A^\rho_\epsilon\right)$$
$$= \sum_{x\in A^\rho} \mathbb{P}\left(x \in A^\rho_\epsilon\right) + \sum_{\substack{x,y\in A^\rho \\ x\neq y}} \mathbb{P}\left(x,y \in A^\rho_\epsilon\right) \leq |A^\rho|^\epsilon + c\,|A^\rho|^{-\epsilon} + |A^\rho|^{2\epsilon},$$

by part a) of Lemma 5.2. Plugging this into (39) yields

$$\mathbb{P}\left(||A^\rho_\epsilon| - |A^\rho|^\epsilon| > |A^\rho|^{2\epsilon/3}\right)$$
$$\leq \frac{|A^\rho|^\epsilon + c\,|A^\rho|^{-\epsilon} + |A^\rho|^{2\epsilon} - |A^\rho|^{2\epsilon}}{|A^\rho|^{4\epsilon/3}} \leq |A^\rho|^{-\epsilon/3} + c\,|A^\rho|^{-7\epsilon/3} \leq c\,|A^\rho|^{-\epsilon/3},$$

and the proof is complete. □

We shall make frequent use of the following inequalities which we state here for convenience. We do not provide any proofs as they are easy exercises in calculus. We have that

$$-x - x^2 \leq \log(1-x) \leq -x \text{ for every } x \in [0, 1/2], \tag{40}$$

and

$$1 - \exp(-x) \leq x \text{ for every } x \geq 0. \tag{41}$$

We can now state and prove the last lemma before proving the main results.

Proposition 5.4. *Let ρ, A satisfy Assumption A4, and let $\epsilon = \rho/50$. We then have that for any $z \geq -\frac{\epsilon}{4}\log|A^\rho|$*

$$\left| \left(1 - \frac{e^{-z}}{|A^\rho|^\epsilon}\right)^{|A^\rho|^\epsilon + |A^\rho|^{2\epsilon/3}} - \exp\left(-e^{-z}\right) \right| \leq 3\,|A^\rho|^{-\epsilon/12}, \tag{42}$$

and similarly,

$$\left| \left(1 - \frac{e^{-z}}{|A^\rho|^\epsilon}\right)^{|A^\rho|^\epsilon - |A^\rho|^{2\epsilon/3}} - \exp\left(-e^{-z}\right) \right| \leq |A^\rho|^{-\epsilon/12}. \tag{43}$$

Proof. Let $a = e^{-z}$ and $b = |A^\rho|^\epsilon$, and note that since $z \geq -\frac{\epsilon}{4}\log|A^\rho|$ we have that

$$a = e^{-z} \leq |A^\rho|^{\epsilon/4}. \tag{44}$$

Therefore,

$$\frac{a}{b} = \frac{e^{-z}}{|A^\rho|^\epsilon} \leq |A^\rho|^{-3\epsilon/4} = |A^\rho|^{-3\rho/200} \leq 1/2,$$

by our choice of ϵ and since $|A^\rho|^{3\rho/200} \geq |A^\rho|^{\rho/200} \geq 2$ by Assumption A4.

We will now consider (42). Note that by (40),

$$\begin{aligned}\left(1-\frac{a}{b}\right)^{b+b^{2/3}} &= \exp\left(\left(b+b^{2/3}\right)\log\left(1-\frac{a}{b}\right)\right)\\ &\le \exp\left(\left(b+b^{2/3}\right)\left(-\frac{a}{b}\right)\right) = \exp\left(-a\left(1+\frac{1}{b^{1/3}}\right)\right) \le \exp\left(-a\right).\end{aligned}$$

This implies that $\exp\left(-a\right) - \left(1-\frac{a}{b}\right)^{b+b^{2/3}} \ge 0$ and so

$$\begin{aligned}&\left|\exp\left(-a\right) - \left(1-\frac{a}{b}\right)^{b+b^{2/3}}\right|\\ &\quad= \exp\left(-a\right) - \exp\left(\left(b+b^{2/3}\right)\log\left(1-\frac{a}{b}\right)\right)\\ &\quad\le \exp\left(-a\right) - \exp\left(\left(b+b^{2/3}\right)\left(-\frac{a}{b}-\frac{a^2}{b^2}\right)\right)\\ &\quad= \exp\left(-a\right)\left(1-\exp\left(-\left(\frac{a^2}{b}+\frac{a}{b^{1/3}}+\frac{a^2}{b^{4/3}}\right)\right)\right),\end{aligned}$$

again by (40). Furthermore, (44) implies that $\frac{a^2}{b}+\frac{a}{b^{1/3}}+\frac{a^2}{b^{4/3}} \le |A^\rho|^{-\epsilon/2} + |A^\rho|^{-\epsilon/12} + |A^\rho|^{-5\epsilon/6} \le 3\left|A^\rho\right|^{-\epsilon/12} = 3b^{-1/12}$. Thus

$$\begin{aligned}&\exp\left(-a\right)\left(1-\exp\left(-\left(\frac{a^2}{b}+\frac{a}{b^{1/3}}+\frac{a^2}{b^{4/3}}\right)\right)\right)\\ &\qquad\le \exp\left(-a\right)\left(1-\exp\left(-3b^{-1/12}\right)\right).\end{aligned}$$

Moreover, since $3b^{-1/12} \ge 0$, we can use (41). This gives

$$\exp\left(-a\right)\left(1-\exp\left(-3b^{-1/12}\right)\right) \le 3\exp\left(-a\right)b^{-1/12} \le 3\left|A^\rho\right|^{-\epsilon/12},$$

since $e^{-a} \le 1$. Thus, (42) is proved.

We now turn to (43). Note that

$$\begin{aligned}\left(1-\frac{a}{b}\right)^{b-b^{2/3}} &= \exp\left(\left(b-b^{2/3}\right)\log\left(1-\frac{a}{b}\right)\right)\\ &\ge \exp\left(\left(b-b^{2/3}\right)\left(-\frac{a}{b}-\frac{a^2}{b^2}\right)\right) = \exp\left(-a\right)\exp\left(-\frac{a^2}{b}+\frac{a}{b^{1/3}}+\frac{a^2}{b^{4/3}}\right)\\ &= \exp\left(-a\right)\exp\left(\frac{a}{b^{1/3}}\left(-\frac{a}{b^{2/3}}+1+\frac{a}{b}\right)\right) \ge \exp\left(-a\right),\end{aligned}$$

which follows since $\frac{a}{b^{2/3}} \le \frac{|A^\rho|^{\epsilon/4}}{|A^\rho|^{2\epsilon/3}} = |A^\rho|^{-5\epsilon/12} \le 1$, by (44). This implies that $\exp\left(-a\right) - \left(1-\frac{a}{b}\right)^{b-b^{2/3}} \le 0$ and therefore

$$\begin{aligned}\left|\exp\left(-a\right) - \left(1-\frac{a}{b}\right)^{b-b^{2/3}}\right| &= \exp\left(\left(b-b^{2/3}\right)\log\left(1-\frac{a}{b}\right)\right) - \exp\left(-a\right)\\ &\le \exp\left(\left(b-b^{2/3}\right)\left(-\frac{a}{b}\right)\right) - \exp\left(-a\right) = \exp\left(-a\right)\left(\exp\left(\frac{a}{b^{1/3}}\right)-1\right)\\ &\le \exp\left(-\frac{a}{b^{1/3}}\right)\left(\exp\left(\frac{a}{b^{1/3}}\right)-1\right) = 1-\exp\left(-\frac{a}{b^{1/3}}\right) \le \frac{a}{b^{1/3}},\end{aligned}$$

where we used (40) in the first inequality, the fact that $\frac{a}{b^{1/3}} \leq a$ in the penultimate inequality and (41) in the last inequality. Furthermore,

$$\frac{a}{b^{1/3}} \leq \frac{|A^\rho|^{\epsilon/4}}{|A^\rho|^{-\epsilon/3}} = |A^\rho|^{-\epsilon/12},$$

as desired. □

We are now ready to prove the main theorem concerning the ρ-well cover time.

Theorem 5.5. *For any ρ and $A \subset \mathbb{R}^d$ satisfying Assumptions A1 through A5 we have that*

$$\sup_{z\in\mathbb{R}} \left|\mathbb{P}\left(\gamma(\rho)T_w^\rho(A) - \log|A^\rho| \leq z\right) - \exp(-e^{-z})\right| \leq c_3 |A^\rho|^{-\rho/600},$$

where c_3 is a constant depending on d only.

Proof. Let $\epsilon = \rho/50$ so that ϵ satisfies the assumptions of Lemma 5.2 and Propositions 5.3 and 5.4. We are going to split the proof into three cases and start with the easy ones first.

Case 1: Consider $z \leq -\frac{\epsilon}{4}\log|A^\rho|$. Then

$$\begin{aligned}
&\mathbb{P}\left(T_w^\rho(A) \leq \gamma^{-1}(\rho)(\log|A^\rho| + z)\right) \\
&\quad\leq \mathbb{P}\left(T_w^\rho(A) \leq \gamma^{-1}(\rho)\left(\log|A^\rho| - \frac{\epsilon}{4}\log|A^\rho|\right)\right) \\
&\quad= \mathbb{P}\left(\left\{x \in A^\rho : T_s^\rho(x) > \gamma^{-1}(\rho)\left(1-\frac{\epsilon}{4}\right)\log|A^\rho|\right\} = \emptyset\right) \\
&\quad= \mathbb{P}\left(A^\rho_{\epsilon/4} = \emptyset\right).
\end{aligned}$$

We have that $\mathbb{P}\left(A^\rho_{\epsilon/4} = \emptyset\right) \leq \mathbb{P}\left(A^\rho_{\epsilon/4} \notin G_{A,\rho,\epsilon/4}\right)$, since $G_{A,\rho,\epsilon/4}$ only contains non-empty sets. Thus, by Proposition 5.3 (applied to $\epsilon/4 = \rho/200$) we have that,

$$\mathbb{P}\left(T_w^\rho(A) \leq \gamma^{-1}(\rho)(\log|A^\rho| + z)\right) \leq \mathbb{P}\left(A^\rho_{\epsilon/4} \notin G_{A,\rho,\epsilon/4}\right) \leq c|A^\rho|^{-\epsilon/12}. \tag{45}$$

Using that $z \leq -\log|A^\rho|^{\epsilon/4}$ we have that $\exp(-e^{-z}) \leq \exp\left(-|A^\rho|^{\epsilon/4}\right)$. Now $|A^\rho|^{\epsilon/4} \geq -1$ and so we can use the inequality $e^{-x} \leq \frac{1}{x}$ true for all $x > 0$, giving $\exp(-|A^\rho|^{\epsilon/4}) \leq |A^\rho|^{-\epsilon/4}$. Then:

$$\begin{aligned}
&\left|\mathbb{P}\left(T_w^\rho(A) \leq \gamma^{-1}(\rho)(\log|A^\rho| + z)\right) - \exp(-e^{-z})\right| \\
&\quad\leq \mathbb{P}\left(T_w^\rho(A) \leq \gamma^{-1}(\rho)(\log|A^\rho| + z)\right) + \exp(-e^{-z}) \\
&\quad\leq c|A^\rho|^{-\epsilon/12} + |A^\rho|^{-\epsilon/4} < c|A^\rho|^{-\epsilon/12}.
\end{aligned} \tag{46}$$

We see that since $\epsilon = \rho/50$ we get that

$$\left|\mathbb{P}\left(T_w^\rho(A) \leq \gamma^{-1}(\rho)(\log|A^\rho| + z)\right) - \exp(-e^{-z})\right| \leq c|A^\rho|^{-\rho/600}.$$

This ends the first case.

Case 2: Assume $z \geq \epsilon \log |A^\rho|$. This gives

$$\begin{aligned}
&\mathbb{P}\left(T_w^\rho(A) > \gamma^{-1}(\rho)(\log |A^\rho| + z)\right) \leq \mathbb{P}\left(T_w^\rho(A) > (1+\epsilon)\frac{\log |A^\rho|}{\gamma(\rho)}\right) \\
&= \mathbb{P}\left(\bigcup_{x \in A^\rho} \left\{T_s^\rho(x) > (1+\epsilon)\frac{\log |A^\rho|}{\gamma(\rho)}\right\}\right) \leq |A^\rho| \, \mathbb{P}\left(T_s^\rho(o) > (1+\epsilon)\frac{\log |A^\rho|}{\gamma(\rho)}\right) \\
&= |A^\rho| \exp(-(1+\epsilon)\log |A^\rho|) = |A^\rho|^{-\epsilon},
\end{aligned}$$

by using (38) in the second equality. Using that $z \geq \epsilon \log |A^\rho|$ we have that $\exp(-e^{-z}) \geq \exp(-|A^\rho|^{-\epsilon}) \geq 1 - |A^\rho|^{-\epsilon}$ by the inequality $e^x \geq 1 + x$, which holds for all x. This, and the above equation gives

$$\begin{aligned}
&\left|\mathbb{P}\left(T_w^\rho(A) \leq \gamma^{-1}(\rho)(\log |A^\rho| + z)\right) - \exp(-e^{-z})\right| \\
&\quad = \left|1 - \mathbb{P}\left(T_w^\rho(A) > \gamma^{-1}(\rho)(\log |A^\rho| + z)\right) - \exp(-e^{-z})\right| \\
&\quad \leq \mathbb{P}\left(T_w^\rho(A) > \gamma^{-1}(\rho)(\log |A^\rho| + z)\right) + \left|1 - \exp(-e^{-z})\right| \\
&\quad \leq |A^\rho|^{-\epsilon} + |A^\rho|^{-\epsilon} = 2\,|A^\rho|^{-\rho/50},
\end{aligned} \tag{47}$$

which proves the case $z \geq \epsilon \log |A^\rho|$.

Case 3: Assume that $z \in (-\frac{\epsilon}{4}\log |A^\rho|, \epsilon \log |A^\rho|)$ and start by observing that

$$\begin{aligned}
&\left|\mathbb{P}\left(T_w^\rho(A) \leq \gamma^{-1}(\rho)(\log |A^\rho| + z)\right) - \exp\left(-e^{-z}\right)\right| \\
&\quad \leq \left|\mathbb{P}\left(T_w^\rho(A) \leq \frac{\log |A^\rho| + z}{\gamma(\rho)}\right) - \mathbb{P}\left(T_s^\rho(A^\rho) \leq \frac{\log |A^\rho| + z}{\gamma(\rho)}, A_\epsilon^\rho \in G_{A,\rho,\epsilon}\right)\right| \\
&\quad + \left|\exp(-e^{-z})\mathbb{P}\left(A_\epsilon^\rho \in G_{A,\rho,\epsilon}\right) - \exp(-e^{-z})\right| \\
&\quad + \left|\mathbb{P}\left(T_s^\rho(A^\rho) \leq \frac{\log |A^\rho| + z}{\gamma(\rho)}, A_\epsilon^\rho \in G_{A,\rho,\epsilon}\right) - \exp(-e^{-z})\mathbb{P}\left(A_\epsilon^\rho \in G_{A,\rho,\epsilon}\right)\right|.
\end{aligned} \tag{48}$$

We will now consider the three terms on the right hand side.

To deal with the first term, note that since $T_w^\rho(A) = T_s^\rho(A^\rho)$ we have that

$$\begin{aligned}
&\left|\mathbb{P}\left(T_w^\rho(A) \leq \frac{\log |A^\rho| + z}{\gamma(\rho)}\right) - \mathbb{P}\left(T_s^\rho(A^\rho) \leq \frac{\log |A^\rho| + z}{\gamma(\rho)}, A_\epsilon^\rho \in G_{A,\rho,\epsilon}\right)\right| \\
&\quad = \mathbb{P}\left(T_s^\rho(A^\rho) \leq \frac{\log |A^\rho| + z}{\gamma(\rho)}, A_\epsilon^\rho \notin G_{A,\rho,\epsilon}\right) \leq \mathbb{P}\left(A_\epsilon^\rho \notin G_{A,\rho,\epsilon}\right) \leq c\,|A^\rho|^{-\epsilon/3},
\end{aligned}$$

by Proposition 5.3, this time applied to ϵ.

For the second term of the right hand side in (48), note that

$$\begin{aligned}
&\left|\exp\left(-e^{-z}\right)\mathbb{P}\left(A_\epsilon^\rho \in G_{A,\rho,\epsilon}\right) - \exp\left(-e^{-z}\right)\right| \\
&\quad = \exp\left(-e^{-z}\right)\mathbb{P}\left(A_\epsilon^\rho \notin G_{A,\rho,\epsilon}\right) \leq c\,|A^\rho|^{-\epsilon/3},
\end{aligned}$$

again by Proposition 5.3.

We now turn to the third term of the right hand side in (48), and this is where all our previous efforts come together. We will show that for any $K \in G_{A,\rho,\epsilon}$,

$$\left|\mathbb{P}\left(T_s^\rho(A^\rho) \leq \frac{\log |A^\rho| + z}{\gamma(\rho)} \Big| A_\epsilon^\rho = K\right) - \exp\left(-e^{-z}\right)\right| \leq c\,|A^\rho|^{-\epsilon/12}. \tag{49}$$

Then, multiplying by $\mathbb{P}(A^\rho_\epsilon = K)$ and summing over all $K \in G_{A,\rho,\epsilon}$ on both sides, we get

$$\left|\mathbb{P}\left(T^\rho_s(A^\rho) \leq \frac{\log|A^\rho| + z}{\gamma(\rho)}, A^\rho_\epsilon \in G_{A,\rho,\epsilon}\right) - \exp\left(-e^{-z}\right)\mathbb{P}\left(A^\rho_\epsilon \in G_{A,\rho,\epsilon}\right)\right|$$
$$\leq c\,|A^\rho|^{-\epsilon/12}.$$

We can then conclude from (48), that

$$\left|\mathbb{P}\left(T^\rho_w(A) \leq \gamma^{-1}(\rho)(\log|A^\rho| + z)\right) - \exp\left(-e^{-z}\right)\right| \leq c\,|A^\rho|^{-\epsilon/12} \tag{50}$$

for all $z \in (-\epsilon/4 \log|A^\rho|, \epsilon \log|A^\rho|)$. Then, by using that $\epsilon = \frac{\rho}{50}$, we get that

$$\left|\mathbb{P}\left(T^\rho_w(A) \leq \gamma^{-1}(\rho)(\log|A^\rho| + z)\right) - \exp\left(-e^{-z}\right)\right| \leq c\,|A^\rho|^{-\rho/600},$$

and the proof will be complete.

In order to prove (49), define

$$\omega^1 = \{L \in \omega_{(1-\epsilon)\frac{\log|A^\rho|}{\gamma(\rho)}} : \exists x \in A^\rho \text{ such that } B(x,\rho) \subset \mathfrak{c}(L)\},$$

so that ω^1 is the subset of $\omega_{(1-\epsilon)\gamma^{-1}(\rho)\log|A^\rho|}$ consisting of the cylinders singularly covering balls $B(x,\rho)$ where $x \in A^\rho$. Similarly, let

$$\omega^2 = \{L \in \omega_{(1-\epsilon)\frac{\log|A^\rho|}{\gamma(\rho)}, \frac{\log|A^\rho|+z}{\gamma(\rho)}} : \exists x \in A^\rho \text{ such that } B(x,\rho) \subset \mathfrak{c}(L)\}$$

so that a cylinder $\mathfrak{c}(L) \in \omega^2$ arrives between times $(1-\epsilon)\gamma^{-1}(\rho)\log|A^\rho|$ and $\gamma^{-1}(\rho)(\log|A^\rho| + z)$, and in addition it must singularly cover a ball $B(x,\rho)$. Finally, let $\omega^3 = \omega^1 \cup \omega^2$. For any $\omega \in \Omega$, let $\mathcal{C}_{s,\rho}(\omega, A^\rho)$ be the set of points in A^ρ that are ρ-singularly covered by the cylinders in ω.

Fix $K \in G_{A,\rho,\epsilon}$ and define the event

$$E_1 := \{A^\rho_\epsilon = K\} = \{A^\rho \setminus K = \mathcal{C}_{s,\rho}(\omega^1, A^\rho)\}.$$

The equality follows since $K \subset A^\rho$ is the set of points which are not ρ-singularly covered if and only if $A^\rho \setminus K$ is the set whose points are ρ-singularly covered.

Furthermore, $\{T^\rho_s(A^\rho) \leq \gamma^{-1}(\rho)(\log|A^\rho| + z)\}$ is the event that all the points of A^ρ are ρ-singularly covered by time $\gamma^{-1}(\rho)(\log|A^\rho| + z)$. Hence, by the definition of ω^3, $\{T^\rho_s(A^\rho) \leq \gamma^{-1}(\rho)(\log|A^\rho| + z)\} = \{A^\rho = \mathcal{C}_{s,\rho}(\omega^3, A^\rho)\}$ so that $E_1 \cap \{T^\rho_s(A^\rho) \leq \gamma^{-1}(\rho)(\log|A^\rho| + z)\} = E_1 \cap \{K \subset \mathcal{C}_{s,\rho}(\omega^2, A^\rho)\}$. Letting $E_2 = \{K \subset \mathcal{C}_{s,\rho}(\omega^2, A^\rho)\}$ and using that ω^1 and ω^2 are independent, we have that $\mathbb{P}(E_1 \cap E_2) = \mathbb{P}(E_1)\,\mathbb{P}(E_2)$. Furthermore, since ω^2 has the same distribution as a Poisson line process with intensity $\gamma^{-1}(\rho)(\epsilon \log|A^\rho| + z)$, we get that

$$\mathbb{P}(E_2) = \mathbb{P}\left(K \subset \mathcal{C}_{s,\rho}(\omega^2, A^\rho)\right) = \mathbb{P}\left(T^\rho_s(K) \leq \gamma^{-1}(\rho)(\epsilon \log|A^\rho| + z)\right).$$

On the other hand,

$$\mathbb{P}(E_2) = \frac{\mathbb{P}(E_1 \cap E_2)}{\mathbb{P}(E_1)}$$
$$= \frac{\mathbb{P}\left(E_1 \cap \left\{T^\rho_s(A^\rho) \leq \frac{\log|A^\rho|+z}{\gamma(\rho)}\right\}\right)}{\mathbb{P}(E_1)} = \mathbb{P}\left(T^\rho_s(A^\rho) \leq \frac{\log|A^\rho| + z}{\gamma(\rho)}\Big| E_1\right),$$

so we conclude that

$$\mathbb{P}\left(T_s^\rho(A^\rho)\leq \frac{\log|A^\rho|+z}{\gamma(\rho)}\Big|A_\epsilon^\rho=K\right)=\mathbb{P}\left(T_s^\rho(K)\leq\frac{\epsilon\log|A^\rho|+z}{\gamma(\rho)}\right). \tag{51}$$

Therefore, using (51) we have

$$\begin{aligned}
&\left|\mathbb{P}\left(T_s^\rho(A^\rho)\leq\gamma^{-1}(\rho)(\epsilon\log|A^\rho|+z)|A_\epsilon^\rho=K\right)-\exp\left(-e^{-z}\right)\right| \qquad (52)\\
&\quad=\left|\mathbb{P}\left(T_s^\rho(K)\leq\gamma^{-1}(\rho)(\epsilon\log|A^\rho|+z)\right)-\exp\left(-e^{-z}\right)\right|\\
&\quad\leq\left|\mathbb{P}\left(T_s^\rho(K)\leq\frac{\epsilon\log|A^\rho|+z}{\gamma(\rho)}\right)-\mathbb{P}\left(T_s^\rho(o)\leq\frac{\epsilon\log|A^\rho|+z}{\gamma(\rho)}\right)^{|K|}\right|\\
&\qquad+\left|\mathbb{P}\left(T_s^\rho(o)\leq\frac{\epsilon\log|A^\rho|+z}{\gamma(\rho)}\right)^{|K|}-\exp\left(-e^{-z}\right)\right|.
\end{aligned}$$

We will deal with the two terms on the right hand side of (52) separately.

For the first term, let $x,y\in K$ be distinct. By the definition of $G_{A,\rho,\epsilon}$ we have $d(x,y)\geq|A^\rho|^{1/(2d)}=(|A^\rho|^\epsilon)^{1/(2\epsilon d)}$. Furthermore, since $||K|-|A^\rho|^\epsilon|\leq|A^\rho|^{2\epsilon/3}$, we have that

$$|A^\rho|^\epsilon-|A^\rho|^{2\epsilon/3}\leq|K|\leq|A^\rho|^\epsilon+|A^\rho|^{2\epsilon/3}. \tag{53}$$

In particular, (53) implies $|K|\leq 2|A^\rho|^\epsilon$, so that $d(x,y)\geq(|A^\rho|^\epsilon)^{1/(2\epsilon d)}\geq(|K|/2)^{1/(2\epsilon d)}$. If we let $m=\frac{1}{2\epsilon}\frac{d-1}{d}-2$, we conclude that $d(x,y)\geq(|K|/2)^{\frac{2+m}{d-1}}$. Moreover, Assumption A2 says that $|A^\rho|^{1/(2d)}\geq 4\geq 4(1-\rho)$ and so we can use Proposition 5.1 with $n=|K|$ together with the fact that $z\leq\epsilon\log|A^\rho|$ to get

$$\begin{aligned}
&\left|\mathbb{P}\left(T_s^\rho(K)\leq\frac{\epsilon\log|A^\rho|+z}{\gamma(\rho)}\right)-\mathbb{P}\left(T_s^\rho(o)\leq\frac{\epsilon\log|A^\rho|+z}{\gamma(\rho)}\right)^{|K|}\right| \qquad (54)\\
&\quad\leq c\epsilon\gamma(\rho)\log|A^\rho|\,(|K|/2)^{-m}.
\end{aligned}$$

Furthermore, since $\epsilon=\rho/50$ we have that $|A^\rho|^{\epsilon/3}=|A^\rho|^{\rho/150}\geq 2$ by Assumption A4. Therefore,

$$\frac{2}{|K|}\leq\frac{2}{|A^\rho|^\epsilon-|A^\rho|^{2\epsilon/3}}\leq\frac{1}{|A^\rho|^{\epsilon/3}},$$

where we use (53) in the first inequality and the fact that $x^3-x^2\geq 2x$ if $x\geq 2$ in the second. Furthermore, by Assumption A1 and our choice of ϵ we have that $\epsilon<\rho<\frac{d-1}{6d}$, and so we get that $m>1$. We conclude that $(|K|/2)^{-m}\leq|A^\rho|^{-m\epsilon/3}\leq|A^\rho|^{-\epsilon/3}$. Thus,

$$c\epsilon\gamma(\rho)\log|A^\rho|\,(|K|/2)^{-m}\leq c\rho(\log|A^\rho|)\,|A^\rho|^{-\epsilon/3} \tag{55}$$

where we also used that $\gamma(\rho)<1$ and that $\epsilon=\rho/50$. Moreover, by Assumption A3, $\rho\log|A^\rho|\leq|A^\rho|^{\rho/200}=|A^\rho|^{\epsilon/4}$ so that $c\rho(\log|A^\rho|)\,|A^\rho|^{-\epsilon/3}\leq c\,|A^\rho|^{\epsilon/4}\,|A^\rho|^{-\epsilon/3}=c\,|A^\rho|^{-\epsilon/12}$. We conclude from (54) and (55) that

$$\left|\mathbb{P}\left(T_s^\rho(K)\leq\frac{\epsilon\log|A^\rho|+z}{\gamma(\rho)}\right)-\mathbb{P}\left(T_s^\rho(o)\leq\frac{\epsilon\log|A^\rho|+z}{\gamma(\rho)}\right)^{|K|}\right|\leq c\,|A^\rho|^{-\epsilon/12}. \tag{56}$$

We can now turn to the second term of (52). By (38),

$$\mathbb{P}\left(T_s^\rho(o) \le \frac{\epsilon \log |A^\rho| + z}{\gamma(\rho)}\right)^{|K|}$$
$$= (1 - \exp(-\epsilon \log |A^\rho| - z))^{|K|} = \left(1 - \frac{e^{-z}}{|A^\rho|^\epsilon}\right)^{|K|}.$$

Then, (53) gives

$$\left(1 - \frac{e^{-z}}{|A^\rho|^\epsilon}\right)^{|A^\rho|^\epsilon + |A^\rho|^{2\epsilon/3}} \tag{57}$$
$$\le \mathbb{P}\left(T_s^\rho(o) \le \frac{\epsilon \log |A^\rho| + z}{\gamma(\rho)}\right)^{|K|} \le \left(1 - \frac{e^{-z}}{|A^\rho|^\epsilon}\right)^{|A^\rho|^\epsilon - |A|^{2\epsilon/3}}.$$

By Proposition 5.4 we get that (with a slight abuse of notation)

$$\left|\mathbb{P}\left(T_s^\rho(o) \le \frac{\epsilon \log |A^\rho| + z}{\gamma(\rho)}\right)^{|K|} - \exp\left(-e^{-z}\right)\right|$$
$$\le \left|\left(1 - \frac{e^{-z}}{|A^\rho|^\epsilon}\right)^{|A^\rho|^\epsilon \pm |A^\rho|^{2\epsilon/3}} - \exp\left(-e^{-z}\right)\right| \le c\,|A^\rho|^{-\epsilon/12}.$$

Combining (52), (56) and (57), the inequality (49) is proved. This completes the proof. □

6 The discrete cover time $T_d^\rho(A)$

The purpose of this section is to establish the analogue of Theorem 5.5 but for $T_d^\rho(A)$. As a consequence, we will then be able to also prove Theorem 1.2. Most of this section is straightforward adjustments of previous results, and so we will not provide proofs (except for Theorem 1.2). However, some changes must be done, and so we will indicate what these are.

Start by noting that ρ is present in two places in the definition of $\beta(\rho, k)$ (see (16)), i.e. in the spacing between the points of A^ρ and also in the radius of the balls $B(x, 1-\rho)$. In our current case, we will consider balls $B(x, 1)$, but since the spacing is the same, it is not the case that the analogue here should simply be $\beta(0, k)$. Instead, we have the following

$$\alpha(\rho, k) := \mu_{d,1}\left(\mathcal{L}_{B(o,1)} \cup \mathcal{L}_{B(2^k\rho e_1, 1)}\right). \tag{58}$$

The next result is the discrete cover case version of Lemma 3.7. The proof is straightforward from that lemma.

Lemma 6.1. *We have that* $1 + \frac{2^k}{12}\rho \le \alpha(k, \rho) \le 2$, *for* $\rho < 2/3$ *and all* k *such that* $2^k \rho \le 2\sqrt{1 - 4^{-1/(d-2)}}$.

In order to distinguish the notation from the ρ-well cover case, we will here let

$$\dot{A}_\epsilon^\rho := \{x \in A^\rho : T_d^\rho(x) > (1-\epsilon)\log|A^\rho|\},$$

so that $\dot{A}_\epsilon^\rho$ is the subset of A^ρ not yet covered at time $(1-\epsilon)\log|A^\rho|$. Note the absence of $\gamma^{-1}(\rho)$ which was present in the corresponding definition of A_ϵ (21).

We next turn to the discrete version of Lemma 5.2. The difference here is due to the change from A_ϵ to $\dot{A}_\epsilon$, and in particular that the time considered has been changed from $\gamma^{-1}(\rho)(1-\epsilon)\log|A^\rho|$ to $(1-\epsilon)\log|A^\rho|$. In the proof itself, one simply needs to replace $\gamma(\rho)$ by 1 and use Lemma 6.1 in place of Lemma 3.7.

Lemma 6.2. *Let A, ρ satisfy Assumptions A1, A4 and A5, and let ϵ be such that $\rho/1000 < \epsilon < \rho/36$. We then have that*

a) $$\sum_{\substack{x,y\in A^\rho \\ x\neq y}} \mathbb{P}\left(x, y \in \dot{A}_\epsilon^\rho\right) < c\,|A^\rho|^{-\epsilon} + |A^\rho|^{2\epsilon}$$

b) $$\sum_{\substack{x,y\in A^\rho \\ 0<d(x,y)<b|A^\rho|^{1/2d}}} \mathbb{P}\left(x, y \in \dot{A}_\epsilon^\rho\right) < cb^d\,|A^\rho|^{-\epsilon},$$

for every $b \geq 1$, and where c is a constant depending only on d.

Our next step is to address the analogue of Proposition 5.3. Recall therefore the definition of $G_{A,\rho,\epsilon}$ in (35), we do not need to adjust this definition. As before, the proof goes through with only minor changes. Of course, one needs to use Lemma 6.2 in place of Lemma 5.2.

Proposition 6.3. *Let ρ, A satisfy Assumptions A1, A4 and A5. Then, for $\rho/1000 < \epsilon < \rho/36$, we have that*

$$\mathbb{P}\left(\dot{A}_\epsilon^\rho \notin G_{A,\rho,\epsilon}\right) \leq c\,|A^\rho|^{-\epsilon/3}.$$

We have arrived at the analogue of Theorem 5.5. Again, we can basically copy the proof from the ρ-well covered case. Of course, we use Proposition 6.3 in place of Proposition 5.3 while Proposition 5.1 can simply be used with $\rho = 0$.

Theorem 6.4. *For any ρ and $A \subset \mathbb{R}^d$ satisfying Assumptions A1 through A5 we have that*

$$\sup_{z\in\mathbb{R}} \left|\mathbb{P}\left(T_d^\rho(A) - \log|A^\rho| \leq z\right) - \exp(-e^{-z})\right| \leq c_4\,|A^\rho|^{-\rho/600},$$

where c_4 is a constant depending on d only.

Remark. It is possible to arrive at Theorem 6.4 through a different route. We here give a brief and somewhat informal outline of how this is done.

Firstly, observe that the ρ-well cover time of A^ρ is the same as the discrete cover time of A^ρ, provided we use cylinders of radius $1-\rho$ in place of cylinders of radius 1.

Secondly, we can scale space by a factor of $(1-\rho)^{-1}$ in order to re-obtain a cylinder process of radius 1. However, the scaling results in a cylinder process of rate $\gamma(\rho)$ (rather than 1). This easily follows by using Lemma 3.1 telling us that for any compact $K \subset \mathbb{R}^d$,

$$\mu(\mathcal{L}_K) = \gamma(\rho)\mu(\mathcal{L}_{K/(1-\rho)}).$$

Furthermore, the set A^ρ is then also scaled by a factor of $(1-\rho)^{-1}$ yielding

$$\left(\frac{1}{1-\rho}\right) A^\rho = \left(\frac{A}{1-\rho}\right)^{\rho/(1-\rho)}.$$

Therefore,

$$T_w^\rho(A) = \tilde{T}_d^{\rho/(1-\rho)}\left(\frac{A}{1-\rho}\right) = \gamma^{-1}(\rho) T_d^{\rho/(1-\rho)}\left(\frac{A}{1-\rho}\right)$$

where $\tilde{T}_d$ represents the discrete cover time of a cylinder process with intensity $\gamma(\rho)$, and where the final equality follows from the fact that the rate is scaled by a factor $\gamma(\rho)$.

Using Theorem 5.5 one then obtains Theorem 6.4 but with $T_d^{\rho/(1-\rho)}\left(\frac{A}{1-\rho}\right)$ in place of $T_d^\rho(A)$. Intuitively, there should be little difference between these two cover times when ρ is very small, and indeed after some additional work, one obtains the exact statementof Theorem 6.4.

We are now ready to prove Theorem 1.2. Since this theorem considers only fixed ρ, we will not have to optimize as we shall be doing in Section 7.

Proof of Theorem 1.2. It is straightforward to verify that for n large enough, A_n, ρ satisfies Assumptions A1 through A5 (see also Proposition 7.2). By Theorem 6.4 we then have that for any $z \in \mathbb{R}$,

$$\limsup_{n\to\infty} |\mathbb{P}(T_d^\rho(A_n) - \log|A_n^\rho| \leq z) - \exp(-e^{-z})| \leq \limsup_{n\to\infty} c_4 |A_n^\rho|^{-\rho/600} = 0.$$

Therefore,

$$\mathbb{P}(T_d^\rho(A_n) - \log|A_n^\rho| \leq z) \to \exp(-e^{-z}),$$

as $n \to \infty$. □

7 Main results

In this section we will finally discuss the asymptotics of the cover time $\mathcal{T}(A)$. Our first preliminary result will be to prove Proposition 7.1 discussed already in the Introduction (see in particular (4)). For this, recall the canonical coupling of $T_d^\rho(A), \mathcal{T}(A)$ and $T_w^\rho(A)$ as described in (8).

Proposition 7.1. *We have that for any $\rho > 0$,*

$$T_d^\rho(A) \leq \limsup_{\delta\to 0} T_d^\delta(A) = \mathcal{T}(A) \ \ a.s.$$

Proof. Obviously, $T_d^\rho(A, (\omega_t)_{t\geq 0}) \leq \mathcal{T}(A, (\omega_t)_{t\geq 0})$ for every $\rho > 0$. Assume that $\mathcal{T}(A, (\omega)_{t\geq 0}) > \tau$ and let

$$\mathcal{C}_\tau := \bigcup_{(L,s)\in\Psi: s\leq\tau} \mathfrak{c}(L).$$

Then, there exists a point $x \in A \setminus \mathcal{C}_\tau$ and some $\delta > 0$ such that $B(x,\delta) \subset A \setminus \mathcal{C}_\tau$. This follows since a.s. $\mathcal{C}_\tau$ is a closed set. Therefore we must have that $T_d^{\delta/2}(A) > \tau$. We conclude that $\limsup_{\delta \to 0} T_d^\delta(A) \geq \mathcal{T}(A)$. $\square$

Proposition 7.1 establishes a natural result, but perhaps more importantly it tells us that in order to obtain the best bounds on $\mathcal{T}(A)$, we should pick ρ as small as possible. However, Theorems 5.5 and 6.4 shows that if we take ρ too small, the estimates that we have obtained (of the form $|A^\rho|^{-\rho}$) will not be small, but rather they will be close to one. In addition, we need to make sure that ρ is picked in such a way that Assumptions A1 through A5 are satisfied.

Consider therefore a sequence of sets $(A_n)_{n\geq 1}$ such that $|A_n^1| \to \infty$ as $n \to \infty$, and let $\rho_n = D/\log|A_n^1|$ where $D \geq 1$ is some constant. The purpose of the next proposition is to verify Assumptions A1 through A5 for our choice of ρ_n whenever n is large enough.

Proposition 7.2. *Consider sequences $(A_n)_{n\geq 1}$ and $(\rho_n)_{n\geq 1}$ as above. If $D \geq 1$ is such that $De^{-D/200} \leq 1/2$, there exists an $N = N(D)$ such that Assumptions A1 through A5 are all satisfied for every $n \geq N$.*

Proof. Start by observing that $|A_n^1|^{\rho_n} = |A_n^1|^{D/\log|A_n^1|} = e^D$. We will deal with the assumptions in order.

A1: Since $0 < \rho_n$ for every n, and $\rho_n \to 0$, Assumption A1 is obviously satisfied for n larger than some N_1.

A2: By Lemma 4.1 part *d*), we have that $|A_n^{\rho_n}|^{1/(2d)} \geq |A_n^1|^{1/(2d)} \geq 4$ for every n larger than some N_2, since $|A_n^1| \to \infty$.

A3: By Lemma 4.1 part *d*) we have that, for large n

$$\begin{aligned}
\rho_n \log|A_n^{\rho_n}|\,|A_n^{\rho_n}|^{-\rho_n/200} &\leq \rho_n \log(6^d \rho_n^{-d} \left|A_n^1\right|) \left|A_n^1\right|^{-\rho_n/200} \\
&= \frac{D}{\log|A_n^1|} \left(d\log 6 - d\log D + d\log\log\left|A_n^1\right| + \log\left|A_n^1\right|\right) e^{-D/200} \\
&\leq De^{-D/200} \left(\frac{d\log 6}{\log|A_n^1|} + \frac{d\log\log\left|A_n^1\right|}{\log|A_n^1|} + 1\right) \leq 1
\end{aligned}$$

since $|A_n^1| \to \infty$ and $De^{-D/200} \leq 1/2$ by the condition on D. Therefore, Assumption A3 holds for all n larger than some N_3.

A4: Again we use Lemma 4.1 part *d*) to conclude that

$$|A_n^{\rho_n}|^{\rho_n/200} \geq |A_n^1|^{\rho_n/200} = e^{D/200} \geq 2$$

for every n by our condition on D.

A5: Finally, in order to prove that Assumption A5 is satisfied for large enough n, we yet again use Lemma 4.1 part *d*) to get that

$$\begin{aligned}
\rho_n^{-d}(\log|A_n^{\rho_n}|)^d\,|A_n^{\rho_n}|^{-\tilde{C}_d/2} &\leq \rho_n^{-d}(\log(6^d\rho_n^{-d}\left|A_n^1\right|))^d \left|A_n^1\right|^{-\tilde{C}_d/2} \\
&= \left(\frac{\log\left|A_n^1\right|}{D}\right)^d \left(d\log 6 - d\log D + d\log\log|A_n^1| + \log|A_n^1|\right)^d \left|A_n^1\right|^{-\tilde{C}_d/2} \\
&\leq (\log\left|A_n^1\right|)^d \left(d\log 6 + d\log\log|A_n^1| + \log|A_n^1|\right)^d \left|A_n^1\right|^{-\tilde{C}_d/2} \to 0
\end{aligned}$$

since $D \geq 1$ and $\left|A_n^1\right| \to \infty$. Thus, there is an N_5 such that Assumption A5 is satisfied for all $n \geq N_5$.

By letting $N = \max_{1\leq i\leq 5} N_i$, we see that all the assumptions are satisfied for $n \geq N$. $\square$

We are now ready to prove the main theorem of the paper.

Proof of Theorem 1.1. We need to show that for any $\epsilon > 0$, there exists a constant C_ϵ independent of n, such that for any set A satisfying (1) we have that

$$\mathbb{P}(|\mathcal{T}(nA) - \dim_B(A)(\log n + \log\log n)| \geq C_\epsilon) \leq \epsilon,$$

for every n.

For fixed $\epsilon > 0$, we start by picking $D \geq 1$ such that $c_3 e^{-D/600} \leq \epsilon/4$ and $c_4 e^{-D/600} \leq \epsilon/4$, where c_3, c_4 are the constants given in Theorems 5.5 and 6.4. In addition, assume that D satisfies the conditions of Proposition 7.2. Thereafter, we choose $C_\epsilon \geq 12dD$ such that $e^{-C_\epsilon/8} \leq \epsilon/4$ and $\exp(-e^{C_\epsilon/8}) \leq \epsilon/4$. The reason for these choices will become clear later.

In order to illustrate the importance of the constant c_A of (2), we shall assume that in fact (2) is satisfied and then make the necessary adjustments at the end. We will let $\tilde{c}_A, := \dim_B(A)^{\dim_B(A)} c_A$.

Write A_n for nA and recall from (7) that for every $n \geq 1$, $|A_n^{\rho_n}| = \left|A^{\rho_n/n}\right|$. Therefore,

$$\lim_{n\to\infty} \frac{\log|A_n^1|}{\log n} = \lim_{n\to\infty} \frac{\log|A^{1/n}|}{\log n} = \dim_B(A),$$

by the definition of $\dim_B(A)$ (see Section 2.3). We then let $\rho_n = \frac{D}{\log|A_n^1|}$ to get that

$$\begin{aligned}\left(\frac{D}{n\log n}\right)^{\dim_B(A)} |A_n^{\rho_n}| &= \left(\frac{\log|A_n^1|}{\log n}\right)^{\dim_B(A)} \left(\frac{D}{n\log|A_n^1|}\right)^{\dim_B(A)} |A_n^{\rho_n}| \\ &= \left(\frac{\log|A_n^1|}{\log n}\right)^{\dim_B(A)} \left(\frac{\rho_n}{n}\right)^{\dim_B(A)} |A^{\rho_n/n}| \to \dim_B(A)^{\dim_B(A)} c_A = \tilde{c}_A,\end{aligned}$$

by (2) and the fact that $\rho_n/n \to 0$.

Thus,

$$\frac{1}{\tilde{c}_A}\left(\frac{D}{n\log n}\right)^{\dim_B(A)} |A_n^{\rho_n}| \to 1. \tag{59}$$

Note further that $|\log(D^{\dim_B(A)})| \leq |d\log D| \leq dD \leq C_\epsilon/12$ so that by (59) we have that

$$|\log(D^{\dim_B(A)})| + \left|\log\left(\frac{1}{\tilde{c}_A}\left(\frac{D}{n\log n}\right)^{\dim_B(A)} |A_n^{\rho_n}|\right)\right| \leq C_\epsilon/2,$$

for every n larger than some $N = N(D, C_\epsilon)$. Thus, for $n \geq N(D, C_\epsilon)$ we get

$$\begin{aligned}
&\mathbb{P}(|\mathfrak{T}(A_n) - \dim_B(A)(\log n + \log\log n) - \log \tilde{c}_A| \geq C_\epsilon) \qquad (60)\\
&= \mathbb{P}\Bigg(\Bigg|\mathfrak{T}(A_n) - \log|A_n^{\rho_n}| - \log(D^{\dim_B(A)}) \\
&\qquad + \log\left(\frac{1}{\tilde{c}_A}\left(\frac{D}{n\log n}\right)^{\dim_B(A)}|A_n^{\rho_n}|\right)\Bigg| \geq C_\epsilon\Bigg)\\
&\leq \mathbb{P}\left(|\mathfrak{T}(A_n) - \log|A_n^{\rho_n}|| \geq C_\epsilon/2\right)\\
&\leq \mathbb{P}\left(T_d^{\rho_n}(A_n) - \log|A_n^{\rho_n}| \leq -C_\epsilon/2\right) + \mathbb{P}\left(T_w^{\rho_n}(A_n) - \log|A_n^{\rho_n}| \geq C_\epsilon/2\right),
\end{aligned}$$

where the last inequality follows since $T_d^{\rho_n}(A_n) \leq \mathfrak{T}(A_n) \leq T_w^{\rho_n}(A_n)$.

We shall now address the two probabilities of the right hand side of (60) separately. First, by Theorem 6.4 and Lemma 4.1 part d) we have that

$$\begin{aligned}
&\mathbb{P}\left(T_d^{\rho_n}(A_n) - \log|A_n^{\rho_n}| \leq -C_\epsilon/2\right) \leq \exp(-e^{C_\epsilon/2}) + c_4|A_n^{\rho_n}|^{-\rho_n/600} \qquad (61)\\
&\quad \leq \exp(-e^{C_\epsilon/2}) + c_4|A_n^1|^{-\rho_n/600} = \exp(-e^{C_\epsilon/2}) + c_4 e^{-D/600} \leq \epsilon/2,
\end{aligned}$$

because of our choices of D and C_ϵ.

We now turn to the second term of the right hand side of (60). By picking N perhaps even larger than before, we have that $\rho_n = D/\log|A_n^1| \leq 1 - 2^{-1/d}$ for $n \geq N$. We can then use the inequality $(1-x)^{-d+1} \leq 1 + 2dx$ which holds for $0 < x \leq 1 - 2^{-1/d}$ to conclude that for such n,

$$\frac{1}{\gamma(\rho_n)} - 1 = \frac{1}{(1-\rho_n)^{d-1}} - 1 \leq 2d\rho_n.$$

Therefore, by Lemma 4.1 part d) we get that

$$\begin{aligned}
\log|A_n^{\rho_n}|\left(\frac{1}{\gamma(\rho_n)} - 1\right) &\leq \log(6^d \rho_n^{-d}|A_n^1|)2d\rho_n\\
&= (d\log 6 + \log|A_n^1| + d\log\log|A_n^1| - d\log\log D)2d\frac{D}{\log|A_n^1|}.
\end{aligned}$$

Clearly, there exists an N perhaps even larger than before, such that for every $n \geq N$,

$$\log|A_n^{\rho_n}|\left(\frac{1}{\gamma(\rho_n)} - 1\right) \leq 3dD \leq C_\epsilon/4,$$

where we use the fact that $C_\epsilon \geq 12dD$ by assumption.

Hence, for $n \geq N$,

$$\begin{aligned}
&\mathbb{P}\left(T_w^{\rho_n}(A_n) - \log|A_n^{\rho_n}| \geq C_\epsilon/2\right)\\
&= \mathbb{P}\left(T_w^{\rho_n}(A_n) - \frac{1}{\gamma(\rho_n)}\log|A_n^{\rho_n}| + \frac{1}{\gamma(\rho_n)}\log|A_n^{\rho_n}| - \log|A_n^{\rho_n}| \geq C_\epsilon/2\right)\\
&\leq \mathbb{P}\left(T_w^{\rho_n}(A_n) - \frac{1}{\gamma(\rho_n)}\log|A_n^{\rho_n}| \geq C_\epsilon/4\right)\\
&= \mathbb{P}\left(\gamma(\rho_n)T_w^{\rho_n}(A_n) - \log|A_n^{\rho_n}| \geq \gamma(\rho_n)C_\epsilon/4\right)\\
&\leq \mathbb{P}\left(\gamma(\rho_n)T_w^{\rho_n}(A_n) - \log|A_n^{\rho_n}| \geq C_\epsilon/8\right),
\end{aligned}$$

where the last inequality holds for every n such that $\gamma(\rho_n) \geq 1/2$. This clearly holds for every $n \geq N$ where N might be even larger than before. Similarly to the first case, we have that by Theorem 5.5 and Lemma 4.1 part d),

$$\begin{aligned}
&\mathbb{P}\left(\gamma(\rho_n)T_w^{\rho_n}(A_n) - \log|A_n^{\rho_n}| \geq C_\epsilon/8\right) \qquad (62)\\
&\quad = 1 - \mathbb{P}\left(\gamma(\rho_n)T_w^{\rho_n}(A_n) - \log|A_n^{\rho_n}| \leq C_\epsilon/8\right)\\
&\quad \leq 1 - \exp(-e^{-C_\epsilon/8}) + c_3|A_n^{\rho_n}|^{-\rho_n/600} \leq e^{-C_\epsilon/8} + c_3 e^{-D/600} \leq \epsilon/2.
\end{aligned}$$

Finally, combining (60), (61) and (62) we conclude that

$$\mathbb{P}(|\mathfrak{T}(A_n) - \dim_B(A)(\log n + \log\log n) - \log \tilde{c}_A)| \geq C_\epsilon) \leq \epsilon \qquad (63)$$

for every $n \geq N$. Then, it follows that (63) holds for every $n \geq 1$ by (perhaps) increasing C_ϵ even further.

The full statement (i.e. assuming (1) instead of (2)) is proved in a very similar way, and therefore we will only indicate the changes. Again we get that $\lim_{n\to\infty} \frac{\log|A_n^1|}{\log n} = \dim_B(A)$ and the first change is that (59) is replaced by

$$0 < \liminf_{n\to\infty} \left(\frac{D}{n\log n}\right)^{\dim_B(A)} |A_n^{\rho_n}| \leq \limsup_{n\to\infty} \left(\frac{D}{n\log n}\right)^{\dim_B(A)} |A_n^{\rho_n}| < \infty.$$

Then, one can pick C_ϵ perhaps even larger so that

$$|\log(D^{\dim_B(A)})| + \limsup_{n\to\infty} \left|\log\left(\left(\frac{D}{n\log n}\right)^{\dim_B(A)} |A_n^{\rho_n}|\right)\right| \leq C_\epsilon/3.$$

This can then be inserted into a slightly modified version of (60) in order to obtain the statement

$$\begin{aligned}
&\mathbb{P}(|\mathfrak{T}(A_n) - \dim_B(A)(\log n + \log\log n)| \geq C_\epsilon)\\
&\quad \leq \mathbb{P}\left(T_d^{\rho_n}(A_n) - \log|A_n^{\rho_n}| \leq -C_\epsilon/2\right)\\
&\qquad + \mathbb{P}\left(\gamma(\rho_n)T_w^{\rho_n}(A_n) - \log|A_n^{\rho_n}| \geq C_\epsilon/2\right),
\end{aligned}$$

for every n large enough, and then we can proceed as above. □

We now address Theorem 2.1.

Proof of Theorem 2.1. The proof is again similar to the proof of Theorem 1.1 and so we shall be brief. One of the main differences is that here we let

$$\rho_n := \frac{\log\log n}{\log|A_n^1|},$$

where $A_n = nA$ as before. We remark that the choice of $\log\log n$ in the numerator is somewhat arbitrary. Indeed, any function that goes to infinity sufficiently slow as $n \to \infty$ would do. The purpose is to make sure that $|A^{\rho_n}|^{-\rho_n/600}$ vanishes in the limit and the Assumptions A1 to A5 are satisfied.

Observe also that by (7),

$$\liminf_{n\to\infty} \frac{\log|A_n^1|}{\log n} = \liminf_{n\to\infty} \frac{\log|A^{1/n}|}{-\log(1/n)} \geq \liminf_{\rho\to 0} \frac{\log|A^\rho|}{-\log\rho} = \underline{\dim}_B(A).$$

Letting $\delta := (\underline{\dim}_B(A) - \underline{\alpha})/2$ we then get that

$$\begin{aligned}
&\left(\frac{\log\log n}{n\log n}\right)^{\underline{\dim}_B(A)-\delta} |A_n^{\rho_n}| \\
&= \left(\frac{\log|A_n^1|}{\log n}\right)^{\underline{\dim}_B(A)-\delta} \left(\frac{\log\log n}{n\log|A_n^1|}\right)^{\underline{\dim}_B(A)-\delta} |A_n^{\rho_n}| \\
&= \left(\frac{\log|A_n^1|}{\log n}\right)^{\underline{\dim}_B(A)-\delta} \left(\frac{\rho_n}{n}\right)^{\underline{\dim}_B(A)-\delta} |A^{\rho_n/n}| \to \infty,
\end{aligned}$$

by the definition of $\underline{\dim}_B(A)$.

We then see that for n larger than some N,

$$\begin{aligned}
&\mathbb{P}(\mathcal{T}(A_n) - \underline{\alpha}\log n \leq z) \\
&= \mathbb{P}\Big(\mathcal{T}(A_n) - \log|A_n^{\rho_n}| + \delta\log n + (\underline{\dim}_B(A) - \delta)\log\log n \\
&\quad - \log((\log\log n)^{\underline{\dim}_B(A)-\delta}) + \log\left(\left(\frac{\log\log n}{n\log n}\right)^{\underline{\dim}_B(A)-\delta} |A_n^{\rho_n}|\right) \leq z\Big) \\
&\leq \mathbb{P}\left(\mathcal{T}(A_n) - \log|A_n^{\rho_n}| + \delta\log n \leq z\right) \\
&\leq \mathbb{P}\left(T_d^{\rho_n}(A_n) - \log|A_n^{\rho_n}| \leq z - \delta\log n\right).
\end{aligned}$$

where the last inequality follows since $\mathcal{T}(A_n) \geq T_d^{\rho_n}(A_n)$. Using Theorem 6.4 and Lemma 4.1 part d), we then get that

$$\begin{aligned}
&\mathbb{P}(\mathcal{T}(A_n) - \underline{\alpha}\log n \leq z) \leq c_4|A_n^{\rho_n}|^{-\rho_n/600} + \exp(-e^{-z+\delta\log n}) \\
&\quad \leq c_4|A_n^1|^{-\rho_n/600} + \exp(-e^{-z}n^\delta) = c_4 e^{-\log\log n/600} + \exp(-e^{-z}n^\delta) \to 0.
\end{aligned}$$

The second statement is proved in the same way so we omit the proof. □

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