TENSOR PRODUCTS AND HIGHER AUSLANDER-REITEN THEORY

This licentiate dissertation consists of two articles:


INTRODUCTION

Higher Auslander-Reiten theory is a generalisation of the techniques developed by Auslander and Reiten in the late 1970’s to study modules over finite-dimensional algebras. Broadly stated, the problem is the following: let $\Lambda$ be a finite-dimensional algebra over a field. Can we describe the category $\text{mod} \, \Lambda$ of finitely generated $\Lambda$-modules? In general the answer is no, but there are some classes of algebras for which it is possible to have complete control over both the indecomposable modules and the morphism spaces between them. One class which is particularly suitable to study is that of hereditary algebras, that is algebras over which every submodule of a projective module is projective. The methods and the viewpoint of Auslander-Reiten theory involve making strong use of the powerful machinery of homological algebra in order to study module categories, and in this language one can say that hereditary algebras are just algebras of global dimension at most one. In the early 2000’s, Iyama introduced a version of Auslander-Reiten theory for algebras of higher global dimension, imitating the constructions that work for hereditary algebras ([Iya07b], [Iya07a], [Iya08]). One possibility is to restrict one’s attention to a subcategory (called $d$-cluster tilting subcategory) of $\text{mod} \, \Lambda$ with nice homological properties, and try to describe this subcategory completely. Here if one takes $\text{gl.} \dim \Lambda = d$, then the $d$-cluster tilting subcategory behaves in many ways like the module category of a hereditary algebra. Since its inception, this higher-dimensional version of the theory has been very fruitful, and has found applications and connections in many different contexts.

In this dissertation we consider the following problem: let $A$ and $B$ be algebras over a field $k$, and let us consider the algebra $\Lambda = A \otimes_k B$. If $k$ is perfect, then all the homological constructions in $\text{mod} \, \Lambda$ are naturally coming from $\text{mod} \, A$ and $\text{mod} \, B$, and moreover global dimension is additive over tensor products. Thus it makes sense to study $n$-Auslander-Reiten theory on $A$, $m$-Auslander-Reiten theory on $B$, and connect them to $(n+m)$-Auslander-Reiten theory on $\Lambda$. A step in this direction is made in [HI11], where Herschend and Iyama study how $d$-representation finiteness behaves with respect to tensor products (an algebra is $d$-representation finite if it admits a $d$-cluster tilting subcategory which has finitely many indecomposable objects). They find explicit conditions on $A$ and $B$ that are equivalent to
A being \((n + m)\)-representation finite, provided that \(A\) and \(B\) are \(n\)- respectively \(m\)-representation finite.

In the first article we investigate the relationship between the \(d\)-almost split sequences in \(A\), \(B\) and \(\Lambda\). These are some particular exact sequences that encode much of the information of the \(d\)-cluster tilting subcategory. We explicitly provide formulas for the \((n + m)\)-almost split sequences over \(\Lambda\) in case \(A\) is \(n\)-representation finite, \(B\) is \(m\)-representation finite, and \(\Lambda\) is \((n + m)\)-representation finite. To do this, we realise every such sequence as the mapping cone of a suitable chain map, and verify that the tensor products of chain maps gives the correct result. It is worth remarking that the full assumptions about \(\Lambda\) being \((n + m)\)-representation finite are never used in the construction.

In the second article we try to approach the problem from a different angle, and look for a weaker condition than that of \(d\)-representation finiteness which might be preserved by taking tensor products. We prove that the notion of \(d\)-completeness, introduced in [Iya11], has this property. A \(d\)-complete algebra has a subcategory which locally has nice homological properties, but which roughly speaking fails to satisfy a boundary condition (i.e. contain all projective modules), so it is not \(d\)-cluster tilting. However, it is \(d\)-cluster tilting in a suitable exact subcategory of \(\text{mod}\ \Lambda\). The notion of \(d\)-completeness was introduced as a weakening of \(d\)-representation finiteness that is preserved under taking so-called higher Auslander algebras. Our result is thus in the same spirit, proving the same kind of property for tensor products.

The proof builds on the results of the first paper, namely on the contruction of \((n + m)\)-almost split sequences in \(\text{mod}\ \Lambda\). This construction still works in the weaker setting, and this has two consequences. On the one hand, it guarantees that \(\text{mod}\ \Lambda\) has the correct local homological properties. On the other hand, it allows us to prove that a certain module \(T\) is tilting. This tilting module plays the role of projective generator in the exact category \(T^\perp\). Inside this category we can prove existence of an \((n + m)\)-cluster tilting subcategory, as is required in the definition of \((n + m)\)-completeness.

It is worth noting that this result gives a new way of constructing \(d\)-complete algebras for any \(d\) (for instance starting from a hereditary representation finite algebra and tensoring it with itself).

ACKNOWLEDGMENTS

I would like to thank my supervisor Martin Herschend for the invaluable help and sustain he has given me throughout the production of this work.

REFERENCES

Tensor products of higher almost split sequences

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A R T I C L E   I N F O

Article history:
Received 8 July 2015
Received in revised form 22
February 2016
Available online 4 August 2016
Communicated by S. Koenig

A B S T R A C T

We investigate how the higher almost split sequences over a tensor product of algebras are related to those over each factor. Herschend and Iyama give in [6] a criterion for when the tensor product of an n-representation finite algebra and an m-representation finite algebra is (n + m)-representation finite. In this case we give a complete description of the higher almost split sequences over the tensor product by expressing every higher almost split sequence as the mapping cone of a suitable chain map and using a natural notion of tensor product for chain maps.

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1. Introduction and conventions

In the context of Auslander–Reiten theory one can study almost split sequences of modules over a finite-dimensional algebra A. These are certain short exact sequences

$$0 \rightarrow M \rightarrow N \rightarrow L \rightarrow 0$$

such that M and L are indecomposable, and it turns out that every nonprojective indecomposable module over A appears as the last term of such a sequence (and every noninjective indecomposable appears as the first term). Moreover, such sequences are determined up to isomorphism by either the first or the last term (see for reference [2]). One can do a similar construction in the context of higher dimensional Auslander–Reiten theory, at the cost of restricting to a suitable subcategory C of mod A that contains all injectives and all projectives. Then one gets longer so called n-almost split sequences

$$0 \rightarrow M \rightarrow X_1 \rightarrow \cdots \rightarrow X_n \rightarrow L \rightarrow 0$$

in C, and again every nonprojective module in C appears at the end of such a sequence and every noninjective at the start of one. Again, these sequences are determined by their first or last term (see [8,9]). One of the most basic cases where such a situation appears is when A is n-representation finite (cf. [6,8]).
**Definition.** Let $A$ be a finite-dimensional $k$-algebra, and let $n \in \mathbb{Z}_{>0}$. An $n$-cluster tilting module for $A$ is a module $M_A \in \text{mod} \ A$ such that

$$\text{add } M_A = \{ X \in \text{mod} \ A \mid \text{Ext}^i_A(M_A, X) = 0 \text{ for every } 0 < i < n \} = \{ X \in \text{mod} \ A \mid \text{Ext}^i_A(X, M_A) = 0 \text{ for every } 0 < i < n \}.$$ 

We say that $A$ is $n$-representation finite if $\text{gl. dim } A \leq n$ and there exists an $n$-cluster tilting module for $A$. Then $\text{gl. dim } A = 0$ or $\text{gl. dim } A = n$.

For such algebras it is known that $\text{add } M_A$ is a subcategory of $\text{mod } A$ that admits $n$-almost split sequences. We call $D$ the functor $D = \text{Hom}_k(-, k) : \text{mod } A \to A \text{mod}$. The (higher) Auslander–Reiten translations $\tau_n, \tau_n^-$ are defined as follows:

$$\tau_n = D \text{Ext}^n_A(-, A) : \text{mod } A \to \text{mod } A$$

$$\tau_n^- = \text{Ext}^n_A(DA, -) : \text{mod } A \to \text{mod } A.$$ 

It is immediate from this definition that

$$\tau_n A = 0 = \tau_n^- DA.$$ 

These higher Auslander–Reiten translations behave similarly to the classical ones.

**Theorem.** Let $A$ be an $n$-representation finite $k$-algebra. Let $P_1, \ldots, P_a$ be nonisomorphic representatives of the isomorphism classes of indecomposable projective right $A$-modules, and $I_1, \ldots, I_a$ the corresponding indecomposable injective modules. Then:

1. There exist positive integers $l_1, \ldots, l_a$ and a permutation $\sigma \in S_a$ (the symmetric group over $a$ elements) such that $P_i \cong \tau_{l_i - 1}^- I_{\sigma(i)}$ for every $i$.
2. There exists a unique (up to isomorphism) basic $n$-cluster tilting module $M_A$, which is given by

$$M_A = \bigoplus_{i=1}^a \bigoplus_{j=0}^{l_i - 1} \tau_n^j I_{\sigma(i)}.$$ 

3. The Auslander–Reiten translations induce mutually quasi-inverse equivalences

$$\text{add}(M_A/P) \xrightarrow{\tau_n^-} \text{add}(M_A/I)$$ 

where $P = \bigoplus_{i=1}^a P_i$ and $I = \bigoplus_{i=1}^a I_i$.

**Proof.** See [9, 1.3(b)]. □

From the last point it follows in particular that the $n$-cluster tilting module can be equally described by

$$M_A = \bigoplus_{i=1}^a \bigoplus_{j=0}^{l_i - 1} \tau_n^{-j} P_i.$$
**Definition ([6]).** An $n$-representation finite algebra $A$ is said to be $l$-homogeneous if with the above notation we have $l_1 = \cdots = l_a = l$.

If $A$ is $n$-representation finite, the category $\text{add} M_A$ decomposes into “slices”, in the sense that every $X \in \text{add} M_A$ can be written uniquely as $X \cong \bigoplus_{i \geq 0} X_i$, where each $X_i \in \text{add} \tau^{-i} A$. If $A$ is $l$-homogeneous, then every slice $\text{add} \tau^{-j} A$, where $0 \leq j \leq l - 1$, has the same number of isomorphism classes of indecomposables.

We denote by $\mathcal{D}^b(\text{mod} A)$ the bounded derived category of $\text{mod} A$, and denote by $\varepsilon : \text{mod} A \to \mathcal{D}^b(\text{mod} A)$ the natural inclusion. The Nakayama functors

$$
\nu = - \frac{L}{A} DA \cong D \circ R \text{Hom}_A(-, A) : \mathcal{D}^b(\text{mod} A) \to \mathcal{D}^b(\text{mod} A)
$$

$$
\nu^{-1} = R \text{Hom}_{A^{op}}(D-, A) \cong R \text{Hom}_A(DA, -) : \mathcal{D}^b(\text{mod} A) \to \mathcal{D}^b(\text{mod} A)
$$

are quasi-inverse equivalences that make the diagram

$$
\xymatrix{ \mathcal{D}^b(\text{mod} A) \ar[rr]^-{\nu} \ar[u]^{\nu^{-1}} & & \mathcal{D}^b(\text{mod} A) \ar[u]_{\nu^{-1}} \ar[d]_{\varepsilon} \\
\mathcal{K}^b(\text{proj} A) \ar[u]_{\nu} \ar[r]^-{\nu^{-1}} & \mathcal{K}^b(\text{inj} A) \ar[u]_{\varepsilon}
}
$$

commute ($\mathcal{K}^b$ denotes the bounded homotopy category). If $A$ is $n$-representation finite, there is a natural isomorphism of functors $\text{mod} A \to \text{mod} A$

$$
\tau_n \cong H_0 \circ \nu_n \circ \varepsilon
$$

where $\nu_n = \nu \circ [-n]$. For every $i$ and for every $0 \leq j \leq l_i$, we have that $\varepsilon \tau_n^{-j} P_i = \nu_n^{-j} \varepsilon P_i$. From now on, explicit mentions of $\varepsilon$ will be omitted for simplicity.

The definition of higher almost split sequences that is convenient to take is the following:

**Definition.** Let $A$ be an $n$-representation finite $k$-algebra, and let $M_A$ be the corresponding basic $n$-cluster tilting module. Let

$$
0 \longrightarrow C_{n+1} \stackrel{f_{n+1}}{\longrightarrow} C_n \longrightarrow \cdots \longrightarrow C_1 \stackrel{f_1}{\longrightarrow} C_0 \longrightarrow 0
$$

be an exact sequence with terms in $\text{add} M_A$. Such a sequence is an $n$-almost split sequence if the following holds:

1. For every $i$, we have $f_i \in \text{rad}(C_i, C_{i-1})$.
2. The modules $C_{n+1}$ and $C_0$ are indecomposable.
3. The sequence of functors from $\text{add} M_A$ to $k$ mod

$$
0 \longrightarrow \text{Hom}_A(-, C_{n+1}) \stackrel{f_{n+1} \circ -}{\longrightarrow} \text{Hom}_A(-, C_n) \longrightarrow \cdots \\
\cdots \longrightarrow \text{Hom}_A(-, C_1) \stackrel{f_1 \circ -}{\longrightarrow} \text{rad}_A(-, C_0) \longrightarrow 0
$$

is exact (i.e. it is an exact sequence when evaluated at any $X \in \text{add} M_A$).
Theorem. Let $A$ be an $n$-representation finite $k$-algebra, and let $M_A$ be the corresponding basic $n$-cluster tilting module. Then we have the following:

(1) For every indecomposable nonprojective module $N \in \text{add} M_A$ there exists an $n$-almost split sequence

$$0 \longrightarrow \tau_n N \longrightarrow \cdots \longrightarrow N \longrightarrow 0,$$

and any $n$-almost split sequence whose last term is $N$ is isomorphic to this one.

(2) For every indecomposable noninjective module $M \in \text{add} M_A$ there exists an $n$-almost split sequence

$$0 \longrightarrow M \longrightarrow \cdots \longrightarrow \tau_m M \longrightarrow 0,$$

and any $n$-almost split sequence whose first term is $M$ is isomorphic to this one.

Proof. See [7, Theorem 3.3.1]. Notice that the term “$n$-cluster tilting subcategory” has replaced “$(n-1)$-orthogonal subcategory” in recent literature. \(\square\)

Remark. The usual, more general definition of $n$-almost split sequences that one takes requires that the condition dual to (3) holds as well (as in [9, Definition 2.1]). However, in the case we are considering (module categories over an $n$-representation finite algebra), the two definitions are equivalent (see [8, Proposition 2.10]).

In their paper [6], Herschend and Iyama construct a class of examples of $n$-representation finite algebras via tensor products, in the setting where the ground field $k$ is perfect. Namely, they find a necessary and sufficient condition (being $l$-homogeneous for the same value of $l$) under which the tensor product $A \otimes B = A \otimes_k B$ of an $n$-representation finite algebra $A$ with an $m$-representation finite algebra $B$ is $(n+m)$-representation finite. They also show that in this case every indecomposable of $\text{add} M_{A \otimes B}$ is of the form $L \otimes N$ for some indecomposables $L \in \text{add} M_A$ and $N \in \text{add} M_B$, and that $\tau_{n+m}^+ L \otimes N \cong \tau_n^+ L \otimes \tau_m^+ N$. Moreover, in this case the algebra $A \otimes B$ is itself $l$-homogeneous.

Remark. Even though not explicitly stated in [6], necessity of the condition comes from the following observation. Let

$$M = \bigoplus_{i,j} \bigoplus_d \tau_{n+m}^{-d} P_i \otimes Q_j$$

where $P_i$ and $Q_j$ run over the indecomposable summands of $A, B$ respectively. If $A$ and $B$ are not $l$-homogeneous for the same value of $l$, then $M$ has either an indecomposable summand of the form $S = L \otimes J$ where $J$ is injective and $L$ is not, or one of the form $S = I \otimes N$ where $I$ is injective and $N$ is not. On the other hand, if $A \otimes B$ is $(n+m)$-representation finite, then $M$ is an $(n+m)$-cluster tilting module, and hence the indecomposable injective $A \otimes B$-modules are precisely those indecomposable direct summands $I \otimes J$ of $M$ such that $\tau_{n+m}^- I \otimes J = 0$. Thus we reach a contradiction, since $S$ is not injective, but $\tau_{n+m}^- S = 0$.

In this setting, if

$$0 \rightarrow L \otimes N \rightarrow \cdots \rightarrow \tau_{n+m}^- L \otimes N \rightarrow 0$$

is an $(n+m)$-almost split sequence, then $\tau_{n+m}^- L \otimes N \cong \tau_n^- L \otimes \tau_m^- N$. On the other hand, there are $n$- respectively $m$-almost split sequences
\[ 0 \to L \to \cdots \to \tau_n L \to 0 \]

and
\[ 0 \to N \to \cdots \to \tau_m N \to 0, \]

so the starting and ending points behave well with respect to tensor products. It is then a natural question to describe the relation between the sequence starting in \( L \otimes N \) and the sequences starting in \( L \) and \( N \). This is the question that we address, and we answer it in the setting where \( A \) is \( n \)-representation finite, \( l \)-homogeneous and \( B \) is \( m \)-representation finite, \( l \)-homogeneous.

For a precise statement, we need some more notation. For a preadditive category \( \mathcal{A} \), we denote by \( \mathcal{C}(\mathcal{A}) \) the category of chain complexes of \( \mathcal{A} \). If \( A \) is a \( k \)-algebra and \( \mathcal{A} \) is a full subcategory of \( \text{mod} \, A \), we denote by \( \mathcal{C}_r(\mathcal{A}) \) the full subcategory of \( \mathcal{C}(\mathcal{A}) \) whose objects are chain complexes where the differentials are radical morphisms (i.e. \( d_i \in \text{rad}(A_i, A_{i-1}) \) for every \( i \)). Let \( \mathcal{B} \) be a full subcategory of \( \mathcal{C}(\mathcal{A}) \). We denote by \( \text{Mor}(\mathcal{B}) \) the category whose objects are chain maps \( A_\bullet \to B_\bullet \) for \( A_\bullet, B_\bullet \in \mathcal{B} \), and whose morphisms are the obvious commutative diagrams. We denote by \( \text{Mor}_r(\mathcal{B}) \) the full subcategory of \( \text{Mor}(\mathcal{B}) \) whose objects are radical chain maps \( A_\bullet \to B_\bullet \) for \( A_\bullet, B_\bullet \in \mathcal{B} \) (meaning that for every \( i \) the map \( A_i \to B_i \) is radical). We often view finite (exact) sequences as bounded chain complexes, and unless otherwise specified the degree-0 term is the rightmost nonzero term. With this point of view in mind, we denote by \( \mathcal{B}^n \) the full subcategory of \( \mathcal{B} \) whose objects are complexes \( C_\bullet \) satisfying \( C_i = 0 \) for every \( i < 0 \) and \( i > n \).

**Definition.** Let \( A \) be an \( n \)-representation finite \( k \)-algebra, and let \( i \in \mathbb{Z}_{\geq 0} \). Let \( L \in \text{add} \, M_A \) be indecomposable noninjective, and let \( C_\bullet \) be the corresponding \( n \)-almost split sequence. Then we say that \( C_\bullet \) *starts in slice* \( i \) if \( L \in \text{add} \, \tau_n^{-1} A \).

We denote by \( \text{Cone} \) the mapping cone (see Definition 2.1). We use the symbol \( \otimes^T \) for the usual “total tensor product” bifunctor
\[ - \otimes^T - : \mathcal{C}(\text{mod} \, A) \times \mathcal{C}(\text{mod} \, B) \to \mathcal{C}(\text{mod} \, A \otimes B) \]
induced by \( \otimes \) (see Section 3 for details). Our main result is the following:

**Theorem 1.1.** Let \( k \) be a perfect field. Let \( A \) and \( B \) be \( n \)- respectively \( m \)-representation finite \( k \)-algebras. Suppose that \( A \) and \( B \) are \( l \)-homogeneous for some common \( l \). Let \( \varphi \in \text{Mor}_r(\mathcal{C}_r(\text{add} \, M_A)) \) and let \( \psi \in \text{Mor}_r(\mathcal{C}_r(\text{add} \, M_B)) \). Suppose that \( \text{Cone}(\varphi) \) and \( \text{Cone}(\psi) \) are \( n \)- respectively \( m \)-almost split sequences starting in slice \( i \) for some common \( i \geq 0 \). Then \( \text{Cone}(\varphi \otimes^T \psi) \) is an \((n + m)\)-almost split sequence.

**Remark.** In Theorem 2.4 we show that every \((n + m)\)-almost split sequence is isomorphic to \( \text{Cone}(\varphi) \) for some suitable \( \varphi \), so all the \((n + m)\)-almost split sequences in \( \text{mod} \, (A \otimes B) \) are obtained by this procedure.

**Remark.** The sequence \( \text{Cone}(\varphi \otimes^T \psi) \) starts in slice \( i \). This is because we have (see [6])
\[ \tau_n^{-i} A \otimes \tau_m^{-i} B = \tau_{n+m}^{-i} A \otimes B. \]

On the other hand, if \( L \in \text{add} \, \tau_n^{-i} A \) and \( N \in \text{add} \, \tau_m^{-j} B \) with \( i \neq j \), then \( L \otimes N \notin \text{add} \, M_{A \otimes B} \), so there is in principle no \((n + m)\)-almost split sequence starting in \( L \otimes N \).

**Remark.** If we drop the condition guaranteeing that \( A \otimes B \) is \((n + m)\)-representation finite, then we can perform the same construction, and we still get some sequences in \( \text{mod} \, (A \otimes B) \) which retain some interesting properties. Similarly, one could tensor sequences that do not start in the same slice. This is a possible topic for future investigation.
In Section 2 we show that every $n$-almost split sequence over an $n$-representation finite algebra is isomorphic to the mapping cone of a suitable chain map of complexes, then we relate the property of being $n$-almost split to a property of the chain map. In Section 3 we define the functor $\otimes^T$ that we have mentioned above, and we prove the main theorem. In Section 4 we compute an example where we explicitly construct a 2-almost split sequence and a 3-almost split sequence starting from a 1-representation finite algebra.

Conventions. Throughout this paper, we denote by $k$ a perfect field (cf. [6]). All $k$-algebras are associative and unitary. For a ring $R$, we denote by $\text{mod } R$ (resp. $R\text{mod}$) the category of finitely generated right (resp. left) $R$-modules. Unless otherwise specified, modules are right modules. Subcategory means full subcategory. For a $k$-algebra $A$ we denote by $\text{rad}_A(-, -)$ the subfunctor of $\text{Hom}_A(-, -)$ defined by

$$\text{rad}_A(X, Y) = \{ f \in \text{Hom}_A(X, Y) \mid \text{id}_X - g \circ f \text{ is invertible } \forall g \in \text{Hom}_A(Y, X)\}$$

for all $A$-modules $X, Y$ (see [2, Appendix 3]). Thus $\text{rad}_A(-, -)$ is biadditive, and for two indecomposable modules $X \not\cong Y$ we have $\text{rad}_A(X, Y) = \text{Hom}_A(X, Y)$. Moreover, for an indecomposable module $X$ we have that $\text{rad}_A(X) := \text{rad}_A(X, X)$ is the Jacobson radical of the algebra $\text{End}_A(X)$. We denote by $S_A(X, Y)$ the quotient $\text{Hom}_A(X, Y)/\text{rad}_A(X, Y)$ (and sometimes write only $S_A(X)$ instead of $S_A(X, X)$). To simplify the notation, we sometimes omit the reference to the algebra when this is clear from the context (writing for instance $\text{Hom}$ instead of $\text{Hom}_A$). For the rest of this paper, fix finite-dimensional $k$-algebras $A$ and $B$, where $A$ is $n$-representation finite and $B$ is $m$-representation finite. Set $\mathcal{A} = \text{add } M_A$, $\mathcal{B} = \text{add } M_B$, $\mathcal{A}_i = \text{add } \tau^{-i}A$ for $i \geq 0$, and $\mathcal{B}_j = \text{add } \tau^{-j}B$ for $j \geq 0$.

2. $n$-Almost split sequences as mapping cones

2.1. Preliminaries

If $A$ is $n$-representation finite, then the morphisms in $\mathcal{A}$ are “directed” with respect to the action of $\tau^n$. More precisely, we have the following:

Proposition 2.1. Let $A$ be an $n$-representation finite $k$-algebra. Let $M \in \mathcal{A}_i$ and $N \in \mathcal{A}_j$ with $i > j$. Then

$$\text{Hom}_A(M, N) = 0.$$

Proof. It is enough to check the result for $M, N$ indecomposable, i.e. $M \cong \tau^{-i}P_1$ and $N \cong \tau^{-j}P_2$ for some indecomposable projectives $P_1, P_2 \in \text{add } A$. We have

$$\text{Hom}_A(M, N) = \text{Hom}_{\mathcal{D}^b(\text{mod } A)}(M, N) = \text{Hom}_{\mathcal{D}^b(\text{mod } A)}(\nu^{-i}P_1, \nu^{-j}P_2) = \text{Hom}_{\mathcal{D}^b(\text{mod } A)}(P_1, \nu^{-i-j}P_2).$$

In particular, $\text{Hom}_A(M, N)$ is a direct summand of (with the previous notation)

$$\bigoplus_{i=1}^n \text{Hom}_{\mathcal{D}^b(\text{mod } A)}(P_i, \nu^{-i-j}P_2) = \text{Hom}_{\mathcal{D}^b(\text{mod } A)}(A, \nu^{-i-j}P_2) = \text{Hom}_{\mathcal{D}^b(\text{mod } A)}(A, \nu^{-i-j}P_2) = H_0(\nu^{-i-j}P_2) = \tau^{-i-j}P_2 = 0$$

since $i > j$ and $P_2$ is projective, so we are done. \qed

Remark 2.1. For $n = 1$, this is a special case of [1, Corollary VIII.1.4], since “1-representation finite” means “hereditary and representation finite”.
We will be interested in checking whether a given complex is an \( n \)-almost split sequence, and for this purpose it is convenient to take a slightly different point of view on the definition of \( n \)-almost splitness. Namely, fix an object \( X \in \mathcal{A} \). We can define a functor \( F_X : \mathcal{C}_r(\mathcal{A}) \to \mathcal{C}(k \text{ mod}) \) by mapping

\[
\mathcal{C}_\bullet = \cdots \to C_i \xrightarrow{f_{i+1}} C_{i+1} \to \cdots \to C_0 \xrightarrow{f_0} \cdots
\]

to

\[
F_X(\mathcal{C}_\bullet) = \cdots \to \text{Hom}(X, C_i) \xrightarrow{f_{i+1}} \text{Hom}(X, C_{i+1}) \to \cdots \to \text{rad}(X, C_0) \xrightarrow{f_0} \cdots
\]

(that is, \( F_X \) is the subfunctor of \( \text{Hom}(X, -) \) given by replacing \( \text{Hom}(X, C_0) \) with \( \text{rad}(X, C_0) \)). This is well defined since \( f_1 \) is a radical morphism, hence the image of \( f_1 \circ - \) lies in \( \text{rad}(X, C_0) \). Then for a complex \( \mathcal{C}_\bullet \in \mathcal{C}_r(\mathcal{A}) \) such that \( C_i = 0 \) for \( i > n + 1 \) and \( i < 0 \), saying that it is an \( n \)-almost split sequence is equivalent to saying that \( C_{n+1} \) and \( C_0 \) are indecomposable, \( \mathcal{C}_\bullet \) is exact, and \( F_X(\mathcal{C}_\bullet) \) is exact for every \( X \in \mathcal{A} \) (or equivalently, for every indecomposable \( X \in \mathcal{A} \)). Similarly, we can define a subfunctor \( G_X \) of the contravariant functor \( \text{Hom}(-, X) : \mathcal{C}_r(\mathcal{A}) \to \mathcal{C}(k \text{ mod}) \) by mapping \( \mathcal{C}_\bullet \) to

\[
G_X(\mathcal{C}_\bullet) = \cdots \to \text{rad}(C_0, X) \xrightarrow{- \circ f_0} \text{Hom}(C_0, X) \xrightarrow{- \circ f_{1}} \cdots \to \text{rad}(C_{n+1}, X) \xrightarrow{- \circ f_{n+2}} \cdots
\]

This is again well defined, and if \( \mathcal{C}_\bullet \in \mathcal{C}_r(\mathcal{A}) \) is \( n \)-almost split then \( G_X(\mathcal{C}_\bullet) \) is exact for every \( X \in \mathcal{A} \) (cf. \cite[Proposition 2.10]{8}).

2.2. From sequences to cones

**Definition 2.1.** Let \( \mathcal{D} \) be an abelian category. Let \( A_\bullet \in \mathcal{C}(\mathcal{D}) \) with differentials \( d_i : A_i \to A_{i-1} \). For any \( m \in \mathbb{Z} \), the **shift** \( A[m]_\bullet \) of \( A_\bullet \) is the complex with objects \( A[m]_i = A_{i+m} \) and differentials \( d[m]_i : A[m]_i \to A[m]_{i-1} \) given by \( d[m]_i = (-1)^m d_{i+m} \) for every \( i \).

Let \((A_\bullet, d^A_\bullet)\) and \((B_\bullet, d^B_\bullet)\) be complexes in \( \mathcal{C}(\mathcal{A}) \). Let \( f : A_\bullet \to B_\bullet \) be a morphism of complexes with components \( f_i : A_i \to B_i \). The **shift** of \( f \) is the morphism \( f[m] : A[m]_\bullet \to B[m]_\bullet \) with components \( f[m]_i = f_{i+m} \). Thus \( [m] \) is an endofunctor on \( \mathcal{C}(\mathcal{D}) \). The **mapping cone** \( \text{Cone}(f) \) of \( f \) is the complex with objects

\[
\text{Cone}(f)_i = A[-1]_i \oplus B_i
\]

and differentials

\[
d^\text{Cone}(f)_i = \begin{bmatrix}
  d[-1]^{A}_i & 0 \\
  f[-1]^{B}_i & d^B_i
\end{bmatrix}.
\]

**Lemma 2.2.** Let \( \mathcal{D} \) be an abelian category, and let \( f \) be a morphism of complexes in \( \mathcal{C}(\mathcal{D}) \). Then \( \text{Cone}(f) \) is exact if and only if \( f \) is a quasi-isomorphism.

**Proof.** This follows straight from \cite[III.18]{5}. \( \square \)

Let \( A \) be \( n \)-representation finite, and let

\[
\mathcal{C}_\bullet = 0 \to C_{n+1} \xrightarrow{f_{n+1}} C_n \to \cdots \to C_1 \xrightarrow{f_1} C_0 \to 0
\]
be an \( n \)-almost split sequence starting in slice \( i_0 \) for some \( i_0 \in \mathbb{Z}_{\geq 0} \). Then we can decompose the modules appearing in the sequence according to the slice decomposition of \( A \), i.e. we write

\[
C_m = \bigoplus_{j \geq 0} B^j_m
\]

with \( B^j_m \in A_j \) for every \( m, j \). We know that \( C_{n+1} \in A_{i_0} \) and \( C_0 \in A_{i_0+1} \) are indecomposable. A first result, which can be seen as a generalisation of \([1, \text{Lemma VIII.1.8(b)}]\), is the following:

**Lemma 2.3.** With the above notation, we have

\[
B^j_m = 0 \text{ for any } m \text{ and for } j \notin \{i_0, i_0 + 1\}.
\]

**Proof.** To reach a contradiction, suppose that the claim is false. Then there is \( B^j_q \neq 0 \) with \( j \notin \{i_0, i_0 + 1\} \). Suppose \( j > i_0 + 1 \), and pick \( j \) maximal such. We can assume \( q \) minimal for that value of \( j \), i.e. \( B^j_{q-p} = 0 \) for all \( p > 0 \). Notice that since \( C_0 = B^{i_0+1}_0 \) it follows that \( q > 0 \). We want to prove that \( C_\bullet \) cannot be \( n \)-almost split in this case, and it is enough to show that \( F_{B^j_q}(C_\bullet) \) is not exact. By **Proposition 2.1**,

\[
\text{Hom}(B^j_q, B^i_p) = 0
\]

for every \( p, p' \) and for every \( i < j \). By maximality of \( j \), we get that \( B^j_q \) is a subcomplex of \( C_\bullet \), and

\[
F_{B^j_q}(C_\bullet) = F_{B^j_q}(B^j_q).
\]

Since \( q \) is minimal and \( q > 0 \) we can write explicitly

\[
F_{B^j_q}(C_\bullet) = \cdots \to \text{Hom}(B^j_q, B^j_m) \to \cdots \xrightarrow{d} \text{Hom}(B^j_q, B^j_q) \to 0.
\]

The map \( d \) in this sequence is composition with a radical morphism, so in particular it cannot be surjective on \( \text{Hom}(B^j_q, B^j_q) \). The sequence is then not exact and we have proved that \( B^j_m = 0 \) for \( j > i_0 + 1 \).

Suppose now that \( j < i_0 \), and pick \( j \) minimal such. We can assume that \( q \) is maximal for that \( j \), i.e. \( B^j_{q+p} = 0 \) for all \( p > 0 \). Notice that since \( C_{n+1} = B^{i_0+1}_{n+1} \) it follows that \( q < n + 1 \). We prove that \( C_\bullet \) is not \( n \)-almost split in this case by showing that \( G_{B^j_q}(C_\bullet) \) is not exact. Again by **Proposition 2.1** we know that

\[
\text{Hom}(B^j_{p'}, B^j_{p'}) = 0
\]

for all \( p, p' \) if \( i > j \). Then by minimality of \( j \) and maximality of \( q \) we get

\[
G_{B^j_q}(C_\bullet) = \cdots \to \text{Hom}(B^j_m, B^j_q) \to \cdots \xrightarrow{d'} \text{Hom}(B^j_q, B^j_q) \to 0
\]

and \( d' \) cannot be surjective, contradiction. Hence we have proved that \( B^j_m = 0 \) for \( j < i_0 \), which completes the proof. \( \square \)

**Theorem 2.4.** Let \( A \) be an \( n \)-representation finite \( k \)-algebra, and let \( i_0 \in \mathbb{Z}_{\geq 0} \). Let \( C_{n+1} \in A_{i_0} \) be indecomposable noninjective, and let

\[
C_\bullet = 0 \to C_{n+1} \xrightarrow{f_{n+1}} C_n \to \cdots \to C_1 \xrightarrow{f_1} C_0 \to 0
\]
be the corresponding $n$-almost split sequence. Then there are complexes $A^0\in C_r(A_{i_0})$, $A^1\in C_r(A_{i_0+1})$, and a radical morphism of complexes $\varphi : A^0\to A^1$, such that $C_r \cong \text{Cone}(\varphi)$ in $\mathcal{C}(A)$.

**Proof.** By Lemma 2.3 we can rewrite the complex $C_r$ as

$$C_m = B^{i_0}_m \oplus B^{i_0+1}_m$$

where $B^{i_0}_m \in A_{i_0}$ and $B^{i_0+1}_m \in A_{i_0+1}$ for every $m$. Moreover,

$$f_m = \begin{bmatrix} b^{i_0}_m & \xi_m \\ \gamma_m & b^{i_0+1}_m \end{bmatrix} : C_m \to C_{m-1}$$

has components $b^{i_0}_m : B^{i_0}_m \to B^{i_0}_{m-1}$, $\xi_m : B^{i_0+1}_m \to B^{i_0}_{m-1}$, $\gamma_m : B^{i_0}_m \to B^{i_0+1}_{m-1}$, and $b^{i_0+1}_m : B^{i_0+1}_m \to B^{i_0+1}_{m-1}$.

Notice that by Proposition 2.1 it follows that $\xi_m = 0$ for all $m$. Define $\psi_0 = b^{i_0}_m + \gamma_m b^{i_0+1}_m$ and $\varphi_m = -\gamma_m : A^0_0 \to A^1_0$. Then $\varphi : A^0_0 \to A^1_0$ is a chain map since

$$d^1_m \varphi_m = -b^{i_0+1}_m \gamma_m = \gamma_m b^{i_0+1}_m = \varphi_m d^0_m$$

where the equality

$$b^{i_0+1}_m \gamma_m + \gamma_m b^{i_0+1}_m = 0$$

comes from the fact that $C_r$ is a complex. Moreover, $C_r \cong \text{Cone}(\varphi)$ and we are done. \qed

**Remark 2.2.** In [9, Proposition 3.23] Iyama constructed certain $n$-almost split sequences as mapping cones of chain maps. Our Theorem 2.4 states that in the $n$-representation finite case, every $n$-almost split sequence can in fact be realised as a mapping cone.

Given that $n$-almost split sequences are determined up to isomorphism by their endpoints, it is interesting to address the issue of uniqueness of the map $\varphi$. Since we are not going to need it in what follows, we do not investigate this in detail. We present however a result:

**Proposition 2.5.** Let $A$ be an $n$-representation finite algebra. Let $A^0_0, B^0_0 \in \mathcal{C}(A_{i_0})$, $A^1_1, B^1_1 \in \mathcal{C}(A_{i_0+1})$. Let $\varphi : A^0_0 \to A^1_0$ and $\psi : B^0_0 \to B^1_1$ be chain maps. Then the following are equivalent:

1. $\text{Cone}(\varphi) \cong \text{Cone}(\psi)$ in $\mathcal{C}(A)$.
2. There are isomorphisms of complexes $f : A^0_0 \to B^0_0, g : A^1_1 \to B^1_1$ such that the diagram

$$
\begin{array}{ccc}
A^0_0 & \xrightarrow{f} & B^0_0 \\
\downarrow{\varphi} & & \downarrow{\psi} \\
A^1_1 & \xrightarrow{g} & B^1_1 
\end{array}
$$

commutes in the homotopy category $\mathcal{K}(A)$.

**Proof.** Let us begin by some observations. Let

$$\alpha_m = \begin{bmatrix} a_m & r_m \\ q_m & b_m \end{bmatrix} : A^0_{m-1} \oplus A^1_m \to B^0_{m-1} \oplus B^1_m$$
be a morphism of modules. Notice that by Proposition 2.1, we have \( r_m = 0 \). Observe now that

\[
(\alpha_m) \text{ defines a chain map } \alpha : \text{Cone}(\varphi) \to \text{Cone}(\psi)
\]

\[
\iff \begin{bmatrix} a_{m-1} & 0 \\ q_{m-1} & b_{m-1} \end{bmatrix} \begin{bmatrix} -dA^0_{m-1} & 0 \\ \varphi_{m-1} & dA^1_{m} \end{bmatrix} = \begin{bmatrix} -dB^0_{m-1} & 0 \\ \psi_{m-1} & dB^1_{m} \end{bmatrix} \begin{bmatrix} a_m & 0 \\ q_m & b_m \end{bmatrix} \text{ for all } m
\]

\[
\iff \begin{cases}
  a_{m-1}dA^0_{m-1} = dB^0_{m-1}a_m & \text{for all } m \\
  b_{m-1}dA^1_{m} = dB^1_{m}b_m & \text{for all } m \\
  b_{m-1}\varphi_{m-1} = \psi_{m-1}a_m + dB^1_{m}q_m + q_m-1dA^0_{m-1} & \text{for all } m
\end{cases}
\]

\[
\iff \begin{cases}
  (a_m) \text{ defines a chain map } a : A^0[-1] \to B^0[-1]. \\
  (b_m) \text{ defines a chain map } b : A^1 \to B^1
\end{cases}
\]

\[
\begin{bmatrix} b_{m-1}\varphi_{m-1} = \psi_{m-1}a_m + dB^1_{m}q_m + q_m-1dA^0_{m-1} & \text{for all } m
\end{bmatrix}
\]

Now let us prove (1) \(\iff\) (2). Use the same notation as above, and assume that \( \alpha \) is an isomorphism. That means that \( \alpha_m \) is an isomorphism for every \( m \). Since \( A^0_{m-1} \in A_{i_0} \) and \( B^1_m \in A_{i_0+1} \), it follows that no indecomposable direct summand of \( A^0_{m-1} \) can be isomorphic to a direct summand of \( B^1_m \), hence \( \text{Hom}(A^0_{m-1}, B^1_m) = \text{rad}(A^0_{m-1}, B^1_m) \). In particular we have that \( q_m \) is a radical map. Since \( \alpha_m \) has an inverse, both \( a_m \) and \( b_m \) have inverses modulo radical morphisms. This means that there are \( x : B^0_{m-1} \to A^0_{m-1}, y : B^1_m \to A^1_m \) such that

\[
a_m x - \text{id}_{B^0_{m-1}},
\]

\[
x a_m - \text{id}_{A^0_{m-1}},
\]

\[
b_m y - \text{id}_{B^1_m},
\]

\[
y b_m - \text{id}_{B^1_m}
\]

are radical morphisms. In particular \( a_m x, x a_m, b_m y, y b_m \) are all invertible, hence \( a_m \) and \( b_m \) are isomorphisms. By the above observations, \( a[1] \) and \( b \) are well-defined isomorphisms of complexes, and since

\[
\left( dB^1_{m}q_m + q_m-1dA^0_{m-1} \right) : A^0 \to B^1
\]

is null-homotopic we obtain that the diagram

\[
\begin{array}{ccc}
A^0 \xrightarrow{a[1]} & B^0 \\
\varphi \downarrow & & \downarrow \psi \\
A^1 \xrightarrow{b} & B^1
\end{array}
\]

commutes in \( K(A) \) as required.

Let us now prove (2) \(\implies\) (1). Since the diagram commutes in \( K(A) \), there is a homotopy \( (q_m : A^0_{m-1} \to B^1_m) \) such that

\[
b_{m-1} \varphi_{m-1} = \psi_{m-1} a_m + dB^1_{m} q_m + q_m-1dA^0_{m-1} \text{ for all } m.
\]

By the above observations, setting for every \( m \)
\[
\alpha_m = \begin{bmatrix}
    f_{m-1} & 0 \\
    q_m & g_m
\end{bmatrix}: A_{m-1}^0 \oplus A_m^1 \to B_{m-1}^0 \oplus B_m^1
\]

defines a chain map \( \alpha : \text{Cone}(\varphi) \to \text{Cone}(\psi) \). It remains to check that \( \alpha \) is an isomorphism, which amounts to checking that \( \alpha_m \) is invertible for all \( m \). Since we are assuming that \( f \) and \( g \) are isomorphisms, we can define for every \( m \)

\[
\beta_m = \begin{bmatrix}
    f_{m-1}^{-1} & 0 \\
    -g_m^{-1} f_{m-1}^{-1} & g_m^{-1}
\end{bmatrix}: B_{m-1}^0 \oplus B_m^1 \to A_{m-1}^0 \oplus A_m^1.
\]

It is then a straightforward computation to check that \( \beta_m \) is the inverse of \( \alpha_m \), and we are done. \( \square \)

### 2.3. From cones to sequences

Since we can realise any \( n \)-almost split sequence as \( \text{Cone}(\varphi) \) for some \( \varphi \), it makes sense to relate the property of being \( n \)-almost split to the properties of \( \varphi \). Let us introduce some more notation. For a given \( X \in \mathcal{A} \), we define a functor \( \tilde{F}_X : \text{Mor}_r(C_r(\mathcal{A})) \to \text{Mor}(\mathcal{C}(k \mod)) \) by mapping \( \varphi : A_\bullet \to B_\bullet \) to

\[
\tilde{F}_X(\varphi) = \varphi \circ - : \text{Hom}(X, A_\bullet) \to F_X(B_\bullet),
\]

where \( \text{Hom}(X, A_\bullet) \) denotes the complex \( \cdots \to \text{Hom}(X, A_i) \to \text{Hom}(X, A_{i-1}) \to \cdots \). This is well defined because \( \varphi_0 \in \text{rad}(A_0, B_0) \).

Consider the mapping cone functor \( \text{Cone} : \text{Mor}_r(C_r(\mathcal{A})) \to \mathcal{C}(\mathcal{A}) \). By definition, this factors through the inclusion \( C_r(\mathcal{A}) \to \mathcal{C}(\mathcal{A}) \), and we still denote by \( \text{Cone} \) the corresponding functor \( \text{Cone} : \text{Mor}_r(C_r(\mathcal{A})) \to \mathcal{C}(\mathcal{A}) \). We also denote by \( \text{Cone} \) the mapping cone functor \( \text{Cone} : \text{Mor}(\mathcal{C}(k \mod)) \to \mathcal{C}(k \mod) \).

**Lemma 2.6.** With the above notation, we have that the diagram

\[
\begin{array}{ccc}
\text{Mor}_r(C^n_r(\mathcal{A})) & \xrightarrow{\tilde{F}_X} & \text{Mor}(\mathcal{C}(k \mod)) \\
\text{Cone} \downarrow & & \downarrow \text{Cone} \\
C_r(\mathcal{A}) & \xrightarrow{F_X} & \mathcal{C}(k \mod)
\end{array}
\]

commutes for every \( X \in \mathcal{A} \) and for any choice of \( n \in \mathbb{Z}_{\geq 0} \).

**Proof.** Pick a morphism \( \varphi : A_\bullet \to B_\bullet \in \text{Mor}_r(C^n_r(\mathcal{A})) \). Then

\[
\text{Cone}(\tilde{F}_X(\varphi))_i = \text{Hom}(X, A_{i-1}) \oplus F_X(B_i) =
\]

\[
= \begin{cases}
    \text{Hom}(X, A_{i-1}) \oplus \text{Hom}(X, B_i) & \text{if } i \neq 0 \\
    \text{Hom}(X, A_{i-1}) \oplus \text{rad}(X, B_0) = \text{rad}(X, B_0) & \text{if } i = 0
\end{cases}
\]

and the differential \( d_i : \text{Cone}(\tilde{F}_X(\varphi))_i \to \text{Cone}(\tilde{F}_X(\varphi))_{i-1} \) is given by

\[
d_i = \begin{bmatrix}
    -d^A_{i-1} \circ - & 0 \\
    -\varphi_{i-1} \circ - & d^B_i \circ -
\end{bmatrix}.
\]

On the other hand, we have

\[
F_X(\text{Cone}(\varphi))_i = \begin{cases}
    \text{Hom}(X, A_{i-1} \oplus B_i) = \text{Hom}(X, A_{i-1}) \oplus \text{Hom}(X, B_i) & \text{if } i \neq 0 \\
    \text{rad}(X, A_{i-1} \oplus B_0) = \text{rad}(X, B_0) & \text{if } i = 0
\end{cases}
\]
and the differential \( d'_i : F_X(\text{Cone}(\varphi))_i \to F_X(\text{Cone}(\varphi))_{i-1} \) is given by
\[
d'_i = d^\text{Cone}(\varphi)_i = \begin{bmatrix} -d^A_{i-1} & 0 \\ -\varphi_{i-1} & -d^B_i \end{bmatrix}.
\]

We get a useful criterion for checking whether the cone of a chain map is an \( n \)-almost split sequence.

**Lemma 2.7 (Criterion for \( n \)-almost splitness).** Let \( A^0_i \in C^n_i(A_{i_0}), A^1_i \in C^n_i(A_{i_0+1}) \) for some \( i_0 \). Let \( \varphi : A^0_i \to A^1_i \) be a chain map. Then the following are equivalent:

1. \( \text{Cone}(\varphi) \) is an \( n \)-almost split sequence.
2. \( A^0_n \) and \( A^1_n \) are indecomposable, and \( \tilde{F}_X(\varphi) \) is a quasi-isomorphism for every \( X \in \mathcal{A} \).

**Proof.** (1) \( \Rightarrow \) (2). Suppose that \( \text{Cone}(\varphi) \) is \( n \)-almost split. Then by definition \( A^0_n = \text{Cone}(\varphi)_{n+1} \) and \( A^1_n = \text{Cone}(\varphi)_0 \) are indecomposable and \( F_X(\text{Cone}(\varphi)) \) is exact for every \( X \in \mathcal{A} \). By Lemma 2.6 we know that \( F_X(\text{Cone}(\varphi)) = \text{Cone}(\tilde{F}_X(\varphi)) \), and by Lemma 2.2 exactness of \( \text{Cone}(\tilde{F}_X(\varphi)) \) implies that \( \tilde{F}_X(\varphi) \) is a quasi-isomorphism.

(2) \( \Rightarrow \) (1). If \( \tilde{F}_X(\varphi) \) is a quasi-isomorphism for every \( X \in \mathcal{A} \), then by Lemma 2.2 we know that \( \text{Cone}(\tilde{F}_X(\varphi)) \) is exact, so by Lemma 2.6 we get that \( F_X(\text{Cone}(\varphi)) \) is exact for every \( X \in \mathcal{A} \). Then by observing that \( \text{Cone}(\varphi)_{n+1} = A^0_n \) and \( \text{Cone}(\varphi)_0 = A^1_0 \) are indecomposable, we can conclude that \( \text{Cone}(\varphi) \) is \( n \)-almost split. \( \square \)

3. Tensor product of mapping cones

3.1. Construction

All tensor products are understood to be over \( k \), even when it is not specified to simplify the notation. The tensor product bifunctor
\[- \otimes - : \text{mod } k \times \text{mod } k \to \text{mod } k\]
induces (for a general construction, see [4, IV.4.5]) a bifunctor
\[- \otimes^T - : \mathcal{C}(\text{mod } k) \times \mathcal{C}(\text{mod } k) \to \mathcal{C}(\text{mod } k)\]
(for clarity, we use the symbol \( \otimes \) for modules and \( \otimes^T \) for complexes). Moreover, since the tensor product defines a bifunctor
\[- \otimes - : \text{mod } A \times \text{mod } B \to \text{mod}(A \otimes B)\]
we can consider \( \otimes^T \) as a bifunctor
\[- \otimes^T - : \mathcal{C}(\text{mod } A) \times \mathcal{C}(\text{mod } B) \to \mathcal{C}(\text{mod } A \otimes B).\]

For convenience, we give the explicit formulas: on objects, we have
\[(A \otimes^T B)_m = \bigoplus_{j \in \mathbb{Z}} A_j \otimes B_{m-j} \]
with differential \( d \) given on an element \( v \otimes w \in A_j \otimes B_{m-j} \) by
\[ d_m(v \otimes w) = d^A_j(v) \otimes w + (-1)^j v \otimes d^B_{m-j}(w). \]

On morphisms, if \( \varphi : A^0_\bullet \rightarrow A^1_\bullet \) and \( \psi : B^0_\bullet \rightarrow B^1_\bullet \) are chain maps, then
\[
(\varphi \otimes^T \psi)_m = \bigoplus_{j \in \mathbb{Z}} \varphi_j \otimes \psi_{m-j} : \bigoplus_{j \in \mathbb{Z}} A^0_j \otimes B^0_{m-j} \rightarrow \bigoplus_{j \in \mathbb{Z}} A^1_j \otimes B^1_{m-j}.
\]

**Lemma 3.1.** Let \( A, B \) be finite-dimensional \( k \)-algebras, let \( A, B \) be subcategories of \( \text{mod} \, A \) and \( \text{mod} \, B \) respectively, and let \( \varphi : A^0_\bullet \rightarrow A^1_\bullet \) and \( \psi : B^0_\bullet \rightarrow B^1_\bullet \) be objects of \( \text{Mor}(\mathcal{C}(A)) \) and \( \text{Mor}(\mathcal{C}(B)) \) respectively. Suppose that both \( \varphi \) and \( \psi \) are quasi-isomorphisms. Then \( \varphi \otimes^T \psi \) is a quasi-isomorphism.

**Proof.** This follows from the Künneth formula over a field (see [4, VI.3.3.1]). That is, for complexes \( A_\bullet \) and \( B_\bullet \) there is for every \( n \) a functorial isomorphism
\[
H_n(A_\bullet \otimes B_\bullet) \cong \bigoplus_{i+j=n} H_i(A_\bullet) \otimes H_j(B_\bullet).
\]

In our case, this gives for every \( n \) an isomorphism
\[
H_n(\varphi \otimes^T \psi) \cong (H_i(\varphi) \otimes H_j(\psi))_{i+j=n}.
\]

Since \( \varphi \) and \( \psi \) are quasi-isomorphisms, the right-hand side is an isomorphism, hence \( \varphi \otimes^T \psi \) is a quasi-isomorphism. \( \Box \)

### 3.2. Preparation

We now focus on the tensor product of homogeneous algebras. In this case the tensor product behaves well (recall that we are assuming \( k \) to be perfect). More precisely, we have the following classical result:

**Proposition 3.2.** Let \( A, B \) be finite-dimensional \( k \)-algebras. Then
\[
\text{gl. dim}(A \otimes_k B) = \text{gl. dim}(A) + \text{gl. dim}(B).
\]

**Proof.** Using a result by Auslander ([3, Theorem 16]), we can assume that \( A \) and \( B \) are semisimple. Then the claim is a special case of [10, Corollary 5.7]. \( \Box \)

In our setting, perfectness of the ground field and homogeneity are enough to guarantee that higher representation finiteness is preserved by tensor products:

**Theorem 3.3.** Let \( A \) be an \( n \)-representation finite \( k \)-algebra, and let \( B \) be an \( m \)-representation finite \( k \)-algebra. If \( A \) and \( B \) are \( l \)-homogeneous, then the algebra \( A \otimes_k B \) is \((n + m)\)-representation finite, \( l \)-homogeneous. Moreover, an \((n + m)\)-cluster tilting module for \( A \otimes_k B \) is given by
\[
M_{A \otimes B} = \bigoplus_{i=0}^{i-1} \tau_{m}^{-i} A \otimes \tau_{m}^{-i} B.
\]

**Proof.** See [6, 1.5]. \( \Box \)

**Proposition 3.4.** Let \( A \) and \( B \) be two finite-dimensional \( k \)-algebras. Let \( M, N \in \text{mod} \, A \) and \( M', N' \in \text{mod} \, B \). Then the canonical map
\[ \text{Hom}_A(M, N) \otimes_k \text{Hom}_B(M', N') \rightarrow \text{Hom}_{A \otimes_k B}(M \otimes_k M', N \otimes_k N') \]

given by \( f \otimes g \mapsto f \otimes g \) is an isomorphism of \( k \)-vector spaces.

**Proof.** See Proposition XI.1.2.3 and Theorem XI.3.1 in [4]. \( \square \)

We will use the above identification quite freely from now on. We need two more lemmas:

**Lemma 3.5.** Let \( R \) and \( S \) be finite-dimensional \( k \)-algebras. Then we have

\[ \text{rad}(R) \otimes_k S + R \otimes_k \text{rad}(S) = \text{rad}(R \otimes_k S) \]

as ideals of \( R \otimes_k S \).

**Proof.** This is [10, Corollary 5.8], combined with the observation that for finite-dimensional algebras the Baer radical and the Jacobson radical coincide (see [11, Proposition 10.27]). \( \square \)

**Lemma 3.6.** Let \( A \) and \( B \) be two finite-dimensional \( k \)-algebras. Let \( M, N \in \text{mod} A \) and \( M', N' \in \text{mod} B \). Then we have

\[ \text{rad}(M, N) \otimes \text{Hom}(M', N') + \text{Hom}(M, N) \otimes \text{rad}(M', N') = \text{rad}(M \otimes M', N \otimes N') \]

as subspaces of \( \text{Hom}(M \otimes M', N \otimes N') \). Moreover, there is an exact sequence

\[ 0 \longrightarrow \text{rad}(M) \otimes \text{rad}(M') \xrightarrow{[\alpha\ -\ \alpha]} \frac{\text{rad}(M) \otimes \text{End}(M')}{\text{End}(M) \otimes \text{rad}(M')} \oplus \frac{\text{rad}(M) \otimes \text{End}(M')}{\text{End}(M) \otimes \text{rad}(M')} \longrightarrow \text{rad}(M \otimes M') \longrightarrow 0 \]

where

\[ \alpha : f \otimes g \mapsto f \otimes g. \]

**Proof.** Let \( R = \text{End}_A(M \oplus N) \) and \( S = \text{End}_B(M' \oplus N') \). By Proposition 3.4 we have

\[ R \otimes S \cong \text{End}_{A \otimes B}((M \oplus N) \otimes (M' \oplus N')). \]

Let \( p, q \in R \) be the projections onto \( M, N \) respectively, and let \( p', q' \in S \) be the projections onto \( M', N' \) respectively. Then we have

\[ (q \otimes q')(\text{rad}(R \otimes S))(p \otimes p') = \text{rad}(M \otimes M', N \otimes N'). \]

By Lemma 3.5,

\[ \text{rad}(R \otimes S) = \text{rad}(R) \otimes S + R \otimes \text{rad}(S) \]

so that

\[ \text{rad}(M \otimes M', N \otimes N') = (q \otimes q')(\text{rad}(R) \otimes S + R \otimes \text{rad}(S))(p \otimes p') = \]

\[ = \text{rad}(M, N) \otimes \text{Hom}(M', N') + \text{Hom}(M, N) \otimes \text{rad}(M', N'), \]
which proves the first claim. Moreover, in the case $M = N, M' = N'$ we easily get the exact sequence by looking at the kernel of the map

$$[\alpha, \alpha] : \frac{\text{rad}(M) \otimes \text{End}(M')}{\text{End}(M) \otimes \text{rad}(M')} \to \text{rad}(M \otimes M').$$

3.3. Proof of main result

We are ready to prove Theorem 1.1:

**Proof of Theorem 1.1.** We fix $\varphi : A_i^0 \to A_i^1$ and $\psi : B_i^0 \to B_i^1$. By definition $C_\bullet = \text{Cone}(\varphi \otimes^T \psi)$ is a complex bounded between 0 and $n + m + 1$, and it is exact by Lemma 2.2 and Lemma 3.1. It follows from Lemma 3.6 that

$$(\varphi \otimes^T \psi)_i \in \text{rad}((A_i^0 \otimes^T B_i^0)_i, (A_i^1 \otimes^T B_i^1)_i)$$

for every $i$, and so $C_\bullet \in C_r(A \otimes B)$. Fix an indecomposable $M \otimes N \in A \otimes B$. We can consider the maps

$$\tilde{F}_M(\varphi) \otimes^T \tilde{F}_N(\psi) : \text{Hom}(M, A_i^0) \otimes^T \text{Hom}(N, B_i^0) \to F_M(A_i^1) \otimes^T F_N(B_i^1)$$

and

$$\tilde{F}_{M \otimes N}(\varphi \otimes^T \psi) : \text{Hom}(M \otimes N, A_i^0 \otimes^T B_i^0) \to F_{M \otimes N}(A_i^1 \otimes^T B_i^1).$$

By Lemma 3.6, the map

$$\iota : \text{Hom}(M, A_i^1) \otimes^T \text{Hom}(N, B_i^1) \to \text{Hom}(M \otimes N, A_i^1 \otimes^T B_i^1), \ f \otimes g \mapsto f \otimes g$$

induces a monomorphism

$$\iota' : F_M(A_i^1) \otimes^T F_N(B_i^1) \to F_{M \otimes N}(A_i^1 \otimes^T B_i^1)$$

so there is a commutative diagram

$$\begin{array}{ccc}
\text{Hom}(M, A_i^0) \otimes^T \text{Hom}(N, B_i^0) & \xrightarrow{\iota} & \text{Hom}(M \otimes N, A_i^0 \otimes^T B_i^0) \\
\tilde{F}_M(\varphi) \otimes^T \tilde{F}_N(\psi) & \downarrow & \tilde{F}_{M \otimes N}(\varphi \otimes^T \psi) \\
F_M(A_i^1) \otimes^T F_N(B_i^1) & \xrightarrow{\iota'} & F_{M \otimes N}(A_i^1 \otimes^T B_i^1). \\
\end{array}$$

By Proposition 3.4, the map $\iota$ is an isomorphism. Moreover, since $\text{Cone}(\varphi)$ is $n$-almost split, it follows by Lemma 2.7 that $\tilde{F}_M(\varphi)$ is a quasi-isomorphism, and similarly $\tilde{F}_N(\psi)$ is a quasi-isomorphism because $\text{Cone}(\psi)$ is $m$-almost split. Then by Lemma 3.1 it follows that $\tilde{F}_M(\varphi) \otimes^T \tilde{F}_N(\psi)$ is a quasi-isomorphism. Again by Lemma 2.7, the claim that $C_\bullet$ is $(m+n)$-almost split will follow if we prove that $\tilde{F}_{M \otimes N}(\varphi \otimes^T \psi)$ is a quasi-isomorphism (since $M \otimes N$ is an arbitrary indecomposable). By the above observations, it is enough to show that $\iota'$ is a quasi-isomorphism. This is in turn equivalent to $\text{coker} \ i'$ being exact, which is what we prove. We claim that we have
coker $\iota' = F_{M}(A_{k}^{1}) \otimes_{k} S(N, B_{0}^{1}) \oplus S(M, A_{0}^{1}) \otimes_{k} F_{N}(B_{1}^{*})$.  

(1)

Assume that this claim holds, and let us prove the theorem. Notice that $S(N, B_{0}^{1}) = 0$ unless $N \cong B_{0}^{1}$ since $B_{0}^{1}$ and $N$ are indecomposable. Suppose that $N \cong B_{0}^{1}$. Then in particular $N \in \text{add } \tau_{n}^{-1}B_{n}^{1}$ and so $M \in \text{add } \tau_{n}^{-1}A$ since $M \otimes N \in \mathbb{A} \otimes \mathbb{B}$ (see Theorem 3.3). Then by Proposition 2.1 we get

$$\text{Hom}(M, A_{k}^{1}) = 0.$$ 

In this case $F_{M}(A_{k}^{1}) \cong \text{Cone}(\bar{F}_{M}(\phi))$ which by Lemma 2.7 is exact if and only if Cone($\phi$) is $n$-almost split, which we are assuming. Tensoring over $k$ is exact, so it follows that the first summand in (1) is exact. By symmetry, the second summand is exact as well and we are done.

It remains to prove the equality (1). Call $D_{*} = F_{M}(A_{*}^{1}) \otimes T F_{N}(B_{*}^{1})$. We have that

$$D_{p} = \bigoplus_{i + j = p} F_{M}(A_{i}^{1}) \otimes F_{N}(B_{j}^{1})$$

and we are interested in computing the cokernels of the maps

$$\iota'_{p} : D_{p} \rightarrow F_{M \otimes N} \left( A_{*}^{1} \otimes T B_{*}^{1} \right)_{p}.$$ 

We proceed by first considering the case $p \neq 0$. Then the codomain of $\iota'_{p}$ is

$$\text{Hom} \left( M \otimes N, \bigoplus_{i + j = p} A_{i}^{1} \otimes B_{j}^{1} \right) \cong \bigoplus_{i + j = p} \text{Hom}(M, A_{i}^{1}) \otimes \text{Hom}(N, B_{j}^{1})$$

and $\iota'_{p}$ is just the canonical diagonal immersion with components

$$\iota'_{ij} : F_{M}(A_{i}^{1}) \otimes F_{N}(B_{j}^{1}) \rightarrow \text{Hom}(M, A_{i}^{1}) \otimes \text{Hom}(N, B_{j}^{1})$$

given by $f \otimes g \mapsto f \otimes g$. In particular, $\iota'_{ij}$ is the identity unless either $i = 0$ and $M \cong A_{0}^{1}$ or $j = 0$ and $N \cong B_{0}^{1}$. It follows that

$$\text{coker } \iota'_{p} = \bigoplus_{i + j = p} \text{coker } \iota'_{ij} = \text{coker } \iota'_{p0} \oplus \text{coker } \iota'_{p0}.$$ 

Let us then suppose $N \cong B_{0}^{1}$, and focus on terms of the form $\text{coker } \iota'_{p0}$, where

$$\iota'_{p0} : \text{Hom}(M, A_{p}^{1}) \otimes \text{rad}(B_{0}^{1}) \rightarrow \text{Hom}(M \otimes B_{0}^{1}, A_{p}^{1} \otimes B_{0}^{1}).$$

We know by Proposition 3.4 that the right-hand side is canonically isomorphic to $\text{Hom}(M, A_{p}^{1}) \otimes \text{End}(B_{0}^{1})$, so from the exact sequence

$$0 \rightarrow \text{rad}(B_{0}^{1}) \rightarrow \text{End}(B_{0}^{1}) \rightarrow S(B_{0}^{1}) \rightarrow 0$$

we conclude that $\text{coker } \iota'_{p0} = \text{Hom}(M, A_{p}^{1}) \otimes S(B_{0}^{1})$. By symmetry we conclude that if $p \neq 0$ then

$$\text{coker } \iota'_{p} = \text{Hom}(M, A_{p}^{1}) \otimes S(N, B_{0}^{1}) \oplus \text{Hom}(M, A_{0}^{1}) \otimes \text{Hom}(N, B_{p}^{1}).$$

Let us analyse the case $p = 0$. Under the identification given by Proposition 3.4, the map
\( \iota'_0 : \text{rad}(M, A^1_0) \otimes \text{rad}(N, B^1_0) \to \text{rad}(M \otimes N, A^1_0 \otimes B^1_0) \)

is the identity if \( M \not\cong A^1_0 \) and \( N \not\cong B^1_0 \), and the inclusion otherwise. If \( M \not\cong A^1_0 \) and \( N \cong B^1_0 \), then we are in the same situation as in the previous case, and

\[
\text{coker } \iota'_0 = \text{Hom}(M, A^1_0) \otimes S(B^1_0)
\]

and similarly for the symmetric case. If both \( M \cong A^1_0 \) and \( N \cong B^1_0 \), then we claim that

\[
\text{coker } \iota'_0 = \text{rad}(A^1_0) \otimes S(B^1_0) \oplus S(A^1_0) \otimes \text{rad}(B^1_0).
\]

Indeed (for simplicity, write \( E = A^1_0 \) and \( F = B^1_0 \)), in the commutative diagram

\[
\begin{array}{ccc}
0 & \to & \text{rad}(E) \otimes \text{rad}(F) \\
& & [\alpha \ -\alpha] \downarrow \\
& & \text{rad}(E) \otimes \text{rad}(F) \\
0 & \to & \text{rad}(E) \otimes \text{rad}(F) \\
& & [\alpha \ 0] \downarrow \\
& & \text{rad}(E) \otimes \text{End}(F) \\
& & \oplus \\
& & \text{End}(E) \otimes \text{rad}(F) \\
& & [\alpha \ -\alpha] \downarrow \\
& & \text{rad}(E) \otimes \text{rad}(F) \\
& & \oplus \\
& & \text{rad}(E) \otimes S(F) \\
& & \oplus \\
& & S(E) \otimes \text{rad}(F) \\
& & 0 \\
& & \downarrow \\
& & \text{coker } \iota'_0 \\
0 & \to & 0
\end{array}
\]

the first row is exact, as well as all the columns (\( \alpha \) denotes the canonical map \( f \otimes g \mapsto f \otimes g \)). The second row is exact by Lemma 3.6. Hence we get an isomorphism

\[
\text{rad}(E) \otimes S(F) \oplus S(E) \otimes \text{rad}(F) \cong \text{coker } \iota'_0
\]

by the \( 3 \times 3 \) lemma. We have shown that

\[
\text{coker } \iota'_p = F_M(A^1_p) \otimes S(N, B^1_0) \oplus S(M, A^1_0) \otimes F_N(B^1_p)
\]

for every value of \( p = 0, \ldots, m + n \).

It remains to show that the differentials \( \text{coker } \iota'_{p+1} \to \text{coker } \iota'_p \) are diagonal, to conclude that the direct-sum decomposition of the objects is actually a direct-sum decomposition into the two complexes appearing in equation (1). The only degree where this poses problems is \( p = 0 \) in the case \( M \cong E = A^1_0, \ N \cong F = B^1_0 \).

For this, consider the following diagram:
\[
\begin{align*}
\text{Hom}(E, A_1^1) \otimes \text{rad}(F) & \quad \xrightarrow{\beta} \quad \text{rad}(E) \otimes \text{rad}(F) \\
\text{rad}(E) \otimes \text{Hom}(F, B_1^1) & \quad \xrightarrow{\iota'_1} \quad \text{rad}(E) \otimes \text{rad}(F) \\
\text{rad}(E) \otimes \text{Hom}(F, B_1^1) & \quad \xrightarrow{\beta} \quad \text{rad}(E \otimes F) \\
\text{End}(E) \otimes \text{Hom}(F, B_1^1) & \quad \xrightarrow{\iota'_0} \quad \text{coker } \iota'_0 \\
\text{End}(E) \otimes \text{Hom}(F, B_1^1) & \quad \xrightarrow{\beta} \quad \text{rad}(E \otimes F) \\
\text{End}(E) \otimes \text{rad}(F) & \quad \xrightarrow{\iota'_0} \quad \text{coker } \iota'_0 \\
\text{End}(E) \otimes \text{rad}(F) & \quad \xrightarrow{\beta} \quad \text{rad}(E \otimes F) \\
\text{coker } \iota'_0 & \quad \xrightarrow{\beta} \quad \text{rad}(E \otimes F) \\
\end{align*}
\]

where the horizontal maps are induced by

\[
\beta = \begin{bmatrix} (d_1^A \circ -) \otimes \text{id} & \text{id} \otimes (d_1^B \circ -) \end{bmatrix},
\]

which is the last map appearing in the sequence \( F_E(A_1^1) \otimes^T F_F(B_1^1) \). The map \( \beta \) factors as

\[
\beta = \begin{bmatrix} \alpha & \alpha \end{bmatrix} \begin{bmatrix} (d_1^A \circ -) \otimes \text{id} & 0 \\ 0 & \text{id} \otimes (d_1^B \circ -) \end{bmatrix},
\]

hence the diagram above can be completed to a diagram

\[
\begin{align*}
\text{Hom}(E, A_1^1) \otimes \text{rad}(F) & \quad \xrightarrow{\iota'_1} \quad \text{rad}(E) \otimes \text{rad}(F) \\
\text{End}(E) \otimes \text{Hom}(F, B_1^1) & \quad \xrightarrow{\beta} \quad \text{rad}(E \otimes F) \\
\text{End}(E) \otimes \text{rad}(F) & \quad \xrightarrow{\iota'_0} \quad \text{coker } \iota'_0 \\
\end{align*}
\]

where the horizontal maps on the left-hand side are diagonal. Hence the induced map

\[
\begin{align*}
\text{Hom}(E, A_1^1) \otimes \text{Hom}(F, B_1^1) & \quad \xrightarrow{\beta} \quad \text{rad}(E) \otimes \text{rad}(F) \\
\text{End}(E) \otimes \text{Hom}(F, B_1^1) & \quad \xrightarrow{\beta} \quad \text{rad}(E \otimes F) \\
\text{coker } \iota'_0 & \quad \xrightarrow{\beta} \quad \text{rad}(E \otimes F) \\
\end{align*}
\]

factors through the diagonal map

\[
\begin{bmatrix} (d_1^A \circ -) \otimes \text{id}_{S(F)} & 0 \\ 0 & \text{id}_{S(E)} \otimes (d_1^B \circ -) \end{bmatrix} : \quad \begin{bmatrix} \text{Hom}(E, A_1^1) \otimes \text{Hom}(F, B_1^1) \\ \text{End}(E) \otimes \text{Hom}(F, B_1^1) \end{bmatrix} \quad \text{rad}(E) \otimes \text{rad}(F) \\
\]

and we are done. \( \square \)

### 4. Examples

As an example, consider the quiver

\[
1 \rightarrow 2 \rightarrow 3 \leftarrow 4 \leftarrow 5
\]
and the corresponding path algebra $A = kQ$. Thus $A$ is 3-homogeneous, (1-) representation finite (see [2,6]). We want to consider the algebra $B = A \otimes A$, which is then 3-homogeneous, 2-representation finite. There are 15 nonisomorphic indecomposables in mod $A$, which have the following dimension vectors:

- $P_1: (11100)$  
- $M_1: (01111)$  
- $I_1: (10000)$  
- $P_2: (01100)$  
- $M_2: (01000)$  
- $I_2: (11000)$  
- $P_3: (00100)$  
- $M_3: (01110)$  
- $I_3: (11111)$  
- $P_4: (00110)$  
- $M_4: (00010)$  
- $I_4: (00011)$  
- $P_5: (00111)$  
- $M_5: (11110)$  
- $I_5: (00001)$.

The Auslander–Reiten quiver of $A$ is the following:

\[ P_1 \to M_4 \to I_5 \]
\[ P_2 \to M_5 \to I_4 \]
\[ P_3 \to M_3 \to I_3 \]
\[ P_4 \to M_1 \to I_2 \]
\[ P_5 \to M_2 \to I_1 \]

where the dotted lines represent $\tau_1^-$. Inside mod $B$ we have the 2-cluster tilting subcategory $C = \text{add } M$, where

\[
M = \bigoplus_{1 \leq i, j \leq 5} P_i \otimes P_j \oplus \bigoplus_{1 \leq i, j \leq 5} M_i \otimes M_j \oplus \bigoplus_{1 \leq i, j \leq 5} I_i \otimes I_j.
\]

Let us consider for instance the (1-)almost split sequences

\[
C_\bullet = 0 \to P_2 \to P_1 \oplus M_3 \to M_5 \to 0
\]

and

\[
D_\bullet = 0 \to P_5 \to M_1 \to M_2 \to 0
\]

in mod $A$. Notice that both these sequences start in slice 0. The sequence $C_\bullet$ is isomorphic to the cone of

\[
\cdots \to 0 \to P_2 \xrightarrow{-a} P_1 \xrightarrow{0} \cdots
\]
\[
\xrightarrow{-b} \downarrow \xrightarrow{-c}
\]
\[
\cdots \to 0 \to M_3 \xrightarrow{d} M_5 \xrightarrow{0} \cdots
\]

and $D_\bullet$ is isomorphic to the cone of
where these diagrams should be seen as morphisms $\varphi, \psi$ of chain complexes. Then we can construct the morphism $\varphi \otimes T \psi$:

$$
\begin{array}{c}
\cdots \longrightarrow 0 \longrightarrow P_3 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots \\
\downarrow -e \downarrow \quad \downarrow f \\
\cdots \longrightarrow 0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow 0 \longrightarrow \cdots 
\end{array}
$$

The cone $E_* = \text{Cone}(\varphi \otimes T \psi)$ is then the sequence

$$
0 \longrightarrow P_2 \otimes P_5 \longrightarrow P_1 \otimes P_5 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots
$$

which is 2-almost split in $\mathcal{C}$ by Theorem 1.1.

Now we can go further, and consider the algebra $B \otimes A$, which is then 3-homogeneous, 3-representation finite. Let us write for simplicity $P_{abc} = P_a \otimes P_b \otimes P_c$ and $M_{abc} = M_a \otimes M_b \otimes M_c$. We look at the 3-almost split sequence starting in $P_{254}$, which is obtained from $E_*$ together with the sequence

$$
0 \longrightarrow P_4 \longrightarrow P_3 \oplus M_3 \longrightarrow M_1 \longrightarrow 0
$$

in mod $A$. By applying the formula we get the sequence

where each arrow is the natural morphism up to sign.
Acknowledgements

The author is thankful to his advisor Martin Herschend for the constant supervision and helpful discussions, as well as for useful ideas about some proofs. The author would also like to thank an anonymous referee for the helpful suggestions. The author was funded by Uppsala University.

References

TENSOR PRODUCTS OF $n$-COMPLETE ALGEBRAS

ANDREA PASQUALI

ABSTRACT. If $A$ and $B$ are $n$- and $m$-representation finite $k$-algebras, then their tensor product $\Lambda = A \otimes_k B$ is not in general $(n + m)$-representation finite. However, we prove that if $A$ and $B$ are acyclic and satisfy the weaker assumption of $n$- and $m$-completeness, then $\Lambda$ is $(n + m)$-complete. This mirrors the fact that taking higher Auslander algebra does not preserve $d$-representation finiteness in general, but it does preserve $d$-completeness. As a corollary, we get the necessary condition for $\Lambda$ to be $(n + m)$-representation finite which was found by Herschend and Iyama by using a certain twisted fractionally Calabi-Yau property.

1. Introduction

Higher Auslander-Reiten theory was developed in a series of papers [Iya07b], [Iya07a], [Iya08] as a tool to study module categories of finite-dimensional algebras. The idea is to replace all the homological notions in classical Auslander-Reiten theory with higher-dimensional analogs. Some early results can be found in [IO11], [HI11b]. This approach has been fruitful in the context of noncommutative algebraic geometry, see for instance [AIR15], [HIO14], [HIMO14]. Higher Auslander-Reiten theory is also deeply tied with $d$-homological algebra ([GKO13], [Jas16], [Jør15]). A presentation of the theory from this point of view can be found in [JK16].

In this setting, $d$-representation finite algebras were introduced in [Iya11] as a generalisation of hereditary representation finite algebras. They are algebras of global dimension at most $d$ that have a $d$-cluster tilting module $M$. The category $\text{add} M$ has nice homological properties and behaves in many ways like the module category of a hereditary representation finite algebra. While classification of $d$-representation finite algebras seems far from being achieved, it makes sense to look for examples, and to try to understand how $d$-representation finiteness behaves with respect to reasonable operations. Notice that in this setting we have more freedom than in the hereditary case, since we are allowed to increase the global dimension and still fall within the scope of the theory.

For instance, in [Iya11] Iyama investigates whether the endomorphism algebra of the $d$-cluster tilting module (called the higher Auslander algebra) is $(d + 1)$-representation finite. This turns out to be false in general, but a necessary and sufficient condition is given: the only case where it is true is within the tower of iterated higher Auslander algebras of the upper triangular matrix algebra, so this construction gives only a specific family of examples. On the other hand, in the same paper the weaker notion of $d$-complete algebra is introduced and studied. A $d$-complete algebra is an algebra of global dimension at most $d$ that has a module which is $d$-cluster tilting in a suitable exact subcategory of the module category. It turns out that this weaker notion is preserved under taking higher Auslander algebras, thereby producing many examples of $d$-complete algebras for any $d$. 

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Another operation one might investigate is that of taking tensor products over the base field \( k \). Indeed, if \( k \) is perfect then \( \text{gl. dim } A \otimes_k B = \text{gl. dim } A + \text{gl. dim } B \), so it makes sense to ask whether the tensor product of an \( n \)- and an \( m \)-representation finite algebras is \((n+m)\)-representation finite. This is false in general, and in [HI11a] Herschend and Iyama give a necessary and sufficient condition (\( l \)-homogeneity) for it to be true.

In this paper we prove that the same weaker notion of \( d \)-completeness which is used in [Iya11] is preserved under tensor products, under the assumption of acyclicity. Namely, if \( A \) is \( n \)-complete and acyclic and \( B \) is \( m \)-complete and acyclic, then \( A \otimes_k B \) is \((n+m)\)-complete and acyclic. If we assume that \( A \) and \( B \) are \( l \)-homogeneous, we recover the result by Herschend and Iyama. This gives a new way of producing \( d \)-complete algebras for any \( d \).

The proof we give is structured as follows. We prove that over the tensor product there are \((n+m)\)-almost split sequences (using the same construction as in [Pas17]), and moreover that injective modules have source sequences. Then we use these sequences, combined with the assumption of acyclicity, to prove that the module \( T \) in the definition of \((n+m)\)-completeness is tilting. By [Iya11, Theorem 2.2(b)], the existence of the above sequences in \( T^\perp \) is equivalent to \( M \) being \((n+m)\)-cluster tilting in \( T^\perp \), which is the key point of \((n+m)\)-completeness.

In Sections 2 we lay down notation, conventions, and preliminary definitions. Section 3 contains the statement of our main result. Section 4 contains the results about \( d \)-almost split sequences and tensor products which we want to use. Section 5 is dedicated to proving the main theorem, which amounts to checking that the tensor product satisfies the defining properties of \((n+m)\)-complete algebras. In Section 6 we present some examples.

2. Notation and conventions

Throughout this paper, \( k \) denotes a fixed perfect field. All algebras are associative, unital, and finite dimensional over \( k \). For an algebra \( \Lambda \), \( \text{mod } \Lambda \) (respectively \( \Lambda \text{ mod} \)) denotes the category of finitely generated right (left) \( \Lambda \)-modules. We denote by \( D \) the duality \( D = \text{Hom}_k(\cdot, k) \) between \( \text{mod } \Lambda \) and \( \Lambda \text{ mod} \) (in both directions). Subcategories are always assumed to be full and closed under isomorphisms, finite direct sums and summands. For \( M \in \text{mod } \Lambda \), we denote by \( \text{add } M \) the subcategory of \( \text{mod } \Lambda \) whose objects are all modules isomorphic to finite direct sums of summands of \( M \). We write \( \text{rad } \Lambda(\cdot, \cdot) \) for the subfunctor of \( \text{Hom}_\Lambda(\cdot, \cdot) \) defined by

\[
\text{rad}_\Lambda(X,Y) = \{ f \in \text{Hom}_\Lambda(X,Y) \mid \text{id}_X - g \circ f \text{ is invertible } \forall g \in \text{Hom}_\Lambda(Y,X) \}.
\]

Moreover, for \( X,Y \in \text{mod } \Lambda \), we write \( \text{top }_\Lambda(X,Y) = \text{Hom}_\Lambda(X,Y)/\text{rad}_\Lambda(X,Y) \).

We often write \( \text{Hom} \) instead of \( \text{Hom}_\Lambda \) and similarly for \( \text{rad} \) and \( \text{top} \) when the context allows it. We denote by \( D^b(\Lambda) \) the bounded derived category of \( \text{mod } \Lambda \). For a subcategory \( \mathcal{C} \) of \( D^b(\Lambda) \), we denote by \( \text{thick } \mathcal{C} \) the smallest triangulated subcategory of \( D^b(\Lambda) \) containing \( \mathcal{C} \). If \( \mathcal{C} = \text{add } M \) for some \( M \in \text{mod } \Lambda \subseteq D^b(\Lambda) \), we write \( \text{thick } M = \text{thick}(\text{add } M) \). All tensor products are over \( k \), even when the specification is omitted to simplify the notation.
Throughout this section, let $\text{gl.dim} \Lambda \leq d$. Then we can define the higher Auslander-Reiten translations by
\begin{align*}
\tau_d &= D \text{Ext}^d(\cdot, \Lambda) : \text{mod} \Lambda \to \text{mod} \Lambda \\
\tau_d^- &= \text{Ext}^d(\cdot, \Lambda) : \text{mod} \Lambda \to \text{mod} \Lambda.
\end{align*}
We are interested in categories associated to tilting modules.

**Definition 2.1.** A $\Lambda$-module $T$ is **tilting** if the following conditions are satisfied:
\begin{enumerate}
\item $\text{Ext}^i(T, T) = 0$ for all $i \neq 0$.
\item there is an exact sequence $0 \to \Lambda \to T_0 \to \cdots \to T_m \to 0$ for some $m$ with $T_i \in \text{add} T$ for all $i$.
\end{enumerate}
The second condition in the definition can be replaced by $\text{thick} T = D^b(\Lambda)$.

For a tilting module $T$, we have an exact subcategory of $\text{mod} \Lambda$
\[ T^\perp = \{ X \in \text{mod} \Lambda \mid \text{Ext}^i(T, X) = 0 \text{ for every } i \neq 0 \} \]
We are interested in $d$-cluster tilting subcategories of $T^\perp$.

**Definition 2.2.** Let $T$ be a tilting module. A subcategory $\mathcal{C}$ of $T^\perp$ is called **$d$-cluster tilting** if
\[ \mathcal{C} = \{ X \in T^\perp \mid \text{Ext}^i(\mathcal{C}, X) = 0 \text{ for every } 0 < i < d \} = \{ X \in T^\perp \mid \text{Ext}^i(X, \mathcal{C}) = 0 \text{ for every } 0 < i < d \}. \]

We follow [Iya11, Definition 1.11] and define the following subcategories of $\text{mod} \Lambda$:
\begin{enumerate}
\item $\mathcal{M} = \mathcal{M}(\Lambda) = \text{add} \{ \tau_d^i D\Lambda \mid i \geq 0 \}$,
\item $\mathcal{P} = \{ X \in \mathcal{M} \mid \tau_d X = 0 \}$,
\item $\mathcal{M}_P = \{ X \in \mathcal{M} \mid X \text{ has no nonzero summands in } \mathcal{P} \}$.
\item $\mathcal{M}_I = \{ X \in \mathcal{M} \mid X \text{ has no nonzero summands in } \text{add} D\Lambda \}$.
\end{enumerate}
Let $T_\Lambda$ be a basic module such that $\text{add} T_\Lambda = \mathcal{P}$.

**Definition 2.3.** An algebra $\Lambda$ is **$d$-complete** if the following conditions hold:
\begin{enumerate}
\item[(A$_d$)] $T_\Lambda$ is a tilting module.
\item[(B$_d$)] $\mathcal{M}$ is a $d$-cluster tilting subcategory of $T_\Lambda^\perp$.
\item[(C$_d$)] $\text{Ext}^i(\mathcal{M}_P, \Lambda) = 0$ for every $0 < i < d$.
\end{enumerate}

Note that condition (A$_d$) implies that $\tau_d^i = 0$ for large $l$ ([Iya11, Proposition 1.12(d) and 1.3(c)]). Note moreover that if $\Lambda$ is $d$-complete then since $\text{gl.dim} \Lambda \leq d$ it follows that $\text{gl.dim} \Lambda \in \{0,d\}$. This is a generalisation of the notion of $d$-representation finiteness which we use in [Pas17]. Without loss of generality, from now on we assume that $\Lambda$ is basic. We write $T$ for $T_\Lambda$ when the context allows it. Then [Iya11, Proposition 1.13] says that “$d$-representation finite” is the same as “$d$-complete with $T = \Lambda$”.

If $\Lambda$ is $d$-complete, then for every indecomposable injective $I_i$ there is a unique $l_i \in \mathbb{N}$ such that $\tau_d^{l_i-1} I_i \in \mathcal{P}$, and
\[ T_\Lambda = \bigoplus_i \tau_d^{l_i-1} I_i. \]

**Definition 2.4** ([HI11a]). We say that a basic $k$-algebra $\Lambda$ of global dimension $d$ is **$l$-homogeneous** if $\tau_d^{l-1} D\Lambda = T_\Lambda$. 
If $\Lambda$ is $d$-complete, this means that $l_i = l$ for every $i$.

## 3. Main result

We now consider the case where $A$ is $n$-complete, $B$ is $m$-complete, and $\Lambda = A \otimes \mathbb{k} B$. Since $\mathbb{k}$ is perfect, we have that $\text{gl. dim } \Lambda = \text{gl. dim } A + \text{gl. dim } B$. Moreover, by the K"unneth formula we have $\tau_{n+m} X \otimes Y = \tau_{n} X \otimes \tau_{m} Y$. Since indecomposable injective $\Lambda$-modules are of the form $X \otimes Y$, it follows that all indecomposable modules in $\mathcal{M}$ are of this form. Our main result is the following:

**Theorem 3.1.** Let $A, B$ be $n$- respectively $m$-complete acyclic $k$-algebras, with $k$ perfect. Then $A \otimes \mathbb{k} B$ is $(n + m)$-complete and acyclic.

Note that as far as the author is aware, there are no known examples of $d$-complete algebras which are not acyclic (this is Question 5.9 in [HIO14]).

This result can be applied inductively to construct $d$-complete algebras starting for example from hereditary representation finite algebras and taking tensor products. In this sense, it is similar in spirit to [Iya11, Theorem 1.14 and Corollary 1.16], where Iyama constructs towers of $d$-complete algebras (with increasing $d$) by taking iterated higher Auslander algebras. The algebra $A \otimes B$ is almost never $(n + m)$-representation finite by the characterisation given by Herschend and Iyama in [HI11a]. Our result specialises to their characterisation in the acyclic case:

**Corollary 3.2.** Let $A, B$ be $n$- respectively $m$-representation finite acyclic $k$-algebras, with $k$ perfect. Then the following are equivalent:

1. $A \otimes \mathbb{k} B$ is $(n + m)$-representation finite.
2. $\exists l \in \mathbb{N}$ such that $A$ and $B$ are $l$-homogeneous.

Moreover, in this case $A \otimes \mathbb{k} B$ is also $l$-homogeneous.

It should be noted that there is a choice involved in the definition we gave of $d$-completeness, namely that we take $\mathcal{M}$ to be the $\tau_d$-completion of $\text{add } D \Lambda$. We might as well take $\mathcal{M}$ to be the $\tau_{-d}$-completion of $\text{add } \Lambda$, and call $\Lambda$ $d$-cocomplete if it satisfies the dual conditions to $(A_d), (B_d), (C_d)$. Then $\Lambda$ is $d$-complete if and only if $\Lambda^{op}$ is $d$-cocomplete. Notice that $d$-representation finite is the same as $d$-complete and $d$-cocomplete with the same $\mathcal{M}$. However, if $A$ and $B$ are $n$- and $m$-representation finite, then $A \otimes B$ is $(n + m)$-complete and cocomplete, but in general not with the same $\mathcal{M}$.

## 4. Preparation

### 4.1. $d$-complete algebras.

Following [Iya11], we make some observations about $d$-complete algebras in general. Fix a finite-dimensional algebra $\Lambda$.

**Lemma 4.1.** If $\text{gl. dim } \Lambda \leq d$, the following are equivalent:

1. $\text{Ext}^i(M_P, \Lambda) = 0$ for $0 < i < d$
2. $\text{Ext}^i(M_P, \Lambda) = 0$ for $0 \leq i < d$.

**Proof.** The only direction to prove follows from [Iya11, Lemma 2.3(b)].

**Proposition 4.2.** If $\Lambda$ is $d$-complete, then

$$\text{Hom}(\tau^d_i \Lambda, \tau^d_j \Lambda) = 0$$

if $i < j$. 

Proof. This follows from [Iya11, Lemma 2.4(e)]. □

We can define slices $S(i)$ on $\mathcal{M}$ by saying that $S(i) = \text{add } \tau_d^i DA$. Thus

$$\mathcal{M} = \bigvee_{i \geq 0} S(i)$$

(meaning that every object $X \in \mathcal{M}$ can be written uniquely as $X = \bigoplus_{i \geq 0} X_i$ with $X_i \in S(i)$) and moreover $\text{Hom}(S(i), S(j)) = 0$ if $i < j$.

Lemma 4.3. If $\Lambda$ is $d$-complete then $\tau_d^\pm$ induce quasi-inverse equivalences $\mathcal{M}_P \leftrightarrow \mathcal{M}_I$.

Proof. This is [Iya11, Lemma 2.4(b)]. □

4.2. $d$-almost split sequences. In the spirit of generalising Auslander-Reiten theory, it is natural to define the higher analog of almost split sequences as follows.

Definition 4.1 (Iyama). A complex with objects in a subcategory $\mathcal{C}$ of $\text{mod } \Lambda$

$$C_d \xrightarrow{f_d} C_{d-1} \xrightarrow{f_{d-1}} C_{d-2} \xrightarrow{f_{d-2}} \cdots$$

is a source sequence (in $\mathcal{C}$) of $C_d$ if the following conditions hold:

1. $f_i \in \text{rad}(C_i, C_{i-1})$ for all $i$,
2. The sequence of functors

$$\cdots \xrightarrow{-\circ f_{d-2}} \text{Hom}(C_{d-2}, -) \xrightarrow{-\circ f_{d-1}} \text{Hom}(C_{d-1}, -) \xrightarrow{-\circ f_d} \text{rad}(C_d, -) \xrightarrow{} 0$$

is exact on $\mathcal{C}$.

Dually we can define sink sequences. An exact sequence

$$0 \longrightarrow C_{d+1} \longrightarrow C_{d-1} \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0$$

is a $d$-almost split sequence if it is a source sequence of $C_{d+1}$ and a sink sequence of $C_0$. We say that such $d$-almost split sequence starts in $C_{d+1}$ and ends in $C_0$.

Definition 4.2. We say that $\mathcal{M} = \mathcal{M}(\Lambda)$ has $d$-almost split sequences if for every indecomposable $X \in \mathcal{M}_I$ (respectively $Y \in \mathcal{M}_P$) there is an $d$-almost split sequence in $\mathcal{C}$

$$0 \rightarrow X \rightarrow C_d \rightarrow \cdots \rightarrow C_1 \rightarrow Y \rightarrow 0.$$  

In this case we must have $X \cong \tau_d Y, Y \cong \tau_d^- X$. This holds for $d$-complete algebras ([Iya11, Theorem 2.2(a)(i)]):

Theorem 4.4. If $\Lambda$ is $d$-complete, then $\mathcal{M}$ has $d$-almost split sequences.

To apply the methods introduced in [Pas17], we need to rephrase Definition 4.1 as follows: for any indecomposable $X \in \mathcal{C}$ we can define a functor $F_X$ on complexes of radical maps by mapping

$$C_\bullet = \cdots \xrightarrow{f_{i+1}} C_i \xrightarrow{f_i} \cdots \xrightarrow{f_1} C_0 \xrightarrow{f_0} \cdots$$

to

$$F_X(C_\bullet) = \cdots \xrightarrow{f_{i+1}^\circ} \text{Hom}(X, C_i) \xrightarrow{f_i^\circ} \cdots \xrightarrow{f_1^\circ} \text{rad}(X, C_0) \xrightarrow{f_0^\circ} \cdots$$
(that is, $F_X$ is the subfunctor of $\text{Hom}(X, -)$ given by replacing $\text{Hom}(X, C_0)$ with $\text{rad}(X, C_0)$). Similarly, we can define a subfunctor $G_X$ of the contravariant functor $\text{Hom}(-, X)$ by mapping $C_\bullet$ to

$$G_X(C_\bullet) = \cdots \xrightarrow{-\circ f_0} \text{Hom}(C_0, X) \xrightarrow{-\circ f_1} \cdots \xrightarrow{-\circ f_{d+1}} \text{rad}(C_{d+1}, X) \xrightarrow{-\circ f_{d+2}} \cdots$$

**Lemma 4.5.** Let $C_\bullet$ be a complex in $\mathcal{C}$. Then

1. If $C_i = 0$ for all $i > d + 1$, then $C_\bullet$ is a sink sequence if and only if $F_X(C_\bullet)$ is exact for every $X \in \mathcal{C}$.
2. If $C_i = 0$ for all $i < 0$, then $C_\bullet$ is a source sequence if and only if $G_X(C_\bullet)$ is exact for every $X \in \mathcal{C}$.
3. If $C_i = 0$ for all $i > d + 1$ and $i < 0$, then $C_\bullet$ is d-almost split if and only if $F_X(C_\bullet)$ and $G_X(C_\bullet)$ are exact for every $X \in \mathcal{C}$.

**Proof.** Direct check using the definitions.

By additivity, in the above Lemma we can replace “every $X \in \mathcal{C}$” by “every indecomposable $X \in \mathcal{C}$”.

Notice that since d-almost split sequences come from minimal projective resolutions of a functor $\text{rad}(C_0, -)$, they are uniquely determined by $C_0$ up to isomorphism of complexes. Moreover, we have

**Lemma 4.6.** Any map $f_0 : C_0 \to D_0$ between indecomposables in $\mathcal{M}_P$ induces a map of complexes $f_\bullet : C_\bullet \to D_\bullet$, where

$$C_\bullet = \cdots \xrightarrow{g_{d+1}} C_{d+1} \xrightarrow{g_i} \cdots \xrightarrow{g_1} C_0 \xrightarrow{0} 0,$$

$$D_\bullet = \cdots \xrightarrow{h_{d+1}} D_{d+1} \xrightarrow{h_i} \cdots \xrightarrow{h_1} D_0 \xrightarrow{0} 0$$

are the d-almost split sequences ending in $C_0$ and $D_0$ respectively, if these exist.

**Proof.** The map $f_0g_i : C_i \to D_0$ is a radical morphism, and since

$$\text{Hom}(C_1, D_1) \xrightarrow{h_1 \circ -} \text{rad}(C_1, D_0)$$

is surjective, there is a map $f_1 : C_1 \to D_1$ such that $h_1f_1 = f_0g_1$. Now assume we have constructed maps $f_j : C_j \to D_j$ that make all diagrams commute, for all $0 \leq j < i$ for some $i \geq 2$. We have that

$$\text{Hom}(C_i, D_i) \xrightarrow{h_i \circ -} \text{Hom}(C_i, D_{i-1}) \xrightarrow{h_{i-1} \circ -} \text{Hom}(C_i, D_{i-2})$$

is exact in the middle term by assumption. Since $h_{i-1}f_{i-1}g_i = f_{i-2}g_{i-1}g_i = 0$, we have that $f_{i-1}g_i \in \ker(h_{i-1} \circ -) = \text{im}(h_i \circ -)$, that is there is a map $f_i : C_i \to D_i$ such that $f_{i-1}g_i = h_if_i$. The $f_i$’s we have defined recursively give by construction a map of complexes $f_\bullet : C_\bullet \to D_\bullet$. □

The following is a result which appeared in [Pas17] in the setting of $d$-representation finite algebras, and which can be reformulated in the setting of $d$-complete algebras.

**Theorem 4.7.** Let $\Lambda$ be $d$-complete. Let $X \in \mathcal{S}(i)$ with $i > 0$. Then the $d$-almost split sequence starting in $X$ is isomorphic as a complex to $\text{Cone} \varphi$, where $\varphi : E_\bullet \to F_\bullet$ is a map of complexes, such that:

1. All the maps appearing in $E_\bullet$, $F_\bullet$, and the components of $\varphi$ are radical,
2. $E_j \in \mathcal{S}(i)$ and $F_j \in \mathcal{S}(i - 1)$ for every $j$. 


Lemma 4.10. Immediately get isomorphism.

VI.3.3.1): computations for tensor products is the Künneth formula over a field ([CE56, 4.3. Tensor products].

Since tensor products of projective resolutions are projective resolutions, we immediately get

Lemma 4.9. If $X_\bullet, Y_\bullet$ are complexes, then there is a functorial isomorphism

$$H_i(X_\bullet \otimes Y_\bullet) \cong \bigoplus_{p+q=i} H_p(X_\bullet) \otimes H_q(Y_\bullet).$$

We will need a technical lemma:

Lemma 4.8. Let

$$0 \longrightarrow C_{d+1} \xrightarrow{f_{d+1}} C_d \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{f_1} C_0 \longrightarrow 0$$

be a $d$-almost split sequence. Then for any choice of decomposition of the modules $C_i$ into indecomposables, the corresponding matrices of the maps $f_i$ have no zero column and no zero row.

Proof. We argue by contradiction. Assume $f_i$ has a zero column for $i < d$. Then there is a complex

$$C_{i+1} \xrightarrow{\begin{bmatrix} f_{i+1}^1 & \cdots & f_{i+1}^j \end{bmatrix}} C_i \oplus C_i^2 \xrightarrow{\begin{bmatrix} f_i^1 & 0 \end{bmatrix}} C_{i-1}$$

such that

$$\text{Hom}(C_i^2, C_{i+1}) \xrightarrow{\begin{bmatrix} f_i^1 \circ - \end{bmatrix}} \text{Hom}(C^2_i, C_i^1) \xrightarrow{\begin{bmatrix} f_i^1 \circ - \end{bmatrix}} \text{Hom}(C_i^2, C_{i-1})$$

is exact in the middle, which implies that $f_i^2 \circ -$ is surjective on $\text{Hom}(C_i^2, C_i^2)$, and so there is $h \in \text{Hom}(C_i^2, C_{i+1})$ such that $f_i^2 \circ h = \text{id}_{C_i^2}$. Since $f_i^2 \circ h \in \text{rad}(C_{i+1}, C_i)$, it follows that $C_i^2 = 0$ and we are done. For proving the case $i = d$, just replace $\text{Hom}(C_i^2, C_{i+1})$ with $\text{rad}(C_i^2, C_{i+1})$, and the argument goes through.

The dual argument, using the fact that $d$-almost split sequences are source, yields the second claim.

4.3. Tensor products. The main tool which allows us to perform homological computations for tensor products is the Künneth formula over a field ([CE56, VI.3.3.1]):

Lemma 4.9. If $X_\bullet, Y_\bullet$ are complexes, then there is a functorial isomorphism

$$H_i(X_\bullet \otimes Y_\bullet) \cong \bigoplus_{p+q=i} H_p(X_\bullet) \otimes H_q(Y_\bullet).$$

Since tensor products of projective resolutions are projective resolutions, we immediately get

Lemma 4.10. If $M_1, M_2 \in \text{mod } A$ and $N_1, N_2 \in \text{mod } B$, then there is a functorial isomorphism

$$\text{Ext}^i_A(M_1 \otimes N_1, M_2 \otimes N_2) \cong \bigoplus_{p+q=i} \text{Ext}^p_A(M_1, M_2) \otimes \text{Ext}^q_B(N_1, N_2).$$
The total tensor product of complexes is a functor in a natural way, so we can speak of tensor products of maps of complexes (for a very general treatment of how this is done, see [CE56, IV.4 and IV.5]). An important result which is proved in [Pas17] for $d$-representation finite algebras is also true for $d$-complete algebras, namely:

**Theorem 4.11.** Let $A, B$ be $n$- respectively $m$-complete algebras. Let $\text{Cone}(\varphi)$ and $\text{Cone}(\psi)$ be $n$- respectively $m$-almost split sequences starting in $\text{add} \tau_n^1 DA$ respectively $\text{add} \tau_m^1 DB$ for some common $i > 0$. Then $\text{Cone}(\varphi \otimes \psi)$ is an $(n + m)$-almost split sequence in $\mathcal{M}(A \otimes B)$.

**Proof.** This is proved in the same way as in [Pas17, Section 3.3]. For convenience, we present the main points of the proof. By definition $\text{Cone}(\varphi \otimes \psi)$ is a complex bounded between 0 and $n + m + 1$, it is exact by the Künneth formula, and it is easy to check that all maps appearing are radical. Now $\varphi : A^0_0 \to A_1^1$ and $\psi : B_0^0 \to B_1^1$, and by assumption we have that $A^0_j \in \text{add} \tau^1_n DA$, $A^1_1 \in \text{add} \tau^{1-1} DA$, $B^0_j \in \text{add} \tau^0_n DB$ and $B^1_1 \in \text{add} \tau^{1-1} DB$ for every $j$ since $A_j \otimes B_j \in \mathcal{M}(A \otimes B)$. Let now $M \otimes N$ be any indecomposable in $\mathcal{M}(A \otimes B)$. We need to prove that $F_{M \otimes N}(\text{Cone}(\varphi \otimes \psi))$ is exact. As in [Pas17, Section 2.3], for a radical map of radical complexes $\eta : A_0^0 \to B_1^1$ and a module $X$ we can define $\tilde{F}_X(\eta) = \eta \circ - : \text{Hom}(X, A_0^0) \to F_X(B_1^1)$. Then in our setting there is a commutative diagram

$$
\begin{array}{ccc}
\text{Hom}(M, A^0_0) \otimes \text{Hom}(N, B^0_0) & \rightarrow & \text{Hom}(M \otimes N, A^0_0 \otimes B^0_0) \\
\tilde{F}_M(\varphi) \otimes \tilde{F}_N(\psi) & & F_{M \otimes N}(\varphi \otimes \psi) \\
F_M(A^1_0) \otimes F_N(B^1_1) & \rightarrow & F_{M \otimes N}(A^1_0 \otimes B^1_1).
\end{array}
$$

Now $F_{M \otimes N}(\text{Cone}(\varphi \otimes \psi))$ is exact if and only if $\tilde{F}_{M \otimes N}(\varphi \otimes \psi)$ is a quasi-isomorphism. The left map in the diagram $\tilde{F}_M(\varphi) \otimes \tilde{F}_N(\psi)$ is a quasi-isomorphism since $\text{Cone}(\varphi)$ and $\text{Cone}(\psi)$ are $n$- respectively $m$-almost split sequences. Then it is enough to prove that the bottom map is a quasi-isomorphism, and this is done by showing that its cokernel is isomorphic to

$$
F_M(A^1_0) \otimes \text{top}(N, B^1_0) \oplus \text{top}(M, A^1_0) \otimes F_N(B^1_1)
$$

and then by easy verification that the above cokernel is exact. The computation of the cokernel is done explicitly in [Pas17, Section 3.3, pp.660–662].

**Corollary 4.12.** Let $A, B$ be $n$- respectively $m$-complete algebras. Then $\mathcal{M}(A \otimes B)$ has $(n + m)$-almost split sequences.

Notice that the above theorem does not require the algebra $A \otimes B$ to be $(n + m)$-representation finite (in which case we know a priori that $(n + m)$-almost split sequences must exist). In the setting of [Pas17], this result is about describing the structure of such sequences. In the setting of $d$-complete algebras, this result is used to prove that $(n + m)$-almost split sequences exist, whereas it is a priori not clear that they should.

One can also say something about injective modules (which are not the starting point of any $d$-almost split sequence).
Proposition 4.13. Let $A, B$ be $n$- respectively $m$-complete algebras, and let $\Lambda = A \otimes B$. Then for every injective $\Lambda$-module $X \otimes Y$ there is a source sequence

$$X \otimes Y \to E_{n+m} \to \cdots \to E_1 \to 0$$

in $\mathcal{M}(\Lambda)$.

Proof. Since $X$ and $Y$ are injective, we have sequences in $\mathcal{M}(A)$ respectively $\mathcal{M}(B)$

$$X_\bullet = X \to C_n \to \cdots \to C_1 \to 0$$
$$Y_\bullet = Y \to D_m \to \cdots \to D_1 \to 0$$

such that

$$0 \to \text{Hom}(C_1, M) \to \cdots \to \text{Hom}(X, M) \to \text{top}(X, M) \to 0,$$
$$0 \to \text{Hom}(D_1, N) \to \cdots \to \text{Hom}(Y, N) \to \text{top}(Y, N) \to 0$$

are exact for all indecomposables $M, N$. Now consider the homology of $X_\bullet \otimes Y_\bullet$.

$$H_i(X_\bullet \otimes Y_\bullet) = \bigoplus_{p+q=i} H_p(X_\bullet) \otimes H_q(Y_\bullet) = \begin{cases} H_0(X_\bullet) \otimes H_0(Y_\bullet) & \text{if } i = n + m + 2 \\ 0 & \text{else.} \end{cases}$$

So we have at least an exact sequence

$$X_\bullet \otimes Y_\bullet = X \otimes Y \to \cdots \to C_1 \otimes D_1 \to 0.$$

Apply $\text{Hom}(\_, M \otimes N)$ to this sequence and compute homology.

$$H_i(\text{Hom}(X_\bullet \otimes Y_\bullet, M \otimes N)) = H_i(\text{Hom}(X_\bullet, M) \otimes \text{Hom}(Y_\bullet, M)) =$$

$$= \bigoplus_{p+q=i} H_p(\text{Hom}(X_\bullet, M)) \otimes H_q(\text{Hom}(Y_\bullet, M)) =$$

$$= \begin{cases} \text{top}(X, M) \otimes \text{top}(Y, N) & \text{if } i = 0 \\ 0 & \text{else.} \end{cases}$$

We will be done if we prove that $X_\bullet \otimes Y_\bullet$ is source, which amounts now to prove that

$$\text{top}(X \otimes Y, M \otimes N) = H_0(\text{Hom}(X_\bullet \otimes Y_\bullet, M \otimes N)) = \text{top}(X, M) \otimes \text{top}(Y, N).$$

By tensoring the complexes

$$0 \to \text{rad}(X, M) \to \text{Hom}(X, M)$$

and

$$0 \to \text{rad}(Y, N) \to \text{Hom}(Y, N)$$

and looking at homology, one finds an exact sequence

$$0 \to \text{rad}(X, M) \otimes \text{Hom}(Y, N) + \text{Hom}(X, M) \otimes \text{rad}(Y, N) \to$$

$$\to \text{Hom}(X, M) \otimes \text{Hom}(Y, N) \to \text{top}(X, M) \otimes \text{top}(Y, N) \to 0.$$

Now the middle term is isomorphic to $\text{Hom}(X \otimes Y, M \otimes N)$, and this isomorphism induces an isomorphism between the first term and $\text{rad}(X \otimes Y, M \otimes N)$, hence by
looking at cokernels we get
\[ \text{top}(X \otimes Y, M \otimes N) \cong \frac{\text{Hom}(X \otimes Y, M \otimes N)}{\text{rad}(X \otimes Y, M \otimes N)} \cong \frac{\text{Hom}(X, M) \otimes \text{Hom}(Y, N)}{\text{rad}(X, M) \otimes \text{Hom}(Y, N) + \text{Hom}(X, M) \otimes \text{rad}(Y, N)} \cong \text{top}(X, M) \otimes \text{top}(Y, N) \]
and we are done.

\[ \square \]

**Lemma 4.14.** Let $A$, $B$ be $n$-respectively $m$-complete algebras. Then the following are equivalent:

1. $T_{A \otimes B} \cong T_A \otimes T_B$.
2. \( \exists \ell \in \mathbb{N} \) such that $A$ and $B$ are $\ell$-homogeneous.

**Proof.** (2) \( \Rightarrow \) (1) is clear by definition.

To prove (1) \( \Rightarrow \) (2), assume it does not hold, that is $T_{A \otimes B} \cong T_A \otimes T_B$ but there are $i, j$ such that $l_i \neq l_j$ for the corresponding indecomposable injectives $E_i \in \text{add} DA$ and $F_j \in \text{add} DB$. We can assume that $l_i > l_j$, otherwise the proof is similar. Call $X_{ij} = \tau_n^{i-1} E_i \otimes \tau_m^{j-1} F_j \in \text{add} T_{A \otimes B}$. Then
\[ \tau_{m+n}^{-l_i+1}(X_{ij}) = \tau_n^{i-1} E_i \otimes F_j \]
is not injective, since by assumption $\tau_n^{i-1} E_i$ is not injective. On the other hand, modules in $\mathcal{M}(A \otimes B)$ which satisfy $\tau_{m+n} X = 0$ are precisely the injective $A \otimes B$-modules, and so $\tau_{m+n}^{-l_i+1}(X_{ij})$ is not in $\mathcal{M}$, contradiction.

\[ \square \]

4.4. **Acyclicity.** We need to discuss what we mean by acyclic algebras. Let $M \in \text{mod} \Lambda$, and let $C = \text{add} M$. We want to define a preorder on the indecomposable objects $\text{ind} C$ of $C$. For $X, Y \in \text{ind} C$, we say $X < Y$ if there is a sequence $(X = X_0, X_1, \ldots, X_{m+1} = Y)$ for some $m \geq 0$, such that $X_i \in \text{ind} C$ and $\text{rad}_\Lambda(X_i, X_{i+1}) \neq 0$ for all $i$. This defines a transitive relation $< \text{on ind} C$. Notice that we can replace $\text{rad}_\Lambda(X_i, X_{i+1}) \neq 0$ with $\text{rad}_C(X_i, X_{i+1}) \neq 0$ above.

**Definition 4.3.** The category $C$ is directed if $< \text{is antisymmetric}, \text{that is if no indecomposable module } X \in C \text{satisfies } X < X$. If $C = \text{add} M$, we say that $M$ is directed. We call the algebra $\Lambda$ acyclic if $\Lambda_{\Lambda}$ is directed.

**Lemma 4.15.** The module $\Lambda_{\Lambda}$ is directed if and only if the module $D_{\Lambda} \Lambda$ is directed.

**Proof.** The Nakayama functor induces an equivalence $\nu : \text{add} \Lambda_{\Lambda} \rightarrow \text{add} D_{\Lambda} \Lambda$, and the definition of directedness is invariant under equivalence.

\[ \square \]

**Lemma 4.16.** Let $\Lambda$ be $d$-complete. Then $\Lambda$ is acyclic if and only if $\mathcal{M}$ is directed.

**Proof.** If $\mathcal{M}$ is directed, then so is $\text{add} D \Lambda \subseteq \mathcal{M}$. By Lemma 4.15, $\Lambda$ is then acyclic.

Conversely, if $\Lambda$ is acyclic then $\text{add} D \Lambda$ is directed by Lemma 4.15, and then so is $\tau_i^{d} D \Lambda$ for any $i$ by Lemma 4.3. Any nonzero map between indecomposables in $\mathcal{M}$ is either within a slice $S(i) = \text{add} \tau_i^{d} D \Lambda$ or from $S(i)$ to $S(j)$ with $j < i$. Therefore there can be no cycles within a slice nor cycles that contain modules from different slices and $\mathcal{M}$ is directed.

\[ \square \]

The relation we have introduced is well suited to study $d$-almost split sequences.
Lemma 4.17. Let $\Lambda$ be $d$-complete, and let
\[ 0 \rightarrow \tau_d X \rightarrow C_d \rightarrow \cdots \rightarrow C_1 \rightarrow X \rightarrow 0 \]
be a $d$-almost split sequence in mod $\Lambda$. Then for every indecomposable summand $Y$ of $\bigoplus_{i=1}^{d} C_i$, we have
\[ \tau_d X < Y < X. \]

Proof. This follows directly from Lemma 4.8 and the definition of $<$. \hfill $\Box$

Let us now consider acyclicity in relation to tensor products.

Lemma 4.18. The algebras $A$ and $B$ are acyclic if and only if $\Lambda = A \otimes B$ is acyclic.

Proof. Let us first remark that for $X, X' \in \text{mod } A$ and $Y, Y' \in \text{mod } B$ we have
\[ \text{rad}(X \otimes Y, X' \otimes Y') = \text{rad}(X, X') \otimes \text{Hom}(Y, Y') + \text{Hom}(X, X') \otimes \text{rad}(Y, Y') \]
by [Pas17, Lemma 3.6]. Assume $X < X$ in add $A$ via $X_1, \ldots, X_m$. Then for an indecomposable $P \in \text{add } B$ we have that $X \otimes P < X \otimes P$ via $X_1 \otimes P, \ldots, X_m \otimes P$ since
\[ \text{rad}(X_i \otimes P, X_{i+1} \otimes P) \supseteq \text{rad}(X_i, X_{i+1}) \otimes \text{End}(P) \neq 0 \]
for all $i$. Therefore if $\Lambda$ is acyclic then $A$ is acyclic. By symmetry, if $\Lambda$ is acyclic then $B$ is acyclic as well.

Let us now prove the converse implication. Assume that $X \otimes Y < X \otimes Y$ in add $\Lambda$ via $X_1 \otimes Y_1, \ldots, X_m \otimes Y_m$. We can assume that $\text{rad}(X, X) = 0 = \text{rad}(Y, Y)$. Moreover, it cannot be that $X_i \cong X$ for all $i$ and that $Y_j \cong Y$ for all $j$. Without loss of generality, assume that $X_i \not\cong X$ for some $i$. We will prove that $X < X$ via a subsequence $(Z_j)$ of the $X_i$'s. We have that $\text{Hom}(X_i, X_{i+1}) \neq 0$ for all $i$ by assumption. Set $Z_0 = X$ and $Z_j = X_i$, where $i = \min \{l \mid X_l \not\cong Z_{j-1} \}$ for $j > 0$. By construction, $Z_p = X$ for some $p$ (and for $j > p$, $Z_j$ is not defined). Then we are done, since by construction $\text{Hom}(Z_i, Z_{i+1}) \neq 0$ and $Z_i \not\cong Z_{i+1}$ so that $\text{rad}(Z_i, Z_{i+1}) \neq 0$ since $Z_i, Z_{i+1}$ are indecomposable. \hfill $\Box$

5. Proof of Main Result

From now on, let $A$ be $n$-complete acyclic, let $B$ be $m$-complete acyclic and let $\Lambda = A \otimes_k B$. We use the notation of Definition 2.3. There are three conditions that need to be checked to prove the main theorem (since we saw in Lemma 4.18 that $\Lambda$ is acyclic), namely that properties $(A_d), (B_d), (C_d)$ in Definition 2.3 are preserved under tensor products.

Proposition 5.1. $\text{Ext}^i_{\Lambda}(M, M) = 0$ for $0 < i < n + m$.

Proof. Let $X \otimes Y \in M_P$. We have for $i < n + m$
\[ \text{Ext}^i(X \otimes Y, A \otimes B) = \bigoplus_{p+q=i} \text{Ext}^p(X, A) \otimes \text{Ext}^q(Y, B) = 0 \]
so we conclude by [Iya11, Proposition 2.5 (a)]. \hfill $\Box$

By the same formula, $\Lambda$ satisfies condition $(C_{n+m})$:

Lemma 5.2. $\text{Ext}^i(M_P, \Lambda) = 0$ for all $0 < i < n + m$. 

\[ \square \]
Proof. Use the same formula as in Proposition 5.1.

Notice that since \( \tau_{n+m} = \tau_n \otimes \tau_m \) on \( \mathcal{M} \), for sufficiently big \( l \) we have \( \tau^{i_l}_{n+m}DA = 0 \), so \( \mathcal{M} \) has an additive generator.

We now start proving that condition \((A_{n+m})\) holds.

For \( S = S_1 \oplus S_2 \) with \( S_1 \in \text{add} T \) and \( S_2 \in \mathcal{M}_P \), define \( ES = S_1 \oplus \tau_{n+m}S_2 \). Note that \( E^lDA = T \) for \( l \gg 0 \). Now fix \( S = E^lDA \) for some \( i \geq 0 \). To check condition \((A_{n+m})\) for \( \Lambda \), we need some preliminaries.

**Lemma 5.3.** If \( \text{Ext}^i(S, S) = 0 \) for all \( i \neq 0 \), then \( \text{Ext}^i(ES, ES) = 0 \) for all \( i \neq 0 \).

**Proof.** Since \( \text{Ext}_A^i(\mathcal{M}, \mathcal{M}) = 0 \) for \( 0 < i < n+m \), it suffices to check that \( \text{Ext}^{n+m}(ES, ES) = 0 \). Since \( ES = S_1 \oplus \tau_{n+m}S_2 \), consider first \( M_1 \otimes N_1 \in \text{add} S_1 \) and \( M_2 \otimes N_2 \in \text{add} ES \). Then

\[
\text{Ext}^{n+m}(M_1 \otimes N_1, M_2 \otimes N_2) = \text{Ext}^n(M_1, M_2) \otimes \text{Ext}^m(N_1, N_2) = 0
\]

since \( M_1 \otimes N_1 \in \text{add} S_1 \subseteq \text{add} T \) implies that either \( M_1 \) or \( N_1 \) is relative projective in \( T^+_B \) respectively \( T^-_B \). This proves that \( \text{Ext}^{n+m}(S_1, ES) = 0 \). Now let \( Y \) be an indecomposable summand of \( ES \), and consider \( \text{Ext}^{n+m}(\tau_{n+m}S_2, Y) \). If \( Y \) is injective, then this is 0. Otherwise, \( Y = \tau_{n+m}^{-1}Y \) and

\[
\text{Ext}^{n+m}(\tau_{n+m}S_2, Y) = \text{Ext}^{n+m}(S_2, \tau_{n+m}Y) = 0
\]

by the assumption. \( \square \)

**Lemma 5.4.** If \( S \) is tilting then thick \( ES = \mathcal{D}^b(\Lambda) \).

**Proof.** Set \( S = \text{add} S \). For \( X \in \text{ind} S \), define \( h(X) \) to be the height of \( X \) with respect to the partial order introduced in Section 4.4 on \( \text{ind} S \) (here it is crucial that \( \Lambda \) be acyclic, which follows from the assumptions on \( A \) and \( B \) and Lemma 4.18), that is

\[
h(X) = \max \{ n \mid \exists Y_0 < \cdots < Y_n = X, Y_i \in \text{ind} S \}.
\]

Notice that \( X > Y \) implies \( h(X) > h(Y) \), and the reverse implication holds provided that \( X \) and \( Y \) are comparable. Call \( C_i = \text{add} \left( \{ ES \} \cup \{ Y \in \text{ind} S \mid h(Y) < i \} \right) \). For \( X \in \text{ind} S \), if \( \tau_{n+m}X = 0 \) then \( X \in \text{add} ES \). Otherwise, there is an \((n+m)\)-almost split sequence

\[
0 \rightarrow \tau_{n+m}X \rightarrow \cdots \rightarrow X \rightarrow 0
\]

whose middle terms are in \( \text{add} \left( \{ ES \} \cup \{ Y \in \text{ind} S \mid Y < X \} \right) \) by Lemma 4.17. In particular if \( h(X) \leq i \) then the middle terms in the sequence are in

\[
\text{add} \left( \{ ES \} \cup \{ Y \in \text{ind} S \mid h(Y) < i \} \right) = C_i.
\]

It follows that thick \( C_{i+1} \subseteq \text{thick} C_i \), so thick \( C_j \subseteq \text{thick} C_0 \) for every \( j \). Now \( C_0 = \text{add} ES \), and \( C_j = \text{add} (ES \oplus S) \) for some \( j \), so we get that thick \( ES = \text{thick} C_0 = \text{thick} C_j = \mathcal{D}^b(\Lambda) \) as claimed. \( \square \)

**Theorem 5.5.** \( T = T_{A \otimes B} \) is tilting.

**Proof.** By Lemma 5.3 and Lemma 5.4, if \( S = E^lDA \) is tilting then \( ES = E^{i+1}DA \) is tilting. Since \( DA \) is tilting, and \( T = E^lDA \) for some \( l \), it follows that \( T \) is tilting. \( \square \)

Now we start proving that condition \((B_{n+m})\) holds. We will use the following result (this is [Iya11, Theorem 2.2(b)]):

\[

\]
Theorem 5.6. Let $\Lambda$ be a finite-dimensional $k$-algebra, $d \geq 1$ and $T \in \operatorname{mod} \Lambda$ a tilting module with $\operatorname{proj} \dim T \leq d$. Let $C = \operatorname{add} C$ be a subcategory of $T^\perp$ such that $\operatorname{Ext}^i_{\Lambda}(C, C) = 0$ for $0 < i < d$ and $T \oplus D\Lambda \in C$. Then the following are equivalent:

1. $C$ is a $d$-cluster tilting subcategory in $T^\perp$.
2. Every indecomposable $X \in C$ has a source sequence of the form
   \[ X \to C_d \to \cdots \to C_0 \to 0 \]
   with $C_i \in C$ for all $i$.

We want to apply this to $\Lambda = A \otimes B$, $C = M$, $T = T_{A \otimes B}$ and $d = n + m$.

Lemma 5.7. $M \subseteq T^\perp$.

Proof. By Proposition 5.1, it is enough to check that $\operatorname{Ext}^{n+m}(T, M) = 0$. Let $M_1 \otimes M_2 \in \operatorname{add} T$. Then either $M_1$ or $M_2$ is relative projective in $T^+_A$ respectively $T^+_B$, so
   \[ \operatorname{Ext}^{n+m}(M_1 \otimes M_2, N_1 \otimes N_2) = \operatorname{Ext}^n(M_1, N_1) \otimes \operatorname{Ext}^m(M_2, N_2) = 0 \]
   for any $N_1 \otimes N_2 \in T$. \qed

Theorem 5.8. $M$ is an $(n+m)$-cluster tilting subcategory of $T^\perp$.

Proof. By Proposition 5.1 and Lemma 5.7, we can take $\Lambda = A \otimes B$, $C = M$, $T = T_{A \otimes B}$ and $d = n + m$ in the assumptions of Theorem 5.6. By Corollary 4.12 and Proposition 4.13, condition (2) is satisfied. Our claim is then the equivalent statement (1). \qed

Now we have established everything we need to prove the main result.

Proof of Theorem 3.1. By Theorem 5.5, Theorem 5.8, and Lemma 5.2, we have that $A \otimes B$ satisfies the conditions $(A_{n+m}, (B_{n+m}), (C_{n+m})$ in the definition of $(n+m)$-complete algebra. By Lemma 4.18, $A \otimes B$ is acyclic. \qed

Proof of Corollary 3.2. By Theorem 3.1, $A \otimes B$ is $(n+m)$-complete. By [Iya11, Proposition 1.13], we have that $T_A \cong A$, $T_B \cong B$ and that $A \otimes B$ is $(n+m)$-representation finite if and only if $T_{A \otimes B} \cong A \otimes B$. By Lemma 4.14, this happens if and only if $A$ and $B$ are $l$-homogeneous for some common $l$. \qed

6. Examples

Let us consider one of the simplest non-trivial examples. Let $A = B = kQ$, where $Q$ is the quiver

\[
\begin{array}{c}
1 \\
\hline \\
2
\end{array}
\]

Then $\Lambda = A \otimes B$ is the quiver algebra of a commutative square. This algebra is 2-complete, since the factors are 1-representation finite. It is not 2-representation finite since the factors are not homogeneous. However, $\Lambda$ is representation finite, so we can draw the entire Auslander-Reiten quiver of $\Lambda$. We represent modules by their dimension vector.
In this case,

\[ T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \]

and

\[ M = \text{add } M = \text{add}(T \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}). \]

One can explicitly compute all Ext-groups of all pairs of indecomposables, since we have only finitely many. If we represent by \( \otimes \) the indecomposables in \( \text{add } T \), by \( \odot \) the ones in \( M \) but not in \( \text{add } T \), by \( \blacksquare \) the ones in \( T^{\perp} \) but not in \( M \), and by \( \cdot \) the ones outside \( T^{\perp} \), we get the following picture:

It can be checked that both the indecomposable modules in \( T^{\perp} \setminus M \) have extensions with \( M \) on both sides, as it is required by the definition of 2-cluster tilting. Here we find that \( M \) is 2-cluster tilting in \( T^{\perp} \).

The Auslander-Reiten quiver of \( \text{add}(M) \) is given by

and this is also a picture of the only 2-almost split sequence we have.

As a second example, consider the quiver \( Q' \):

\[ 2 \]

\[ 3 \rightarrow 1 \]

\[ 4 \]

\[ \]
and the corresponding path algebra \( A' = kQ' \). The Auslander-Reiten quiver of \( A' \) looks like

\[
\begin{array}{c}
P_2 \\
\downarrow \\
P_1 \\
\downarrow \\
P_3 \\
\downarrow \\
N_1 \\
\downarrow \\
I_1 \\
\downarrow \\
I_2 \\
\end{array}
\quad
\begin{array}{c}
P_4 \\
\downarrow \\
P_4 \\
\downarrow \\
P_3 \\
\downarrow \\
N_3 \\
\downarrow \\
I_3 \\
\downarrow \\
I_4
\end{array}
\]

We take \( B' = kQ'' \), where \( Q'' \) is the quiver

\[
a \leftarrow b \leftarrow c
\]

The Auslander-Reiten quiver of \( B' \) looks like

\[
P_c = I_a
\]

These algebras are both 1-representation finite, so in particular they are 1-complete. Their tensor product \( \Lambda' = A' \otimes B' \) is therefore 2-complete. It is not 2-representation finite since \( B' \) is not homogeneous. In this example, we cannot draw the entire module category of \( \Lambda' \), but we still have complete control over the “higher Auslander-Reiten quiver” of \( \Lambda' \), that is the Auslander-Reiten quiver of \( \text{add}(M) \):

Here the dashed arrows represent \( \tau_2 \), and we have drawn them only between some modules to avoid clogging the picture. We have again written \( \otimes \) for indecomposable
summands of $T$, and $\odot$ for the other indecomposable summands of $M$. It should be clear from the picture which module corresponds to which node.

Notice that this example presents some regularity which is not to be expected in general, since we have taken $\Lambda'$ to be homogeneous. Moreover, in this example (and in general) we cannot directly check that arbitrary modules in $\text{mod} \Lambda'$ which are in $T^\perp$ have extensions on both sides with $M$.

Acknowledgement. The author is especially thankful to his advisor Martin Herschend for the many helpful comments and suggestions.

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