Basal boundary conditions, stability and verification in glaciological numerical models

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Abstract
To increase our understanding of how ice sheets and glaciers interact with the climate system, numerical models have become an indispensable tool. However, the complexity of these systems and the natural limitation in computational power is reflected in the simplifications of the represented processes and the spatial and temporal resolution of the models. Whether the effect of these limitations is acceptable or not, can be assessed by theoretical considerations and by validating the output of the models against real world data. Equally important is to verify if the numerical implementation and computational method accurately represent the mathematical description of the processes intended to be simulated. This thesis concerns a set of numerical models used in the field of glaciology, how these are applied and how they relate to other study areas in the same field.

The dynamical flow of glaciers, which can be described by a set of non-linear partial differential equations called the Full Stokes equations, is simulated using the finite element method. To reduce the computational cost of the method significantly, it is common to lower the order of the used elements. This results in a loss of stability of the method, but can be remedied by the use of stabilization methods. By numerically studying different stabilization methods and evaluating their suitability, this work contributes to constraining the values of stabilization parameters to be used in ice sheet simulations. Erroneous choices of parameters can lead to oscillations of surface velocities, which affects the long term behavior of the free-surface ice and as a result can have a negative impact on the accuracy of the simulated mass balance of ice sheets.

The amount of basal sliding is an important component that affects the overall dynamics of the ice. A part of this thesis considers different implementations of the basal impenetrability condition that accompanies basal sliding, and shows that methods used in literature can lead to a difference in velocity of 1% to 5% between the considered methods.

The subglacial hydrological system directly influences the glacier's ability to slide and therefore affects the velocity distribution of the ice. The topology and dominant mode of the hydrological system on the ice sheet scale is, however, ill constrained. A third contribution of this thesis is, using the theory of R-channels to implement a simple numerical model of subglacial water flow, to show the sensitivity of subglacial channels to transient processes and that this limits their possible extent. This insight adds to a cross-disciplinary discussion between the different sub-fields of theoretical, field and paleo-glaciology regarding the characteristics of ice sheet subglacial hydrological systems. In the study, we conclude by emphasizing areas of importance where the sub-fields have yet to unify: the spatial extent of channelized subglacial drainage, to what degree specific processes are connected to geomorphic activity and the differences in spatial and temporal scales.

As a whole, the thesis emphasizes the importance of verification of numerical models but also acknowledges the natural limitations of these to represent complex systems. Focusing on keeping numerical ice sheet and glacier models as transparent as possible will benefit end users and facilitate accurate interpretations of the numerical output so it confidently can be used for scientific purposes.

Keywords: Glaciology, subglacial hydrology, ice sheet modeling, basal boundary conditions, non-linear Stokes flow.

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If you try and take a cat apart to see how it works, the first thing you have on your hands is a non-working cat.

Douglas Adams
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To increase our understanding of how ice sheets and glaciers interact with the climate system, numerical models have become an indispensable tool. However, the complexity of these systems and the natural limitation in computational power is reflected in the simplifications of the represented processes and the spatial and temporal resolution of the models. Whether the effect of these limitations is acceptable or not, can be assessed by theoretical considerations and by validating the output of the models against real world data. Equally important is to verify if the numerical implementation and computational method accurately represent the mathematical description of the processes intended to be simulated. This thesis concerns a set of numerical models used in the field of glaciology, how these are applied and how they relate to other study areas in the same field.

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Sammanfattning

För att öka förståelsen för hur inlandsisar och glaciärer interagerar med klimatsystemet, har numeriska modeller blivit ett ombärligt verktyg. Komplexiteten hos dessa system och den naturliga begränsningen av tillgänglig datorkraft återspeglas i förenklingar av de representerade processerna och den rumsliga och temporalna upplösningen i modellerna. Om effekten av dessa begränsningar är acceptabla eller inte, kan bedömas genom teoretiska överväganden och genom att validera modellresultat mot verkliga data. Lika viktigt är att kontrollera om de numeriska modellerna tillräckligt noggrant representerar den matematiska beskrivning av de processer som är avsedda att simuleras. Denna avhandling berör en uppsättning av numeriska modeller som används inom glaciologi, hur dessa tillämpas och hur de förhåller sig till andra studieområden inom samma fält.

Glaciärers dynamiska flöde, som kan beskrivas av en uppsättning av partiella differentialekvationer som kallas Full Stokes ekvationerna, simuleras genom att använda finita element-metoden. För att reducera metodens beräkningskostnader betydligt är det vanligt att minska ordeningen av de element som används. Detta resulterar i att metoden blir instabil, men detta kan åtgärdas med hjälp av stabiliseringsmetoder. Genom att numeriskt studera olika stabiliseringsmetoder och utvärdera deras lämplighet, bidrar detta arbete till att begränsa värden för de stabiliseringsparametrar som används i syftet att simulera inlandsisar. Felaktiga val av sådana parametrar kan leda till oscillationer i ythastigheten vilket påverkar den långsiktiga utvecklingen av isens yta, och kan resultera i en negativ inverkan på nöggspanheten av inlandsisars simulera massbalans.

Hur mycket en glaciär glider mot sitt underlag är en viktig komponent i isens övergripande dynamik. En del av denna avhandling betraktar olika implementeringar av det villkor av ogenomtränglighet vid glaciärens bädd som åtfoljer basal glidning, och visar att det mellan metoder som används i litteraturen kan skilja 1 % to 5 % i hastighet.


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Som helhet betonar avhandlingen betydelsen av att verifiera numeriska modeller, men medger också de naturliga begränsningarna som dessa modeller har för att representera complexa system. Ett fokus på att behålla numeriska ismodeller transparenta kommer att vara till nytta för slutanvändare och underlätta riktiga tolkningar av det numeriska resultatet så att dessa tillitsfullt kan användas i ett vetenskapligt syfte.
Thesis content

This doctoral compilation dissertation consists of a summarizing text and the four articles listed below, which will be referred to by their Roman numerals in the text.


IV Helanow, C. Effects of numerical implementations of the impenetrability condition on non-linear Stokes flow: applications to ice dynamics. *Manuscript*. 
Author contributions

The contributions from listed authors are divided as follows for each article.

I  The study was conceived and designed by CH. The model was implemented by CH, with feedback from TM who provided insight regarding the original model. CH wrote the manuscript, aided by constructive comments from TM and the expertise and guidance of PJ.

II The study was conceived and designed by all authors. The three sections of the review were written by CH (theoretical), CC (contemporary observational) and SG and MM (paleo) respectively. CH performed all simulations and produced the relevant output. All authors jointly and in equal parts contributed to the Discussion and conceptualization of the figures. SG and MM amalgamated the separate parts into the final manuscript.

III CH and JA jointly conceived and designed the study and wrote the manuscript. CH implemented, set-up and ran the simulations using FEniCS while JA did the same for the simulations using Elmer/Ice.

IV CH conceived and designed the study, implemented all methods and set up and ran all simulations.
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1 Introduction

Not too long ago, large parts of the Northern Hemisphere were covered withcontinental scale ice sheets. At the last glacial maximum (LGM; when the ice
sheets were at their greatest extent), about 26.5 ka to 19 ka ago (Clark et al.,
2009), the Laurentide Ice Sheet (LIS) covered most of what today is Canada
and stretched as far south as the northern U.S. (e.g. Denton and Hughes, 1981;
Dyke et al., 2002), while the Fennoscandian Ice Sheet (FIS) overlying northern
Europe extended to northern Germany and northwestern Russia and merged
with the British-Irish Ice Cap (Svendsen et al., 2004; Stroeven et al., 2016). The
reason for these extensive glaciations may not come as a surprise; the climate was
significantly colder compared to the climate that has been observed in modern
times, with a global mean temperature about 4°C below pre-industrial values
(Annan and Hargreaves, 2013). Due to the vast amount of water stored in these
ice sheets, the sea level was approximately 120 m to 135 m lower than at present
(Clark and Mix, 2002). The rise of average global temperatures lead to the mass-
wasting of the LIS and FIS, eventually ending the glacial period about 11 ka ago
(e.g. Fairbridge, 1982; Walker et al., 2009). Today only two ice sheets remain:
Antarctica and the Greenland Ice Sheet (GrIS).

Ice sheets and glaciers are not as static as they seem upon first glance, but
are dynamically active features the exhibit movement and grow and decay as a
result of climate change. On a long term average, the mass added to a glacier
must be greater than or equal to the mass lost for the glacier to not disappear.
This is particularly true during the inception phase of glaciers. Periods of cold
climate, when more snow falls during winter than melts away in summer, result
in the build-up of ice in a region. Eventually, when the body of ice is large
enough, it starts to deform under its own weight and advance by flowing across
the landscape. This deformation of ice is called creep flow, and understanding
this process is an important part of being able to reconstruct and predict glacier
and ice sheet extent. However, it is not only the climate that affects the state of
glaciers and ice sheets, but also the other way around. The sheer magnitude of
ice sheets influence weather patterns (e.g. Clark et al., 1999). During deglaciation
everous quantities of fresh water can be released into the ocean and affect the
salinity of surface waters of for instance the North Atlantic. This can disrupt the
formation of deep water making the Gulf Stream to terminate further north than
today thereby altering the heat distribution (Marshall and Clarke, 1999).

Needless to say, understanding how the dynamics of glaciers and ice sheets in-
teract with a changing climate is of importance if accurate predictions regarding
the global climate are to be made. Even though the world’s smaller ice masses,
like alpine glaciers, are the fastest to react to a warming climate, they collectively
would contribute “only” about 0.5 m to a global sea level rise if they were all to
melt away. This can be compared to GrIS that holds the sea level equivalent of
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∼7 m, which is still dwarfed by Antarctica’s ∼60 m (Vaughan et al., 2013). Sudden changes in the dynamical characteristics of the LIS and FIS indicate that the behavior of ice sheets as a whole responds to climate in a non-linear manner (e.g. MacAyeal, 1993a; Greve and MacAyeal, 1996; Denton et al., 2010). Similar instabilities have been proposed for the West Antarctic Ice Sheet (e.g. Mercer, 1978; Schoof, 2007; Vaughan, 2008), highlighting the importance of understanding the link between the past and the present, to be able predict the future. To establish such a link, not only the internal controls on the deformation of ice masses must be well established, but also the external controls and possible feedback effects.

1.1 Internal and external controls on ice dynamics

The internal dynamics that result in the creeping flow of ice differ from the flow of water, not only in the time it takes for the ice to deform, but also in the ways it does so. Ice as a material is, just like water, incompressible, but has a very high viscosity and is a shear-thinning fluid which, contrary to water, makes it a non-linear fluid. The higher the stress applied to the ice is, the easier it deforms. This property of ice is characterized by a constitutive relation, called Glen’s flow law, which relates the stress to the strain rate in a non-linear manner (Glen, 1955). Ice deformation is accurately described by a set of non-linear partial differential equations (PDE), called in the glaciological community the Full Stokes (FS) equations (e.g. Greve and Blatter, 2009). These are a simplification of the famous Navier-Stokes equations that together with equations describing the evolution of the glacier surface and the temperature of the ice, are believed to describe well the dynamics of glaciers and ice sheets. However, these equations only characterize the general physics. To make these situation specific, that is to be able to find a unique solution, it is necessary to specify initial and boundary conditions. The initial conditions can, for instance, be the initial geometry and temperature distribution of the ice, while the boundary conditions are related to external conditions at the glacier surface and bed.

An intuitively important external forcing acting at the ice-atmosphere interface, is the surface mass balance. For instance, the temperature distribution (available energy) at the surface influences how much mass is lost while precipitation patterns determine the amount of mass added. At the bed, the geothermal heat flux affects the temperature gradient of the ice at the bed or melts the ice if at the melting point. The above conditions relate to the evolution of the surface and temperature field. For the FS equations, stress or velocity components need to be specified. It is commonly assumed that the stresses due to wind at the surface can be neglected, which results in a stress-free condition at the part of the ice surface in contact with the atmosphere. At the bed, a variety of conditions can occur. Frozen basal conditions restrict the velocities to be (very) small, so in this case a no-slip condition is applied (velocity equals zero). However, if the glacier is temperate at the bed (i.e. not frozen), water can exist at the ice-bed interface and act as either a lubricant in the case of a hard bed or by changing the characteristics of the underlying substrate. This is especially significant if the water pressure at the bed is high, in which case sliding can be facilitated significantly. The sliding boundary condition is more common, and results in an impenetrability condition (no flow of ice perpendicular to the bed) and sliding law that relates tangential stresses to the tangential velocities. In fact, in many cases sliding can amount to a substantial part of the total observed ice velocity (Alley
et al., 1986; Humphrey et al., 1993; Hooke et al., 1997; Truffer and Harrison, 2006; Sole et al., 2013). Ice flowing into a body of water is affected by that water exerting a normal pressure on the ice, while tangential stresses are negligible. That is, at the submerged boundary the ice strives to be in a state of flotation with little to no frictional resistance at the base. In this case, if no topographical constraint is present to support the ice, it will calve (break off) into the water resulting in a negative contribution to the mass balance. If a bay or similar is present, that can confine and support the ice, ice shelves can form. This is the case in many parts of Antarctica where, even though much of the negative mass balance occurs in these areas (Zwally et al., 2002; Joughin et al., 2012; Shepherd et al., 2012), the shelves have a buttressing effect on the ice that would otherwise flow into the ocean (Rignot et al., 2004; Dupont and Alley, 2005). In contrast, marine outlet glaciers on Greenland lack the buttressing generated by ice shelves, consequently calving close to the margin and resulting in very high ice velocities (e.g. Sundal et al., 2013).

Specifying a set of boundary conditions representative of a scenario is not an easy task. One important reason for this is that whatever sliding law is used, it is dependent on the subglacial conditions, such as subglacial water pressure, substrate or bottom topography, either explicitly or implicitly (e.g. Budd et al., 1979; Weertman, 1979; Schoof, 2005). The distribution of water (and its local pressure) is in turn determined by how it is transported at the bed, which depends on water delivery from sub-, en- or supraglacial sources. The latter links the mass balance boundary condition at the surface and the sliding condition at the bed. Understanding how water is routed from the surface to the bed, at what rate, and how this affects the subglacial hydrological system becomes of vital importance to understand the dynamical response of the ice to climate forcing (Bell, 2008). This is in particular true in areas where the mass loss is primarily determined by the surface mass balance.

In the case of smaller glaciers, such as alpine glaciers, this link has been studied extensively both in the field (e.g. Iken, 1981; Jansson, 1995; Harper et al., 2007) and theoretically by considering various conceptual subglacial systems to explain observations (e.g. Röthlisberger, 1972; Nye, 1976; Walder, 1986; Kamb, 1987). The prevailing conceptualization of a subglacial hydrological system that inefficiently transports water along the bed is that of the linked cavity system (Walder, 1986; Kamb, 1987). The basic characteristic of this system is that it is distributed and that water pressures are relatively high and rise with discharge. Therefore, an increase in water discharge to the subglacial system should result in a higher sliding velocity. Indeed, such spring speed-up events are a common feature for alpine glaciers at the onset of substantial melting. This speed-up does not persist throughout the melting season, due to the linked cavity system becoming unstable at a high discharge, triggering a switch to an efficient system. This consists of channels transporting water at relatively low water pressures, commonly called R-channels (Röthlisberger, 1972; Nye, 1976). Contrary to the linked cavities, the water pressure in a channel at steady-state will decrease with a higher discharge. This is envisioned to lead to water piracy (channels with a high discharge have lower pressure than channels with low discharge) and a dendritic type topology of discrete channels. Due to the active part of the system covering a small area and having relatively low pressure, sliding velocities decrease back to a background level. At the end of the season when melt decreases and not enough water is available to keep the channels open, they collapse and a linked
A great challenge in attempting to unite not only the fields of contemporary observational and paleo-glaciology, but also the sub-fields of subglacial hydrology and ice dynamics, is the large contrast in operational scales. For instance, the cavity system takes over again.

1.2 Future and past scenarios

With recent observations of increasing temperatures and projections of a warming climate in the Arctic regions (e.g. Serreze and Francis, 2006; Collins et al., 2013; Kirtman et al., 2013), an amplification of surface melt of the GrIS has been observed (e.g. Nghiem et al., 2012) and a continued enlargement of the ablation area can be expected. Ice velocity speed-up related to both seasonal melting (Zwally et al., 2002), and events such as supraglacial lake drainages (Das et al., 2008; Doyle et al., 2013), can in turn lead to an increase in ice flux and more mass lost. This has led to the question whether additional surface melt will result in a subglacial lubricating effect such that melt correlates with seasonal speed-up during the summer season (Parizek and Alley, 2004; Bartholomew et al., 2010; Sole et al., 2013) or if the subglacial hydrological system adapts earlier to an efficient system leading to a decrease in velocity (van de Wal et al., 2008; Sundal et al., 2011). However, direct observations of the subglacial system have been limited to the near margin area, and using methods such as borehole measurements result in spatially limited information (e.g. Meierbachtol et al., 2013; Andrews et al., 2014), with the topology of the system often inferred from measuring efficiency, with for instance tracers studies (Chandler et al., 2013).

This can be contrasted with the field of paleo-glaciology, where the geomorphology of past ice sheets is studied. Here, the spatial distributions and patterns of subglacial hydrological landforms can be studied over the whole domain of the ice sheet bed, since the bed is unobstructed by the presence of the ice sheet. However, observing the active formation is of course not possible, so the time-transgressive nature and the connection to specific hydrological events, as well as the efficiency or discharge in a system, is left for interpretation. One of the most distinct and abundant glaciofluvial landforms are those of eskers, which are long linear sediment deposits in the shape of ridges whose genesis have traditionally been interpreted to be due to deposition in (mainly) subglacial channels or directly where channels exit at the ice margin (e.g. Warren and Ashley, 1994). Esker systems have been used as an indicator of the flow direction and retreat dynamics of former ice sheets (e.g. Shreve, 1985a,b). These systems can be quite extensive, in segments extending hundreds of kilometers in the regions previously covered by the LIS (northern Canada; e.g. Storrar et al. (2014)) and FIS (Sweden; Lundqvist (1997)), sometimes exhibiting large-scale dendritic patterns (Brennand, 2000). This has been hypothesized to reflect an active low-pressure channelized system in the form of R-channels. However, the size of many esker features are not easily reconciled with the size of presently observable R-channels. Consequently, the genesis of eskers systems as a whole is not certain, and if it is concomitant of a large scale and active efficient hydrological system (Brennand, 1994; Brennand and Shaw, 1994) or if it is time-transgressively formed with a depositional environment in the near-margin area (Mäkinen, 2003; Storrar et al., 2014; Livingstone et al., 2015), and if the processes that are at work in these systems are at all representative of the current glacio-hydrological environment in e.g. Greenland.

A great challenge in attempting to unite not only the fields of contemporary observational and paleo-glaciology, but also the sub-fields of subglacial hydrology and ice dynamics, is the large contrast in operational scales. For instance, the
velocity field in ice sheets does not vary abruptly and point-wise, but rather changes over kilometers. On the other hand, a supraglacial lake draining through a moulin to the bed of the ice sheet is a very local event, and a resulting subglacial channel may have a cross section of tens of meters. However, this may affect a larger part of the subglacial hydrological system and therefore affect sliding properties at the bed and the velocity distribution in parts of the ice sheet. This is also reflected in the different use and formulations of numerical models, which can simulate flow patterns on the ice sheet scale or focus on a specific and local process of interest. For instance, simulating the surface velocities of the GrIS is the result of a three-dimensional model, while the evolution of the water pressure in a subglacial channel can, with certain simplifications, be represented by a one-dimensional model. Figure 1.1 shows the simulated velocity field as a result of a three-dimensional ice sheet model and a moulin at the surface of the GrIS, to contrast the scales. Even if a conceptual model would describe the interaction of subglacial hydrological processes and ice dynamical behavior, the small scale information necessary to resolve these processes numerically is not easily included on the ice sheet scale. This difference in spatial and temporal scale and interaction between (sub-)systems makes it difficult to assess whether numerical models accurately represent the considered processes.

### 1.3 Thesis development

In order to draw meaningful conclusions regarding a natural system, whether regarding the past, present or the future, a first step is to specify a conceptual model deemed valid for the considered scenario. Secondly, to be able to constructively apply the model, a validation procedure against the data representing the natural system should be made. For instance, in the case of ice dynamics, the conceptual model is the FS and evolution equations together with appropriate boundary conditions, and the solutions of these should be validated against collected data.

Figure 1.1. Contrasting the ice sheet and hydrological scale. (a) Simulated surface velocity of the GrIS on a mesh with horizontal resolution 5km to 25km. (b) A large moulin on the surface of the GrIS, presumably establishing a direct connection to the subglacial hydrological system. The opening of the moulin at the ice surface is approximately approximately 20 m.
from a glacier or an ice sheet, e.g. the surface velocity field, ice temperature and position of the free surface. Of course, as briefly mentioned above, parts of such a data set can be very hard to obtain for a significant part of an ice sheet.

However, when investigating natural systems an additional step is often required. The complexity inherent in such systems usually entails that a closed form solution to the (mathematical) conceptual model cannot be found. Therefore, the equations must be solved with the aid of numerical methods. In this case, the numerical model must be verified to accurately reproduce the conceptual model, since it is the numerical output that will represent the natural system. The Sargent Circle, shown in Fig. 1.2, describes the above process for a closed system, called “Reality” in the figure. Many great challenges come with glaciers and ice sheets not being closed systems, and the difficulty of not being able to strictly control the verification and validation process in the Sargent Circle.

![Figure 1.2. The Sargent Circle (after Schlesinger et al. (1979)).](image)

This thesis was initiated as a part of an international collaboration called the Greenland Analogue Project (GAP), which aims to increase the knowledge related to how ground-water flow and chemistry is affected by glaciations, and if this knowledge can be transferred to areas that have previously witnessed large-scale glacial influence (e.g. Canada, Sweden and Finland) and how these areas would be affected in a future glaciation scenario. In the cross-disciplinary spirit of this project, the core idea for this thesis was to investigate the connections between hydrology, specifically subglacial hydrology, and ice dynamics. By analyzing and linking the topology of hydrological systems found in the paleo-record (e.g. eskers) to numerical simulations and contemporary records (e.g. supraglacial lake drainages and input points) collected under the GAP, the ideal result would be to be able to increase the insight into the governing type, topology and stability of subglacial hydrological systems of future glaciations. In the larger scheme of things, the simulated distribution of water pressure at the ice bed interface for various climate scenarios, could then serve as a boundary condition for numerical models of ground-water flow.

My approach to the project was to start by implementing a simple model
of water flow in a subglacial channel (Paper I), which was to serve as a basis for studying the possible extent and topology of subglacial hydrological networks from both paleo and contemporary perspectives, and how these relate to the existing theory (Paper II). Following this, focus would be on implementing a subglacial hydrological model, with the purpose to investigate what parameters affect the general topology of subglacial channels. In addition, an ice sheet model would be used to investigate how the sliding and surface velocities were numerically affected by resolving frequencies present in bottom topography data, collected and presented in a GAP sister study by Lindbäck et al. (2014). It became apparent that the necessary additional data was either not available or did not offer the quality needed over a spatially relevant area for the studies. To constructively contribute to the original idea of the study, I decided to shift focus towards examining some numerical aspects of the model, which had shown possible limitations when used for the intended scenarios.

During the process of implementing the models, questions came to light regarding how to best numerically implement the sliding condition at the base and stabilize the FS equations. More specifically, together with J. Ahlkrona, we noticed that the choice of stabilization method and user specified parameters could potentially induce velocity oscillations at surface in a way that introduces errors in the surface topography in transient simulations. In addition, I observed that different formulations of the impenetrability condition at the base, associated with the tangential sliding velocity, resulted in a change in the velocity field in areas of high sliding or varying bottom topography which could be important for accurate simulation of the coastal regions of the GrIS. As a result, the latter part of the thesis concerns the numerical implementation of various stabilization methods (Paper III) and impenetrability formulations (Paper IV), and the process of verifying these numerical methods.

In the light of the above, the overarching aim of this thesis has become to examine the challenges that come with glaciers and ice sheets not being easily represented by Fig. 1.2, and how this is reflected in the links between the sub-disciplines of numerical glaciology, glacial hydrology and paleo-glaciology. In particular the thesis attempts to contribute to the areas of interdisciplinary understanding, internal and external controls by

- pinpointing the connects and disconnects between theoretical subglacial hydrology, observational hydrology and paleo-glaciology by reviewing the literature in each field, using a numerical model of subglacial channels to elucidate the theoretical channel extent compared to contemporary and paleo-records and discussing the the link between former, contemporary and future ice sheet scenarios from a cross-disciplinary point of view.

- studying the stability of the numerical implementation of the Full Stokes equations using the finite element method, how this issue is remedied with various stabilization methods, and how these in turn perform and affect the final solution (velocity and pressure) in the context of ice sheets.

- investigating the effects of numerical implementations of the impenetrability condition, which in turn affects the sliding velocity and subsequently the velocity field in glaciers and ice sheets.

This thesis summary is intended as an introduction to the processes considered (Chapter 2) and the methods used (Chapter 3) in the thesis work (attached
Papers I to IV). Together with some additional work regarding the verification procedure of numerical models (Chapter 4) and a summary of the papers (Chapter 5), this is meant to serve as a basis for the synthesis and discussion provided in Chapter 6.
2 Processes

On time scales longer than days, the stresses in the ice caused by the pull of gravity makes it deform in a viscous manner. When this behavior dominates, ice can be regarded as a very viscous fluid that exhibits creep deformation. Applying stress to an ice crystal causes both slip along basal planes and crystal dislocations (e.g. Duval et al., 1983). Laboratory and field studies of ice have indicated that it is an anisotropic material (e.g. Pimienta et al., 1987; Lipenkov et al., 1989; Thorsteinsson et al., 1997), and that various properties of individual crystals as well as the crystal lattice (such as crystal axis orientation, defects and impurities in the ice) affect how easily the ice deforms in different directions. However, in glaciology the most commonly used constitutive relation relates the dominant shear stress, \( \sigma \), to the strain rate, \( \dot{\varepsilon} \), through a power-law as

\[
\dot{\varepsilon} = k \sigma^n,
\]

where \( k \) and \( n \) are constants. This general form of the constitutive relation is usually called Glen’s flow law, after the studies conducted by Glen (1955). The flow law treats ice as an isotropic material, i.e. the rate of deformation is independent of the direction in which the stress is applied. Similar to relations for other materials (e.g. metal close to the melting point), the flow law is a non-linear function, in this case meaning that ice deformation increases non-linearly with an increase in applied stress. Moreover, Glen’s flow law is such that the material becomes infinitely viscous when no deformation occurs, making ice a singular power-law fluid.

The dynamics of fluids are often described by the Navier-Stokes equation, which balances the change of (linear) momentum and the stresses acting on an arbitrary fluid volume. Denoting the velocity and (constant) density of the fluid by \( \mathbf{u} \) and \( \rho \) and the Cauchy (total) stress tensor and gravitational acceleration as \( \mathbf{T} \) and \( \mathbf{g} \) respectively, the balance equation can be written as

\[
\rho \frac{D\mathbf{u}}{Dt} = \nabla \cdot \mathbf{T} + \rho \mathbf{g},
\]  

(2.1)

where \( \frac{D}{Dt} \) is the total time derivative. Intuitively, the expression above can be seen as a version of Newton’s second law, which states that the force equals the mass times the acceleration (time derivative of the velocity), but considering the body forces acting on the volume. Together with the conservation of mass, these equations capture well the characteristics of fluid motion, for instance wave-propagation and turbulence. However, due to its viscous nature ice flows relatively slowly, and it can be shown by scaling arguments that acceleration is negligible (see e.g. Greve and Blatter (2009)). In this case, the left-hand side of the balance of momentum is zero (\( \frac{D\mathbf{u}}{Dt} = 0 \) in (2.1)), and the expression can be
simplified to its steady-state version. Together with the assumption that ice is incompressible, this gives rise to the *Stokes equations*. In the ice sheet modeling community these are considered to be the most complete set of equations that describe ice deformation, and are when using Glen’s flow law often called the *Full Stokes (FS)* equations.

Further simplifications to the FS equations can be made on the basis of aspect ratio of the typical thickness and horizontal scale of an ice sheet leading to certain stress components being excluded in the momentum balance. Such approximations, for instance the Shallow Ice (assuming that only the horizontal plane shear stresses are non-zero, which gives a hydrostatic pressure distribution) and Shallow Shelf (neglecting shear stresses in the horizontal plane resulting in a “plug”-type flow) approximations and the Blatter-Pattyn (allowing for decoupling the vertical component of the balance of momentum) approximation (Hutter, 1983; Morland, 1987; Blatter, 1995; Pattyn, 2003), have been and are still widely used in ice sheet modeling with good results. The main reason for using these approximative methods is that they are computationally much cheaper than the expensive FS equations. Nevertheless, since computational power is rapidly increasing, the FS equations are used more often (e.g. Gagliardini and Zwinger, 2008; Larour et al., 2012; Brinkerhoff and Johnson, 2013).

It is not only the crystal properties of ice that govern its general velocity, but also how the body of ice interacts with its surrounding environment. How an ice sheet or glacier slides over its underlying bed can account for substantial parts of the total velocity field. For instance, around 80% of the total velocity was reported to be due to basal sliding at Storglaciären (valley glacier in northern Sweden) and at a Greenland outlet glacier (Hooke et al., 1997; Sole et al., 2013). Of particular importance is the *glacial hydrological system* and how water is routed beneath glaciers and ice sheets. Depending on the type of subglacial hydrological system prevailing, high water pressures can act to lubricate the bed or even locally decouple the ice (e.g. Iken and Bindschadler, 1986; Jansson, 1995). If the ice is underlain by a deformable substrate, such as sediment, water can act to saturate this and change its yield strength so that that only minor stresses are needed to cause a high basal slip (Boulton, 1979; Kamb, 1991; Iverson et al., 1995). Basal sliding in turn acts as a feedback and can change the morphology of the subglacial system which then can switch character from a *slow* system conceptualized to consist mainly of high-pressure cavities (Walder, 1986; Kamb, 1987) to a *fast* system consisting of discrete channels (Röthlisberger, 1972) operating at relatively low pressure.

In the following sections the above mentioned processes are presented in more detail, but are not meant to be comprehensive. Rather, they are intended to be an introduction to the theoretical aspects of the processes that are considered in this thesis and to serve as a background for Chapters 5 and 6. Section 2.1 mainly considers the FS equations that govern the flow of ice. These, in their isothermal form, are the focus of Paper III and Paper IV, but for completeness the evolutionary equations for the free surface and internal energy are stated as well. An overview of the glacial hydrological system, which is studied in Paper I and Paper II, is given in Section 2.2. However, since Paper II in part is a review paper on ice sheet hydrology, I only here highlight the main processes.

In literature the topic of glacier dynamics is well described in several books on glaciology (e.g. Hooke, 2005; Greve and Blatter, 2009; Cuffey and Paterson, 2010; Van der Veen, 2013), most of which also cover glacial hydrology. In addition,
excellent reviews on the glacial hydrological system are given in Fountain and Walder (1998); Clarke (2005); Flowers (2015).

2.1 Glacier dynamics

In what follows, a $d$-dimensional domain specified in the Cartesian coordinate system $(x, y, z)$, most commonly an ice sheet or a glacier, is denoted by $\Omega \subset \mathbb{R}^d$, and its boundary by $\Gamma$. The unit normal, $\mathbf{n}$, is defined as being outward pointing when associated with $\Gamma$. In general, scalars are given by normal font symbols, vectors by a bold-face lower case letter $\mathbf{a}$ and (second order) tensors by a upper case letter in bold-face $\mathbf{A}$. For the mathematical notation and descriptions of the symbols used in this thesis summary, see Appendix A.

In general, the presentation below is made for the three-dimensional case making, for instance, $\Gamma$ (a two-dimensional surface in three-dimensional space) a function of $x$ and $y$. If considering the same in two dimensions, the boundary instead becomes a line and only a function $x$.

2.1.1 The Full Stokes equations

The FS equations are given by the balance of momentum and conservation of mass as

$$\nabla \cdot \mathbf{T} + \rho \mathbf{g} = 0 \quad \text{(momentum),}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{(mass),}$$

where $\rho \mathbf{g}$ the gravitational body force and $\mathbf{u} = (u_x, u_y, u_z)$ is the velocity. The stress tensor is the linear map $\mathbf{T} : \mathbb{R}^d \to \mathbb{R}^d$ that defines the stress (vector) at a point of the domain. Since the stress is the force per unit area, $\mathbf{T}$ not only depends on the location of the point but also on what orientation the plane that the stress acts on has. Such a plane can be associated with its unit normal, $\mathbf{n}$. It is a postulate of Cauchy that the stress vector then is defined by $\mathbf{n} \cdot \mathbf{T}$. Below, $\mathbf{n}$ will most commonly represent the unit normal that defines the “real surface” that is the boundary of the domain.

To arrive at (2.2a) the assumption that the change of momentum and inertial forces are negligible compared to the viscous forces has been made. This means that the mass times acceleration term in Newton’s second law and the forces that occur due to a non-inertial system (e.g. Coriolis and centrifugal) can be neglected, resulting in a balance between the (divergence) of the stress and gravitational body force. The conservation of mass (2.2b) reflects that for an incompressible material (having constant density) the deformation of a fluid volume must be such that the total volume is unchanged.

Generally, the primary variables considered are $\mathbf{u}$ and $p$, the pressure. To arrive at FS equations stated in the primary variables, we need to relate the Cauchy stress to the velocity and pressure. In fluid dynamics the pressure is normally defined as the average compressive stress, which is the negative sum of the diagonal elements (the trace) of $\mathbf{T}$. This makes it possible to split the Cauchy stress into a traceless deviatoric part and the pressure part as

$$\mathbf{T} = \mathbf{S} - p \mathbf{I},$$

(2.3)
where $I$ is the identity tensor, and $p = -\frac{1}{3} \text{tr} T$. The non-linearity of the FS equations may not be obvious from (2.2), but becomes clearer when relating the deviatoric stress components to the primary variable $u$. This is done by using the strain rate tensor, $D = \frac{1}{2} (\nabla u + \nabla u^T)$, in the constitutive equation as

$$S(u) = 2\eta(D_e) D.$$  \hspace{1cm} (2.4)

The effective strain-rate is defined as $D_e = \sqrt{\frac{1}{2} D : D} = \sqrt{\frac{1}{2} D_{ij} D_{ij}}$ and $\eta$ is the viscosity which is the non-linear function

$$\eta(D_e) = \frac{1}{2} A^{-1/n} D_e^{(1-n)/n},$$  \hspace{1cm} (2.5)

called the generalized Glen’s flow law. Here Glen’s parameter is commonly taken as $n = 3$ and $A$ is the temperature dependent deformation rate factor (Glen, 1955), that can additionally depend on the water content of the ice if it is temperate (Duval, 1977). Note that, since $n > 1$, (2.5) gives

$$\lim_{D_e \to 0} \eta(D_e) = \infty.$$  \hspace{1cm} (2.6)

The above is not necessarily a problem in the continuous case, where an infinite viscosity occurs at zero deformation. However, it can cause issues when the problem is discretized since too large values are not optimal when using numerical methods.

We can now state the momentum equation (2.2a) in the primary variables as

$$-\nabla p + \nabla \cdot S + \rho g = -\nabla p + \nabla \cdot \left( \eta(\nabla u + \nabla u^T) \right) + \rho g = 0,$$  \hspace{1cm} (2.7)

where $\eta$ is understood to be the velocity dependent function in (2.5). Finally, a balance equation for the angular momentum also exists. This does not result in an equation, but rather a relation of the (transpose) symmetricity of the Cauchy tensor, i.e.

$$T = T^T.$$  \hspace{1cm} (2.8)

To be able to solve the above system for $u$ and $p$, we need to close it with a set of appropriate boundary conditions, which are statements that specify how the solution acts on the boundary of the domain.

### 2.1.2 Boundary conditions

The boundary conditions are connected to the physical scenario that we expect to act at the boundary of the domain. Considering glaciers and ice sheets, it is natural to have the surface (part of the ice in contact with the atmosphere) as a part of the boundary. However, on the rest of the boundary, the “bottom part”, various physical scenarios can be imagined. Most commonly: ice can be frozen to its bed (no sliding occurs), it can slide or potentially be submerged into a body of water (for instance ice shelves). We can divide the boundary $\Gamma$ into parts that represent the surface ($\Gamma_s$), the frozen bed ($\Gamma_f$), bed where sliding is present ($\Gamma_{sl}$) and parts of the ice that are submerged into water ($\Gamma_w$), such that $\Gamma = \Gamma_s \cup \Gamma_f \cup \Gamma_{sl} \cup \Gamma_w$ and $\Gamma_i \cap \Gamma_j = \emptyset$ for $i, j \in \{s, f, sl, w\}$ and $i \neq j$.

For the surface, it is usually assumed that the stresses exerted by the atmospheric pressure and winds as well as the effects of accumulation or ablation,
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are negligible. At $\Gamma_s$ a stress-free condition is therefore specified, i.e. the stress vector at the surface equals zero, $\mathbf{T} \cdot \mathbf{n} = 0$. Here, since $\mathbf{T}$ is a function of $\mathbf{u}$ and $p$, the condition relates the gradient of the velocity to the pressure at the surface. The stress-free condition is a so-called natural or Neumann boundary condition, because it is naturally included in the variational formulation of the FS equations (see Chapter 3). Closely related to the above situation is that of parts of the boundary being in contact with water. Similarly, here one assumes that the shear stresses (tangential part of the stress vector) exerted on the ice by the flowing water are insignificant and that melt and refreezing processes do not affect the total stress. However, the stresses acting normal to $\Gamma_w$ are those of the hydrostatic water pressure, $-p_w \mathbf{n}$.

If there is no sliding at the bed (often called no slip), the velocity simply equals zero. Contrary to the Neumann-type boundary condition above, we here specify the value that the velocity field solution must take on $\Gamma_f$. This is called an essential, or Dirichlet, boundary condition.

Finally, a sliding boundary condition occurs on parts of the bed where non-zero velocity is present ($\Gamma_{sl}$). Normally, a requirement is that the ice cannot penetrate the underlying bed (in particular if it is a hard bed), which amounts to specifying that the velocity component normal to the bed is zero. In the tangential direction, the stresses are related to the velocity by a “sliding law”. In this thesis the sliding law linearly relates the frictional stresses to the tangential velocity, $\mathbf{u}^\parallel$, by a spatially varying coefficient of basal drag, $\beta^2(x, y) \geq 0$. For the sliding condition we therefore combine a Dirichlet condition (in the normal direction) and a Neumann condition (in the tangential direction). The boundary conditions can be summarized as:

$$\mathbf{T} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_s, \quad (2.9a)$$
$$\mathbf{u} = 0 \quad \text{on } \Gamma_f, \quad (2.9b)$$
$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_{sl}, \quad (2.9c)$$
$$\mathbf{T} \cdot \mathbf{n}^\parallel = -\beta^2 \mathbf{u}^\parallel \quad \text{on } \Gamma_{sl}, \quad (2.9d)$$
$$\mathbf{T} \cdot \mathbf{n} = -p_w \mathbf{n} \quad \text{on } \Gamma_w. \quad (2.9e)$$

Most often, the $\beta^2$-parameter is obtained by inversion modeling (e.g. MacAyeal, 1993b; Larour et al., 2005; Gudmundsson and Raymond, 2008; Morlighem et al., 2010; Petra et al., 2012). Thus, processes that affect sliding, such as substrate and specifically how the glacial hydrology determines the distribution of subglacial water pressure, is implicitly included in the $\beta^2$-parameter. It is worth pointing out that other sliding laws exist, such as the Weertman-type sliding law which is a non-linear function of the velocity (Weertman, 1957), and as a function of water pressure at the bed (e.g. Iken, 1981; Schoof, 2005). However, the latter needs to be connected to a separate set of equations that describe the glacial hydrological system.

Note that neither the FS equations (2.2) nor the above boundary conditions show any dependence on the time variable. The FS equations are a steady-state version of the Navier-Stokes equations. This means that the solution to (2.2) with (2.9) depends only on the geometry of the considered domain. The discretization of these equations with related problems is treated in Chapter 3: the effect of various numerical stabilization methods on the solution is investigated in Paper III while different ways on implementing the sliding condition (2.9c) and (2.9d) is studied in Paper IV.
Since glaciers and ice sheets are not static features, any time dependence has to enter by including a different set of equations that evolve the geometry of the domain. These will briefly be described in the next section.

2.1.3 Evolution equations

The free surface, \( z_s(x, y, t) \), of the ice that evolves with time, \( t \), can be described by

\[
\frac{\partial z_s}{\partial t} + \frac{\partial z_s}{\partial x} u_x + \frac{\partial z_s}{\partial y} u_y - u_z = \dot{a},
\]

(2.10)

where \( \dot{a} \) is the accumulation rate (normal to the ice surface), taken to be positive when mass is added. Equation (2.10) is an advection equation and the position of the surface is determined by the velocity field at the surface acquired by solving (2.2).

Finally, the internal energy of the system is balanced as

\[
\rho \frac{du_e}{dt} = -\nabla \cdot \phi_q + \text{tr}(T \cdot D).
\]

(2.11)

Here the change of the internal energy, \( u_e \), is the difference between the energy sink due to the heat flux, \( \phi_q \), and the energy generated by viscous dissipation (any energy source due to radiation has been omitted). Equation (2.11) can be used with Fourier’s law relating the heat flux to the temperature or the enthalpy of the system (Greve and Blatter, 2009). In ice sheet modeling both the temperature and enthalpy formulations have been used (e.g. Zwinger et al., 2007; Aschwanden and Blatter, 2009; Larour et al., 2012; Brinkerhoff and Johnson, 2013). Solving (2.10) and (2.11) changes the geometry of the domain and updates the temperature or enthalpy field. The new domain is then used to again set up the FS equations and the temperature used to update the viscosity through the variable \( A \). Iterating between solving the FS system and the evolution equations result in the transient flow of ice.

Both the evolution of the free surface and the temperature or enthalpy field come with challenges when solved for numerically (issues of stability). However, when considering an isothermal, steady-state ice like I have done in this thesis, both (2.10) and Equation (2.11) are not considered, which is why I will not go into deeper detail regarding the numerics of the evolution equations.

2.2 Hydrology

Below follows a short overview of concepts important in glacial hydrology. The focus will be on the subglacial domain, since this is closely interlinked with the dynamics described in Section 2.1. Importantly, the hydrological conditions at the bed of a glacier or an ice sheet directly affect the mechanisms controlling glacier sliding. Even though supra- and englacial systems are important in their own right, much focus has been on how these deliver water to and connect with the subglacial system. For instance, water collected in supraglacial lakes on the surface of the GrIS can act to propagate fractures in the ice all the way to the bed, resulting in the lake draining to the subglacial system in a possibly catastrophic manner (van der Veen, 1998; Alley et al., 2005; Catania et al., 2008). However, a lake drainage is not necessary for supraglacial water to be able to reach the bed. If enough water is supplied steadily to a crevasse, for instance
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delivery of surface meltwater to the bed. The two scenarios above just highlight some important aspects of the subglacial hydrological system: the ability to accommodate large volumes of water in a short time period and to evolve seasonally with meltwater discharge. This necessitates that the system operates at various degrees of efficiency to be able to transport the water to the front of the glacier.

In Paper II, we adopt the terminology introduced in Raymond et al. (1995): fast and slow subglacial systems. These terms describe the mode of transport rather than the topology or type of subglacial system. The reason for this being that it has long been conceptualized that an efficient subglacial systems consist of high discharge, low water pressure subglacial channels (Röthlisberger, 1972). However, there need not be a one-to-one correspondence, that is, a fast system is not necessarily restricted to mean a channelized system. The physics considered for fast systems still relate closely to the initial studies from the 1970s. Slow systems have been suggested to consist of many different components, that at times may interact or overlap, for instance of interlinked subglacial cavities filled with highly pressurize water (Walder, 1986; Kamb, 1987) or a porous substrate (Flowers and Clarke, 2002a).

Just like overland water generally flows from high to low potential, glacial water moves from areas of high hydraulic potential, \( \Phi \), to areas of low potential. That is, the flow at the bed in a hydraulically connected system follows the negative gradient described by

\[
-\nabla \Phi = -\nabla (p_i - N + \rho_w g z_b) \approx -\rho_i g \nabla z_s - (\rho_w - \rho_i) g \nabla z_b + \nabla N,
\]  

(2.12)

where \( N = p_i - p_w \) is the effective pressure, \( \rho_i \) and \( \rho_w \) the densities of ice and water and \( z_b \) and \( z_s \) are the elevations of the bed and surface of the glacier (for clarity, \( p_i \) is used to denote the pressure of the ice instead of \( p \), cf. (2.3)). The approximation above comes from approximating the pressure of ice by \( p_i \approx \rho_i g (z_s - z_b) \) in the expression for the hydraulic potential, \( \Phi = \Phi_0 + p_w + \rho_w g z_b \) (where \( \Phi_0 \) is an arbitrary constant background potential). Equation (2.12) holds for any type of subglacial water flow, however, it does not specify what type of system the water flows through. Often, \( \Phi \) is related to the discharge in a channel or pressure in a porous system by some empirical relation. The processes that govern the distribution of \( \Phi \) have to be considered separately and ultimately determine whether the system is of fast or slow type.

2.2.1 Fast subglacial systems

During the melt season, a common sight at both alpine glacier fronts and at the terminus of the GrIS are subglacial streams emanating from subglacial tunnels. Such features lend argument to studying the simplified scenario of a water-filled pipe-like structure and how the transport of water is affected by the deformation of ice. Since the stresses in the ice act to close any void, the closure rate must be balanced by a process that increases the radius of the channel. The main process responsible for this is the melt of the channel walls due to the frictional heat produced by the flow of water in the channel. Moreover, if the channel is completely water filled, the water pressure acts as an inhibitor of closure by reducing or balancing the normal stresses at the ice roof. Röthlisberger (1972) and Shreve (1972) analyzed the water pressure in such intraglacial and subglacial channels on glaciers overlying a hard bed. Both studies reached similar qualitative
conclusions; assuming that the cross section of the channel is circular (if englacial) or semi-circular (if at interface between the glacier and a hard bed) and horizontal, they found that the water pressure during a steady high discharge would be lower than during a steady low discharge. Such channels have in the literature been called R-channels after the study by Röthlisberger (1972). The more general time-transient case was later analyzed by Nye (1976) to be

\[
\frac{\partial S}{\partial t} + \frac{\partial Q}{\partial s} = (1 - c_t c_w \rho_w) \frac{Q}{\rho_w L} \left| \frac{\partial \Phi}{\partial s} \right| \quad \text{(continuity), (2.13b)}
\]

where \(S\), \(Q\) and \(s\) are the cross sectional area of the conduit, the discharge and the along conduit coordinate and \(L\) is the latent heat of melt. The factor \((1 - c_t c_w \rho_w)\), where \(c_t\) is the Clapeyron slope and \(c_w\) the specific heat capacity of water, was considered important by Röthlisberger (1972) as he argued that the change of the pressure melting point affects the temperature and therefore about 1/3 of the frictional heat goes toward adjusting this change.

To be able to solve for the variables \(\Phi\), \(S\) and \(Q\) in (2.13), a relation between the discharge and hydraulic potential is needed. Two empirical relationships are mainly used in glaciology: the Manning formula (e.g. Röthlisberger, 1972; Nye, 1976) and the Darcy-Weisbach formula (e.g. Spring and Hutter, 1982; Hewitt et al., 2012; Werder et al., 2013). Since the latter is also used in Paper I, it is stated here as well:

\[
Q = -c_{dw} S^{5/4} \left| \frac{\partial \Phi}{\partial s} \right|^{-1/2} \frac{\partial \Phi}{\partial s} \quad \text{(2.14)}
\]

Here, \(c_{dw}\) is a constant depending on friction. The minus sign indicates the direction of the flow, that is, if the hydraulic gradient is positive in the \(s\)-direction (increasing hydraulic head away from the front of the glacier) the flow (discharge) is in the opposite direction towards the front.

For the numerical model used in Paper I and Paper II we follow Meierbachtol et al. (2013) in assuming a steady-state scenario of a straight conduit on a flat bed at constant discharge, which reduces the variables to solve for to \(S\) and \(\Phi\). The continuity equation (2.13b) is not fulfilled for this scenario, however, the impact of this is deemed to be minor since the amount of water added due to the melted ice in the channel is small compared to the typical discharge considered. Solving for \(S\) in (2.14) and inserting into (2.13a), using that \(N = p_i - p_w\), \(\Phi = p_w\) (due to the flat bottom) and \(\partial S/\partial t = 0\) (steady-state) gives

\[
\frac{dp_w}{ds} = \left[ \frac{2 A \rho_i L}{n^6 (1 - c_t c_w \rho_w)} \right]^{5/7} c_{dw}^{4/7} Q^{-1/7} (p_i - p_w)^{5n/7}. \quad \text{(2.15)}
\]

For a given discharge, specifying the boundary condition \(p_w = 0\) at the front, the above ordinary differential equation can be solved for the water pressure. Noticing the negative power of \(Q\) in (2.15), we can reach the same qualitative understanding that Röthlisberger (1972) and Shreve (1972) did; in the steady-state case, a higher discharge lowers the gradient of the water pressure (or \(\Phi\) in general) resulting in lower pressure along the channel. This important realization has led to the interpretation that larger subglacial channels feed of smaller adjacent high-pressure channels, resulting in a dendritic type network. Another characteristic of the above equation is that the solution will never reach ice overburden
pressure, that is $p_w < p_i$ for a channel of any length and the water pressure will only approach the ice overburden pressure asymptotically. This is not deemed realistic, since we expect channels to have a lower pressure than surrounding subglacial hydrological systems and should be seen as a mathematical construct and not interpreted as the possibility of a channel to exist at “steady-state”.

In the context of alpine glaciers, the above theoretical formulations correspond well to field data that indirectly supports the presence of low pressure channels (e.g. Fountain, 1994; Hubbard et al., 1995; Nienow et al., 1998; Bingham et al., 2005). This widespread evidence has set up an analogy between alpine glaciers and outlet glaciers of the GrIS: the types of subglacial hydrological systems and how they affect dynamics of the ice is similar (Zwally et al., 2002). However, more recently studies have shown that channels such as described by (2.13), are unlikely to exist except in proximity to the ice sheet margin (e.g. Meierbachtol et al., 2013), while other studies indicate that a fast type drainage (interpreted as channelized) still can exist away from the margin (Chandler et al., 2013). Such a fast system could possibly exist under a high-pressure regime or in a different format than the typical R-channelized form. Regardless, if channels are present in any configuration, the scale of these features is small compared to the scales at which the glacier and in particular ice sheets dynamics operate. Therefore, how the fast systems affect and interact with the more widespread slow systems should ultimately be of major importance.

### 2.2.2 Slow subglacial system

Slow systems have been proposed to take several different forms. Even though slow systems have not been a prominent part of this thesis, a conceptual understanding of these help to put the subglacial hydrology into a larger perspective. Therefore this section aims to at least introduce the concepts of slow systems.

Slow systems share a characteristic that can be contrasted to that of a system of R-channels; steady-state water pressures increase with the discharge. The typical slow system is commonly envisioned to consist of cavities at the bed, formed at the lee side of bedrock irregularities, interlinked by minute orifices. These cavities become water filled since an empty cavity would be at a lower pressure (atmospheric) than its surroundings. This type has been coined a “linked cavity system”, first mentioned in Kamb (1987), who studied how a switch from a predominantly efficient system to an high-pressurized inefficient system could explain the episodic fast flow in some glaciers (in particular glaciers going through a surging phase). However, prior to this Lliboutry (1968) considered how cavitations at the bed affect basal sliding and Walder (1986) considered a steady-state scenario conceptually very similar to that of Kamb (1987). Their findings are in Fountain and Walder (1998) summarized as

\[
Q \sim u_b^m \left( \frac{d\Phi}{ds} \right)^{\frac{1}{2}} N^{-n},
\]

where $u_b$ is the basal sliding (of the ice) and $m = 0.5$ in Kamb (1987) $m = 1$ in Walder (1986). If the above equation is compared to (2.15), there are two distinct features to note. First, the relationship between the hydraulic gradient and the discharge is converse to that of the steady-state R-channel and the discharge increases with water pressure (i.e. when $N$ decreases). Second, the basal velocity has been included in the expression. The above reasoning suggests that a linked
cavity system should not result in a dendritic, water capturing system, which is why the term distributed is often used. Both Walder (1986) and Kamb (1987) suggest that the cavity system is stable when sliding is the dominating process of cavity enlargement, but that at high water pressure, the melting becomes the dominant mechanism and acts as a feedback rendering the system unstable. At this point, the authors hypothesize that a typical R-channel system would develop.

Additional types of slow systems have been suggested. Early on Weertman (1957, 1972) suggested that a thin sheet of water, necessary for lubrication would exist at the ice/bed interface, however, it was shown that such a layer could not remain stable and transport any significant amount of water (Walder, 1982). For glaciers underlain by sediment Shoemaker (1986) considered the sediment as a porous medium, which needed the presence of channels to not reach to high water pressures, while Fowler (1987) considered so-called canals cut into the sediment (not to be confused with channels which are cut into the overlying ice).

2.2.3 Modeling the subglacial hydrological system

As mentioned in Section 2.1.2, the friction factor $\beta^2$ used in the sliding boundary condition, is indirectly dependent on the conditions at the bed. From a hydrological perspective, one could intuitively hypothesize that the friction at the glacier-bed interface would decrease with an increase in subglacial water pressure, that is the function that determines the sliding velocity should depend on the inverse of the effective pressure, $N$. Therefore, besides a sliding law that accurately describes the behavior of the ice at the bed, the interaction between the fast and slow hydrological must be accurately modeled to determine the distribution of $N$.

Recent efforts have resulted in progress regarding the above. Models including the switch between a slow and fast system, or a slow distributed system interacting with a discrete channelized system have been successfully applied to both idealized and realistic domains (Schoof, 2010; Hewitt, 2013; Werder et al., 2013; de Fleurian et al., 2014; Hoffman and Price, 2014).

For modeling purposes, slow systems are often thought of as a continuum in adherence with the distributed property. In this case, a water height representative over a spatial area (for instance a cell of the mesh), $h_w$, is described by the conservation of mass using the flux of water through the system, $q$, as

$$\frac{\partial h_w}{\partial t} + \nabla \cdot q = \text{sources} - \text{sinks},$$

(2.17)

where the sources can for instance be the addition of water due to basal melt, melting of the overlying ice, delivery of water from englacial or supraglacial sources (e.g. a moulin). In the case of a co-existing channel network, water can be exchanged between these two systems serving as both a source or sink. If $h_w$ completely fills the available space (pressure is larger than atmospheric), geometrical considerations of how this space evolves should also be taken into account, similarly to the evolution of the cross section of an R-channel, as

$$\frac{\partial h_w}{\partial t} = \text{opening} - \text{closure}.$$  

(2.18)

Here the opening term includes processes such as melt of the overlying ice and basal sliding while the closure term can be sediment deformation and the creep
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closure of ice. The above equations are then combined with an empirical relation, for instance the Darcy-Weisbach in (2.14) or Darcy’s law, to solve the system. The above or related approaches have for instance been used in Flowers and Clarke (2002a,b); Pimentel and Flowers (2010); Hewitt (2011); Schoof et al. (2012).

Due to the scale difference of a channel and the typical resolution of a mesh cell, recent numerical models have incorporated R-channels as the line segments (facets) making up the cell. In this case, (2.13) together with some relation describing the interaction with the distributed system, is solved only on facets (Hewitt, 2013; Werder et al., 2013). Using such an approach, even though the channels spontaneously evolve, the location and topology of the channel system is described by the chosen mesh.

In Chapter 1, I introduced the general aspects of modeling glacier dynamics and how this relates to the climate system. Here the mathematical conceptual models representing the specific processes of ice deformation and subglacial hydrology have been presented. In the next chapter the numerical methods used in this thesis to model the dynamics of ice are discussed.
3 Methods

The FS equations presented in Chapter 2 can rarely be solved analytically. Only for special scenarios, for instance when using a very simple geometry and constant coefficient of basal drag, a closed expression has been found (e.g. Greve and Blatter, 2009). Since the FS equations are considered to accurately describe ice deformation, solving the general case becomes vital. This can be done with the aid of computers by restating the continuous equations (2.2) in a form that is suitable for computation. This process is called discretization, and the specific manner in which this is done depends on the numerical method chosen.

A part of this thesis has been to implement and evaluate numerical methods used to model ice dynamical processes. In particular, the finite difference method (FD) was used to discretize the equations used to describe the water pressure in an R-channel (Paper I and Paper II) while the finite element method (FEM) was used in Paper III and Paper IV to solve the FS equations. Since the former follows the discretization procedure presented in Meierbachtol et al. (2013), I below focus summarizing the FEM.

3.1 The finite element method

This section attempts to introduce the basic approach of the FEM. A vast body of literature exists on the general topic of FEM, both text books treating the fundamentals (e.g. Zienkiewicz et al., 2013; Larson and Bengzon, 2013), as well as more specialized books on for instance the topic of fluid dynamics (e.g. Donea and Huerta, 2003; Zienkiewicz et al., 2014), the more mathematical aspects of FEM (Brenner and Scott, 2008) and the approach of mixed FEM arising in for instance the discretization of the FS equations (Boffi et al., 2013). These texts are beyond the scope of this thesis summary and I will therefore try to take a somewhat light and informal approach which fits more in the line with the thesis as a whole, and try to focus on the FS equations as the underlying example. For simplicity, the non-linearity introduced by $\eta$ will only briefly be touched upon to emphasize the iterative procedure necessary to solve the discrete problem. I believe that considering the linear problem keeps the notation less cumbersome by simplifying the introduced function spaces without losing the principle idea of the method and discretization procedure. The general disposition and notation of this section loosely follows Larson and Bengzon (2013), while subsection Section 3.1.3 is structured to a degree after Hansbo (2002); Mardal and Langtangen (2003).

The general approach of FEM is closely interlinked to the theory of partial differential equations, in that the solution is sought in its \textit{weak form}. By multiplying the governing equations by a \textit{test function} (belonging to a specified set of functions) and integrating by parts, the continuity requirements of the solution can be relaxed. In the continuous case the weak form is often necessary to prove
the existence of a unique solution to the problem. Additionally, since the test functions can be chosen in an appropriate way, it provides a framework to discretize the governing equations. The resulting method is very flexible, allowing unstructured meshes and many different approaches depending on the choice of function spaces used.

3.1.1 Weak form of the Full Stokes problem

For clarity, let us start by restating the problem of an isothermal body of ice with a frozen bed in the relevant strong form (see Section 2.1.1)

\[
\begin{align*}
\nabla \cdot S - \nabla p + \rho g &= 0 \quad \text{in } \Omega \\
\nabla \cdot u &= 0 \quad \text{in } \Omega \\
T \cdot n &= 0 \quad \text{on } \Gamma_s \\
u &= 0 \quad \text{on } \Gamma_f
\end{align*}
\]

(3.1)

Remembering that \( S = \eta(\nabla u + \nabla u^T) \), we see that the above contains second order derivatives of the velocity, \( u \). That is, for a solution \((u, p)\) of (3.1) to make sense, the velocity must be a function that is twice differentiable. The pressure variable on the other hand, is not as strongly constrained since only the first derivative appears in the equation.

To arrive at the weak formulation, also called the variational formulation, of the problem we first note that the balance of momentum remains valid if both sides are multiplied by a vector-valued test function, \( v \), and integrated over the domain as

\[
-\int_\Omega (\nabla \cdot S - \nabla p + \rho g) \cdot v \, dx = \int_\Omega 0 \cdot v \, dx
\]

That is, for an (almost) arbitrary test function \( v \neq 0 \), the above implies that the corresponding equation in (3.1) holds. In this context \( u \) and \( p \) are often called trial functions. Integrating by parts gives

\[
-\int_\Omega S : \nabla v \, dx + \int_\Omega p \nabla \cdot v \, dx + \int_{\Gamma_s} n \cdot (S - pI) \cdot v \, ds + \int_\Omega \rho g \cdot v \, dx = 0. \quad (3.2)
\]

There are a few aspects to note in the above expression. First, only first order derivatives of the velocity appear, which means that the restriction on the solution is reduced from a twice differentiable (for \( u \)) function to a differentiable function.

Second, a boundary integral over \( \Gamma \) is added as part of the integration by parts procedure. Subdividing the boundary of the domain into the free surface and the frozen bed, we have that

\[
\int_{\Gamma_s} n \cdot (S - pI) \cdot v \, ds + \int_{\Gamma_f} n \cdot (S - pI) \cdot v \, ds = 0
\]

must be fulfilled. Since \( v \neq 0 \), the above integrals can only equal zero if \( n \cdot (S - pI) = 0 \) on the boundary. Examining the stress-free boundary condition on \( \Gamma_s \), we see that a solution to the problem (3.1) must be such that the first integral on the right-hand side vanishes (since \( n \cdot T = n \cdot (S - pI) = 0 \)). Therefore, the Neumann-type stress-free boundary condition is naturally incorporated in the weak formulation. Note that this does not specify what the solution is on the surface in the variables \((u, p)\), only that the stress tensor vanishes there. However, at the bed we know the solution of the velocity component beforehand, i.e. we
have a Dirichlet boundary condition on the velocity. In the frozen case we specify that \( \mathbf{u} = \mathbf{0} \) on \( \Gamma_f \), but we could as well have specified a non-zero sliding velocity. If the sliding is non-zero, the integral over \( \Gamma_f \) would not vanish for an arbitrary test function. This implies that we need to require that the function space that the test function is defined on be such that \( \mathbf{v}|_{\Gamma_f} = \mathbf{0} \). Basically, if the velocity is specified, there is no room for a variation of the test function, which in this case must be zero wherever we have a Dirichlet boundary condition. So the type of boundary condition already implies that the test function in the weak formulation cannot be completely arbitrary.

A final characteristic of the weak formulation is that a degree of differentiation in this case need only be reflected in the function spaces, but does not affect the general disposition of the discretization approach presented below.

To specify what type of function space that can be used such that a variation of mass equation. Note that this is a scalar equation and that the test function \( \mathbf{v} \) is present. However, if we extend the above to include the derivatives of the function, \( \frac{\partial \mathbf{v}}{\partial x_j} \), all the terms in (3.2) are well-defined. This function space, denoted by \( H^1(\Omega) \), is a common occurrence in FEM and is also used in the (linear) FS problem. We can denote this by

\[
H^1(\Omega) = \{ v : \int_{\Omega} (\nabla v \cdot \nabla v) \, dx < \infty \} \quad (H^1\text{-space}),
\]

\[
(u, v)_{H^1} = (\nabla u, \nabla v)_{L^2} + (u, v)_{L^2} \quad (H^1\text{-inner product}),
\]

\[
\|v\|_{H^1} = (\|\nabla v\|_{L^2}^2 + \|v\|_{L^2}^2)^{1/2} \quad (H^1\text{-norm}).
\]

I point out that it is here that the assumed linearity (i.e. a constant \( \eta = \eta_0 \)) of the FS problem matters, since in this case no powers of \( \nabla \mathbf{u} \) appear in \( \mathbf{S} \) and the \( H^1\)-inner product is sufficient for \( \int_{\Omega} \mathbf{S} : \nabla \mathbf{v} \, dx = \int_{\Omega} \eta_0 (\nabla \mathbf{u} + \nabla \mathbf{u}^T) : \nabla \mathbf{v} \, dx = \infty \). In the non-linear case, powers of \( \nabla \mathbf{u} \) dependent on \( n \) would appear. This should be reflected in the function spaces, but does not affect the general disposition of the discretization approach presented below.

With these general function spaces established, we can finalize the weak formulation of the FS problem by following the above procedure for the conservation of mass equation. Note that this is a scalar equation and that the test function in this case need only be \( q \in L^2(\Omega) \), since we don’t need to integrate by parts and there are no Dirichlet boundary conditions on \( p \). The two test functions need two function spaces for this specific problem, which given the above considerations are

\[
\mathcal{V}_0 = \{ \mathbf{v} : \mathbf{v} \in [H^1(\Omega)]^d, \mathbf{v}|_{\Gamma_f} = \mathbf{0}\} \quad \text{and} \quad \mathcal{Q} = \{ q : q \in L^2(\Omega) \}.
\]
The complete FS variational problem then reads: find \((u, p) \in V_0 \times Q\) such that
\[
\begin{align*}
\int_\Omega \mathbf{S} : \nabla \mathbf{v} \, dx - \int_\Omega p \nabla \cdot \mathbf{v} \, dx - \int_\Omega \rho g \cdot \mathbf{v} \, dx &= 0 \quad \forall \mathbf{v} \in V_0, \quad (3.5a) \\
\int_\Omega q \nabla \cdot \mathbf{u} \, dx &= 0 \quad \forall q \in Q.
\end{align*}
\]
Here, the boundary integral over the surface, has disappeared since this is zero if the solution is to be stress-free on \(\Gamma_s\). Similarly, the bed counterpart vanished due to the Dirichlet boundary condition on \(\Gamma_f\). In the following section, (3.5) will be used as the starting point for the discretization procedure.

### 3.1.2 Discretization procedure

The main statement from the previous section was that the weak formulation of the FS equations holds for all test functions \((\mathbf{v}, q) \in V_0 \times Q\). This suggests that we can choose not only one test function, but a set of test functions and (3.5) must hold for each test function of that set. To define these sets of functions in a way suitable for discretization, the domain is subdivided into elements that collectively result in a mesh of the domain. In this thesis the mesh is a triangulation, \(T_h\), of the domain consisting of a set of triangles (in two dimensions) or tetrahedrons (in three dimensions), which will be denoted by \(\{K\}\). The triangulation must be such that the intersection of two triangles either be an edge (completely) shared by the triangles, a point (common vertex) or the empty set. The union of the triangles covers the approximation of the domain, \(\Omega_h = \bigcup_{K \in T_h} K\). The approximation of the domain depends on the number of triangles/tetrahedrons and their size, denoted by \(h\) (commonly the maximum diameter of the cells is used to specify the mesh size, but different measures are used). If \(\Omega\) is a polyhedral domain, it can be covered completely by the triangulation (see Fig. 3.1).

![Triangulations of domains](image)

**Figure 3.1.** Triangulations (blue) of domains (red). (a) The Rebel Alliance is out of luck, since the union of the triangles can only approximate the curvature of the “real” domain. (b) The Triforce is polygonal and can be up to machine precision represented by \(\Omega_h\).

Two common function spaces used in FEM are those that contain functions that are piece-wise linear and piece-wise quadratic on each \(K \in T_h\). By denoting the space of polynomial functions of degree \(k\) as \(P_k\) this can be expressed as
\[
\begin{align*}
V_h^k &= \{ \mathbf{v} \in [C^0]^d : \mathbf{v}|_K \in [P_k]^d, k = 1 \text{ or } 2 \} \quad \text{(vector valued)}, \\
Q_h^k &= \{ q \in C^0 : q|_K \in P_k, k = 1 \text{ or } 2 \} \quad \text{(scalar valued)}.
\end{align*}
\]
The spaces are continuous ($C^0$), but have derivatives that are discontinuous across the edges of each $K$. An important property is that $Q_h^k \subset Q$ (with appropriate boundary conditions), so $Q_h^k$ can be viewed as a smaller function space where approximations to the true solution live (the subscript $h$ here is used to indicate that the specific function space is dependent on the size of the mesh). The main point of using these function spaces is that any function belonging to $Q_h^k$ can be represented by a linear combination of basis functions, $\{\varphi_i\}_{i=1}^{n_q}$, where $n_q$ is the number of basis functions necessary to span the space. Commonly, for FEM, a nodal basis is chosen. Nodes in this context are taken to mean specific points of the mesh at which coefficients of the basis functions take the value of the function to be expressed. That is, a node can, but does not necessarily have to, coincide with the vertices of a mesh. For piece-wise linears however, the nodal basis consists of the so-called hat-functions. A hat-function is the piece-wise linear function that takes the value one at a specified node, but is equal to zero at every other node in the mesh (see Fig. 3.2 for the one-dimensional case). Since a combination of linear functions is a linear function, specifying the values at the vertices of a mesh will be enough to define any piece-wise linear function on that particular mesh, so in the case of $Q_h^1$ the nodes are specified to be the vertices. Hence, any $p_h \in Q_h^1$ can be written as

$$p_h(x) = \sum_{i=1}^{n_q} \alpha_i \varphi_i(x),$$

where $n_q$ equals the number of nodes (i.e. vertices in the mesh), and $\alpha_i$ is the nodal value of $q(N_i) = \alpha_i$ at the vertex coinciding with node $N_i$.

**Figure 3.2.** A piece-wise linear approximation (blue) of a function (red) by the FEM. The basis functions ($\varphi_i$, black solid line) multiplied by a constant $\alpha_i$ (dashed line) add up to the $L^2$-projection of $f$ (see text), which is piece-wise linear on each discrete interval, $K_i$. 

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The $Q^2_h$ space can be used to achieve a more accurate approximation, however, this comes at a cost. To specify a function $q \in Q^2_h$, a larger set of nodal basis functions is required, and more nodal evaluations are necessary. In this case, the nodes are not only located at the vertices of the mesh, but also at the mid points of edges (Fig. 3.3). If a quadratic polynomial is to be specified on a single line segment, triangle or tetrahedron, the result is that instead of having 2, 3 and 4 nodal evaluations necessary for a linear polynomial, there are going to be 3, 6 and 10. Therefore, the increased computational cost of a piece-wise quadratic compared to a linear approximation rises with the geometrical dimension. When using the above in the FEM on a triangular mesh, the above spaces are commonly referred to as the $P^1$ and $P^2$ finite elements (see Fig. 3.3).

To represent (approximate) a function $f \in Q$ in the space $Q^k_h$, we can accordingly choose the constants $\alpha_i$ to take the values of $f$ at the nodes of the mesh. This is called interpolation of $f$. For instance, in the case of $Q^1_h$, this is just the linear interpolation of $f$ between the mesh nodes (vertices). However, remembering that our underlying goal is to find the best function in $Q^k_h$ that approximates the true pressure solution to the FS problem, we can try to minimize the distance in some metric that reflects the underlying weak formulation of the problem (3.5). A good approach is to use the $L^2$-inner product over the domain as

$$\int_{\Omega} p_h q \, dx = \int_{\Omega} f q \, dx \quad \forall q \in Q^k_h. \quad (3.7)$$

It turns out that the piece-wise polynomial function $p_h$ that fulfills (3.8) is the one closest to $f$ in the $L^2$-norm. Note that the test function here belongs to the space in which the function is to be approximated. Since $p_h$ can be expanded as in (3.6) and we can choose the test functions as the nodal basis functions, the above becomes

$$\int_{\Omega} \sum_{j=1}^{n} \alpha_j \varphi_j \varphi_i \, dx = \int_{\Omega} f \varphi_i \, dx \quad i = 1, \ldots, n. \quad (3.8)$$

Since we explicitly know the basis functions (their exact shape being dependent on the mesh), and $f$ is known, it is possible to calculate the terms above. Note that for most indices $i$ and $j$, the basis functions do not overlap, and the integral on the left hand side is zero. Moreover, since each $\int_{\Omega} \varphi_j \varphi_i \, dx$ is a scalar, the left hand side can be represented as linear system of equations

$$M \alpha = b, \quad (3.9)$$

where

$$M_{ij} = \int_{\Omega} \varphi_j \varphi_i \, dx, \quad \alpha_i = \alpha_i \quad \text{and} \quad b_i = \int_{\Omega} f \varphi_i \, dx.$$

The above is called the $L^2$-projection of the function $f$. Contrary to the interpolation of $f$, we in (3.9) solve a system for the unknown coefficients $\alpha$. The projection is on average close to the estimated function on a given mesh. An example of the $L^2$-projection of $f = \sin \pi x + 2$ on an unstructured unit interval mesh is given in Fig. 3.2.

Conceptually, the discretization of the FS equations is similar to the above procedure for the $L^2$-projection. We choose function spaces, one for $u$ and one for $p$, in which the approximate solutions, $u_h$ and $p_h$, will live. We express $u_h$ and $p_h$ as a linear combination of basis functions, insert these into the weak formulation (3.5) and choose a set of test function as the basis function for $V^k_h$ for (3.5a) and
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\[ A(u_h, v_h) = \int_{\Omega} S(u_h) : \nabla v_h \, dx = \int_{\Omega} (\eta(\nabla u_h + \nabla u_h^T) : \nabla v_h) \, dx, \quad (3.10a) \]

\[ B(u_h, p_h) = -\int_{\Omega} p_h \nabla \cdot u_h \, dx, \quad (3.10b) \]

\[ F(v_h) = \int_{\Omega} \rho g \cdot v_h \, dx. \quad (3.10c) \]

Figure 3.3. Nodes \((N_i)\) for the \(P_1\) and \(P_2\) elements on a triangle.

\[ Q_h^k \text{ for (3.5b). The difference is that the weak form for the FS equations will include the derivatives of the basis functions as well and that the function spaces approximating the velocity and pressure do not have to be of the same order. We can define the associated variational forms with each part of the FS equations as} \]

\[ A(u_h, \phi_i) = \sum_{l=1}^{d} \sum_{j=1}^{N_v} \int_{\Omega} \eta(\xi_j^l(\nabla \phi_j^l + \nabla (\phi_j^l)^T) : \nabla \phi_i^k) \, dx \]

\[ B(u_h, \varphi_i) = -\sum_{l=1}^{d} \sum_{j=1}^{N_v} \int_{\Omega} \xi_j^l \nabla \cdot \phi_j^l \varphi_i \, dx = -\sum_{l=1}^{d} \sum_{j=1}^{N_v} \nabla \cdot \phi_j^l \varphi_i \, dx, \quad (3.11) \]

\[ B(\phi_i^k, p_h) = -\sum_{j=1}^{n_q} \int_{\Omega} \alpha_j \varphi_j \nabla \cdot \phi_i^k \, dx = -\sum_{j=1}^{n_q} \alpha_j \int_{\Omega} \varphi_j \frac{\partial \phi_i}{\partial x_k} \, dx, \]

\[ F(\phi_i^k) = \int_{\Omega} \rho g \cdot \phi_i^k \, dx = \int_{\Omega} \delta_{ik} \rho g \phi_i \, dx. \]

Note that the form (3.10b) is used for both the pressure related term in (3.5a) and the conservation of mass in (3.5b). The only difference is that in first case the test functions for the velocity space are used while in the latter the test functions for the pressure space is used. Evaluating the integrals results in matrices, which
can be arranged in a block matrix form as

\[
\begin{bmatrix}
A & B^T \\
B & 0
\end{bmatrix}
\begin{bmatrix}
\xi \\
\varphi
\end{bmatrix}
= 
\begin{bmatrix}
f \\
0
\end{bmatrix},
\]

(3.12)

where \( A \) (a \( n_v \times n_v \) matrix), \( B \) (\( n_q \times n_v \)) and \( f \) (\( n_v \times 1 \)) are associated with the variational forms above, and \( \xi \) and \( \varphi \) are the column vectors containing the coefficients to be solved for.

A comment about the matrix \( A \): in the above we have not touched on the issue that the viscosity, \( \eta \), is a non-linear function of \( \mathbf{u}_h \). That is, the entries \( A_{ij} \) are not really scalars, but depend on the coefficients \( \xi^k \), making (3.12) a system of non-linear equations. However, to solve the equations for the coefficients, we need to make \( A \) linear. This process is called *linearizing* the problem. For instance, one approach would be to start of with an *initial guess* of the solution, \( (\mathbf{u}_h^0, p_h^0) \). This would enable us to calculate \( \eta^0 = \eta(\mathbf{u}_h^0) \) and insert this into the variational form \( A(\cdot, \cdot) \), making it a linear approximation of \( A \). Therefore, solving this system results in a first approximation of the solution, \( (\mathbf{u}_h^1, p_h^1) \), which can then be used as a new initial guess repeating the procedure. Continuing until the change between subsequent approximations is small, solves the problem. The above *iterative* process is called a *Piccard or fixed-point iteration*. An alternative is to use *Newton’s method*, which simply described is a numerical approach to find a solution that is a *root* of (3.5). The process is conceptually similar to that of finding a root of a scalar one-dimensional function, in which the Taylor expansion of the function is used to solve for a perturbation of the initial guess with the aid of the function gradient. When considering a PDE, the gradient is instead replaced by a functional derivative which is discretized as the Jacobian of the problem. This is then used to calculate the perturbation of the initial guess that solves the linearized problem. The guess is updated by adding the perturbation that was solved for, in the hope that this is closer to the true root of the problem. Again, this results in an iterative procedure. I will not go into more detail here regarding the specifics of the two different solution processes loosely described above, but only conclude by stating some of their general characteristics. The Piccard iteration is considered to be very robust but often slow to converge while Newton’s method converges faster but may diverge if the problem is hard and the initial guess is not sufficiently close to the true solution.

### 3.1.3 Stability and the mixed formulation

In the previous sections, the relevant function spaces for the FS equations in the continuous setting and function spaces suitable for discretizing the problem were introduced formally. The occurrence of two physical variables opens up the possibility for a *mixed formulation*, that is to use different types of finite element basis functions for \( \mathbf{u}_h \) and \( p_h \). Equation (3.12) is a so-called discrete *saddle-point problem*. A characteristic of such a problem is the 0-block matrix in the lower right corner, which is the result of imposing an additional condition. In the case of the FS equations the pressure acts as a *Lagrange multiplier* enforcing the condition that the velocity solution be divergence-free, or alternatively the Lagrange multiplier that enforces the incompressibility has the physical meaning of pressure.

We have yet to say anything about whether the resulting system of algebraic equations (3.12) is solvable at all. Intuitively, we can already speculate that if the
0-block of the matrix would dominate (i.e. $n_q$ is too large), the system may not have a unique solution. Since there has been some hint that numerical solutions to the steady flow of ice do exist, one can of course guess that, at least under some circumstances, solvability is an option. Below I formally assume that the matrix $A$ is linear or that it represents the matrix resulting from the linearization process (which gives rise to a structurally similar matrix).

If we denote $C = \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix}$, then for the system to have a unique (discrete) solution $(\xi, \varphi)$, the inverse $C^{-1}$ must exist. In this case, we have that $\begin{bmatrix} \xi \\ \varphi \end{bmatrix} = C^{-1} \begin{bmatrix} f \\ 0 \end{bmatrix}$.

Of course, how to best numerically invert the matrix $C$ is a book of its own (I will not attempt to go into any detail regarding this), but if the above is the case we at least know that the system is solvable by, for instance, Gaussian elimination. Using the block structure of (3.12) gives that $A\xi + B^T \varphi = f$. Left multiplying this by $BA^{-1}$ and using $B\xi = 0$ gives

$$BA^{-1}B^T \varphi = BA^{-1}f.$$ 

For the above to make sense, two requirements are needed: the inverses of $A$ and $H = BA^{-1}B^T$ must both exist.

To illustrate problems that may arise, I will just assume that $A^{-1}$ exists, and focus on the second requirement. Since $B$ has $n_q$ rows and $n_v$ columns, the first thing to observe is that if $n_q > n_v$ this matrix becomes singular, making $H$ singular. From this we can conclude that the function space used to approximate the pressure, $Q_h$, cannot be too rich compared to the velocity space $V_h$. For instance, the combination $V_h^1 \times Q_h^2$ becomes impossible. More precisely, we have as a necessary condition that $n_v \geq n_q$. We could hope that this condition would be enough to guarantee that $C$ has an inverse. This, unfortunately, is not the case. As an example, we can think of a situation where $\ker B^T \neq 0$, that is, there exists a set of pressure coefficients $\varphi_{\ker} \neq 0$ such that $B^T \varphi_{\ker} = 0$. To emphasize why this scenario can cause problems, consider the coefficient velocity and pressure pair $(\xi, \varphi)$ that solves the system (3.12) together with an additional coefficient vector consisting of $(0, \varphi_{\ker})$, i.e. the vector consisting of the pressure coefficients for $\varphi_{\ker}$. Since matrices are representations of linear maps, we have that

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \xi + 0 \\ \varphi + \varphi_{\ker} \end{bmatrix} = \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \varphi \end{bmatrix} + \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \varphi_{\ker} \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}.$$ 

That is, for any solution $(\xi, \varphi)$ of the system (3.12), $(\xi, \varphi + \varphi_{\ker})$ also is a solution. This problem gives rise to spurious (and unphysical) pressure oscillations, see Fig. 3.4. Hence, an additional condition is that $\ker B^T = 0$, i.e. $B^T$ is injective.

The above conditions can be summarized by the so-called *inf-sup* or LBB (Ladyzhenskaya-Babuška-Brezzi) condition (Babuška, 1973; Brezzi, 1974), which can be written as

$$\inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{B(v_h, q_h)}{\|v_h\| \|q_h\|} \geq \beta_h,$$
where $\beta_h > 0$. It can happen that the inf-sup condition is fulfilled for the spaces $V \times Q$, but not for the subspaces $V_h \times Q_h$, or that $\beta$ is dependent on the mesh size $h$ in a way that does not ensure optimal accuracy as the mesh is refined.

To directly use the inf-sup condition to deduce what function spaces are appropriate for the FS problem is (at least to me) non-trivial. Therefore, I simply state here that the mixed function space $V^2_h \times Q^1_h$ fulfills the above condition. In the literature this finite element pair is often called the (lowest order) Taylor-Hood (Taylor and Hood, 1973; Boffi, 1997) or $P_2P_1$ (piece-wise quadratic polynomials used for the velocity and piece-wise linears for the pressure). Discretizing the FS equations leads to an invertible matrix $C$ and a stable formulation. However, as mentioned previously, the space $V^2_h$ is quite computationally expensive.

To reduce the computational cost, the question then becomes if it is possible to reduce the polynomials used to approximate the velocity and still have a stable formulation. The space $V^1_h \times Q^1_h$ does not fulfill the inf-sup condition, even though $n_v \geq n_q$. Solving the system resulting from the $P_1P_1$ discretization results in pressure oscillations and is therefore not stable. Fortunately, various methods to stabilize the pressure (making it oscillation-free) have been developed (e.g. Hughes and Franca, 1987; Becker and Braack, 2001; Burman and Hansbo, 2006). Loosely speaking, the general approach to stabilizing the $P_1P_1$ formulation is to add terms to the FS equations in such a way that the resulting discrete system is invertible and therefore circumventing the inf-sup condition. As mentioned above, the 0-block is a part of the problem, and therefore the added terms fill this part of the system. The modified matrix can, for instance, be of the form

$$
\begin{bmatrix}
(A + \hat{A}) & (B + \hat{B})^T \\
(B + \hat{B}) & \hat{K}
\end{bmatrix}
\begin{bmatrix}
\xi \\
\varphi
\end{bmatrix}
= 
\begin{bmatrix}
f + \hat{f}_1 \\
\hat{f}_2
\end{bmatrix}.
$$

Here, the caret denotes the terms added by the stabilization method. For instance, if we would restate the conservation of mass as $\nabla \cdot u_h - \varepsilon p_h = 0$, the discretization would result in $\hat{A} = \hat{B} = 0$ and $\hat{K} = -\varepsilon M$ (where $M = \int_\Omega \varphi_j \varphi_i \, dx$). The idea is that $M$ is invertible, making the resulting problem

$$(BA^{-1}B^T + \varepsilon M)\varphi = BA^{-1}f$$

well posed. The parameter $\varepsilon > 0$ should be chosen small enough (maybe depending on the mesh size) that it does not significantly alter the original problem. This stabilization is usually called a penalty method (e.g. Boffi et al., 2013). In glaciology however, it seems that the most commonly used stabilization method is the Galerkin Least-Squares (GLS) stabilization (Hughes and Franca, 1987). This is a consistent method, in that the added terms vanish for the (smooth) solution of the problem. In general, this leads to contributions to all the additional matrices introduced above, but for the $P_1P_1$ element only $\hat{K}$ and $\hat{f}_2$ become non-zero. In

The GLS method was initially studied for the case of Newtonian fluids and on domain that allowed triangulations of good shape. However, ice as a fluid has additional challenges. Not only is it non-linear, but can come close to being singular for small deformations. Moreover, in the case of ice sheets the domain has a high aspect ratio (thickness is small compared to horizontal extent), which results in mesh triangles of elongated shape. How these affect the characteristics and accuracy of the GLS method and how this compares to other stabilization methods is the topic of Paper III.
3.1.4 Impenetrability boundary condition

So far in this chapter, the boundary conditions considered have been those of a stress-free surface and a no-slip bed, see (3.1). However, it is not often that glaciers and ice sheets are completely frozen to their bed. More commonly, the ice slides against the underlying bed or substrate. The complete boundary conditions, including sliding, were given in (2.9) in Chapter 2. In this section I will focus on the implementation of the impenetrability condition, which for clarity is repeated together with the sliding condition below. For a linear sliding law and an impenetrable bed, the following set of boundary conditions apply on the part of the boundary subject to sliding conditions, $\Gamma_s$:

\[
\begin{align*}
\mathbf{u} \cdot \mathbf{n} &= 0 \quad \text{on } \Gamma_{sl}, \\
\left( (\mathbf{S}(\mathbf{u}) - \rho \mathbf{I}) \cdot \mathbf{n} \right)_\| &= -\beta^2 \mathbf{u}_\| \quad \text{on } \Gamma_{sl}.
\end{align*}
\]  

(3.14a)  

(3.14b)

There are a few aspects to note regarding the conditions in (3.14). First, letting $\beta^2 \to \infty$ forces $\mathbf{u} \to \mathbf{0}$, so by setting a high value for the friction factor, the frozen (no-slip) condition can be included as a subset of the sliding condition. Second, since the pressure dependent part of the total stress tensor is $\rho \mathbf{n}$, its tangential part equals zero. Introducing the orthonormal set of tangent vectors, $\{\mathbf{t}_i\}_{i=1}^{d-1}$
that span the tangential plane (the plane that contains all vectors orthogonal to \( \mathbf{n} \)), we can write the components of (3.14b) in the directions of the tangents as

\[
((\mathbf{S}(\mathbf{u}) - p\mathbf{I}) \cdot \mathbf{n})^\parallel = \mathbf{t}_i \cdot (\mathbf{S}(\mathbf{u}) \cdot \mathbf{n} - p\mathbf{n}) = \mathbf{t}_i \cdot \mathbf{S}(\mathbf{u}) \cdot \mathbf{n} = -\beta^2 \mathbf{u} \cdot \mathbf{t}_i
\]

(3.15)

The last row of (3.15) is known as Navier’s slip, which is the same as the linear sliding law.

Similarly to Section 3.1.1, when the \( \mathbf{V}_0 \) was the restriction of \( [H^1(\Omega)]^d \) so that the Dirichlet condition \( \mathbf{u} = 0 \) was fulfilled, we can define the space

\[
\mathcal{V}_h = \{ \mathbf{v} : [\mathbf{v} \in H^1(\Omega)]^d, \mathbf{v} \cdot \mathbf{n}|_{\Gamma_{sl}} = 0 \}.
\]

(3.16)

Using trial and test functions from \( \mathcal{V}_h \times \mathcal{Q} \) would then result in enforcing (3.14a). Since the boundary condition is a part of the underlying function space it referred to as strongly imposing the condition. To weakly impose a boundary conditions means that the boundary conditions is part of the variational formulation. This, for instance, happens at the stress-free surface with \( \int_{\Gamma_s} (\mathbf{T} \cdot \mathbf{n}) \cdot \mathbf{v} \, ds = 0 \). Similarly, the same integral taken over \( \Gamma_{sl} \) remains in the variational formulation. Decomposing the total normal stress into the normal (\( \perp \)) and tangential (\( \parallel \)) directions gives that

\[
\int_{\Gamma_{sl}} (\mathbf{T} \cdot \mathbf{n}) \cdot \mathbf{v} \, ds = 0 \quad \text{due to } \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma_{sl}
\]

\[
= \int_{\Gamma_{sl}} (\mathbf{T} \cdot \mathbf{n})^\perp \cdot \mathbf{v} \, ds + \int_{\Gamma_{sl}} (\mathbf{T} \cdot \mathbf{n})^\parallel \cdot \mathbf{v} \, ds
\]

\[
= \int_{\Gamma_{sl}} \beta^2 \mathbf{u}^\parallel \cdot \mathbf{v} \, ds - \int_{\Gamma_{sl}} \beta^2 \mathbf{u}^\perp \cdot \mathbf{v} \, ds
\]

\[
= - \int_{\Gamma_{sl}} \beta^2 \mathbf{u} \cdot \mathbf{v} \, ds.
\]

Therefore, if we assume that \( \Gamma = \Gamma_s \cup \Gamma_{sl} \) (i.e. the glacier boundary consists of a stress-free surface and a bed subject to sliding), the last integral needs to be added to (3.5a).

To solve the FS equations with the sliding conditions, function spaces suitable for discretization need to be defined (Section 3.1.2). However, here a difficulty arises in that the possibly smooth boundary \( \Gamma \) of \( \Omega \), becomes a polygonal boundary, \( \Gamma_h \) (see Fig. 3.1). Most commonly, and for all meshes used in this thesis, the boundary is piece-wise linear. The problem becomes how to define the discrete unit normal, \( \mathbf{n}_h \). For triangular and tetrahedral meshes the facets (surfaces of the mesh of dimension one less than \( d \), i.e. edges and faces) are linear, which makes the normal uniquely defined as a constant unit vector. However, at the vertices the normal is not well-defined. Since both the \( P1 \) and \( P2 \) elements for velocity have nodes on the vertices of the triangle (see Fig. 3.3), \( \mathbf{n}_h \) needs to be well-defined over the whole of \( \Gamma_{sl} \). That is, to strongly impose \( \mathbf{u}_h \cdot \mathbf{n}_h = 0 \), a discrete normal \( \mathbf{n}_h \in \mathcal{V}_h \) must be constructed in such a way that it is defined at every node. Different approaches to choosing \( \mathbf{n}_h \) exists (e.g. Dione et al., 2013). What seems most natural is to define the normal as a weighted average of the (well-defined) facet normals on adjacent cells. That is, a facet with large area or length influences the direction of \( \mathbf{n}_h \) more than a smaller facet (see Fig. 3.5).
As mentioned, the impenetrability condition is a Dirichlet condition, but contrary to the no-slip condition, we do not want to set each (or some) of the velocity components equal to zero. Rather, a combination of the velocity components at each node on $\Gamma_{sl}$ is to be set to zero. Specifically, the combination of velocity components that points in the direction of the boundary normal should be set to zero. To accomplish this, a local rotation of the coordinate system at each relevant node is necessary. This can be seen by for instance considering a horizontal flat bed along which the glacier slides. In this case, the intuitive way to define the normal everywhere along $\Gamma_{sl}$ is as $n_h = (0, 0, -1)$ (the negative sign is due to that the normal points out of the domain). In this case we do not need to do much, since we can set the vertical component to zero and let the remaining (tangential) components be solved for. Inclining the domain at an 45° angle in the $x$-direction, would result in a normal $n_h = (\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}})$. One option would be to account for this inclination by adjusting the direction of the body force and proceeding as above. This however is not viable in anything but very simple domains. Another option is to rotate the coordinate system based on $n_h$ in such a way that the velocity is expressed in the orthonormal system $(t_{h1}, t_{h2}, n_h)$. If this is the case, we can proceed to set the last component of the velocity to zero. Doing this for every node on the boundary results in imposing the impenetrability strongly. After the solution is retrieved, the coordinates must be rotated back to the regular Cartesian system to make sense with the rest of the velocity nodes. Note that this procedure does not enforce impenetrability across the facet of a cell, only at the nodes and in the direction of $n_h$.

An alternative approach to imposing the impenetrability conditions strongly is to, just like the sliding, impose it weakly. That is, we would like to have

$$\int_{\Gamma_{sl}} u_h \cdot n_f \, ds = 0,$$

where $n_f$ is taken to be the well-defined normals on the facets of $\Gamma_{sl}$ (see Fig. 3.5). The difference between the above and imposing the condition strongly, is that
no rotation of the coordinate system is necessary and that the normal is unambiguous. As a function space we do not need to use (3.16), but can use $V_h$. This means that the impenetrability will not be enforced at the nodes, but rather that the velocity solution will be such that it on average over each facet has the impenetrability property. This can be seen as a drawback of the method, in particular if the domain is large with a coarse mesh resolution. However, there also is a computational drawback to the method. Just like enforcing the incompressibility (3.5b) weakly by the addition of an unknown in the form of a Lagrange multiplier (pressure), the impenetrability can be imposed by introducing a separate Lagrange multiplier, call it $\lambda$. The use of Lagrange multipliers to enforce a side condition stems from minimizing a functional (functions that take functions as arguments and maps into the reals), related to some property of the system. Not uncommonly, this is the energy of the system. Since the functional for the FS problem with sliding is not trivial, I will therefore not state it here in its entirety. The interested reader is referred to the papers in which such a functional was derived (Dukowicz et al., 2010, 2011). The important part to emphasize here is that $\lambda$ does not need to be defined over the whole of the domain, but rather on the relevant boundary $\Gamma_{sl}$. This does not increase the additional unknowns by a tremendous amount, however, it extends the saddle-point characteristic of the system. The space used for $\lambda$ can loosely be said to be $L^2(\Gamma_{sl})$. Similarly to the form $B(\cdot, \cdot)$ in (3.11) that appears both in the momentum equation (3.5a) with $(v, p)$, and separately to enforce incompressibility (3.5b) with $(u, q)$, the imposing the impenetrability with a Lagrange multiplier results in the form

$$C(u, \lambda) = \int_{\Gamma_{sl}} \lambda u \cdot n ds. \quad (3.17)$$

Given a set of trial and test functions ($\lambda$ and $\varrho$), the form $C(v, \lambda)$ is then added to the balance of momentum and $C(u, \varrho) = 0$ becomes a separate equation that enforces the impenetrability.

Implementing and studying the above methods is done in Paper IV, where an additional method that weakly imposes impenetrability in an approximative way is also investigated. For two dimensional glacier and ice sheet scenarios, the study emphasizes the similarities of the strongly and weakly imposed methods, while the approximative method results in surface velocities that differ 1\% to 5\% from the other methods. However, the weak method is, due to the extended saddle point character of the system, not stable in three dimensions. Differences between the strong and weak method under mesh refinement for an analytical solution is further studied in Section 4.2.

3.2 Note on computational software

All of the FEM simulations in this thesis summary have been made with the aid of the computational platform FEniCS (Logg et al., 2012; Alnaes et al., 2015). FEniCS is a collection of libraries, that can easily be interfaced through a high-level Python or C++ interface. The syntax is similar to that of the weak formulations of PDE, and together with the ability for symbolic differentiation, this facilitates the process of testing a wide variety of problems. Within the ice sheet modeling community, FEniCS has for instance been used in Brinkerhoff and Johnson (2013). FEniCS was used in Paper III and Paper IV.

In addition, the software Elmer/Ice (Gagliardini et al., 2013) was used in Paper III. This is an extension of the multi-physics software Elmer (Räback et al.,
with a focus on advanced ice sheet modeling. Elmer/Ice has been used quite extensively in the ice sheet modeling community (e.g. Zwinger et al., 2007; de Fleurian et al., 2014; Ahlkonra et al., 2016).

For the FD simulations used in Paper I and Paper II, MATLAB and its built-in ODE solvers were used (Matlab, 2014).
4 Methods of verification and associated problems

When attempting to understand complex natural systems a first step is to identify the most important physical processes, possibly limit which of these are to be included given some criteria, and finally state the formal description of the system. The end product of this process is the conceptual mathematical model. For instance, the dynamics of fluid flows are generally described by the Navier-Stokes equations, while a simplification of these, the Stokes equations for incompressible fluids, govern the movement of ice sheets and glaciers. Thus, the mathematical model serves as a symbolic description of reality. However, the result of this is often a set of ordinary or partial differential equations for which an analytical solution may not be available. For these kind of systems, the aid of numerical models, which approximate the true solution of the mathematical model, become indispensable. Numerical models serve as a link between reality, represented by the considered natural system, and the conceptual description of the same (see Fig. 1.2).

To assess how accurate a numerical model is in its representation of a natural system, the process of first verification, and second validation (V&V) should be used (e.g. Oberkampf and Trucano, 2002). More general definitions for the words verification and validation exist, but given the physical setting of glaciology, the following seems to be most appropriate and applicable (e.g. Schlesinger et al., 1979; AAIA, 1998; Babuška and Oden, 2004):

- **Verification**: The process of determining if a numerical model with specified accuracy can represent the mathematical model of the natural system
- **Validation**: The process of determining if and to what extent a mathematical model accurately represents reality in the context of the natural system.

From an Earth Science point of view, reaching the validation state is the goal of any model. In this state the model can be used to for instance predict future scenarios and the output of the model can be interpreted in an critical and constructive manner. Naturally, to be able to do this with confidence the verification of the numerical model must be as thorough as possible. A part of this thesis has been focused on in particular the verification of numerical models of specific glaciological processes represented by mathematical models. This focus is the result of what to me has been a difficulty to confidently verify the implemented models in a general setting.

Two methods of verification that are used in this thesis are the method of manufactured solutions and comparing the numerical model to an analytical solution. The method of manufactured solutions consists of finding an analytical
expression to a problem as similar as possible to the FS problem by modifying some of the terms in the system. Using such manufactured solutions for various glaciological scenarios gave results that, depending on the case, either converged to the solution when it was not expected to, did not converge or converged as expected. This behavior is observed in both Paper III and Paper IV.

As a consequence, to verify the implementation of the numerical methods in Paper IV, I wanted to find an analytical solution for the FS equations on a domain with a curved boundary subject to sliding. The solution found was that of laminar flow between two concentric cylinders with slip boundary conditions. However, depending on the method chosen to enforce impenetrability, refining the mesh non-uniformly resulted in either the method converging toward the analytical solution or converging to a solution that is clearly different from the analytical solution. These experiences are summarize in the following sections.

4.1 Using manufactured solutions as a verification method

One of the main difficulties when it comes to the verification of numerical ice sheet and glacier models for the FS equations, is that an analytical solution for a non-trivial set-up has not been found ruling out verification against an analytical solution. An alternative approach is to specify a solution and adapt the mathematical model slightly to a state such that the solution chosen is fulfilled. In practice this means that the the boundary conditions (2.9) of the FS equations are changed and that the body force is modified and no longer represented by $\rho g$. Of course, the better the chosen solution for a given scenario, the less the mathematical model has to be changed and only minor changes to boundary stresses and gravity are necessary. For the glaciological setting such manufactured solutions were found in Sargent and Fastook (2010) for the two-dimensional case and modified in Leng et al. (2013) to apply to the three-dimensional case. The general solutions derived are for the transient case (ice surface evolves with time), however, since only steady-state flow is considered in this thesis, the equations are restated as such.

For a given ice sheet domain, described by a surface $z_s(x, y)$ and a bed $z_b(x, y)$, the manufactured solutions for the velocity $\tilde{u} = (\tilde{u}, \tilde{v}, \tilde{w})$ and pressure $\tilde{p}$ are given in Leng et al. (2013) as

\begin{align*}
\tilde{u}(x, y, z) &= c_x(z_s - z_b)^{\gamma_1} \left[ 1 - \left( \frac{z_s - z}{z_s - z_b} \right)^{\lambda_1} \right] + c_{bx} \frac{1}{z_s - z_b}, \quad (4.1a) \\
\tilde{v}(x, y, z) &= \frac{c_y}{z_s - z_b} \left[ 1 - \left( \frac{z_s - z}{z_s - z_b} \right)^{\lambda_2} \right] + c_{by} \frac{1}{z_s - z_b} \\
&\quad - \frac{c_x}{z_s - z_b} (1 + \gamma_1) \left[ 1 - \left( \frac{z_s - z}{z_s - z_b} \right)^{\lambda_1} \right] \cdot \int \left( \frac{\partial z_s}{\partial x} - \frac{\partial z_b}{\partial x} \right) (z_s - z_b)^{\gamma_1} \ dy, \quad (4.1b) \\
\tilde{w}(x, y, z) &= \tilde{u}(x, y, z) \left( \frac{\partial z_b}{\partial x} \frac{z_s - z}{z_s - z_b} + \frac{\partial z_s}{\partial x} \frac{z - z_b}{z_s - z_b} \right) \\
&\quad + \tilde{v}(x, y, z) \left( \frac{\partial z_b}{\partial y} \frac{z_s - z}{z_s - z_b} + \frac{\partial z_s}{\partial y} \frac{z - z_b}{z_s - z_b} \right), \quad (4.1c) \\
\tilde{p}(x, y, z) &= -2\eta \frac{\partial \tilde{u}}{\partial x} - 2\eta \frac{\partial \tilde{v}}{\partial y} + \rho g (z_s - z). \quad (4.1d)
\end{align*}
That the term \( \gamma \over the domain, while in the
\text{c}\) with the depth of the domain. Besides the no-slip condition, the parameter \( c \) velocity in the horizontal directions due to deformation) at the surface, \( c_{bx} \) and \( c_{by} \) (typical sliding velocity), \( \lambda_1 \) and \( \lambda_2 \) (influences the velocity profile) and \( \gamma_1 \) and \( \gamma_2 \) (depth scaled velocity). The velocity solution fulfills the incompressibility and impenetrability conditions, see (2.2b) and (2.9c), by construction. However, to determine what the appropriate boundary conditions and body force to be used should be, \( \tilde{u}, \tilde{v}, \tilde{w} \) and \( \tilde{p} \) in (4.1) are inserted into (2.2a) and (2.9a) and, if sliding conditions are used, into (2.9d) as well. The viscosity, \( \eta \), is determined in a similar manner.

Setting \( \gamma_1 = -1 \) and \( c_y = c_{by} = 0 \), the two-dimensional version from Sargent and Fastook (2010) is recovered as

\[
\begin{align*}
\tilde{u}(x, y, z) &= \frac{c_x}{z_s - z_b} \left[ 1 - \left( \frac{z_s - z}{z_s - z_b} \right)^{\lambda_1} \right] + c_{bx} \frac{1}{z_s - z_b}, \\
\tilde{v}(x, y, z) &= \tilde{u}(x, y, z) \left( \frac{\partial z_s}{\partial x} \frac{z_s - z}{z_s - z_b} + \frac{\partial z_s}{\partial y} \frac{z - z_b}{z_s - z_b} \right), \\
\tilde{p}(x, y, z) &= -2\eta \frac{\partial \tilde{u}}{\partial x} + \rho g(z_s - z).
\end{align*}
\]

That the term \( (z_s - z_b) \) enters in the denominators for in (4.1) and (4.2) indicates that geometries that specify a zero thickness are not suitable (for instance, the typical glacier front scenario subject to some sliding). For this reason, a particularly appropriate use of the manufactured solutions is the ISMIP-HOM style geometries (see Pattyn et al. (2008)), defined by

\[
\begin{align*}
z_s(x, y) &= -\tan(\alpha)x, \\
z_b(x, y) &= z_s(x, y) - Z + \frac{Z}{2} \left[ \sin \left( \frac{2\pi x}{L} \right) \sin \left( \frac{2\pi y}{L} \right) \right],
\end{align*}
\]

where \( Z \) and \( L \) are the typical height and horizontal extent in the \( xy \)-direction (for two-dimensional simulation the \( y \)-dependency is omitted). For the ISMIP-HOM A, \( Z = 1 \) km is the typical thickness of the ice, \( L = 80 \) km the horizontal \( (x \) and \( y \) \) extension of the domain and \( \alpha = 0.5^\circ \) is the inclination of the surface. Two dimensional equivalent, the ISMIP-HOM B experiment, is specified as the cross-section at \( y = L/4 \) of the ISMIP-HOM A domain.

The above solutions (or similar manufactured solutions) have on several occasions been used to validate numerical ice flow models in (e.g. Brinkerhoff and Johnson, 2013; Gagliardini et al., 2013; Leng et al., 2013). In this thesis the interest was to use (4.1) with non-zero \( c_{bx} \) and \( c_{by} \) for the propose of evaluating how or if the velocity field is affected by different implementations of impenetrability (Paper IV). However, in Paper III we also used the solutions with a no-slip condition to quantify the effects of stabilization on the vertical surface velocity.

In Leng et al. (2013) the solutions where used with \( \gamma_1 = 0 \), \( \lambda_1 = \lambda_2 = 4 \) and \( c_{bx} = c_{by} = 0 \), which means that the surface velocity in \( x \)-direction is constant over the domain, while in the \( y \)-direction the surface velocity scales (inversely) with the depth of the domain. Besides the no-slip condition, the parameter \( c_x \) is set so that the surface velocity in the \( x \)-direction is approximately \( 45 \) m a\(^{-1}\) and \( c_y = c_z \). The supplementary material to Leng et al. (2013) includes a C code that point-wise calculates the necessary compensation terms. This code was used in Gagliardini et al. (2013) for verification and has also been used by us for the three-dimensional simulations (Paper III). For the two-dimensional case, I wrote a Python module using SymPy (SymPy Development Team, 2016) to generate

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the necessary FEniCS compatible C++ code. This module is meant to be a bit more flexible in that $z_a(x)$, $z_b(x)$ and the relevant parameters can be specified at run-time by the user.

There are some aspects to note about the character of the manufactured solutions. First, with the parameters chosen as above, deformation at points of the surface overlying the peaks and valley bottom parts of the bed topography is zero, i.e. $D_e = 0$. This makes $\eta$ singular at these point (e.g. $x = y = L/4$). From a numerical perspective, this is not ideal, since $\eta$ is used to determine the necessary compensation terms resulting in traction and body force of high magnitude. Intuitively, this problem can be overcome by setting a (large) maximum value for the allowed viscosity as $\eta_{\text{max}} = 10^{10}$ Pa a. However, even with this modification, the resulting compensation terms become very high. In the C code found in the supplementary material to Leng et al. (2013), this has been dealt with by specifying that any expression that has a denominator with a number close to zero (machine precision) is set to zero. This instead results in $\eta = 0$ at the singular points. The experience we gained from the experiments in Paper III indicate that this (from a physical perspective maybe somewhat unintuitive action) is necessary for convergence. The resulting compensation terms at the surface of the domain are shown in Fig. 4.1, where the singular points can be seen as values of zero that clearly differ from the surrounding values. Figure 4.2 shows the two-dimensional body force for $\lambda_1 = 4$ and $\lambda_1 = 1$. The latter choice of $\lambda_1 = 1$ does not capture the natural non-linear behavior of ice deformation, since it specifies a linear profile, but has been used in earlier convergence studies by Leng et al. (2012), as a two-dimensional solution extended to the three-dimensional domain. The benefit is in that the areas adjacent to the singularities do not affect the compensation terms to as an extensive degree as when the non-linear velocity profile is specified.

As the computational cost for proper convergence studies in three dimensions is very high for the FS equations (especially for the P2P1 element), the results in this section are performed with the two-dimensional solutions (4.2) using $\lambda_1 = 4$, $\gamma_1 = -1$ (as presented in Sargent and Fastook (2010)), but also with $\lambda_1 = 1$. Note that these do not necessarily extend (in a good or bad way) to the three-dimensional case strictly, but should rather be seen as an indication.

The changes in the relative errors between the simulated ($u_h, p_h$) and the manufactured ($\tilde{u}, \tilde{p}$) solutions, i.e. $\| \tilde{u} - u_h \|_{L^2} / \| \tilde{u} \|_{L^2}$, with mesh refinement are shown in Fig. 4.3. Convergence rates for the stable P2P1 element for a fluid

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4_1}
\caption{Compensation body force (a) and traction (b) at the surface for the three-dimensional manufactured solution in (4.1), on a $40 \times 40 \times 40$ mesh. The removed high values resulting from the singular compensations terms can be see in both panels, e.g. at $(x, y) = (20 \text{ km}, 20 \text{ km})$. Produced with code modified from Leng et al. (2013).}
\end{figure}
Figure 4.2. Compensation body force for the two-dimensional manufactured solution in (4.2), on a $320 \times 160$ mesh for (a) $\lambda_1 = 4$ and (b) $\lambda_1 = 1$. Panel (a) shows how the body force increases towards the surface and the removed singularity at the surface at $x = 20\text{km}$, while this behavior is not present in panel (b).

obeying a linear Stokes equation (that is a fluid with constant viscosity), would typically be of the form

$$
\|u - u_h\|_{H^1} + \|p - p_h\|_{L^2} \leq C h^2 (\|u\|_{H^3} + \|p\|_{H^2}),
$$

(4.4)

where $h$ is the mesh size and $C$ is a constant independent of $h$. That is, 2nd order convergence for the velocity (gradients) and pressure. However, if the domain is smooth enough, the above can be extended to a 3rd order for the velocity in the $L^2$-norm (e.g. Ern and Guermond, 2004). Finite element estimates for non-linear Stokes flow have been investigated in Hirn (e.g. 2012), and in the particular case of ice in Jouvet and Rappaz (2011). However, these estimates are in a norm dependent on the power ($n$ in Glen’s law) of the fluid and it is indicated in Jouvet and Rappaz (2011) that the estimates likely are sub-optimal. Therefore the convergence is presented in the more commonly used $L^2$-norm (e.g. Brinkerhoff and Johnson, 2013; Gagliardini et al., 2013; Leng et al., 2012, 2013). All simulations ran until a relative tolerance of $10^{-12}$ was reached between consecutive Newton iterations, using a direct linear solver.

As can be seen in Fig. 4.3, the convergence rates for $\lambda_1 = 4$ are poor at best, both for velocity and pressure. This can possibly be attributed to the high magnitude of the body force and surface traction introduced by the manufactured solutions. A likely explanation is that the singularity at the surface, becomes more pronounced with mesh refinement, resulting in higher absolute values at increasingly many points of the mesh. An effort was also made in removing the zero value at the singularities (and letting the viscosity reach a max value of $\eta_{\text{max}} = 1 \times 10^{10}$ Pa a at these points), but apart from the coarsest mesh (which did not converge to the specified relative tolerance), the errors did not changed by this modification. This is taken to indicate that the compensation terms close to the singular points become too high to represent a good verification scenario and that this is not remedied by setting the singularities to zero values. At least not in the two-dimensional case studied.

For $\lambda_1 = 1$, the rates coincide very well to the 2nd order convergence. However, when compared to the ideal for the $P2P1$ element for the linear Stokes equation the rate is of one order lower. Similar rates were obtained when the $H^1$-norm was considered, which seems to coincide well with the estimate (4.4). This could possibly indicate that manufactured solution or domain are not regular enough to benefit from the extra order of convergence in the $L^2$-norm. To compare the convergence to a scenario that does not introduce artificial boundary
and body force terms, I consider the analytical solution for ice flow on a simple domain. The solution (which can be found for a body of ice with parallel bed and surface inclined at an angle \( \alpha \), see e.g. Greve and Blatter (2009)) has a very simplified domain with no along flow changes in velocity and cannot be deemed a realistic scenario, but is an option to examine if convergence rates change. Figure 4.4 shows that the convergence rates for the velocity are of 3rd order while the pressure converges around 2nd order (although somewhat scattered). The simple domain in this case is a convex polygonal domain, which should suffice for the observed convergence (based on the linear Stokes estimate (4.4)), but shows that this can apply given a “nice enough” scenario. Convergence results found in glaciological literature cited in this section using the method of manufactured solutions vary as well. Leng et al. (2012) used the two-dimensional scenario (\( \lambda_1 = 1 \) with \( P2P1 \)) extended to three dimensions by extruding the solution in the \( y \)-direction, for which they found convergence rates of 3 and 1.5 for for \( u \) and \( p \). However, using the same model but with the fully three-dimensional case (\( \gamma_1 = 0, \lambda_1 = \lambda_2 = 4 \)), the values in Leng et al. (2013) seem to be more scattered and slightly lower for the velocity field at fine resolutions. The same manufactured solutions gave 3rd order convergence for the velocity for both the stable MINI element and a lower order stabilized bi-linear element, the latter case indicating super-convergence (Gagliardini et al., 2013). The \( P1P1 \) element was used in Brinkerhoff and Johnson (2013) giving close to ideal 2nd order convergence as predicted by estimates for the linear Stokes. In Paper III, we experience the same

Figure 4.4. Convergence rates for the analytical solutions (anal.) on a simple inclined domain with the \( P2P1 \) element. Velocity and pressure are shown in panel (a) and (b) respectively. Dashed lines show 2nd and 3rd order convergence rates.
type of convergence rates as the two latter cases with the same (Elmer/Ice) or similar (implemented in FEniCS) models. The above indicates that convergence rates with the manufactured solutions may differ depending on numerical implementation, but more importantly may be parameter dependent. In Paper IV, different implementations of impenetrability, some which should differ, converged in almost identical fashion using the manufactured solutions with sliding.

The conclusion of the above is that, depending on the parameters chosen, the manufactured solutions used here may (the case of $\lambda = 1$) or may not ($\lambda = 4$) be an appropriate choice to confidently verify a numerical model. It seems that the compensatory boundary conditions and body force required to fulfill the conservation of momentum can change the underlying problem to a degree such that this results in non-convergence or, as a worst case scenario, the resulting convergence is an artifact.

4.2 An analytical solution for non-linear Stokes flow

In Paper IV the main purpose was to examine the effects of different implementations of impenetrability. Since impenetrability needs to be enforced only in the case of basal sliding, the initial idea was to use the manufactured solutions presented in (4.2) to examine how the three implementations differ and in what way. The convergence results in Paper IV, which served as a basis for the material presented in Section 4.1, ultimately led to a lack of confidence when attempting to verify the different sliding implementations. The interest shifted towards being able to verify the implementations against an analytical solution with the general physics presented by the FS equations (2.2). However, such a solution is not readily available for a meaningful setting for ice. An alternative scenario which includes the qualitative settings (slip on a curved domain) is that of laminar flow of a power-law fluid occupying the space between two concentric (infinitely long) rotating cylinders.

A three-dimensional case with a gravitational body force, but with no-slip boundary conditions, can be found in e.g. Bird et al. (2001). To simplify the setting, the influence of the body force is removed, which then allows the setting to be reduced to a two-dimensional cross section of the cylinder. In the case studied below, both cylinders can have different slip parameters (constant on each cylinder), while rotating at a constant angular velocities ($\omega_i$ and $\omega_o$), see Fig. 4.5 for notation. The solution to the problem was derived starting from Bird et al. (2001), but has also been compared to Hron et al. (2008) in which a more general solution for second-grade fluids (power-law behavior being a sub-case) obeying the steady Navier-Stokes equations is given. A short outline of the derivation is given below, but more complete derivations can be found in the above cited sources.

For this problem, the cylindrical coordinate system $(r, \phi, z)$ is a natural choice, and a reasonable assumption when considering Stokes flow is that only circular motion occurs. Removing the $z$-dependency (no body force) leads to $u = (u_r, v_\phi, w)$ and $p(r)$ with $v_\phi(r)$ and $u_r = w = 0$. The conservation of mass (2.2b) then takes the form

$$\nabla \cdot u = \frac{1}{r} \frac{\partial (ru_r)}{\partial r} + \frac{1}{r} \frac{\partial v_\phi}{\partial \phi} + \frac{\partial w}{\partial z} = 0.$$ 

With the above assumptions, this is fulfilled by default, since $u_r = w = \frac{\partial v_\phi}{\partial \phi} = 0$. 43
In cylindrical coordinates the strain rate tensor, its invariant and the deviatoric stress using Glen’s flow law (2.4) can be expressed as

\[
D = \frac{1}{2} \begin{bmatrix}
0 & -\frac{v_\phi}{r} & 0 \\
-\frac{v_\phi}{r} & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
\]

\[
D_e = \sqrt{\frac{1}{2} D : D} = \sqrt{\frac{1}{4} \left( \frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r} \right)^2},
\]

\[
S = (2A)^{\frac{1}{n}} \left| \frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r} \right|^{(1-n)/n} \begin{bmatrix}
0 & -\frac{v_\phi}{r} \\
\frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r} & 0 \\
0 & 0
\end{bmatrix}.
\]

Inserting these into the conservation of momentum we get

\[
\nabla \cdot S - \nabla p = \left[ \frac{\partial S_\phi}{\partial r} + \frac{2}{r} S_r \right] e_\phi - \frac{dp}{dr} e_r = 0,
\]

which in turn gives two separate ordinary differential equations for the velocity and pressure as

\[
\frac{dp}{dr} = 0,
\]

\[
\frac{d}{dr} \left( S_r r^2 \right) = 0.
\]

We arrived at the last equation by multiplying the expression in brackets in (4.6) by \( r^2 \) and using the chain rule and the fact that the deviatoric stress is symmetric. Integrating the velocity equation and performing a similar action one more time gives the final result as

\[
p = P',
\]

\[
v_\phi = A' r + B' r^{\frac{-2n+1}{n}}.
\]
where $P'$ is an arbitrary constant and

$$B' = \begin{cases} 
-\frac{1}{2n} \left( \frac{C'}{\eta^*} \right)^n \frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r} > 0 \\
\frac{1}{2n} \left( \frac{C'}{\eta^*} \right)^n \frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r} < 0.
\end{cases} \quad (4.8)
$$

In (4.8), the constants $A'$, $B'$ and $C'$ and $\eta^* = (2A)^{-\frac{1}{n}}$ have to be determined by the sliding boundary condition. Note that in the cylindrical coordinate system the unit tangent and unit outward normal are $t = (0, 1, 0)$ and $n = (\pm 1, 0, 0)$ (negative on inner cylinder boundary). Since the slip velocity ($u_i$ in a glacier setting) in this scenario equals the difference of the velocity of the fluid and the angular velocity of the cylinder, we have the general boundary conditions $\pm S_{r\phi}(r) = -\beta^2(v_\phi(r) - \omega r)$, where the sign on the left hand side depends on whether taken over $\Gamma_i$ or $\Gamma_o$. Using the expression for $S_{r\phi}$ from (4.5) and inserting $r = R_i$ and $r = R_o$ separately into the boundary condition yields

$$C'R_{i}^{-2} = \beta^2(A'R_i + B'R_i^{-2n+1} - \omega_i R_i) \quad \text{on} \; \Gamma_i,$$

$$C'R_{o}^{-2} = -\beta^2(A'R_o + B'R_o^{-2n+1} - \omega_o R_o) \quad \text{on} \; \Gamma_o,$$

which together with (4.8) can be used to determine the constants. The sliding parameter $\beta^2$ is such that it is constant on each cylinder but can have different values. The above outline follows that of Hron et al. (2008), however I believe there is a misprint regarding the values of $B'$. In their study the equivalent of (4.8) contains a factor which equals $2n \left( \frac{C'}{\eta^*} \right)^n$ using the notation and Eq. (70) from Hron et al. (2008), which I have as $\frac{C'}{\eta^*}$.

In the following simulations, the parameters have been set to $R_i = 500 \text{ m}$, $R_o = 1000 \text{ m}$, the inner cylinder being rigid ($\omega_i = 0$) and the outer cylinder rotating at a speed of $200 \text{ m a}^{-1}$ ($\omega_o = 200/R_h = 0.2 \text{ rad a}^{-1}$). The sliding parameter was set to $\beta_i^2 = 100 \text{ Pa m}^{-1}$ and $\beta_o^2 = 3000 \text{ Pa m}^{-1}$, that is (close to) a free slip condition on the inner cylinder and a values corresponding to the mean sliding parameter used in Pattyn et al. (2008). Even though the parameters may be somewhat arbitrary, the intent was to examine what could be considered a typical value and the extreme case (free slip). For the simulations the $P2P1$ element was used and the impenetrability was enforced by the methods presented in Paper IV. That is, impenetrability was either enforced strongly or weakly. Results using the approximative method are omitted, since these either did not converge or converged to a solution that on the boundaries would equal the rotational velocities. One explanation for this is that the Lagrange multiplier used to enforce impenetrability in the approximative method is the pressure, which in this scenario is constant over the domain and not uniquely determined. A Newton method with a direct solver was used to solve the non-linear problem. In general, it seems that this problem is very hard or ill conditioned, since convergence was dependent on the relaxation parameter in the Newton method. It seems the strong implementation was more sensitive than the weak in this context. In Fig. 4.7, the output of the simulations is presented. For clarity, the figure shows the difference between the analytical solution and the strong method solution (left) and weak method solution (right). Both methods perform well under a uniform mesh refinement. That is, if the mesh size $h$ (coarse resolution mesh) is refined to $h/2$ (medium resolution mesh), both the strong method (panel a and c) and the weak method (panel b, d) show almost identical results. The velocity
error seems to follow a 2nd order convergence rate for both methods. However, if the medium mesh is refined non-uniformly (panel e, f), that is element sizes vary from a medium size in one region to a finer size ($h/4$) in another region, the errors of the strong solution changes character. The highest absolute value of the error increases compared to the uniform medium mesh and are located in the transition area between the medium and fine cells. Compare this to the result of the weak method (panel f), which follows the expected decrease in error as the mesh is refined. Refining the mesh uniformly again (from the medium mesh to $h/4$) resulted in the expected reduction of errors for both the strong and weak method (not shown).

It is possible that the symmetry of the domain and the simplicity of the solution results in a numerical situation that does not cope well with breaking this symmetry. However, at the time of writing, it is not clear exactly why the non-uniform mesh refinement causes the relatively high errors. Examining the calculated normal function (which is used to rotate the coordinates at the nodes) does not give any indication of the issue (see Fig. 4.6). However it is possible that the relatively small differences in the results of Fig. 4.7 could be caused by minor directional variations in the normals not easily quantifiable from Fig. 4.6. There seems to be a symmetry in the distribution of errors, with areas of high errors depending on how the domain is refined. For instance, if a larger part is refined, the errors will migrate into the coarsest sized part of the domain. The above phenomenon appeared irrespective of the fluid being linear or non-linear (note that a very small value of $A$ makes $B'$ in (4.8) small which effectively results in (4.7) becoming linear). However, numerical experiments showed that one of the critical factors is that of high viscosity. That is, simulating a (non-)linear fluid with a lower viscosity (this was for instance tried with $A = 1$) resulted in nearly identical results for the strong and weak method, independent of type of mesh refinement. Finally, the choice of the sliding parameter $\beta^2$ directly affects the errors in the strong method introduced by non-uniform refinement. For instance, if a high $\beta^2$ (little to no slip) is set on both the inner and outer cylinder, errors were smaller and not as significant. At the other extreme, if a condition close to free slip (low $\beta^2$) is set on both cylinder boundaries, the strong method would converge to...
a solution that does not resemble the analytical solution. The erroneous solution would still enforce impenetrability and incompressibility in a satisfactory manner and the general pattern of the flow would be correct, but in general the velocities would be far too low. Even though the weak method did converge towards the analytical solution, both methods showed difficulties when solved with Newton’s method.

Similar scenarios regarding linear Stokes flow in an annular domain has been studied in e.g. Urquiza et al. (2014); Dione et al. (2013). These studies show that when approximating smooth domains by polygonal domains, enforcing the impenetrability condition weakly (either by a Lagrange multiplier or by a penalty method and depending on parameters chosen in either method) can result in convergence to a different solution than the true solution. Therefore, the above comes as somewhat of a surprise, as a possible outcome would have been for the weak solution not to converge to the correct solution. It would be interesting
to rigorously examine the cause of the above if something similar could appear in a non-symmetrical scenario, in particular in a glacier or ice sheet setting exhibiting high velocities in for instance regions of significant slip, such as an ice stream. However, during the work of this thesis I have not come across anything that would indicate that this could be a problem. Either the scenarios normally present in glaciers and ice sheets only introduce errors that are small compared to other sources, or the above simply does not happen for more realistic flows.

Similar scenarios to the above where also performed using an alternative low-level C++ code that implements impenetrability strongly using the framework of FEniCS (supplied by M. Nazarov, personal communication). The code, which is an improvement of the code presented in Nazarov (2011), gave similar results regarding for non-uniform mesh refinement as the results above using my own Python implementation in FEniCS.
Summary of papers

This thesis consists of four papers focusing on two themes: subglacial hydrology (Paper I and Paper II) and ice dynamics (Paper III and Paper IV). The overarching aim of these papers is to investigate how the underlying physical processes are represented by numerical models, and how the output of such models can have an effect on the interpretation of interconnected processes, such as extent of efficient channelized systems and mass balance.

5.1 Paper I


We present corrections to the model implemented in *Meierbachtol et al.* (2013), that was used to simulate water pressures in subglacial conduits for the steady-state case. Part of their study consisted of recording subglacial water pressures from multiple boreholes taken over a multi-year span at a Greenland outlet glacier. The data was compared to the steady-state and transient pressures generated by the output of a simple numerical model for a subglacial conduit (R-channel) using typical discharge scenarios. The subglacial pressures were interpreted as not coming from a conduit, but from another type of drainage system. Close to the margin (<30 km) the data showed higher pressures than modeled water pressures, suggesting that the existence of a conduit was possible but that the boreholes did not intersect such a feature. In up-glacier regions the converse would hold, both for steady and time varying discharge. This was taken to indicate that the water pressures in a potential subglacial conduit would be higher than the surrounding drainage system. Such an isolated conduit, not interacting with the adjacent system, is not likely and their existence further from the margin was deemed improbable in the study area.

The current study shows that the steady-state model was not implemented correctly, and that a correct implementation generates water pressures that lie below the data presented in *Meierbachtol et al.* (2013). Following the methodology of the original study, we use FD to discretize the steady-state and transient case (shown in (2.15) and (2.13) respectively), using parameters that are taken to be beneficial to the existence of a subglacial conduit. We then use the correct steady-state water pressure as an initial condition to the transient scenarios and find that, although there is an initial difference in simulated water pressures, after a few simulated days the model reproduces the transient output from the original study. That is, over the time span of a melt season, the transient scenarios were insignificantly altered by the erroneous initial condition, and the conclusions...
made in Meierbachtol et al. (2013) based on such scenarios remain unaffected by the changed numerics.

By this we conclude that this simplified subglacial conduit scenario modeled, first and foremost has time-dependency as a controlling factor and that it is this which induces the sensitivity of the system. In particular, the steady-state pressures modeled, despite being relatively moderate up-glacier, should not be taken for granted as a realistic substitute for scenarios of “even and continuous supraglacial input”. This conclusion is made as it takes only small transient changes in discharge for the system to each overburden, emphasizing the importance of proper modeling of subglacial water pressures.

5.2 Paper II


The study concerns the interconnection between the processes and geomorphological products related to subglacial ice-sheet hydrology. In particular, we aim to highlight the areas in which there is a missing link between glaciofluvial landforms and observational glaciohydrology, and if either are supported by existing theory. The field of theoretical subglacial hydrology has been developed much in parallel with field glaciology to explain the observational real-time processes. However, many of these studies concern the alpine glacier scale, and much of the existing theory is closely related to seminal work done in the 1970s and 1980s. The alpine glacier scale contrasts with the many paleo-glaciological studies that have the entire domain of ice sheets at their disposal. Inferring the dynamics of paleo-ice sheets therefore includes an upscaling or interpretation of the processes acting in a glacier. However, more recently field and theoretical studies have shifted their focus to the domain of ice sheets, which has raised the question of whether the processes acting in a glacier setting are also the dominant processes in an ice-sheet. As a basis for discussing the above in a cross-disciplinary context, in the first part of the study, we provide a broad up-to-date review on subglacial hydrology subdivided into three research disciplines: theoretical glacial hydrology, contemporary observations and paleo-glaciology.

In the discussion, we identify a set of areas in which the link between the research fields is uncertain or highly conceptualized:
1) The inferred operational scale of channelized subglacial networks differs between those predicted by theory, contemporary observations and the paleo-record. The theory of R-channels (see Section 2.2.1) indicates that these will be operating at a low-pressure regime only close to the ice-sheet margin and are unlikely to exist further from the margin under high-pressure conditions ($\gtrsim 50$ km). The paleo-record, which in particular shows an abundance of esker systems compared to landforms associated to distributed systems, brings up the possibility of extensive channelization. In some studies this has been interpreted as being the result of a long-distance ($\sim 200$ km) continuous and active system, whose dendritic character would indicate a relative low pressure. Moreover, contemporary studies on Greenland show that large quantities of supraglacial water transported to the bed by moulins, can be efficiently transported to the margin. However, it is uncertain what type of pressure regime and topology this operates under.
2) To what extent is the paleo-record composed of an average of the prevailing mode of drainage system rather than being the product of less frequent high-drainage events, and at what magnitude does an event start to become a factor for the geomorphic activity.

3) Temporal and spatial scales differ when comparing the research fields. Evidence from the paleo-record indicates that the scales necessary can be higher by orders of magnitude, with for instance subglacial channels interpreted to be 10 to 100 times wider.

The main conclusions of the study are that the bi-modal interpretation of efficient/inefficient drainage system as a channelized/distributed system probably is not valid at an ice-sheet scale. Rather, different drainage modes occur and interact on different scales and under different circumstances, with discharge variability being a controlling factor. It is likely that the dominating pressure regime away from the margin is that of high-pressure, which can possibly be efficient. To be able to reliably use the information in the paleo-record to infer the future, a first step is to decide to what extent the paleo-record preserves landforms connected to physical scenarios that are a good analogue to contemporary ice sheets. To see how climatic patterns are reflected in the geomorphology, we must know if it is a composite of landforms that are the result of the normal drainage system or if it is the unusual. A dedicated cross-disciplinary action between the research fields to determine the physical processes that can be reflected in both paleo- and contemporary record will be necessary.

5.3 Paper III


We use two different FEM software packages (FEniCS and Elmer/Ice) applied to the isothermal Full Stokes equations (3.5) with no-slip boundary conditions to investigate the effects of stabilized piece-wise linear ($P_1P_1$ on triangle or tetrahedron meshes, see Section 3.1.3) and bi-linear ($Q_1Q_1$ on rectangular or hexahedron meshes) elements. For the FS equations, the chosen element pairs are an inherently unstable formulation and result in unphysical pressure oscillations. However, the elements remain an attractive choice in ice sheet modeling due to the relatively low computational cost compared to higher order stable elements. The stability issue must then be overcome by the use of stabilization methods. We start by studying the GLS stabilization method (Hughes and Franca, 1987), which to our knowledge, is the most frequently used stabilization method in glaciology. The method depends on a set of parameters, that for the typical ice sheet scenario differ from a linear fluid on a simpler domain, namely:

1) Cell size of the mesh and how this is defined. Ice sheet models often use extruded meshes, which due to the ice geometry results in very high aspect ratios (flat elements).

2) The viscosity, which in the case of ice is a non-linear function of the velocity.

3) A user-defined parameter, for which it is possible to determine an optimal size for linear fluids.

Performing convergence tests with the aid of a set of manufactured solutions,
using a standardized scenario (ISMIP-HOM; Pattyn et al. (2008)) and a more realistic domain, we investigate how the different parameters affect the stability and errors of the numerical solutions. In particular, we focus on how these affect the surface velocity. We reason that, due to the direct influence of the FS solution on the free-surface evolution (2.10), even small instabilities propagate over time. Therefore, determining appropriate parameters for the stabilization method becomes important for transient simulations predicting ice sheet response to climate.

In the latter part of the study, we implement additional stabilization methods developed for Stokes flow: the Pressure Penalty (PP), Interior Penalty (IP; Burman and Hansbo (2006)), Pressure-Global-Projection (PGP; Codina and Blasco (1997)) and Local Procection Stabilization (LPS; Becker and Braack (2001)) methods. We perform the same test as for the GLS method and compare the results. We find that for the studied scenarios, the GLS method performs well when defining the cell size as the minimum edge length, using an iterative procedure that includes the non-linear viscosity and setting the user-defined parameter to the ideal parameter size for linear fluids. However, deviations from these choices result in oscillations of either the pressure field or the surface velocity, making the method relatively sensitive to the (possibly situation dependent) choice of parameters. In contrast to this, the IP method performed equally well over a wider set of parameters than the GLS method for the cases studied (introducing less oscillations at the surface). The IP method is comparative in computational costs to the GLS method, and can be viewed as an alternative to the latter that should be tested in more general ice sheet scenarios.

5.4 Paper IV

Helanow, C. Effects of numerical implementations of the impenetrability condition on non-linear Stokes flow: applications to ice dynamics. Manuscript.

This study examines how different FEM formulations of the impenetrability condition (2.9c) affect the velocity and pressure solutions of the isothermal FS equations with a linear sliding boundary condition. In FEniCS, I implement methods to enforce impenetrability strongly and weakly (as described in Section 3.1.4), and an approximative method in which the pressure variable acts as a Lagrange multiplier for both the incompressibility and impenetrability (i.e. p is used instead of λ in (3.17)). The strong method is the most commonly used method in glaciology. The weak method was suggested in Dukowicz et al. (2011), but is subject to instability in the introduced Lagrange multiplier due to the extension of the saddle-point problem. The approximative method is easy to implement (and has been used in Brinkerhoff and Johnson (2013)) but does not strictly enforce impenetrability and affects the pressure field close to the sliding parts of the bed. The simulations are performed with a P2P1 (strong and approximative) and a P2P1P0 (weak) formulation.

Convergence studies using manufactured solutions with sliding, using parameters and two-dimensional solutions from Sargent and Fastook (2010), show little difference between the methods. Since the approximative method should at least differ from the other methods close to the bed, I interpret this to possibly be due to the by the manufactured solutions introduced compensatory body force and traction. These terms are necessary for convergence, but may be such that they
modify the original problem in a way that over-shadows the differences between the methods. What is evident is, that for coarse resolutions, velocities close to the bed show a weak oscillatory behavior for both the weak and approximative methods. This however is resolved with mesh refinement.

The line of reasoning regarding the compensatory terms is emphasized when simulating ice flow on the benchmark glacier *Haute Glacier d’Arolla*. The simulation specifies a no-slip condition ($\beta^2$ is high) everywhere except in a free-slip zone ($\beta^2 = 0$) in the middle part of the glacier. Here the output of the simulations are very close for the strong and weak formulations, while the approximative solution differs both in the pressure field and in the velocity field in particular (as predicted). At the surface, overlying the area of the slip-zone, velocity differences are in the order of 1%.

As a final comparison, I use the strong and approximative methods to simulate the GrIS, given a specified $\beta^2$-field. For this three-dimensional scenario, the weak method did converge to a sensible solution, which indicates the need of a stabilization procedure compared to the two-dimensional case. The differences between the strong and approximative methods are, for more or less the entire of the interior of ice sheet, insignificant. However, it seems that in regions of high sliding (outlet glaciers) and varying bottom topography, the velocity distributions differ by a significant amount. For instance, in the region of Nioghalvfjerdsbrae in the north-east, the approximative method can differ by about 100 m a$^{-1}$ from the by the strong method modeled surface velocity of $\sim$1300 m a$^{-1}$.

In conclusion, the strong and weak methods do not differ significantly in two dimensions, where the the weak methods does not show any oscillations for finer meshes. In three dimensions, the approximative method can differ by almost 10% compared to the strong method. To further investigate if there are any benefits from using the weak solution in a realistic three-dimensional case, a reliable manufactured solution with sliding must be used. Furthermore, for a stable formulation of the weak method, a modification to the elements or a stabilization as proposed in Verfürth (1986, 1991), should be implemented.
6 Synthesis

Climate variability can be seen as the result of the natural interaction between the climate system’s different subsystems: the atmosphere, hydrosphere, cryosphere, lithosphere and biosphere. Having sufficient knowledge about each subsystem is vital to understand the past and future states of the climate system, but the process of acquiring this knowledge is complicated by not being able to treat each system as isolated. The subsystems can and are in turn divided into further sublevels to focus on specific scenarios or set of processes. The focus of this thesis is on one of the subsystems of the cryosphere; glaciers and ice sheets.

In what follows below, my intention is to reflect upon what has evolved to become the common ground of Papers I to IV. The connecting theme, which was briefly introduced in Chapter 1 and further explored in Chapter 5, is that of conceptualization, verification and validation in glaciology.

Figure 6.1, which is inspired by the Sargent Circle (see Fig. 1.2), summarizes the interaction between the dynamical (here taken to mean the flow of ice described by the FS equations) and hydrological systems of ice sheets and glaciers. The figure attempts to show some of the complexity which can result from considering multiple systems, each with its own set of conceptual models representing the “reality” of interest, that also are interconnected through specific processes. In one direction, the hydrology affects the dynamical system through the water pressure at the bed, $p_w$, which affects the sliding rate. In the other direction, the basal velocity, $u_b$, has an impact on the character of the hydrological system. Such a connection exemplifies that the systems are not closed, and therefore cannot easily be considered as completely separate. This is something that is true of most natural large scale systems and complicates the procedure of verification and validation.

Below the individual topics of this thesis, glaciodynamics and glaciohydrology (left and right sides of Fig. 6.1), are considered in the above context, after which the link between these topics is discussed.

6.1 The hydrological cycle

Like any conceptual model, that of the subglacial hydrological system simplifies reality to some extent. The necessity to do this does not only come from the amount of small scale processes acting in the system, but also due to the need of applicability. For a model to be useful the amount of parameters from the system needed as input must reflect the availability of data. In addition, it is often beneficial to consider an idealized system to investigate its characteristics.

Using the equations for an R-channel with a set of parameters beneficial for channel maintenance, we argue that the large scale topology (>50km) of esker networks does not reflect long subglacial channels (Paper II). For suggested
Figure 6.1. Flowchart, adapted after the Sargent Circle (see Fig. 1.2) to show the glaciological subsystems considered in this thesis, with associated processes.

Assimilation of data (Model inversion)

Boundary condition

Hydro-sliding:

Frictional sliding:

Data:

uantitative comparison

Characteristics

large scale

Validation

Verification

Numerical implementation

Flow model

Subglacial hydrology model

Paleo-record

Hydro-sliding:

Frictional sliding:

Data:

n, u, T, geometry

Paper I, Paper II

Paper III, Paper IV

Black solid arrows indicate assessment activities, while dashed arrows are (numerical) links. The topics treated by the thesis papers are marked in light green (Paper I, Paper II) and light red (Paper III, Paper IV).
surface profiles of paleo-ice sheets, transient discharge scenarios indicate that channels would reach overburden pressure closer to the ice margin. In this case the output of the model, cannot be directly validated against the paleo-record, since it is not a conceptual model of esker genesis that is simulated. Rather, in the context of eskers being sediment fills of subglacial channels, the maintenance of channels as described by the conceptual model is unlikely. Therefore, the assessment of the validity of the conceptual model for R-channels must be put into the interpretative framework of the paleo-record. As such, the output limits the extent to which the eskers should be seen as a direct consequence of a channelized efficient subglacial hydrological system using the theory presented in Röthlisberger (1972). A steady-state scenario results in a profile of $p_w$ asymptotically approaching ice overburden pressure. However, even a small perturbation over time to the discharge through a channel causes the pressure to be at, or in very moderate scenarios close to, overburden.

The above does not preclude the existence of a channelized system, it only argues that the dendritic nature that would be a result of the relatively low-pressure is not viable at pressures close to overburden. The simplification of using a one-dimensional model explicitly disregards all interaction between channels or any other type of hydrological system that may exist, so the behavior of the system at overburden pressure can only be stated to be unknown. However, even if the current understanding is that the inefficient system is of a distributed character operating at high water pressures, it seems to be unlikely that such a system would exist at a relatively higher pressure than close to overburden. It is possible that future confirmation against the paleo-record necessitates an adaptation of the conceptual model of the efficient hydrological to an efficient distributed system or a channelized system that can be maintained for a greater distance away from the margin. However, validation against the paleo-record to explain contemporary observations of hydrological processes assumes that the GrIS is a valid analogue for paleo ice sheets like LIS and FIS. This may not necessarily be the case, since geographical and climate differences between the ice sheets could be such that they affect the dominant mode of hydrological system and possibly resulting effects on the landscape.

In Paper I, the attempt of a verification of the simple numerical implementation of the R-channel highlights the dual nature of the process; the verification of the numerical code and verification of the calculation (Thacker et al., 2004). Previous efforts had verified the calculation by convergence to a steady-state solution that did not reveal the issues with the implementation of the numerical code. Re-implementing the code points out the differences of the numerical output at steady-state, but also re-emphasizes the findings of Paper II and Meierbachtol et al. (2013) for the contemporary setting of the GrIS. However, to my knowledge, convergence to a good scenario representing the general time-transient case has not been shown.

Naturally, the above simplifications are much too severe to in any way be able to describe the general hydrological characteristics. In particular, to reflect the spatial characteristics, a two-dimensional conceptual model including the interaction between inefficient and efficient parts of the system is needed. Recent advances in numerical hydrological models have included this capability and ability to reproduce the general characteristics of data from investigated glaciers (e.g. Werder et al., 2013; de Fleurian et al., 2014) or conceptual switches from inefficient to efficient in an ice sheet setting (Schoof, 2010). Due to the spatial
complexity that can be expected at the bed, these models simplify the underlying conceptual models in order to homogenize the problem to some extent. The linked cavity system proposed by Kamb (1987) is replaced by a continuum formulation of a porous sheet for both the inefficient and efficient system (de Fleurian et al., 2014), while Werder et al. (2013) use the facets of an unstructured finite element mesh to represent the discrete channels with the porous inefficient sheet represented on the interior of the cells. In these cases the numerical model does not really represent a single conceptual model, but rather two models of inefficient and efficient systems designed to numerically interact. Schoof (2010) presents a mathematical formulation which describes both systems as part of the same conceptual channel model, but has a limitation in that these channels must be oriented at an angle towards the direction of ice flow, obviously restricting the topology of the network to a large extent. Assessing whether these conceptual models can be accurately confirmed by reality becomes, due to the complexity of the system and difficulty to retrieve spatially dense data, a question of general reproducible traits. Thus the validation of the numerical output to some extent justifies the conceptual model formulation, and its simplification is qualitative rather than quantitative.

The verification of such models is nearly impossible. Since a manufactured or analytical solution, even for some simple scenario, is not available, the procedure becomes one to investigate certain model variables under mesh refinement, e.g. how mass (amount of water in the domain) changes under steady discharge. In the case of Werder et al. (2013), which uses what can be deemed to be the more realistic discrete channels represented by facets, a refinement of the mesh leads to a change in the solution and also potential topology of the system.

On the paleo-ice sheet scale, it seems that the validation of physically based models like the ones mentioned above is still a future endeavor. Investigating the large scale topology of esker systems from a conceptual model perspective is thus, as of yet, limited to rather simple methods using hydraulic potential surfaces calculated by assuming the water pressure to be at some fraction of overburden pressure (Livingstone et al., 2015). Since the method is a flow routing algorithm, and not a physical model, the model output changes with mesh refinement and the procedure of verification in the sense of convergence is not valid. Nevertheless, by comparing the likely subglacial water pathways, Livingstone et al. (2015) find that the spatial distribution of eskers best correspond to short channels segments close to the ice margin. This is taken to indicate a time-transgressive genesis of eskers, which conforms well with the concluding discussion in Paper II.

6.2 Never go Full Stokes?

The answer to this, of course, depends on the application. Good reasons exist why we use the FS equations as the conceptual model describing the deformation of ice. By scaling arguments a quantitative assessment of the relative importance of specific terms in the Navier-Stokes equations can be made. In particular, using typical values for ice sheets, the contributions accounting for inertia can be neglected (e.g. Greve and Blatter, 2009).

Since no analytical solution to the FS equation for a general scenario has been found, using manufactured solutions to quantitatively verify the numerical implementation becomes the preferred choice. The experience gained in this thesis shows that this process can result in unexpected convergence rates depending
on parameter choices; using the solutions provided in Leng et al. (2013) superconvergence is observed in the three-dimensional case (Paper III, Gagliardini et al. (2013)) and poor convergence in the two-dimensional case (Section 4.1). This may be due to the singularities present in the manufactured solutions which result in large compensatory terms. If these are not numerically adjusted for the process typically results in the numerical solution not converging to the manufactured. Even in the case of numerical implementations of different conceptual models, it is possible that convergence rates are indistinguishable, perhaps giving a false impression of a correct convergence rather than highlighting the difference (Paper IV). Due to the expensive nature of the computations (even for (bi-)linear elements), a three-dimensional convergence study does not reach the resolution at which the (manufactured) compensation terms that are close to the singular points start to become large. This can more easily be seen in two dimensions, where my experience is that a step-wise mesh refinement at some point results in a numerical solution that seems to have converged to a solution that does not coincide with manufactured solution. This casts some shadow over the usability of this particular approach as a method to confidently assess the verification of the FS system, given the different results. It is possible that the best choice for verifying numerical implementations are inter-comparison and benchmark studies such as ISMIP-HOM (Pattyn et al., 2008) and MISMIP (Pattyn et al., 2012), even though the chosen scenarios are not guaranteed to highlight differences between models or detect potential errors and do not provide the means for a proper quantitative verification.

As a result, to evaluate stabilization methods and how inherent parameters affect the solution, comparison studies of different scenarios are the best choice (Paper III). The relative assessments in this case serves as a verification process, indicating proper choice of values for both commonly (GLS) and less commonly used stabilization methods.

The validation process for numerical ice sheet models is to compare the output of models to variables that can be observed on a relevant scale, e.g. surface velocity, \( u_s \), or specific events. To be able to close the system, boundary conditions must be specified. At the bed, a sliding law dependent on the subglacial hydrological system can be applied (option 2 in Fig. 6.1). However, a more common approach for large scale systems is to use a frictional sliding law by supplying a friction factor field, \( \beta^2 \) (option 1 in Fig. 6.1). The most suitable way of specifying this field is through model inversion or assimilation. That is, \( \beta^2 \) is numerically determined in a way to minimize the difference of the numerical and observed surface velocity. Such a procedure is not limited to specifying the friction factor, but can be applied to other parameters as well. Thus the validation procedure involves the adaptation of some parameter to fit observational data. This is natural due to the complexity of the system and its underdetermined nature, but also limits the quantitative assessment of the validation. For instance, using the same field for the friction factor on GrIS, but for different numerical implementations gives velocity results that can differ by \( \sim 10\% \) along coastal regions. However, using an inverse method to determine the \( \beta^2 \)-field would likely result in very similar surface velocities for both methods, but with different friction factors. In this case, the differences of the numerical methods (if not studied explicitly) would likely become evident for time-transient simulations where they would be difficult to quantify. A more reliable approach is to choose to fit parameters that are constants or have limited spatial variation. Otherwise, by tuning the parameters
of models to fit the observed “reality”, we run at risk defeating the purpose of validation.

6.3 A link to the past, present and future

The above sections, of course, only partially treat the considered systems. In addition, more (sub-)systems of the glaciological system can be defined depending on the variables of interest. However, as mentioned, most of such considered systems are hard to treat as closed. An exception to this could be laboratory studies of, for instance, ice deformation, in which the parameters can be somewhat controlled.

In Fig. 6.1, the link between the systems is primarily that through subglacial hydrology affecting the basal sliding of the glacier. A significant difference between the systems is the scale, both spatial and temporal, at which the processes operate. For instance, R-channels are discrete features that most likely don’t have a cross-sectional area much greater than \( \sim 10 \text{ m} \), but affect the discharge and water pressure in adjacent parts of the distributed system. On the other hand, on scales that are below the thickness of the ice, variations in basal velocities do not propagate to the surface velocity (Raymond et al., 1995). The latter, at least from an ice sheet scale perspective, supports the spatial homogenization of the distributed system (Section 6.1).

The main component of linking the systems would be that of a sliding law that depends on, among other things, the water pressure. Conceptual models of glacier sliding were considered early (e.g. Lliboutry, 1968; Kamb, 1970; Nye, 1970), with recent advances proposed in Schoof (2005), and similarly by Gagliardini et al. (2007) where the law was numerically investigated on a cavity scale. Coupling models of the subglacial hydrological systems and ice dynamics through such a sliding law has been done successfully (e.g. Pimentel and Flowers, 2010; de Fleurian et al., 2014; Hoffman and Price, 2014). Additionally, recent research has shown that a very good match between observed and modeled velocity patterns is possible using approximations of the FS equations and only global (domain-wide) tuning of parameters (Aschwanden et al., 2016), further highlighting the importance of the pressure-induced sliding in the link. Models such as these must surely be the way forward when aiming to predict the dynamics of ice sheet based on physical conceptual models.

Since the sliding law can be viewed as a conceptual model in itself, a process of verification and validation should be initiated. However, such a Sargent Circle suffers from similar complexities and difficulties as the hydrological part of Fig. 6.1, with the reproduction of characteristic features serving as validation and lacking scenarios for verification.

Nevertheless, increasing our understanding by better representing hydrological processes, and in particular how these interact with ice dynamics, remains one of the most important topics in numerical ice sheet models. Progress in this field would allow us to constrain the extent and timing of events of paleo ice sheets, improve our understanding of present and short variation of contemporary ice masses and how they interact with other parts of the climate system, and ultimately better predict future response to a changing climate.
6.4 Future perspectives

The use of numerical models to understand the climate system will most certainly increase in the near future. Specific processes will become more accurately represented as model complexity increases, and at the same time the interface to use advanced numerical methods is directed towards end users. This comes with both great opportunity and challenge. Being able to use a state-of-the-art model for a specific purpose without having profound knowledge about the low-level implementation details can be of benefit to a multitude of studies. However, as an end user it is necessary to recognize both the potential and limitations of such a model.

With this in mind, a future direction in numerical ice sheet modeling is to continue to evaluate the methods used. The stabilization methods implemented for the FS equations in this thesis need to be assessed for more general scenarios to confirm the suitability of each method. Furthermore, recent work on non-linear stabilization methods, developed specifically for singular power-law fluids (Hirn, 2012), presents a possibility to further improve the stability properties for the FS equations in the context of ice sheets.

The impact of sliding on the total dynamical behavior of glaciers and ice sheets emphasizes the importance of representing this process as accurately as possible in numerical models.

For coarser parts of meshes, used on the interior of ice sheets or where velocity changes are small, it is likely appropriate to use the method of a strongly imposed impenetrability condition. However, studying and comparing the strong and weak methods for refined meshes in areas of high sliding could be of benefit. A possible scenario is that a combination of imposing the boundary condition strongly where mesh resolution is coarse, and weakly in refined parts, increases accuracy or stability. Since, at least at present, it seems that no solution is available for a true verification procedure, a suitable study would be to perform a benchmark experiment such as the Marine Ice Sheet Model Intercomparison Project (Pattyn et al., 2012).

An approach to increasing our understanding of the topology of subglacial networks would be to use a more process oriented model. For instance, the model presented in Werder et al. (2013), can be used to simulate the subglacial network in various settings of paleo-ice sheets. The output would, using a similar framework as in Livingstone et al. (2015), be compared to the mapped topology of esker systems. Given that the problem is severely underdetermined, the focus of such a study is not to attempt to accurately recreate the topology. Rather, it is to assess to what degree one can explain the topology given the underlying physical processes represented in the model.
Christian Helanow
7 Conclusions

Ultimately, the goal of robust predictions of the climate system’s response to a changing climate, numerical glacier and ice sheet models must be able to interact with other components of the climate system on a multitude of spatial and temporal scales. Ideally, the accuracy and limitations of dynamically evolving glacier system should be assessed. The difficulty of this lies in the near impossibility of delimiting large scale natural systems in a way that is suitable for a true verification and validation process. This, of course, is not something unique for glaciers and ice sheets. The challenge becomes to uphold the integrity that should be inherent in a system of verification and validation.

By studying numerical implementation of well established conceptual mathematical models, I have in this thesis contributed to both evaluating various numerical methods in a glaciological context and by considering the glacial hydrological system from a cross-disciplinary point of view helped to constrain the interpretative framework in paleo- and observational glaciology. Specifically, the main conclusions can be summarized as:

- **Limited extent of low-pressure channelized subglacial hydrological system.** R-channels, thought to be the main mechanism in efficient subglacial systems, are difficult to maintain far away from the ice margin (>50 km) under both a GrIS and proposed LIS surface profiles. In particular, the sensitivity to time transient processes, such as a varying discharge through the channel, makes the interpretation of a steady-state channel misleading. Channels being able to accommodate variations in discharge through for instance the input of a moulin, will likely be limited to the near-margin area. This has implications for both contemporary observational glaciology and paleo-glaciology. Efficient subglacial transport of water from interior regions may be the result of a different subglacial hydrological system, or a channelized system not well represented by the mathematical models in e.g. Röthlisberger (1972); Shreve (1972). As a result, esker genesis is more likely to be of time-transgressive nature, and not a topological reflection of a large scale, simultaneously active system of R-channels.

- **Incorrect application of stabilization methods affect surface velocities.** Using stabilizing methods in equal-order (bi-)linear FEM, can easily result in over-stabilization for certain methods, with oscillations in surface velocities as a result. The GLS method often used when stabilizing the linear Stokes (and FS) equations, is a suitable choice for ice sheet simulations if the parameter from Franca and Frey (1992) is used and the cell size parameter is defined as the minimum edge of the cell. The latter is due to the process of mesh extrusion used for ice sheets. Not defining the cell size in this way, quickly leads to an over-stabilized solution. In addition, the **Interior**
The Penalty method (Burman and Hansbo, 2006) shows less sensitivity to the choice of parameter than the GLS method. Since the choice of parameter can be situation dependent, this is a benefit of the former method.

- **Implementing impenetrability strongly is good.** The in glaciology common way of implementing impenetrability at the bed has proven to be a stable and good method in almost all considered cases. An approximate weak method, although simple to implement, results in surface velocity differences in the order of 1% to 5% in areas of high sliding or irregular bottom topography. A weak implementation of the impenetrability condition gives results comparable to the strong method in two dimensions, but is unstable in three dimensions. However, for certain extreme case scenarios (e.g. rotating cylinders) with close to free slip, the weak method converges to the analytical solution under mesh refinement while the strong method can diverge for non-uniform refinement.

- **The validity of manufactured solutions used in glaciology is parameter dependent.** Depending on what parameters are chosen for the manufactured solutions presented in Sargent and Fastook (2010) and Leng et al. (2013), convergence under mesh refinement is not possible. Numerically modifying the solutions to avoid high values can result in proper convergence for fairly coarse meshes.

The above conclusions are specific to the detailed studies presented in this thesis. In addition to these, during the thesis work a general opinion of a somewhat philosophical character emerged. If it is a conclusion as such or not, I do not know. It stems from asking myself the question: “To what degree do numerical ice sheet models represent reality under a specific scenario?” Based on the discussion in Chapter 6, I cannot supply a rigorous quantitative answer. The complexity of the system makes it only natural that model tuning and parameter adaptation is necessary to generate results that can be validated against observations, but I can’t help but echo the statement of a 25 year old paper regarding the verification, validation and confirmation of Earth System Models: “Models can only be evaluated in relative terms, and their predictive value is always open to question. The primary value of models is heuristic.” (Oreskes et al., 1994).

With this said, the steady progress made by the modeling community is vital to increase the knowledge of the climate system. Models become more and more complex, being able to reproduce characteristics of the system in better ways. At the same time, computational power continues to increase, allowing more realistic simulations and input and assimilation of data. Progress is constantly made in conceptual mathematical models.

As a final thought, I believe that the strength of conceptual and numerical models are when they are used in a manner investigative of the processes or system considered, highlighting limitations in the model or our understanding of those governing processes, rather than as a tool tuned to fit a specific need.
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Basal boundary conditions, stability and verification


Mercer, J. H., 1978. West Antarctic ice sheet and CO2 greenhouse effect: a threat of


Appendix A: Nomenclature

In general, a $d$-dimensional domain will be denoted by $\Omega \subset \mathbb{R}^d$ and its boundary by $\Gamma$. In the main text, the index notation is commonly used, i.e. $A_{ij}$ is the element on the $i$:th row and $j$:th column in the matrix representation of the tensor $A$. The (Einstein) summation convention is also used for any repeated indices, so called dummy indices, so that $A_{ii} = \sum_{i=1}^{d} A_{ii} = A_{11} + A_{22} + \cdots + A_{dd}$. For instance, a matrix acting on a vector $A \cdot a = b$ can be written as $A_{ij}a_j = \sum_{i=1}^{d} A_{ij}a_j = b_i$.

Mathematical notation

- $C^k(\Omega)$ Space of $k$ times continuously differentiable functions on $\Omega$
- $L^2(\Omega)$ Space of Lebesgue square-integrable functions on the domain $\Omega$: $L^2(\Omega) = \{v : \int_{\Omega} |v|^2 \, dx < \infty\}$
- $H^1(\Omega)$ Functions whose first order derivatives are in $L^2(\Omega)$: $\{v : \int_{\Omega} (\nabla v \cdot \nabla v + vv) \, dx < \infty\}$
- $P_k(\Omega)$ Space of polynomials of degree $k$ on $\Omega$
- $\langle u, v \rangle_{L^2}$ Inner product on the space $L^2(\Omega)$: $\int_{\Omega} uv \, dx$
- $\|u\|_{L^2}$ Norm on the space $L^2(\Omega)$: $\int_{\Omega} uv \, dx$
- $\langle u, v \rangle_{H^1}$ Inner product on the space $H^1(\Omega)$: $\langle \nabla u, \nabla v \rangle_{L^2} + \langle u, v \rangle_{L^2}$
- $\|u\|_{H^1}$ Norm on the space $H^1(\Omega)$: $\left(\|\nabla v\|_{L^2}^2 + \|v\|_{L^2}^2\right)^{\frac{1}{2}}$
- $u \cdot v$ Dot product of two vectors: $u_i v_i = u_1 v_1 + \cdots + u_d v_d$
- $A : B$ Double contraction: For $A, B \in \mathbb{R}^{n \times m}$, $A : B = A_{ij}B_{ij}$
- $A^T$ Transpose of matrix $A$: $(A^T)_{ij} = A_{ji}$
- $\text{tr} A$ Trace: $\text{tr} A = A_{ii} = A_{11} + \cdots + A_{dd}$ for $A \in \mathbb{R}^{d \times d}$
- $\ker A$ Kernel (nullspace) of $A$: $\{u : Au = 0\}$
- $\nabla u$ Gradient: $\frac{\partial u}{\partial x}$ if $u \in \mathbb{R}$, and $\frac{\partial u}{\partial x_j}$ if $u \in \mathbb{R}^d$
- $\nabla \cdot u$ Divergence: $\frac{\partial u_i}{\partial x_i}$, if $u \in \mathbb{R}^d$, and $\frac{\partial u_{ij}}{\partial x_j}$ if $u \in \mathbb{R}^{d \times d}$
- $\frac{D}{Dt}$ Total derivative: $\frac{\partial (\cdot)}{\partial t} + u \cdot \nabla (\cdot)$
- $u^\parallel$ Tangential component of $u$: $u^\parallel = u - (u \cdot n)n$
- $u|\Gamma$ Restriction of $u$ to the set $\Gamma$
### List of symbols

- **A**: Deformation rate factor in Glen’s flow law
- **\( \dot{a} \)**: Accumulation rate
- **\( \beta^2 \)**: Coefficient of basal drag
- **\( c_{dw} \)**: Constant (depending on friction) in Darcy-Weisbach relation
- **\( c_t \)**: Clapeyron slope, pressure melting coefficient
- **\( c_w \)**: Specific heat capacity of water
- **\( d \)**: Dimension, \( d = 2 \) or 3
- **\( D \)**: Strain rate tensor: \( \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T) \)
- **\( D_e \)**: Effective strain: \( \sqrt{\frac{1}{2} \mathbf{D} : \mathbf{D}} = \sqrt{\frac{1}{2} D_{ij}D_{ij}} \)
- **\( \dot{\varepsilon} \)**: Strain rate
- **\( \eta \)**: Viscosity
- **\( g \)**: Gravitational acceleration
- **\( \Gamma \)**: Boundary of the domain, \( \Omega \). Subscripts can refer to parts of the boundary of an ice sheet as the free surface \( (\Gamma_s) \), frozen bed \( (\Gamma_f) \), subject to sliding \( (\Gamma_{sl}) \) or submerged into water \( (\Gamma_w) \)
- **\( h \)**: Mesh size (maximum element size of mesh)
- **\( h_w \)**: Height of water column
- **\( \mathbf{I} \)**: Identity tensor: \( \delta_{ij} \)
- **\( K \)**: Triangle or tetrahedron belonging to a mesh, \( \mathcal{T}_h \)
- **\( L \)**: Latent heat of melt
- **\( \mathbf{n} \)**: Unit normal, outward pointing if on the boundary of a domain
- **\( N \)**: Effective pressure: \( p_i - p_w \)
- **\( n \)**: Glen’s parameter: \( n = 3 \)
- **\( \mathbf{n}_h \)**: Discrete unit normal (finite element approximation of \( \mathbf{n} \))
- **\( \Omega \)**: Domain under consideration
- **\( p \)**: Pressure, \( -\frac{1}{3} \text{tr} \mathbf{T} \)
- **\( p^\circ \)**: Pressure for manufactured solutions
- **\( p_h \)**: Finite element approximation of \( p \)
- **\( p_i \)**: Ice pressure
- **\( p_w \)**: Water pressure
- **\( \Phi \)**: Hydraulic potential: \( p_i - N + \rho_w g z_b \)
- **\( \Phi_0 \)**: Background potential (arbitrary)
- **\( \phi_q \)**: Energy sink due to heat flux
- **\( q \)**: Flux (water)
- **\( Q \)**: Discharge in channel
- **\( q \)**: Test function, pressure
- **\( \rho \)**: Density
- **\( \rho_i \)**: Density of ice
Basal boundary conditions, stability and verification

$\rho_w$  Density of water  
$s$  Along conduit coordinate  
$S$  Deviatoric stress tensor: $S = T + pI$  
$S$  Channel cross section  
$\sigma$  Dominant stress  
$t$  Time  
$T$  Cauchy stress tensor  
$\mathcal{T}_h$  Mesh tessellation of a domain  
$u$  Velocity  
$u_b$  Basal (sliding) velocity  
$u_e$  Internal energy  
$u_h$  Finite element approximation of $u$  
$\tilde{u}$  Velocity for manufactured solutions, $\tilde{u} = (\tilde{u}, \tilde{v}, \tilde{w})$  
$v$  Test function, velocity  
$z_b$  Elevation of the ice bed  
$z_s$  Elevation of the (free) ice surface

**Abbreviations**

FD  Finite difference method  
FEM  Finite element method  
FIS  Fennoscandian Ice Sheet  
FS  Full Stokes (equations)  
GAP  Greenland Analogue Project  
GLS  Galerkin Least-Squares  
GrIS  Greenland Ice Sheet  
LGM  Last glacial maximum  
LIS  Laurentide ice sheet  
PDE  Partial differential equation(s)  
V&V  Verification and Validation