Static Analysis of Power Systems

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Preface to third English edition

The first Swedish edition of this compendia was written during the summer 1991. The compendia has since then been used in different courses at the Department of Electric Power Engineering at the Royal Institute of Technology in Stockholm, at the Universities of Luleå and Kalmar, and at ABB T&D University.

The author wants to acknowledge professor Göran Andersson, all graduate and undergraduate students and teachers in Luleå and Kalmar for all valuable comments and suggestions to improvements and extensions.

A special acknowledge to Dr. Stefan Arnborg for his excellent translation from Swedish to English.

Lennart Söder
Preface to third English edition
Chapter 1

Introduction

In this compendium, models and mathematical methods for static analysis of power systems are discussed.

In chapter 2, the design of the power system is described and in chapter 3, the fundamental theory of alternating current is presented. Models of overhead power lines and transformers are given in chapter 4, and in chapter 5 some important theorems in three-phase analysis are discussed. In chapter 6 and 7, power system calculations in symmetrical conditions are performed by using the theorems presented earlier.

Chapter 5–7 are based on the assumption that the power system loads can be modeled as impedance loads. This leads to a linear formulation of the problem, and by that, relatively easy forms of solution methods can be used. In some situations, it is more accurate to model the system loads as power loads. How the system analysis should be carried out in such conditions is elaborated on in chapter 8.

In chapter 9, an overview of linear transformations in order to simplify the power system analysis, is given. In chapter 10–13, the basic concepts of analysis in un-symmetrical conditions are given. The use of symmetrical components is presented in detail in chapter 10. The modeling of lines, cables and transformers must be more detailed in un-symmetrical conditions, this is the topic of chapter 11–12. In chapter 13, an un-symmetrical, three-phase power system with impedance loads is analyzed.

In chapter 3–13, it is assumed that the power system frequency is constant and the system components are linear, i.e. sinusoidal voltages gives sinusoidal currents. With non-linear components in the system, as high power electronic devices, non-sinusoidal currents and voltages will appear. The consequences such non-sinusoidal properties can have on the power system will be discussed briefly in chapter 14.
1. Introduction
Chapter 2
Power system design

2.1 The development of the Swedish power system

The Swedish power system started to develop around a number of hydro power stations, Porjus in Norrland, Ålvkarleby in eastern Svealand, Motala in the middle of Svealand and Trollhättan in Götaaland, at the time of the first world war. Later on, coal fired power plants located at larger cities as Stockholm, Göteborg, Malmö and Västerås came into operation. At the time for the second world war, a comprehensive proposal was made concerning exploitation of the rivers in the northern part of Sweden. To transmit this power to the middle and south parts of Sweden, where the heavy metal industry were located, a 220 kV transmission system was planned.

Today, the transmission system is well developed with a nominal voltage of 220 or 400 kV. In rough outline, the transmission system consists of lines, transformers and sub-stations.

A power plant can have an installed capacity of more than 1000 MW, e.g. the nuclear power plants Forsmark 3 and Oskarshamn 3, whereas an ordinary private consumer can have an electric power need of some kW. This implies that electric power can be generated at some few locations but the consumption, which shows large variations at single consumers, can be spread all over the country. In Figure 2.1, the electricity supply in Sweden during the last 50 years is given. The hydro power was in the beginning of this period the dominating source of electricity until the middle of the 1960s when some conventional thermal power plants (oil fired power plants, industrial back pressure, etc.) were taken into service. In the beginning of the 1970s, the first nuclear power plants were taken into operation and this power source

![Figure 2.1. Electricity supply in Sweden 1947–2000](image-url)
has ever after being the one showing the largest increase in generated electric energy. In recent years, the trend showing a continuous high increase in electric power consumption has been broken. On 30th November 1999, the 600 MW nuclear power plant Barsebäck 1 was closed due to a political decision. In Table 2.1, the electricity supply in Sweden in 2000 is given.

<table>
<thead>
<tr>
<th>Source of power</th>
<th>Energy generation TWh = 10^9 kWh</th>
<th>Installed capacity 00-12-31 MW</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hydro</td>
<td>77.8</td>
<td>16 229</td>
</tr>
<tr>
<td>Nuclear</td>
<td>54.8</td>
<td>9 439</td>
</tr>
<tr>
<td>Industrial back pressure</td>
<td>4.3</td>
<td>932</td>
</tr>
<tr>
<td>Combined heat and power</td>
<td>4.2</td>
<td>2 264</td>
</tr>
<tr>
<td>Oil fired condensing power</td>
<td>0.2</td>
<td>448</td>
</tr>
<tr>
<td>Gas turbine</td>
<td>0.003</td>
<td>1 341</td>
</tr>
<tr>
<td>Wind power</td>
<td>0.4</td>
<td>241</td>
</tr>
<tr>
<td>Total</td>
<td>141.9</td>
<td>30 894</td>
</tr>
</tbody>
</table>

Table 2.1. Electricity supply in Sweden 2000

The total consumption of electricity is usually grouped into different categories. In Figure 2.2, the consumption during the latest 45 years is given for different groups.

As shown in the figure, the major increase in energy need has earlier been dominated by the industry. When the nuclear power was introduced in the early 1970s, the electric space heating increased significantly. Before 1965, the electric space heating was included in the group miscellaneous. Communication, i.e. trains, trams and subway, has increased its consumption from 1.4 TWh/year in 1950 to 2.2 TWh/year in 1995.
In proportion to the total electricity consumption, the communication group has decreased from 7.4 % to 1.6 % during the same period. The losses on the transmission and distribution systems have during the period 1950–1995 decreased from more than 10 % of total consumption to approximately 7 %.

2.2 The structure of the electric power system

A power system consist of generation sources which via power lines and transformers transmits the electric power to the end consumers.

The power system between the generation sources and end consumers is divided into different parts according to Figure 2.3.

![Diagram of the electric power system]

The transmission network, connects the main power sources and transmits a large amount of electric energy. The Swedish transmission system consists of approximately 15250 km power lines, and is connected to other countries on 23 different locations. In Figure 2.4, a general map of the transmission system in Sweden and neighboring countries is given. The primary task for the transmission system is to transmit energy from generation areas to load areas. To achieve a high degree of efficiency and reliability, different aspects must be taken into account. The transmission system should for instance make it possible to optimize the generation within the country and also support trading with electricity with neighboring countries. It is also necessary to withstand different disturbances such as disconnection of transmission lines, lightning storms, outage of power plants as well as unexpected growth in power demand without reducing the quality of the electricity services. As shown in Figure 2.4, the transmission system is meshed, i.e. there are a number of closed loops in the transmission system.
A new state utility, Svenska Kraftnät, was launched on January 1, 1992, to manage the national transmission system and foreign links in operation at date. Svenska Kraftnät owns all 400 kV lines, all transformers between 400 and 220 kV and the major part of the 220 kV lines in Sweden. Note that the Baltic Cable between Sweden and Germany was taken into operation after the day Svenska Kraftnät was launched and is therefore not owned by them.

Sub-transmission network, in Sweden also called regional network, has in each load region the same or partly the same purpose as the transmission network. The amount of energy transmitted and the transmission distance are smaller compared with the transmission network which gives that technical-economical constraints implies lower system voltages. Regional networks are usually connected to the transmission network at two locations.

Distribution network, transmits and distributes the electric power that is taken from the substations in the sub-transmission network and delivers it to the end users. The distribution network is in normal operation a radial network, i.e. there is only one path from the sub-transmission sub-station to the end user.
The electric power need of different end users varies a lot as well as the voltage level where the end user is connected. Generally, the higher power need the end user has, the higher voltage level is the user connected to.

The nominal voltage levels (Root Mean Square (RMS) value for tree-phase phase-to-phase voltages) used in distribution of high voltage electric power is normally lower compared with the voltage levels used in transmission. In Figure 2.5, the voltage levels used in Sweden are given. In special industry networks, except for levels given in Figure 2.5, also the voltage 660 V as well as the non-standard voltage 500 V are used. Distribution of low voltage electric power to end users is usually performed in three-phase lines with a zero conductor, which gives the voltage levels 400/230 V (phase-to-phase/phase-to-ground voltage).

<table>
<thead>
<tr>
<th>Nominal voltage kV</th>
<th>Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>ultra high voltage (UHV)</td>
</tr>
<tr>
<td>800</td>
<td>extra high voltage (EHV)</td>
</tr>
<tr>
<td>400</td>
<td></td>
</tr>
<tr>
<td>220</td>
<td></td>
</tr>
<tr>
<td>132</td>
<td></td>
</tr>
<tr>
<td>66</td>
<td></td>
</tr>
<tr>
<td>45</td>
<td></td>
</tr>
<tr>
<td>33</td>
<td>high voltage</td>
</tr>
<tr>
<td>22</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>only industry network</td>
</tr>
<tr>
<td>6.6</td>
<td>only</td>
</tr>
<tr>
<td>3.3</td>
<td>only</td>
</tr>
<tr>
<td>400/230 V</td>
<td>low voltage</td>
</tr>
</tbody>
</table>

Figure 2.5. Standard voltage level for transmission and distribution. In Sweden, 400 kV is the maximum voltage.
2. Power system design
Chapter 3
Alternating voltage

In this chapter, the fundamental properties of alternating voltage, alternating current and power under symmetrical conditions are summarized.

3.1 Single-phase alternating voltage

Assume that a source of alternating voltage feeds an impedance according to Figure 3.1. The alternating voltage \( u(t) \) gives rise to an alternating current \( i(t) \). These properties are varying with time as

\[
\begin{align*}
    u(t) &= U_M \cos \omega t \\
    i(t) &= I_M \cos(\omega t - \phi)
\end{align*}
\]

where

\[
\begin{align*}
    U_M &= \text{peak value of the voltage} \\
    I_M &= \frac{U_M}{|Z|} = \frac{U_M}{\bar{Z}} = \text{peak value of the current} \\
    \omega &= 2\pi f \text{ where } f \text{ is the frequency} \\
    \phi &= \arctan \frac{X}{R} = \text{phase angle between voltage and current}
\end{align*}
\]

The power consumed by the impedance \( \bar{Z} \) in Figure 3.1 can be calculated as

\[
\begin{align*}
p(t) &= u(t) \cdot i(t) = U_M I_M \cos \omega t \cos(\omega t - \phi) = \\
    &= U_M I_M \cos \omega t \left[ \cos \omega t \cos \phi + \sin \omega t \sin \phi \right] = \\
    &= \frac{U_M I_M}{\sqrt{2} \sqrt{2}} \left[ (1 + \cos 2\omega t) \cos \phi + \sin 2\omega t \sin \phi \right] = \\
    &= P(1 + \cos 2\omega t) + Q \sin 2\omega t
\end{align*}
\]
where

\[ P = \frac{U_M I_M}{\sqrt{2}} \cos \phi = \text{active power} \]

\[ Q = \frac{U_M I_M}{\sqrt{2}} \sin \phi = \text{reactive power} \]

As given in equation (3.2), the power can be divided into two parts. One part with the mean value \( P \) which pulsates with the double frequency and one part with the amplitude \( Q \) which also pulsates with double frequency. In Figure 3.2, the voltage, current and power as a function of time are given. In the figure, the notation \( U = \frac{U_M}{\sqrt{2}} \) and \( I = \frac{I_M}{\sqrt{2}} \) are valid.

![Figure 3.2. Voltage, current and power as a function of time](image)

The RMS-value of the voltage and current are defined as

\[ U = \sqrt{\frac{1}{T} \int_0^T u(t)^2 dt} \]  (3.3)

\[ I = \sqrt{\frac{1}{T} \int_0^T i(t)^2 dt} \]  (3.4)

With a sinusoidal voltage and current, according to equation (3.1), the corresponding RMS-values can be calculated as

\[ U = \sqrt{\frac{1}{T} \int_0^T U_M^2 \cos^2 \omega t = U_M \sqrt{\frac{1}{T} \int_0^T \left( \frac{1}{2} + \frac{\cos 2\omega t}{2} \right) } = \frac{U_M}{\sqrt{2}} \]  \]  (3.5)

\[ I = \sqrt{\frac{1}{T} \int_0^T I_M^2 \cos^2 (\omega t - \phi) = \frac{I_M}{\sqrt{2}} } \]  (3.6)
Example 3.1 Which mean power is consumed by a resistor of 1210 Ω which is fed by an alternating voltage of 50 Hz and 220 V RMS.

Solution

The power consumed in the resistor can be calculated as the time mean value during one period as

\[ P = \frac{1}{T} \int_0^T R \cdot i^2(t) dt = \frac{1}{T} \int_0^T R \frac{u^2(t)}{R^2} dt = \frac{1}{R} \frac{1}{T} \int_0^T u^2(t) dt \]

which can be rewritten according to equation (3.3) as

\[ P = \frac{1}{R} U^2 = \frac{220^2}{1210} = 40 \text{ W} \]

3.2 Complex power

The complex method is a powerful tool for calculation of electrical power and can offer solutions in an elegant manner.

The complex single-phase voltage and current can be expressed as

\[ \bar{U} = U e^{j \arg(U)} \]
\[ \bar{I} = I e^{j \arg(I)} \quad (3.7) \]

where

\[ \bar{U} = \text{complex voltage} \]
\[ U = U_M/\sqrt{2} = \text{voltage RMS-value} \]
\[ \bar{I} = \text{complex current} \]
\[ I = I_M/\sqrt{2} = \text{current RMS-value} \]

The complex power is defined as

\[ \bar{S} = Se^{j \arg(S)} = P + jQ = \bar{U} \bar{I}^* = UI e^{j(\arg(U) - \arg(I))} \quad (3.8) \]

where

\[ \bar{S} = \text{complex power} \]

With phase angles on the voltage and the current as given in equation (3.1), i.e. \( \arg(U) = 0 \) and \( \arg(I) = -\phi \), together with equation (3.8) the result is

\[ \bar{S} = P + jQ = \bar{U} \bar{I}^* = UI e^{j\phi} = UI(\cos \phi + j \sin \phi) \quad (3.9) \]

which implies that

\[ P = S \cos \phi = UI \cos \phi \]
\[ Q = S \sin \phi = UI \sin \phi \quad (3.10) \]

i.e. \( P = \text{active power} \) and \( Q = \text{reactive power} \).
Example 3.2 Calculate the power consumed by an inductor rated 3.85 H which is fed by a sinusoidal voltage with 50 Hz and 220 V RMS.

Solution

The impedance of the inductor can be calculated as
\[ Z = j\omega L = j \cdot 2 \cdot \pi \cdot 50 \cdot 3.85 = j1210 \ \Omega \]

The complex current through the inductor can be calculated as
\[ I = \frac{U}{Z} = \frac{220}{j1210} = -j0.1818 \ A \]

giving that the complex power can be calculated as
\[ S = UI^* = 220(-j0.1818)^* = 220(j0.1818) = j40 \ VA \]
i.e. \( P = 0 \) W, \( Q = 40 \) VAr.

Example 3.3 Two series connected impedances are fed by a voltage having an RMS-value of 1 V according to Figure 3.3.

a) Calculate the power consumed by \( Z_2 \) as well as the power factor (cos\( \phi \)) at bus 1 and 2 where \( \phi_k \) is the phase angle between the voltage and the current at bus \( k \).

b) Calculate \( U_2 \) when \( Z_2 \) is capacitive: \( Z_2 = 0.7 - j0.5 \ \Omega \)

Solution

a)
\[ I = \frac{U_1}{Z_1 + Z_2} = 1.118\angle -26.57^\circ \ A \]
i.e. \( \phi_1 = -26.57^\circ \) and \( \text{cos}\phi_1 = 0.8944 \) lagging, since the current lags the voltage.
\[ U_2 = Z_2 \cdot I = 0.814\angle -10.62^\circ \]
i.e. $\phi_2 = \phi_1 - \theta_2 = -15.95^\circ$ and $\cos \phi_2 = 0.9615$, lagging. The equation above can be written on polar form as

$$U_2 = Z_2 \cdot I, \quad \theta_2 = \arg(Z_2) + \phi_1$$

i.e. $\phi_2 = - \arg(Z_2) = - \arctan \frac{X}{R} = -15.95^\circ$

![Figure 3.4. Solution to example 3.3a](image)

The power consumption in $Z_2$ can be calculated as

$$\bar{S}_2 = P_2 + jQ_2 = Z_2 \cdot I^2 = (0.7 + j0.2)1.118^2 = 0.875 + j0.25 \text{ VA}$$

or

$$\bar{S}_2 = P_2 + jQ_2 = \bar{U}_2 \bar{I}^* = 0.814 \cdot 1.118 \angle -10.62^\circ + 26.57^\circ = 0.875 + j0.25 \text{ VA}$$

![Figure 3.5. Solution to example 3.3b](image)

b)

$$U_2 = \left| \frac{Z_2}{Z_1 + Z_2} \right| U_1 = \frac{|0.7 - j0.5|}{|0.8 - j0.3|} = \frac{\sqrt{0.49 + 0.25}}{\sqrt{0.64 + 0.09}} = \frac{\sqrt{0.74}}{\sqrt{0.73}} = 1.007 \text{ V}$$

Conclusions from this example are that

- a capacitance increases the voltage - so called phase compensation
• active power can be sent towards higher voltage
• \( \cos\phi \) is different in different ends of a line
• the line impedances are \( \ll \) load impedances

### 3.3 Symmetrical three-phase alternating voltage

Symmetrical three-phase alternating voltage means three sinusoidal voltages, with a phase angle of 120° between the voltages and with the same peak value compared with ground. In the time domain the expressions of the three voltages are

\[
\begin{align*}
    u_a(t) &= U_M \cos \omega t \\
    u_b(t) &= U_M \cos(\omega t - 120^\circ) \\
    u_c(t) &= U_M \cos(\omega t + 120^\circ)
\end{align*}
\]  

(3.11)

In Figure 3.6, all three voltages \( u_a(t), u_b(t) \) and \( u_c(t) \) are shown.

![Graphs of the three voltages](image)

**Figure 3.6.** The symmetrical voltages \( u_a(t), u_b(t), u_c(t) \) and \( u_{ab}(t) \), \( f = 50 \text{ Hz}, U_M = 1 \)

When having a three-phase system, the voltage between two phases are often used. This voltage is called *phase-to-phase voltage*. The voltage \( u_{ab} \) between phase \( a \) and \( b \) can be written as

\[
\begin{align*}
    u_{ab}(t) &= u_a(t) - u_b(t) = U_M \cos \omega t - U_M \cos(\omega t - 120^\circ) = \\
    &= \sqrt{3}U_M \cos(\omega t + 30^\circ)
\end{align*}
\]  

(3.12)
3.3. Symmetrical three-phase alternating voltage

i.e. the phase-to-phase voltage has an amplitude (and by that an RMS-value, see equation (3.5)) which is \( \sqrt{3} \) times larger than the amplitude of the phase voltage (phase-to-ground voltage). An example is low voltage distribution where the RMS-value of the phase voltage is 230 V and the RMS-value of the phase-to-phase voltage is \( \sqrt{3} \cdot 230 = 400 \) V.

Equation (3.12) shows also that \( u_{ab} \) is leading the voltage \( u_a \) by 30°. The phase-to-phase voltage \( u_{ab} \) is shown at the bottom of Figure 3.6.

With the assumption that the phase angle between the voltage and the current is \( \phi \) (equal in all phases because of symmetrical conditions), the following expression of the three phase currents can be obtained:

\[
i_a(t) = I_M \cos(\omega t - \phi)
\]
\[
i_b(t) = I_M \cos(\omega t - 120^\circ - \phi)
\]
\[
i_c(t) = I_M \cos(\omega t + 120^\circ - \phi)
\]

For the total symmetrical three-phase power, the following is valid:

\[
p_3(t) = p_a(t) + p_b(t) + p_c(t) = u_a(t)i_a(t) + u_b(t)i_b(t) + u_c(t)i_c(t) =
\]
\[
= \frac{U_M I_M}{\sqrt{2} \sqrt{2}} [(1 + \cos 2\omega t) \cos \phi + \sin 2\omega t \sin \phi] +
\]
\[
+ \frac{U_M I_M}{\sqrt{2} \sqrt{2}} [(1 + \cos 2[\omega t - 120^\circ]) \cos \phi + \sin 2[\omega t - 120^\circ] \sin \phi] +
\]
\[
+ \frac{U_M I_M}{\sqrt{2} \sqrt{2}} [(1 + \cos 2[\omega t + 120^\circ]) \cos \phi + \sin 2[\omega t + 120^\circ] \sin \phi] =
\]
\[
= 3 \frac{U_M I_M}{\sqrt{2} \sqrt{2}} \left[ \cos \phi + \left( \cos 2\omega t + \cos 2[\omega t - 120^\circ] + \cos 2[\omega t + 120^\circ] \right) \right] +
\]
\[
+ \left( \sin 2\omega t + \sin 2[\omega t - 120^\circ] + \sin 2[\omega t + 120^\circ] \right) = 3 \frac{U_M I_M}{\sqrt{2} \sqrt{2}} \cos \phi
\]

i.e. it is constant and does not pulsate as the single-phase power does. This is one very important reason why the electric power usually is transmitted by a three-phase system.

Corresponding complex properties of the voltages are:

\[
\mathcal{U}_a = U_f \angle 0^\circ
\]
\[
\mathcal{U}_b = U_f \angle -120^\circ
\]
\[
\mathcal{U}_c = U_f \angle 120^\circ
\]

The symmetrical voltages are shown in Figure 3.7.

In the figure also the phase-to-phase voltages \( \mathcal{U}_{ab} \), \( \mathcal{U}_{bc} \) and \( \mathcal{U}_{ca} \) are indicated. These phase-to-phase voltages form together a symmetrical tree-phase system, i.e. they have the same amplitude and are individually separated by a 120° phase angle. In the same way as for the time domain expressions, the complex formulation of the phase-to-phase voltage \( \mathcal{U}_{ab} \) can be written as

\[
\mathcal{U}_{ab} = \mathcal{U}_a - \mathcal{U}_b = U_f (1 - e^{-j120^\circ}) = \sqrt{3} U_f \angle 30^\circ
\]
where the RMS-value of the phase-to-phase voltage is $\sqrt{3}$ times larger than the RMS-value of the phase-to-ground voltage.

The complex expressions of the currents are:

$$
\begin{align*}
I_a &= I \angle (0^\circ - \phi) \\
I_b &= I \angle (-120^\circ - \phi) \\
I_c &= I \angle (120^\circ - \phi)
\end{align*}
$$

and the total three-phase power:

$$
\overline{S}_3 = \overline{U}_a \overline{I}_a^* + \overline{U}_b \overline{I}_b^* + \overline{U}_c \overline{I}_c^* = 3U_f I \cos \phi + j3U_f I \sin \phi = 3U_f I e^{j\phi}
$$

In equation (3.18), $U_f$ is the RMS-value of the phase-to-ground voltage. If the expression is rewritten in order to use the absolute value of the phase-to-phase voltage $U = \sqrt{3}U_f$ the following is obtained

$$
\overline{S}_3 = \sqrt{3}U I e^{j\phi} = \sqrt{3}U e^{j \arg(U)} I e^{-j (\arg(U) - \delta)} = \sqrt{3}UI^*
$$

When discussing a voltage in a three-phase system, e.g. 10 kV, normally it is assumed that it is the RMS-value of the phase-to-phase voltage that is discussed. This will also apply in this compendium. It is also common that an angle is attached to the absolute value of the phase-to-phase voltage. That angle is normally the phase-to-ground angle. This will also apply in this compendium.

**Example 3.4** The student Elektra lives in a house situated 2 km from a transformer having a completely symmetrical three-phase voltage $\overline{U}_a = 220V \angle 0^\circ, \overline{U}_b = 220V \angle -120^\circ, \overline{U}_c =$
220V \( \angle 120^\circ \). The house is connected to this transformer via a three-phase cable (EKKJ, \( 3 \times 16 \, \text{mm}^2 + 16 \, \text{mm}^2 \)). A cold day, Elektra switches on two electrical radiators to each phase, each radiator is rated 1000 W (at 220 V with \( \cos \phi = 0.995 \) lagging (inductive)). Assume that the cable can be modeled as four impedances connected in parallel (\( Z_L = 1.15 + j0.08 \, \Omega/\text{phase,km} \), \( Z_{L0} = 1.15 + j0.015 \, \Omega/\text{km} \)) and that the radiators also can be considered as impedances. Calculate the total thermal power given by the radiators.

![Single line diagram for example](image)

**Figure 3.8.** Single line diagram for example

**Solution**

\[
\begin{align*}
U_a &= 220 \angle 0^\circ \, \text{V, } U_b = 220 \angle -120^\circ \, \text{V, } U_c = 220 \angle 120^\circ \, \text{V} \\
Z_L &= 2(1.15 + j0.08) = 2.3 + j0.16 \, \Omega \\
Z_{L0} &= 2(1.15 + j0.015) = 2.3 + j0.03 \, \Omega \\
P_a = P_b = P_c &= 2000 \, \text{W (at 220 V, } \cos \phi = 0.995) \\
\sin \phi &= \sqrt{1 - \cos^2 \phi} = 0.0999 \\
Q_a = Q_b = Q_c &= S \sin \phi = \frac{P}{\cos \phi} \sin \phi = 200.8 \, \text{VAR} \\
Z_a = Z_b = Z_c &= \frac{U^2}{S} = U^2/(P_a - jQ_a) = 23.96 + j2.40 \, \Omega \\
I_a = \frac{U_a - U_0}{Z_a} = \frac{U_b - U_0}{Z_b} = \frac{U_c - U_0}{Z_c} \\
I_a + I_b + I_c &= \frac{U_a - U_0}{Z_{L0}} = \frac{U_b - U_0}{Z_{L0}} = \frac{U_c - U_0}{Z_{L0}} \\
\Rightarrow \, U_0' = 0.0 \\
\Rightarrow \, I_a = 8.34 \angle -5.58^\circ \, \text{A, } I_b = 8.34 \angle -125.58^\circ \, \text{A, } I_c = 8.34 \angle 114.42^\circ \, \text{A} \\

\text{The voltage at the radiators can be calculated as:} \\
U'_a = U_0' + I_a Z_a = 200.78 \angle 0.15^\circ \, \text{V} \\
U'_b = 200.78 \angle -119.85^\circ \, \text{V} \\
U'_c = 200.78 \angle 120.15^\circ \, \text{V} \\

\text{Finally, the power to the radiators can be calculated as}
\[ \overline{S}_{za} = \overline{Z}_a I_a^2 = 1666 + j167 \text{ VA} \]
\[ \overline{S}_{zb} = \overline{Z}_b I_b^2 = 1666 + j167 \text{ VA} \]
\[ \overline{S}_{zc} = \overline{Z}_c I_c^2 = 1666 + j167 \text{ VA} \]

The total amount of power consumed is
\[ \overline{S}_{za} + \overline{S}_{zb} + \overline{S}_{zc} = 4998 + j502 \text{ VA}, \text{i.e. the thermal power} = 4998 \text{ W} \]

The total transmission losses are
\[ \overline{Z}_L (I_a^2 + I_b^2 + I_c^2) = 480 + j33 \text{ VA} \]
i.e. the active losses are 480 W, which means that the efficiency is 91.2 %. As shown in example 3.4, a symmetrical voltage that feeds a symmetrical impedance gives rise to a symmetrical current. As a consequence, no current will flow in the neutral conductor. Since the voltage at the neutral connection point at the house = 0, it is possible to perform the calculations one phase at the time without paying attention to the other phases. The total amount of power consumed can be calculated as three times the power consumption per phase.

**Example 3.5** Use the same example as 3.4 but with the difference that the student Elektra connects one 1000 W radiator (at 220 V with \( \cos \phi = 0.995 \) lagging) to phase a, three radiators to phase b and two to phase c. Calculate the total thermal power given by the radiators, as well as the system losses.

**Solution**
\[ \overline{U}_a = 220\angle 0^\circ \text{ V,} \overline{U}_b = 220\angle -120^\circ \text{ V,} \overline{U}_c = 220\angle 120^\circ \text{ V} \]
\[ \overline{Z}_L = 2(1.15 + j0.08) = 2.3 + j0.16 \Omega \]
\[ \overline{Z}_{L0} = 2(1.15 + j0.015) = 2.3 + j0.03 \Omega \]
\[ P_a = 1000 \text{ W (at 220 V,} \cos \phi = 0.995) \]
\[ \sin \phi = \sqrt{1 - \cos^2 \phi} = 0.0999 \]
\[ Q_a = S \sin \phi = \frac{P}{\cos \phi} \sin \phi = 100.4 \text{ VAr} \]
\[ \overline{Z}_a = U_a^2 / \overline{S}_a = U_a^2 / (P_a - jQ_a) = 47.9 + j4.81 \Omega \]
\[ \overline{Z}_b = \overline{Z}_a / 3 = 15.97 + j1.60 \Omega \]
\[ \overline{Z}_c = \overline{Z}_a / 2 = 23.96 + j2.40 \Omega \]
\[ \overline{U}_0' = U_0' + \bar{I}_a Z_a = 209.45\angle 0.02^\circ \text{ V} \]
\[ \overline{U}_b' = U_0' + \bar{I}_b Z_b = 193.60\angle -120.05^\circ \text{ V} \]
3.3. Symmetrical three-phase alternating voltage

\[
U'_c = U'_0 + I_c Z_c = 200.91 \angle 129.45^\circ \text{ V}
\]

Observe that these voltages are not local phase voltages since they are calculated as \(U'_a - U'_0\) etc. The power to the radiators can be calculated as:

\[
S_{za} = Z_a I_a^2 = 1004 + j101 \text{ VA}
\]
\[
S_{zb} = Z_b I_b^2 = 2095 + j210 \text{ VA}
\]
\[
S_{zc} = Z_c I_c^2 = 1655 + j166 \text{ VA}
\]

The total amount of power consumed is

\[
S_{za} + S_{zb} + S_{zc} = 4754 + j477 \text{ VA}, \text{ i.e. the thermal power is } 4754 \text{ W}
\]

The total transmission losses are

\[
Z_L (I_a^2 + I_b^2 + I_c^2) + Z_{Lo} |I_a + I_b + I_c|^2 = 572.1 + j36 \text{ VA}, \text{ i.e. } 572.1 \text{ W}
\]

which gives an efficiency of 89.3\%. As shown in this example, an un-symmetrical impedance will give an un-symmetrical current. As a consequence, a voltage can be detected in the neutral conductor which gives rise to a current in the neutral conductor. The total thermal power obtained was reduced by approximately 5\% and the line losses increased partly due to the losses in the neutral conductor. The efficiency of the transmission decreased. It can also be noted that the power per radiator decreased with the number of radiators connected to the same phase. This owing to the fact that the voltage in the neutral connection point will be closest to the voltage in the phase with the lowest impedance, i.e. the phase with the largest number of radiators connected.
3. Alternating voltage
Chapter 4
Models of power system components

Electric energy is transmitted from power plants to consumers via overhead lines, cables and transformers. In the following, these components will be discussed and mathematical models to be used in the analysis of symmetrical three-phase systems will be derived. In chapter 11 and 12, analysis of power systems under un-symmetrical conditions will be discussed.

4.1 Electrical characteristic of an overhead line

Overhead transmission lines need large surface area and are mostly suitable to be used in rural areas and in areas with low population density. In areas with high population density and urban areas cables are a better alternative. For a certain amount of power transmitted, a cable transmission system is about ten times as expensive as an overhead transmission system.

Power lines have a *resistance* $r$ owing to the resistivity of the conductor and a *shunt conductance* $g$ because of leakage currents in the insulation. The lines also have an *inductance* $\ell$ owing to the magnetic flux surrounding the line as well as a *shunt capacitance* $c$ because of the electric field between the lines and between the lines and ground. These quantities are given per unit length and are continuously distributed along the whole length of the line. Resistance and inductance are in series while the conductance and capacitance are shunt quantities.

![Figure 4.1. A line with distributed quantities](image)

Assuming symmetrical three-phase, a line can be modeled as shown in Figure 4.1. The quantities $r$, $g$, $\ell$, and $c$ determines the characteristics of a line. Power lines can be modeled by simple equivalent circuits which, together with models of other system components, can be formed to a model of a complete system or parts of it. This is important since such models are used in power system analysis where active and reactive power flows in the network, voltage levels, losses, power system stability and other properties at disturbances as e.g. short circuits, are of interest.

For a more detailed derivation of the expressions of inductance and capacitance given below, more fundamental literature in electro-magnetic theory has to be studied.
4. Models of power system components

4.1.1 Resistance

The resistance of a conductor with the cross-section area $A$ $mm^2$ and the resistivity $\rho$ $\Omega mm^2$/km is

$$r = \frac{\rho}{A} \Omega/km \quad (4.1)$$

The conductor is made of copper with the resistivity at $20^\circ C$ of 17.2 $\Omega mm^2$/km, or aluminum with the resistivity at $20^\circ C$ of 27.0 $\Omega mm^2$/km. The choice between copper or aluminum is related to the price difference between the materials.

The effective alternating current resistance at normal system frequency (50–60 Hz) for lines with a small cross-section area is close to the value for the direct current resistance. For larger cross-section areas, the current density will not be equal over the whole cross-section. The current density will be higher at the peripheral parts of the conductor. That phenomena is called current displacement or skin effect and depends on the internal magnetic flux of the conductor. The current paths that are located in the center of the conductor will be surrounded by the whole internal magnetic flux and will consequently have an internal self inductance. Current paths that are more peripheral will be surrounded by a smaller magnetic flux and thereby have a smaller internal inductance.

The resistance of a line is given by the manufacturer where the influence of the skin effect is taken. Normal values of the resistance of lines are in the range 10–0.01 $\Omega$/km.

The resistance plays, compared with the reactance, often a minor role when comparing the transmission capability and voltage drop between different lines. For low voltage lines and when calculating the losses, the resistance is of significant importance.

4.1.2 Shunt conductance

The shunt conductance of an overhead line represents the losses owing to leakage currents at the insulators. There are no reliable data over the shunt conductances of lines and these are very much dependent on humidity, salt content and pollution in the surrounding air. For cables, the shunt conductance represents the dielectric losses in the insulation material and data can be obtained from the manufacturer.

The dielectric losses are e.g. for a 12 kV cross-linked polyethylene (XLPE) cable with a cross-section area of 240 $mm^2$/phase 7 W/km,phase and for a 170 kV XLPE cable with the same area 305 W/km,phase.

The shunt conductance will be neglected in all calculations throughout this compendium.

4.1.3 Inductance

The inductance is in most cases the most important parameter of a line. It has a large influence on the line transmission capability, voltage drop and indirectly the line losses. The inductance of a line can be calculated by the following formula:
4.1. **Electrical characteristic of an overhead line**

\[ \ell = 2 \cdot 10^{-4} \left( \ln \frac{a}{d/2} + \frac{1}{4n} \right) \text{ H/km.phase} \]  

where

- \( a = \sqrt{a_{12}a_{13}a_{23}} \text{ m} = \text{geometrical mean distance according to Figure 4.2.} \)
- \( d = \text{diameter of the conductor, m} \)
- \( n = \text{number of conductors per phase} \)

**Figure 4.2.** The geometrical quantities of a line in calculations of inductance and capacitance

The calculation of the inductance according to equation (4.2), is made under some assumptions, viz. the conductor material must be non-magnetic as copper and aluminum together with the assumption that the line is transposed. The majority of the long transmission lines are transposed, see Figure 4.3.

**Figure 4.3.** Transposing of three-phase overhead line

This implies that each one of the conductors, under a transposing cycle, has had all three possible locations in the transmission line. Each location is held under equal distance which implies that all conductors in average have the same distance to ground and to the other
conductors. This gives that the mutual inductance between the three phases are equalized so that the inductance per phase is equal among the three phases.

In many cases, the line is constructed as a multiple conductor, i.e. more than one conductor is used for each phase, see Figure 4.4. Multiple conductors implies both lower reactance of the line and reduced corona effect (glow discharge). The radius $d/2$ in equation (4.2) must in these cases be replaced with the equivalent radius

$$\left(\frac{d}{2}\right)_{\text{eqv}} = \sqrt[n]{n\left(\frac{D}{2}\right)^{n-1} \cdot \left(\frac{d}{2}\right)} \quad (4.3)$$

where

$n = \text{number of conductors per phase}$

$D/2 = \text{radius in the circle formed by the conductors}$

By using the inductance, the reactance of a line can be calculated as

$$x = \omega \ell = 2\pi f \ell \quad \Omega/\text{km,phase} \quad (4.4)$$

and is only dependent on the geometrical design of the line if the frequency is kept constant. The relationship between the geometrical mean distance $a$ and the conductor diameter $d$ in equation (4.2) varies within quite small limits for different lines. This due to the large distance between the phases and the larger conductor diameter for lines designed for higher system voltages. The term $\frac{1}{4n}$ has, compared with $\ln\left(\frac{a}{d/2}\right)$, usually a minor influence on the line inductance.

At normal system frequency, the reactance of an overhead line can vary between 0.3 and 0.5 $\Omega/\text{km,phase}$ with a typical value of 0.4 $\Omega/\text{km,phase}$. For cables, the reactance vary between 0.08 and 0.17 $\Omega/\text{km,phase}$ where the higher value is valid for cables with a small cross-section area. The reactance for cables is considerably lower than the reactance of overhead lines. The difference is caused by the difference in distance between the conductors. The conductors are more close to one another in cables which gives a lower reactance. See equation (4.2) which gives the inductance of overhead lines.

**Example 4.1** Determine the reactance of a 130 kV overhead line where the conductors are located in a plane and the distance between two closely located conductors is 4 m. The conductor diameter is 20 mm. Repeat the calculations for a line with two conductors per phase, located 30 cm from one another.
4.1. Electrical characteristic of an overhead line

Solution

\[ a_{12} = a_{23} = 4, \quad a_{13} = 8 \]

\[ d/2 = 0.01 \text{ m} \]

\[ a = \sqrt[3]{4 \cdot 4 \cdot 8} = 5.04 \]

\[ x = 2\pi \cdot 50 \cdot 2 \cdot 10^{-4} \left( \ln \frac{5.04}{0.01} + \frac{1}{4} \right) = 0.0628 \left( \ln(504) + 0.25 \right) = 0.41 \Omega/\text{km,phase} \]

Multiple conductor (duplex)

\[ (d/2)_{ekv} = \sqrt[3]{2(0.3)/0.01} = 0.055 \]

\[ x = 0.0628 \left( \ln \frac{5.04}{0.055} + \frac{1}{8} \right) = 0.29 \Omega/\text{km,phase} \]

The reactance is in this case reduced by 28%.

4.1.4 Shunt capacitance

For a three-phase transposed overhead line, the capacitance to ground per phase can be calculated as

\[ c = \frac{10^{-6}}{18 \ln \left( \frac{2H}{A} \cdot \frac{a}{(d/2)_{ekv}} \right)} \text{ F/km,phase} \quad (4.5) \]

where

\[ H = \sqrt[3]{H_1 H_2 H_3} = \text{geometrical mean height for the conductors according to Figure 4.2.} \]

\[ A = \sqrt[3]{A_1 A_2 A_3} = \text{geometrical mean distance between the conductors and their image conductors according to Figure 4.2.} \]

As indicated in equation (4.5), the ground has some influence on the capacitance of the line. The capacitance is determined by the electrical field which is dependent on the characteristics of the ground. The ground will form an equipotential surface which has an influence on the electric field.

The degree of influence the ground has on the capacitance is determined by the factor \( 2H/A \) in equation (4.5). This factor has usually a value near 1.

Assume that a line mounted on relatively high poles (\( \Rightarrow A \approx 2H \)) is considered and that the term \( \frac{1}{2n} \) can be neglected in equation (4.2). By multiplying the expressions for inductance and capacitance, the following is obtained

\[ \ell \cdot c = 2 \cdot 10^{-4} \left( \ln \frac{a}{(d/2)_{ekv}} \right) \cdot \frac{10^{-6}}{18 \ln \left( \frac{a}{(d/2)_{ekv}} \right)} = \frac{1}{(3 \cdot 10^5)^2} \left( \frac{\text{km}}{s} \right)^{-2} = \frac{1}{v^2} \quad (4.6) \]

where \( v = \text{speed of light in vacuum in km/s.} \) Equation (4.6) can be interpreted as the inductance and capacitance are the inverse of one another for a line. Equation (4.6) is a good approximation for an overhead line.

The shunt susceptance of a line is

\[ b = 2\pi f \cdot c \quad \text{S/km,phase} \quad (4.7) \]
A typical value of the shunt susceptance of a line is \( 3 \cdot 10^{-6} \text{ S/km,phase} \). Cables have considerable higher values between \( 3 \cdot 10^{-5} \) – \( 3 \cdot 10^{-4} \text{ S/km,phase} \).

**Example 4.2** Assume that a line has a shunt susceptance of \( 3 \cdot 10^{-6} \text{ S/km,phase} \). Use equation (4.6) to estimate the reactance of the line.

**Solution**

\[
x = \frac{\omega \ell}{c v^2} \approx \frac{\omega^2}{b v^2} = \frac{(100\pi)^2}{3 \cdot 10^{-6} (3 \cdot 10^{5})^2} = 0.366 \text{ \Omega/km}
\]

which is near the standard value of \( 0.4 \text{ \Omega/km} \) for the reactance of an overhead line.

### 4.2 Model of a line

Both overhead lines and cables have their electrical quantities \( r, x, g \) and \( b \) distributed along the whole length. Figure 4.1 shows an approximation of the distribution of the quantities. Generally, the accuracy of the calculation result will increase with the number of distributed quantities.

At a first glance, it seems possible to form a line model where the total resistance/inductance is calculated as the product between the resistance/inductance per length unit and the length of the line. This approximation is though only valid for short lines and lines of medium length. For long lines, the distribution of the quantities \( r, x, g \) and \( b \) must be taken into account. Such analysis can be carried out with help of differential calculus.

There are no absolute limits between short, medium and long lines. Usually, lines shorter than 100 km are considered as short, between 100 km and 300 km as medium long and lines longer than 300 km are classified as long. For cables, having considerable higher values of the shunt capacitance, the distance 100 km should be considered as medium long. In the following, models for short and medium long lines are given.

#### 4.2.1 Short lines

In short line models, the shunt parameters are neglected, i.e. conductance and susceptance. This because the current flowing through these components is less than one percent of the rated current of the line. The short line model is given in Figure 4.5. This single-phase model of a three-phase system is valid under the assumption that the system is operating under symmetrical conditions.

The impedance of the line can be calculated as

\[
\mathcal{Z} = R + jX = (r + jx)s \text{ \Omega/phase}
\]

where \( s \) = the length of the line in km.
4.2. Model of a line

The relationship between voltage and current in Figure 4.5 is

$$U_j = U_k - \sqrt{3}(R + jX)I_k$$  \hspace{1cm} (4.9)

where $U_k = \sqrt{3}U_{k-phase}$ and $U_j = \sqrt{3}U_{j-phase}$.

4.2.2 Medium long lines

For lines having a length between 100 and 300 km, the shunt capacitance cannot be neglected. The model shown in Figure 4.5 has to be extended with the shunt susceptance, which results in a model called the $\pi$-equivalent shown in Figure 4.6. The impedance is calculated according to equation (4.8) and the admittance to ground per phase is obtained as

$$Y = jB = \frac{jbs}{2} \quad S$$  \hspace{1cm} (4.10)

i.e. the total shunt capacitance of the line is divided into two equal parts, one at each end of the line. The $\pi$-equivalent is a very common and useful model in power system analysis.

The electrical behavior of the model is described by

$$U_j = U_k - \sqrt{3}ZI$$  \hspace{1cm} (4.11)

$$I = I_k - Y\frac{U_k}{\sqrt{3}}$$  \hspace{1cm} (4.12)

which results in

$$U_j = (1 + ZY)U_k - \sqrt{3}ZI_k$$  \hspace{1cm} (4.13)
4.3 Single-phase transformer

The principle diagram of a two winding transformer is shown in Figure 4.7. The fundamental principles of a transformer are given in the figure. In a real transformer, the demand of a strong magnetic coupling between the primary and secondary sides must be taken into account in the design.

![Figure 4.7. Principle design of a two winding transformer](image)

Assume that the magnetic flux can be divided into three components. There is a core flux $\Phi_m$ passing through both the primary and the secondary windings. There are also leakage fluxes, $\Phi_{l1}$ passing only the primary winding and $\Phi_{l2}$ which passes only the secondary winding. The resistance of the primary winding is $r_1$ and for the secondary winding $r_2$. According to the law of induction, the following relationships can be given for the voltages at the transformer terminals:

$$u_1 = r_1 i_1 + N_1 \frac{d(\Phi_{l1} + \Phi_m)}{dt}$$  \hspace{1cm} (4.14)

$$u_2 = r_2 i'_2 + N_2 \frac{d(\Phi_{l2} + \Phi_m)}{dt}$$

Assuming linear conditions, the following is valid

$$N_1 \Phi_{l1} = L_{l1} i_1$$  \hspace{1cm} (4.15)

$$N_2 \Phi_{l2} = L_{l2} i'_2$$

where

$L_{l1} =$ inductance of the primary winding

$L_{l2} =$ inductance of the secondary winding
Equation (4.14) can be rewritten as

\[
\begin{align*}
    u_1 &= r_1 i_1 + L_{i1} \frac{di_1}{dt} + N_1 \frac{d\Phi_m}{dt} \\
    u_2 &= r_2 i'_2 + L_{i2} \frac{di'_2}{dt} + N_2 \frac{d\Phi_m}{dt}
\end{align*}
\] (4.16)

With the reluctance \( R \) of the iron core and the definitions of the directions of the currents according to Figure 4.7, the magnetomotive forces \( N_1 i_1 \) and \( N_2 i'_2 \) can be added as

\[
N_1 i_1 + N_2 i'_2 = R \Phi_m
\] (4.17)

Assume that \( i'_2 = 0 \), i.e. the secondary side of the transformer is not connected. The current now flowing in the primary winding is called the *magnetizing current* and the magnitude can be calculated using equation (4.17) as

\[
i_m = \frac{R \Phi_m}{N_1}
\] (4.18)

If equation (4.18) is inserted into equation (4.17), the result is

\[
i_1 = i_m - \frac{N_2}{N_1} i'_2 = i_m + \frac{N_2}{N_1} i_2
\] (4.19)

where

\[
i_2 = -i'_2
\] (4.20)

Assuming linear conditions, the induced voltage drop \( N_1 \frac{d\Phi_m}{dt} \) in equation (4.16) can be expressed by using an inductor as

\[
N_1 \frac{d\Phi_m}{dt} = L_m \frac{di_m}{dt}
\] (4.21)

i.e. \( L_m = N_1^2 / R \). By using equations (4.16), (4.19) and (4.21), the equivalent diagram of a single-phase transformer can be drawn, see Figure 4.8.

**Figure 4.8.** Equivalent diagram of a single-phase transformer

In Figure 4.8, one part of the *ideal* transformer is shown, which is a lossless transformer without leakage fluxes and magnetizing currents.
The equivalent diagram in Figure 4.8 has the advantage that the different parts represents different parts of the real transformer. For example, the inductance $L_m$ represents the assumed linear relationship between the core flux $\Phi_m$ and the magnetomotive force of the iron core. Also the resistive copper losses in the transformer are represented by $r_1$ and $r_2$.

In power system analysis, where the transformer is modeled, a simplified model is often used where the magnetizing current is neglected.

### 4.4 Three-phase transformer

There are three fundamental ways of connecting single-phase transformers into one three-phase transformer. The three combinations are Y-Y-connected, Δ-Δ-connected and Y-Δ-connected. In Figure 4.9, the different combinations are shown.

![Three-phase transformer diagrams](image)

**Figure 4.9.** Standard connections for three-phase transformers

The different consequences that these different connections imply, are discussed in chapter 12.
Chapter 5
Important theorems in power system analysis

In many cases, the use of theorems can simplify the analysis of electrical circuits and systems. In the following sections, some important theorems will be discussed and proofs will be given.

5.1 Bus analysis, admittance matrices

Consider an electric network which consists of four buses as shown in Figure 5.1. Each bus is connected to the other buses via an admittance $y_{mn}$ where the subscript indicates which buses the admittance is connected to. Assume that there are no mutual inductances between

\[ y_{12} \]
\[ y_{23} \]
\[ y_{13} \]
\[ y_{14} \]
\[ y_{24} \]
\[ y_{34} \]

Figure 5.1. Four bus network

the admittances and that the buses voltages are $U_1$, $U_2$, $U_3$ and $U_4$. The currents $I_1$, $I_2$, $I_3$ and $I_4$ are assumed to be injected into the buses from external current sources. A balance equation for bus 1 is

\[ I_1 = y_{12}(U_1 - U_2) + y_{13}(U_1 - U_3) + y_{14}(U_1 - U_4) \] (5.1)

or

\[ I_1 = (y_{12} + y_{13} + y_{14})U_1 - y_{12}U_2 - y_{13}U_3 - y_{14}U_4 = Y_{11}U_1 + Y_{12}U_2 + Y_{13}U_3 + Y_{14}U_4 \] (5.2)

where

\[ Y_{11} = y_{12} + y_{13} + y_{14} \] and \[ Y_{12} = -y_{12}, Y_{13} = -y_{13}, Y_{14} = -y_{14} \] (5.3)

Corresponding equations can be formed for the other buses. These equations can be put
together to a matrix equation as:

\[
I = \begin{bmatrix}
I_1 \\
I_2 \\
I_3 \\
I_4
\end{bmatrix} = \begin{bmatrix}
Y_{11} & Y_{12} & Y_{13} & Y_{14} \\
Y_{21} & Y_{22} & Y_{23} & Y_{24} \\
Y_{31} & Y_{32} & Y_{33} & Y_{34} \\
Y_{41} & Y_{42} & Y_{43} & Y_{44}
\end{bmatrix}
\begin{bmatrix}
U_1 \\
U_2 \\
U_3 \\
U_4
\end{bmatrix} = YU
\]

This matrix is called the bus admittance matrix or Y-bus matrix. The following properties are valid:

- It can be uniquely determined from a given admittance network.
- The diagonal element \( Y_{kk} \) = the sum of all admittances connected to bus \( k \).
- Non-diagonal element \( Y_{ik} = -y_{ik} \) where \( y_{ik} \) is the admittance between bus \( i \) and bus \( k \).
- This gives that the matrix is symmetric, i.e. \( Y_{ik} = Y_{ki} \) (one exception is when the network includes phase shifting transformers).
- It is singular since \( I_1 + I_2 + I_3 + I_4 = 0 \)

If the potential in one bus is assumed to be 0, the corresponding line and column in the admittance matrix can be taken away which results in a non-singular matrix. Bus analysis using the Y-bus matrix is the method most often used when studying larger, meshed networks in a systematic manner.

Example 5.1 Re-calculate example 3.5 by using the Y-bus matrix of the network in order to calculate the power given by the radiators.

Solution

According to the task and to the calculations performed in example 3.5, the following is valid;

\[
\begin{align*}
\bar{Z}_L &= 2.3 + j0.16 \ \Omega, \bar{Z}_{L0} = 2.3 + j0.03 \ \Omega, \bar{Z}_a = 47.9 + j4.81 \ \Omega, \bar{Z}_b = 15.97 + j1.60 \ \Omega, \bar{Z}_c =
\end{align*}
\]
23.96 + j2.40 Ω. Start with forming the Y-bus matrix. \( I_0 \) and \( U_0 \) are neglected since the system otherwise will be singular.

\[
\mathbf{I} = \begin{bmatrix}
\bar{I}_1 \\
\bar{I}_2 \\
\bar{I}_3 \\
\bar{I}_4
\end{bmatrix} = \begin{bmatrix}
\frac{1}{Z_L + Z_a} & 0 & 0 & -\frac{1}{Z_L + Z_a} \\
0 & \frac{1}{Z_L + Z_b} & 0 & -\frac{1}{Z_L + Z_b} \\
0 & 0 & \frac{1}{Z_L + Z_c} & -\frac{1}{Z_L + Z_c} \\
-\frac{1}{Z_L + Z_a} & -\frac{1}{Z_L + Z_b} & -\frac{1}{Z_L + Z_c} & \frac{1}{Y_{44}}
\end{bmatrix} \begin{bmatrix}
\bar{U}_1 \\
\bar{U}_2 \\
\bar{U}_3 \\
\bar{U}_4
\end{bmatrix} = \mathbf{YU} \quad (5.5)
\]

where

\[
Y_{44} = \frac{1}{Z_L + Z_a} + \frac{1}{Z_L + Z_b} + \frac{1}{Z_L + Z_c} + \frac{1}{Z_{L0}} \quad (5.6)
\]

In the matrix equation above, \( \bar{U}_1, \bar{U}_2, \bar{U}_3 \) and \( \bar{I}_4 \) \((\bar{I}_4=0)\) as well as all impedances, i.e. the Y-bus matrix, are known. If the given Y-bus matrix is inverted, the corresponding Z-bus matrix is obtained:

\[
\mathbf{U} = \begin{bmatrix}
\bar{U}_1 \\
\bar{U}_2 \\
\bar{U}_3 \\
\bar{U}_4
\end{bmatrix} = \mathbf{ZI} = \mathbf{Y}^{-1}\mathbf{I} = \begin{bmatrix}
Z_{11} & Z_{12} & Z_{13} & Z_{14} \\
Z_{21} & Z_{22} & Z_{23} & Z_{24} \\
Z_{31} & Z_{32} & Z_{33} & Z_{34} \\
Z_{41} & Z_{42} & Z_{43} & Z_{44}
\end{bmatrix} \begin{bmatrix}
\bar{I}_1 \\
\bar{I}_2 \\
\bar{I}_3 \\
\bar{I}_4
\end{bmatrix} \quad (5.7)
\]

Since the elements in the Y-bus matrix are known, all the elements in the Z-bus matrix can be calculated. Since \( \bar{I}_4=0 \) the voltages \( \bar{U}_1, \bar{U}_2 \) and \( \bar{U}_3 \) can be expressed as a function of the currents \( \bar{I}_1, \bar{I}_2 \) and \( \bar{I}_3 \) by using only a part of the Z-bus matrix:

\[
\mathbf{U}' = \begin{bmatrix}
\bar{U}_1 \\
\bar{U}_2 \\
\bar{U}_3
\end{bmatrix} = \mathbf{Z'I} = \begin{bmatrix}
Z_{11} & Z_{12} & Z_{13} \\
Z_{21} & Z_{22} & Z_{23} \\
Z_{31} & Z_{32} & Z_{33}
\end{bmatrix} \begin{bmatrix}
\bar{I}_1 \\
\bar{I}_2 \\
\bar{I}_3
\end{bmatrix} \quad (5.8)
\]

Since the voltages \( \bar{U}_1, \bar{U}_2 \) and \( \bar{U}_3 \) are known, the currents \( \bar{I}_1, \bar{I}_2 \) and \( \bar{I}_3 \) can be calculated as :

\[
\mathbf{I}' = \begin{bmatrix}
\bar{I}_1 \\
\bar{I}_2 \\
\bar{I}_3
\end{bmatrix} = (\mathbf{Z'})^{-1}\mathbf{U}' = \begin{bmatrix}
19.0 - j1.83 & -1.95 - j0.324 & -1.36 + j0.227 \\
-1.95 + j0.324 & 48.9 - j4.36 & -3.73 - j0.614 \\
-1.36 + j0.227 & -3.73 + j0.614 & 35.1 - j3.25
\end{bmatrix} \begin{bmatrix}
220\angle0^\circ \\
220\angle-120^\circ \\
220\angle120^\circ
\end{bmatrix}
\]

\[
= \begin{bmatrix}
4.58\angle-4.39^\circ \\
11.5\angle-123.6^\circ \\
8.31\angle111.3^\circ
\end{bmatrix} \text{ A}
\]

By using these currents, the power given by the radiators can be calculated as :

\[
\bar{S}_{za} = \bar{Z}_a \bar{I}_1^2 = 1004 + j101 \text{ VA} \\
\bar{S}_{zb} = \bar{Z}_b \bar{I}_2^2 = 2095 + j210 \text{ VA} \\
\bar{S}_{zc} = \bar{Z}_c \bar{I}_3^2 = 1655 + j166 \text{ VA}
\]

\[
\sum = 4754 + j477 \text{ VA} \quad (5.10)
\]

i.e. the thermal power obtained is 4754 W.
5.2 Millman’s theorem

Millman’s theorem (the parallel generator-theorem) gives that if a number of admittances \( Y_1, Y_2, Y_3, \ldots, Y_n \) are connected to a common bus \( k \), and the voltages to a reference bus \( U_{10}, U_{20}, U_{30}, \ldots, U_{n0} \) are known, the voltage between bus \( k \) and the reference bus, \( U_{k0} \) can be calculated as

\[
U_{k0} = \frac{\sum_{i=1}^{n} Y_i U_{i0}}{\sum_{i=1}^{n} Y_i} \quad (5.11)
\]

Assume a Y-connection of admittances as shown in Figure 5.3. The Y-bus matrix for this network can be formed as

\[
\begin{bmatrix}
I_1 \\
I_2 \\
\vdots \\
I_n \\
I_k
\end{bmatrix} =
\begin{bmatrix}
Y_1 & 0 & \ldots & 0 & -Y_1 \\
0 & Y_2 & \ldots & 0 & -Y_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & Y_n & -Y_n \\
-Y_1 & -Y_2 & \ldots & -Y_n & (Y_1 + Y_2 + \ldots + Y_n)
\end{bmatrix}
\begin{bmatrix}
U_{10} \\
U_{20} \\
\vdots \\
U_{n0} \\
U_{k0}
\end{bmatrix}
\]

\[
(5.12)
\]

This equation can be written as

\[
\begin{bmatrix}
I_1 \\
I_2 \\
\vdots \\
I_n \\
I_k
\end{bmatrix} =
\begin{bmatrix}
(U_{10}Y_1 - U_{k0}Y_1) \\
(U_{20}Y_2 - U_{k0}Y_2) \\
\vdots \\
(-U_{10}Y_1 - U_{20}Y_2 - \ldots + \sum_{i=1}^{n} Y_i U_{k0})
\end{bmatrix}
\]

\[
(5.13)
\]

Since no current is injected at bus \( k \) \((I_k = 0)\), the last equation can be written as

\[
I_k = 0 = -U_{10}Y_1 - U_{20}Y_2 - \ldots + \sum_{i=1}^{n} Y_i U_{k0}
\]

\[
(5.14)
\]
5.2. Millman’s theorem

This equation can be written as

\[ U_{k0} = \frac{U_{10}Y_1 + U_{20}Y_2 + \ldots + U_{n0}Y_n}{\sum_{i=1}^{n} Y_i} \]  \hspace{1cm} (5.15)

and by that, the proof of the Millman’s theorem is completed.

Example 5.2 Find the solution to example 3.5 by using Millman’s theorem, which will be the most efficient method to solve the problem so far.

![Figure 5.4. Diagram of the network used in the example](image)

Solution

According to the task and to the calculations performed in example 3.5, the following is valid; 
\[ Z_L = 2.3 + j0.16 \, \Omega, Z_{L0} = 2.3 + j0.03 \, \Omega, Z_a = 47.9 + j4.81 \, \Omega, Z_b = 15.97 + j1.60 \, \Omega, Z_c = 23.96 + j2.40 \, \Omega. \]

By using Millman’s theorem, the voltage at bus 4 can be calculated as

\[ U_{40} = \frac{\sum_{i=1}^{n} Y_i U_{i0}}{\sum_{i=1}^{n} Y_i} = \frac{U_{0} \frac{1}{Z_{L0}} + U_{1} \frac{1}{Z_a + Z_L} + U_{2} \frac{1}{Z_b + Z_L} + U_{3} \frac{1}{Z_c + Z_L}}{\frac{1}{Z_{L0}} + \frac{1}{Z_a + Z_L} + \frac{1}{Z_b + Z_L} + \frac{1}{Z_c + Z_L}} \]  \hspace{1cm} (5.16)

\[ = 12.08 \angle -155.1^\circ \, \text{V} \]

The currents through the impedances can be calculated as

\[ I_1 = \frac{U_1 - U_4}{Z_a + Z_L} = 4.58 \angle -4.39^\circ \, \text{A} \]

\[ I_2 = \frac{U_2 - U_4}{Z_b + Z_L} = 11.5 \angle -123.6^\circ \, \text{A} \]  \hspace{1cm} (5.17)

\[ I_3 = \frac{U_3 - U_4}{Z_c + Z_L} = 8.31 \angle 111.3^\circ \, \text{A} \]
By using these currents, the power from the radiators can be calculated in the same way as earlier:

\[
\begin{align*}
\mathcal{S}_{za} &= Z_a I_1^2 = 1004 + j101 \text{ VA} \\
\mathcal{S}_{zb} &= Z_b I_2^2 = 2095 + j210 \text{ VA} \\
\mathcal{S}_{zc} &= Z_c I_3^2 = 1655 + j166 \text{ VA} \\
\sum &= 4754 + j477 \text{ VA} \tag{5.18}
\end{align*}
\]

i.e. the thermal power is 4754 W.

### 5.3 Superposition theorem

According to section 5.1, each admittance network can be described by a Y-bus matrix, i.e.

\[
\mathbf{I} = \mathbf{YU} \tag{5.19}
\]

where

\[
\begin{aligned}
\mathbf{I} &= \text{ vector with currents injected into the buses} \\
\mathbf{U} &= \text{ vector with the bus voltages}
\end{aligned}
\]

The superposition theorem can be applied to variables with a linear dependence, as shown in equation (5.19). This implies that the solution is obtained piecewise, e.g. for one generator at the time. The total solution is obtained by adding all the part solutions found:

\[
\mathbf{I} = \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_n \end{bmatrix} = \mathbf{Y} \begin{bmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \\ \vdots \\ \mathbf{U}_n \end{bmatrix} = \mathbf{Y} \begin{bmatrix} \mathbf{U}_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \mathbf{Y} \begin{bmatrix} 0 \\ \mathbf{U}_2 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \mathbf{Y} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \mathbf{U}_n \end{bmatrix} \tag{5.20}
\]

It can be noted that the superposition theorem cannot be applied to calculations of the power flow since they cannot be considered as linear properties since they are the product between voltage and current.

**Example 5.3** Use the conditions given in example 5.1 and assume that a fault at the feeding transformer gives a short circuit of phase 2. Phase 1 and 3 are operating as usual. Calculate the thermal power obtained in the house of Elektra.

**Solution**

According to equation (5.9) in example 5.1, the phase currents can be expressed as a function of the feeding voltages as

\[
\mathbf{I}' = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = (\mathbf{Z}')^{-1} \begin{bmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \\ \mathbf{U}_3 \end{bmatrix} \tag{5.21}
\]
5.4. Reciprocity theorem

A short circuit in phase 2 is equivalent with connecting an extra voltage source in reverse direction in series with the already existing voltage source. The phase currents in the changed system can be calculated as:

\[
\mathbf{I}' = \begin{bmatrix}
\tilde{I}_1 \\
\tilde{I}_2 \\
\tilde{I}_3
\end{bmatrix} = (\mathbf{Z}')^{-1} \left[ \begin{bmatrix}
U_1 \\
U_2 \\
U_3
\end{bmatrix} - \begin{bmatrix}
0 \\
-\tilde{U}_2 \\
0
\end{bmatrix} \right] = \begin{bmatrix}
4.58 \angle -4.39^\circ \\
11.5 \angle -123.6^\circ \\
8.31 \angle 111.3^\circ
\end{bmatrix} + \\
10^{-3} \begin{bmatrix}
19.0 - j1.83 \\
-1.95 - j0.324 \\
-1.36 + j0.227
\end{bmatrix} = 0
\]

\[
\begin{bmatrix}
4.34 \angle -9.09^\circ \\
0.719 \angle -100.9^\circ \\
7.94 \angle -116.5^\circ
\end{bmatrix} \text{ A}
\]

\[
\begin{align*}
\mathbf{S}_{za} &= \mathbf{Z}_a \mathbf{I}_a^2 = 904 + j91 \text{ VA} \\
\mathbf{S}_{zb} &= \mathbf{Z}_b \mathbf{I}_b^2 = 8.27 + j0.830 \text{ VA} \\
\mathbf{S}_{zc} &= \mathbf{Z}_c \mathbf{I}_c^2 = 1509 + j151 \text{ VA}
\end{align*}
\]

\[
\sum = 2421 + j243 \text{ VA}
\]

i.e. the thermal power is 2421 W

As shown in this example, the superposition theorem can, for instance, be used when studying changes in the system. But it should once again be pointed out that this is valid under the assumption that the loads (the radiators in this example) can be modeled as impedances.

5.4 Reciprocity theorem

Assume that a voltage source is connected to a terminal \(a\) in a linear reciprocal network and is giving rise to a current at terminal \(b\). According to the reciprocity theorem, the voltage source will cause the same current at \(a\) if it is connected to \(b\). The Y-bus matrix (and by that also the Z-bus matrix) are symmetrical matrices for a reciprocal electric network.
Important theorems in power system analysis

Assume that an electric network with \( n \) buses can be described by a symmetric Y-bus matrix, i.e.

\[
\begin{bmatrix}
I_1 \\
I_2 \\
\vdots \\
I_n
\end{bmatrix} = \mathbf{I} = \mathbf{YU}
\]

\[
\begin{bmatrix}
Y_{11} & Y_{12} & \cdots & Y_{1n} \\
Y_{21} & Y_{22} & \cdots & Y_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
Y_{n1} & Y_{n2} & \cdots & Y_{nn}
\end{bmatrix}
\begin{bmatrix}
U_1 \\
U_2 \\
\vdots \\
U_n
\end{bmatrix}
\]

(5.24)

Assume that all voltages are = 0 except \( U_a \). The current at \( b \) can now be calculated as

\[
I_b = Y_{ba} U_a
\]

(5.25)

Assume instead that all voltages are = 0 except \( U_b \). This means that the current at \( a \) is

\[
I_a = Y_{ab} U_b
\]

(5.26)

If \( U_a = U_b \), the currents \( I_a \) and \( I_b \) will be equal since the Y-bus matrix is symmetric, i.e. \( Y_{ab} = Y_{ba} \). By that, the proof of the reciprocity theorem is completed.

5.5 Thévenin-Helmholtz’ theorem

This theorem is often called the Thévenin’s theorem (after Léon Charles Thévenin, telegraph engineer and teacher, who published the theorem in 1883). But 30 years earlier, Hermann von Helmholtz published the same theorem in 1853, including a simple proof. The theorem can be described as

- Assume that a linear electric network has two terminals \( a \) and \( b \). When looking into the system from these two terminals, the rest of the system can be expressed as a voltage source \( U_T \) in series with an impedance \( Z_T \). The voltage \( U_T \) has the same amplitude as the voltage between the terminals \( a \) and \( b \), whereas the impedance \( Z_T \) is the impedance between \( a \) and \( b \) assuming that all voltage sources are short circuited and all current sources are disconnected.
Proof:
Assume that a linear network has two terminals, \( a \) and \( b \), where the voltage between the terminals equals \( U_T \). Between \( a \) and \( b \) an impedance \( Z \) is connected giving a current \( I \) through the impedance. This is equal with having a network with a voltage source \( U_T \) between \( a \) and \( b \) in series with the impedance \( Z \), together with having a network with the voltage source \(-U_T\) and the other voltage sources in the network shortened. By using the superposition theorem, the current \( I \) can be calculated as the sum of \( I_1 \) and \( I_2 \). The current \( I_1 = 0 \) since the voltage is equal on both sides of the impedance \( Z \). The current \( I_2 = 0 \) can be calculated as \( I_2 = -(\frac{-U_T}{Z + Z_T}) \) since the network impedance between \( a \) and \( b \) is \( Z_T \). The conclusion is that

\[
I = I_1 + I_2 = \frac{U_T}{Z + Z_T}
\]

(5.27)

which is the same as obtained by the Thévenin-Helmholtz’ theorem, viz.
5. Important theorems in power system analysis
Chapter 6
Analysis of symmetric three-phase systems

The analysis of a three-phase system operating in symmetric conditions can be carried out by studying only one phase. In Figure 6.1, a simple three-phase system is shown. Since the system is operating in symmetric conditions, the neutral connection points at the generator and at the load have both the same potential which means that \( I_n = 0 \). The impedance in the neutral conductor \( Z_n \) is therefore of no importance. The equations for phase \( a \) can be formulated as:

\[
U_a = (Z_G + Z_L)I_a
\]

(6.1)

The currents and voltages in the other phases have all the same amplitude as in phase \( a \) but they have a phase displacement of \( 120^\circ \). Equation (6.1) corresponds to the single-phase modeling of the network in Figure 6.2. The solution to this system gives also the complete solution to the whole system in Figure 6.1.

Assume that one part of a network consists of a Y-Y-connected three-phase transformer, see Figure 6.3. The single-phase equivalent is given in Figure 6.4 having the same turns-ratio as the three-phase transformer.

If the transformer is Y-∆-connected according to Figure 6.5, the ∆-side must be replaced by an equivalent Y-connection as indicated with dashed windings in the figure. The single-
6. Analysis of symmetric three-phase systems

Figure 6.3. Three-phase Y-Y-connected transformer

Figure 6.4. Single-phase equivalent of a three-phase Y-Y-connected transformer

phase equivalent of the Y-Δ-transformer is shown in Figure 6.6. It is important to notice that a phase shift of both the current and voltage takes place in this type of transformer (YΔ11 ⇒ 30°). This owing to the fact that a phase-to-phase voltage on one side corresponds to a phase-to-ground voltage on the other side. The phase shift between a phase-to-phase voltage and a phase-to-ground voltage is 30°, which is given in equation (3.12) and (3.16).

If this phase shift is of interest, the calculations have first to follow the single-phase equivalent and then a phase shift according to the actual transformer connection has to be added. It should be pointed out that the voltage ratio is not dependent on the type of transformer connection and the voltage ratio is the same for the single-phase equivalent as for the three-phase transformer.

Figure 6.5. Three-phase Y-Δ-connected transformer
6.1 Single-line diagram and impedance network

A single-line diagram of a power system shows the main components as well as the connections between them. A component is only given in the diagram if the component is of interest in the analysis. In Figure 6.7, an example of a single-line diagram is given. Breakers are drawn as squares, power lines as lines, generators and transformers are given with their different types of connections and finally, loads are drawn as arrows. The impedance network of the system (more precisely the positive sequence network, see chapter 13), to be used under symmetrical conditions, can be constructed by using the single-line diagram. In Figure 6.8, the impedance network of the system given by its single-line diagram in Figure 6.7 is drawn. Single-phase transformer equivalents are given as ideal transformers with the actual impedance of the correct side. The magnetization reactance has been neglected.

**Figure 6.6.** Single-phase equivalent of a three-phase Y-Δ-connected transformer

**Figure 6.7.** Single-line diagram of a small power system

*Generator 1*: 30 MVA, 10 kV, $X = 1.6 \Omega$

*Generator 2*: 15 MVA, 6 kV, $X = 1.2 \Omega$

*Generator 3*: 25 MVA, 6 kV, $X = 0.56 \Omega$

*Transformer $T_1$* (three-phase): 15 MVA, 30/10 kV, $X = 15.2 \Omega$/phase on high voltage side

*Transformer $T_2$* (three-phase): 15 MVA, 30/6 kV, $X = 16 \Omega$/phase on high voltage side

*Power line*: $R = 5 \Omega$/phase, $X = 7 \Omega$/phase

*Load A*: 15 MW, 10 kV, $\cos \phi = 0.9$ inductive

*Load B*: 40 MW, 6 kV, $\cos \phi = 0.85$ inductive

*Generator data are given as three-phase MVA, phase-to-phase voltage and the reactance per phase (Y-connected). Transformer data are given as three-phase MVA, phase-to-phase voltage ratio and per-phase impedance on one side. Load data are given as three-phase MW, phase-to-phase voltage and power factor.*
Generators are represented as a voltage source behind an impedance. Power lines are given as Π-equivalents and the loads are represented as impedances. The different connections of transformers and earthing impedances are not represented as symmetrical conditions apply.

The system has three different voltage levels (6, 10 and 30 kV). The analysis of the system can be carried out due to a transformation of the different voltage levels and impedances to one specific voltage level, e.g. the voltage of the power line, 30 kV. This method gives often quite extensive calculations. The normal procedure when analyzing a system with different voltage levels is to take advantage of the per-unit system.

### 6.2 Per-unit (pu) system

A common method to express voltages, currents, powers and impedances in an electric network is in per-unit (or percent) of a certain base or reference value. The per-unit value of a certain quantity is defined as

\[
\text{Per-unit value} = \frac{\text{true value}}{\text{base value of the quantity}} \quad (6.2)
\]

The per-unit method is very suitable for power systems with several voltage levels and transformers. In a three-phase system, the per-unit value can be calculated using the corresponding base quantity. By using the base voltage

\[ U_b = \text{phase-to-phase voltage} = \text{base voltage, kV} \quad (6.3) \]

and a base power,

\[ S_b = \text{three-phase base power, MVA} \quad (6.4) \]

the base current

\[ I_b = \frac{S_b}{\sqrt{3}U_b} = \text{base current/phase, kA} \quad (6.5) \]

as well as a base impedance

\[ Z_b = \frac{U_b^2}{S_b} = \text{base impedance, } \Omega \quad (6.6) \]
can be calculated. In expressions given above, the units kV and MVA have been assumed, which implies units in kA and Ω. Of course, different combinations of units can be used, e.g. V, VA, A, Ω or kV, kVA, A, kΩ.

The reasons why using the per-unit system are (among other things)

- The percentage voltage drop is directly given in the per-unit voltage.
- It is possible to analyze power systems having different voltage levels in a more efficient way.
- When having different voltage levels, the relative importance of different impedances is directly given by the per-unit value.
- When having large systems, numerical values of the same magnitude are obtained which increase the numerical accuracy of the analysis.

6.2.1 Per-unit representation of transformers

As mentioned above, a three-phase transformer connected to a power system operating in symmetrical conditions, can be represented by a single-phase transformer. A Δ-connected transformer has to be replaced with a Y-connected transformer with the same turns-ratio.

Figure 6.9 shows a model of a single-phase transformer with leakage reactances on both the primary and the secondary side together with an ideal transformer with the turns-ratio 1:a. The magnetization impedance is neglected. Assume a base power $S_b$ and choose base voltages on each side of the transformer in relation to the transformer ratio, i.e.

$$\frac{U_{1b}}{U_{2b}} = \frac{1}{a}$$  \hspace{1cm} (6.7)

This gives that

$$\frac{I_{1b}}{I_{2b}} = a \text{ (since } S_b \text{ is the same)}$$  \hspace{1cm} (6.8)
and
\[ Z_{1b} = \frac{U_{1b}}{\sqrt{3}I_{1b}}, \quad Z_{2b} = \frac{U_{2b}}{\sqrt{3}I_{2b}} \] (6.9)

By using Figure 6.9, the voltage \( U_2 \) can be expressed as
\[ U_2 = [U_1 - \sqrt{3}I_1 Z_p]a - \sqrt{3}I_2 Z_s \] (6.10)

By using the definition (6.2), the equation (6.10) can be rewritten as
\[ U_2(pu)U_{2b} = [U_1(pu)I_{1b} - \sqrt{3}I_1(pu)I_{1b}Z_p(pu)Z_{1b}]a - \sqrt{3}I_2(pu)I_{2b}Z_s(pu)Z_{2b} \] (6.11)

By dividing with \( U_{2b} \) and by using the expressions 6.7, 6.8 and 6.9, the following is obtained
\[ U_2(pu) = U_1(pu) - I_1(pu)Z_p(pu) - I_2(pu)Z_s(pu) \] (6.12)

But since
\[ \frac{T_1}{T_2} = \frac{I_{1b}}{I_{2b}} = a \quad \text{or} \quad \frac{T_1}{T_{1b}} = \frac{T_2}{T_{2b}} \] (6.13)

the following is valid
\[ T_1(pu) = T_2(pu) = T(pu) \] (6.14)

The equation (6.12) can thereby be rewritten as
\[ U_2(pu) = U_1(pu) - T(pu)Z(pu) \] (6.15)

where
\[ Z(pu) = Z_p(pu) + Z_s(pu) \] (6.16)

Equation (6.15) can now be represented by a simple circuit as given in Figure 6.10, which do not need the model of the ideal transformer. This means that a power system transformer can

![Figure 6.10. Per-unit representation of a symmetrically loaded transformer](image)

be represented as an impedance if the per-unit system with a common power base together with two voltage bases according to the ratio, is used.

The impedance \( Z(pu) \) can directly be determined by using the value of the impedance given in ohm from the primary or the secondary side of the transformer together with the corresponding base values.
6.2. Per-unit (pu) system

On the primary side:

\[
Z_1 = Z_p + Z_s/a^2 \quad (6.17)
\]

\[
Z_{1\,(pu)} = \frac{Z_1}{Z_{1b}} = \frac{Z_p}{Z_{1b}} + \frac{Z_s}{Z_{1b}} \cdot \frac{1}{a^2}
\]

But

\[
a^2 Z_{1b} = Z_{2b} \quad (6.18)
\]

which gives that

\[
Z_{1\,(pu)} = Z_{p\,(pu)} + Z_{s\,(pu)} = \mathcal{Z}(pu) \quad (6.19)
\]

Of course, the same calculations can be performed using the secondary side of the transformer with corresponding result. The conclusions is that the per-unit value of the impedance of the transformer is independent on the side used in the calculations. It must be pointed out that if the transformer impedance is given in percent, i.e. internal per-unit, the impedance must first be converted to system per-unit if another base power and/or base voltage is used.

**Example 6.1** Assume that a 15 MVA transformer has a voltage ratio of 6 kV/30 kV and a short circuit reactance of 8%. Calculate the pu-impedance when the base power of the system is 20 MVA and the base voltage on the 30 kV-side is 33 kV.

**Solution**

The first value to calculate is the transformer impedance in ohm on the 30 kV-side and after that, the per-unit value.

\[
Z_{30\,kv} = \frac{Z\%}{100} Z_{trafo-bas} = \frac{Z\%}{100} \frac{U_{trafo-30kV}^2}{S_{trafo}} = \frac{j8 \cdot 30^2}{100 \cdot 15} = j4.8 \ \Omega
\]

\[
Z_{pu} = \frac{Z_{30kV}}{Z_{b-30kV}} = \frac{Z_{30kV} \cdot S_b}{U_{b-30kV}^2} = \frac{j4.8 \cdot 20}{33^2} = j0.088 \ pu
\]

It is possible to calculate the pu value directly by transforming to a new base value as

\[
Z_{pu-ny} = \frac{Z_{pu}}{S_b} \cdot \frac{S_{b-ny}}{S_b} \cdot \frac{U_{b-ny}^2}{U_{b-30kV}^2} = \frac{j8 \cdot 20 \cdot 30^2}{100 \cdot 15 \cdot 33^2} = j0.088 \ pu \quad (6.20)
\]

### 6.2.2 Calculations by using the per-unit system

As shown earlier, based on a single-line diagram, an impedance network can be drawn. This network contains transformers which can be replaced by impedances by using the per-unit system. The algorithm to follow when analyzing a power system is as follows:

1. Choose a suitable base power for the system. It should be in the same range as the rated power of the installed system equipments.
2. Choose a base voltage at one section of the system. The system is divided into different sections by the transformers.
3. Calculate the base voltages in all sections of the system by using the transformer ratios.

4. Calculate all per-unit values of all system components that are connected.

5. Draw a single-line diagram of the system.

6. Perform the analysis asked for.

7. Transform the result back to nominal values.

Example 6.2 A power system is given in Figure 6.11 where a load is fed by a generator via a transmission line and two transformers. Calculate the load voltage as well as the active power of the load.

![Figure 6.11. Power system in example 6.2](image)

**Generator G**: 13.8 kV phase-to-phase voltage

**Transformer T1**: 10 MVA, 13.8/69 kV, $j1.524 \Omega$ (13.8 kV-side)

**Line L**: 10 km, $j0.8 \Omega$ /phase.km

**Transformer T2**: 5 MVA, 66/13.2 kV, $x = 8\%$

**Load LD**: 4 MW, $\cos \phi = 0.8$, 13.2 kV, impedance characteristic

Solution

1. Assume base power 10 MVA.

2. Assume base voltage 13.8 kV at the generator.

3. The transformer ratio gives a base voltage of 69 kV for the line and $69 \cdot 13.2 / 66 = 13.8$ kV for the load. In Figure 6.12, the different sections of the system are given.

![Figure 6.12. Different sections of the system given in example 6.2](image)
4. Calculate the per-unit values of the system components.

\[
G: \quad U_g = \frac{U_{g-nom}}{U_{1-bas}} = \frac{13.8}{13.8} = 1.0 \text{ pu}
\]

\[
T1: \quad \mathcal{Z}_{t1} = \frac{Z_{t-nom}}{Z_{1-bas}} = j1.524 \cdot \frac{10}{13.8^2} = j0.080 \text{ pu}
\]

\[
L: \quad \mathcal{Z}_L = \frac{s\mathcal{Z}_L}{Z_{2-bas}} = \frac{10 \cdot j0.8 \cdot 10}{69^2} = j0.017 \text{ pu}
\]

\[
T2: \quad \mathcal{Z}_{T2-13.2kV}(\Omega) = j\frac{8}{100} \frac{13.2^2}{5} = j2.788 \Omega
\]

\[
\mathcal{Z}_{T2} = \frac{\mathcal{Z}_{T2-13.2kV}(\Omega)}{Z_{3-bas}} = \frac{j2.788 \cdot 10}{13.8^2} = 0.146 \text{ pu}
\]

\[
LD: \quad \mathcal{Z}_{LD}(\Omega) = \frac{U_{LD}^2}{S_{LD}} = \frac{U_{LD}^2}{S_{LD}(\cos \phi - j \sin \phi)} = \frac{U_{LD}^2}{S_{LD}(\cos \phi + j \sin \phi)} =
\]

\[
= \frac{13.2^2}{4/0.8} (0.8 + j0.6) = 27.88 + j20.91 \Omega
\]

\[
\mathcal{Z}_{LD} = \frac{\mathcal{Z}_{LD}(\Omega)}{Z_{3-bas}} = \frac{(27.88 + j20.91) \cdot 10}{13.8^2} = 1.464 + j1.098 \text{ pu}
\]

5. By using these values, an impedance network can be drawn as given in Figure 6.13.

\[\text{Figure 6.13. Impedance network in per-unit}\]

6. The current through the network can be calculated as

\[
I = \frac{1 + j0}{j0.08 + j0.017 + j0.146 + 1.464 + j1.098} = 0.3714 - j0.3402 \text{ pu}
\]

The load voltage is

\[
U_{LD} = I\mathcal{Z}_{LD} = 0.9173 - j0.0903 \text{ pu}
\]

The load power is

\[
S_{LD} = U_{LD}^*I_{LD} = 0.3714 + 0.2785 \text{ pu}
\]

7. The nominal values of the load voltage and active load power can be obtained by multiplying with corresponding base quantities.

\[
U_{LDkV} = U_{LD}U_{3-bas} = \sqrt{0.9173^2 + 0.0903^2} \cdot 13.8 = 12.72 \text{ kV}
\]

\[
P_{LDMW} = \text{Real}(S_{LD})S_{bas} = 0.371 \cdot 10 = 3.71 \text{ MW}
\]
6. Analysis of symmetric three-phase systems
Chapter 7

Power transmission to impedance loads

Power lines and cables are normally operating in symmetrical conditions, i.e. with a symmetrical three-phase. At such conditions, a line may be represented with a single-phase equivalent. This equivalent can be described by a twoport. (Or more precisely, with a positive sequence equivalent, see chapter 11.)

7.1 Twoport theory

Assume that a linear, reciprocal twoport is of interest, where the voltage and current in one end are $U_a$ and $I_a$ whereas the voltage and current in the other end are $U_b$ and $I_b$. The conditions valid for this twoport can be described by constants $ABCD$ as

$$\begin{bmatrix} U_a \\ I_a \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} U_b \\ I_b \end{bmatrix} \quad (7.1)$$

Assume that the twoport is shortened in the receiving end, (i.e. $U_b = 0$) according to Figure 7.1, and that the voltage $U$ is applied to the sending end. The system in Figure 7.1 can be written as

$$U = B \cdot I_b \quad (7.2)$$

$$I_a = D \cdot I_b \quad (7.3)$$

If it is assumed that the twoport is shortened in the sending end instead, ($U_a = 0$) as given in Figure 7.2, and the voltage $U$ is applied to the receiving end. The system can according

$$\begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix}$$

Figure 7.1. Twoport, shortened in the receiving end

written as

$$\begin{align*}
\bar{U} &= \bar{B} \cdot \bar{I}_{b1} \\
\bar{I}_{a1} &= \bar{D} \cdot \bar{I}_{b1}
\end{align*} \quad (7.2)$$

If it is assumed that the twoport is shortened in the sending end instead, ($U_a = 0$) as given in Figure 7.2, and the voltage $U$ is applied to the receiving end. The system can according

$$\begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix}$$

Figure 7.2. Twoport, shortened in the sending end
to Figure 7.2 be written as

\[ 0 = \overline{A} \cdot U - \overline{B} \cdot I_{b2} \] (7.4)
\[ -I_{a2} = \overline{C} \cdot U - \overline{D} \cdot I_{b2} \] (7.5)

The reciprocity theorem gives that

\[ I_{a2} = I_{b1} = \overline{I} \] (7.6)

From the equations given above, the following expressions can be derived:

\[ \text{eq. 7.4} \Rightarrow I_{b2} = \frac{\overline{A}}{\overline{B}} \cdot U \] (7.7)
\[ \text{eq. 7.7+7.6+7.2} \Rightarrow I_{b2} = \overline{A} \cdot I \] (7.8)
\[ \text{eq. 7.2+7.5+7.8} \Rightarrow -\overline{I} = \overline{C} \cdot \overline{B} \cdot I - \overline{D} \cdot \overline{A} \cdot \overline{I} \] (7.9)
\[ \text{eq. 7.9, } \overline{I} \neq 0 \Rightarrow \overline{A} \cdot \overline{D} - \overline{B} \cdot \overline{C} = 1 \] (7.10)

i.e. the determinant of a reciprocal twoport = 1. This implies that if several reciprocal twoports are connected after one another, the determinant of the total twoport obtained is also = 1. With three reciprocal twoports \( F_1, F_2 \) and \( F_3 \) connected after one another, the following is always valid:

\[ \det(F_1 F_2 F_3) = \det(F_1) \det(F_2) \det(F_3) = 1 \cdot 1 \cdot 1 = 1 \] (7.11)

### 7.1.1 Symmetrical twoports

Assume that a symmetrical linear reciprocal twoport is of interest. If the definitions of directions given in Figure 7.3 is used, a current injected in the sending end \( I_a \) at the voltage \( U_a \) gives rise to a current \( I_1 \) at the voltage \( U_1 \) in the receiving end. This can be written in an equation as

\[ \begin{bmatrix} U_a \\ I_a \end{bmatrix} = \begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & \overline{D} \end{bmatrix} \begin{bmatrix} U_1 \\ I_1 \end{bmatrix} \] (7.12)

Suppose that the circuit is fed in the opposite direction, i.e. \( U_1 \) and \( I_1 \) are obtained in the sending end according to Figure 7.4. This connection can mathematically be formulated as:

\[ \begin{bmatrix} U_1 \\ -I_1 \end{bmatrix} = \begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & \overline{D} \end{bmatrix} \begin{bmatrix} U_b \\ -I_b \end{bmatrix} \] (7.13)
By changing the position of the minus sign inside the matrix, equation (7.13) can be rewritten as
\[
\begin{bmatrix}
U_1 \\
I_1
\end{bmatrix}
=\begin{bmatrix}
A & -B \\
-C & D
\end{bmatrix}
\begin{bmatrix}
U_b \\
I_b
\end{bmatrix}
\]
(7.14)
The matrix in equation (7.14) can be inverted which gives that
\[
\begin{bmatrix}
U_b \\
I_b
\end{bmatrix}
=\frac{1}{A \cdot D - B \cdot C}
\begin{bmatrix}
D & B \\
C & A
\end{bmatrix}
\begin{bmatrix}
U_1 \\
I_1
\end{bmatrix}
\]
(7.15)
Since the twoport is symmetrical, the following is valid
\[
\begin{bmatrix}
U_b \\
I_b
\end{bmatrix}
=\begin{bmatrix}
U_a \\
I_a
\end{bmatrix}
\]
(7.16)
The equations (7.12), (7.15) and (7.16) gives together that
\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
=\begin{bmatrix}
D & B \\
C & A
\end{bmatrix}
\]
(7.17)
This concludes that for symmetrical twoports \( A = D \).

### 7.2 Model of a power transmission line

A power transmission line operating under symmetrical conditions can be modeled with a single-phase equivalent (more precisely a positive sequence equivalent) which consists of series and parallel connected inductances, resistances, conductances and capacitances according to Figure 7.5, presented earlier in chapter 4. For a more accurate model of a transmission line or cable, see chapter 11.

A common way of simplifying this transmission line equivalent is to
**Power transmission to impedance loads**

- Neglect the shunt conductance \(g\).
- Divide the shunt capacitance \(c\) into two parts, located at each end of the line.
- Connect all series inductances \(l\) and resistances \(r\) into one long series impedance component.

The result of these simplifications is the commonly used \(\Pi\)-equivalent of a transmission line according to Figure 7.6. The *nominal* values of the parameters in Figure 7.6 are:

\[
\begin{align*}
\bar{Z}_n &= \ell (r + j2\pi fl) \\
\bar{Y}_n &= \ell (j2\pi fc)
\end{align*}
\]

where

- \(\ell\) = the length of the line in km
- \(r\) = the resistance of the line in \(\Omega/\text{km}\)
- \(l\) = the inductance of the line in \(\text{H/\text{km}}\)
- \(f\) = frequency (50 or 60 Hz)
- \(c\) = the capacitance of the line in \(\text{F/\text{km}}\)

With a per-unit representation of the line \((\bar{Y} = \bar{Y}_n/Y_{\text{bas}}, \bar{Z} = \bar{Z}_n/Z_{\text{bas}})\) together with notations according to Figure 7.7, the following equations can be written for the voltages and currents at the ends of the transmission line.

\[
\begin{align*}
\bar{U}_a &= \bar{U}_b + (\bar{I}_b + \frac{1}{2}\bar{U}_b \cdot \bar{Y}) \bar{Z} \\
\bar{I}_a &= \frac{1}{2}\bar{U}_a \cdot \bar{Y} + \bar{I}_b + \frac{1}{2}\bar{U}_b \cdot \bar{Y}
\end{align*}
\]
These equations can be rewritten as

\[
\begin{align*}
U_a &= (1 + \frac{1}{2}Z \cdot Y) U_b + Z \cdot I_b \\
I_a &= \frac{1}{2} Y(Y + \frac{1}{2}Z \cdot Y) U_b + (\frac{1}{2}Z \cdot Y + 1) I_b
\end{align*}
\]  

(7.19)

by using the matrix notation, this can be written as a twoport equation

\[
\begin{bmatrix} U_a \\ I_a \end{bmatrix} = \begin{bmatrix} \frac{\overline{A}}{1 + \frac{1}{2}Y \cdot Z} & \frac{\overline{B}}{Y(1 + \frac{1}{4}Y \cdot Z)} \\ \frac{\overline{C}}{1 + \frac{1}{2}Y \cdot Z} & \frac{\overline{D}}{Z} \end{bmatrix} \begin{bmatrix} U_b \\ I_b \end{bmatrix}
\]  

(7.20)

As shown in equation (7.20), a line is symmetrical which gives that \( \overline{A} = \overline{D} \). A line is also reciprocal which gives that \( \overline{A} \cdot \overline{D} - \overline{B} \cdot \overline{C} = 1 \).

### 7.2.1 Model of simplified line and transformer

In some cases, the shunt capacitance of the line may be neglected which implies that the line can be modeled as a series impedance \( Z \) only. This is the same model as earlier used for transformers (see Figure 6.10). By using this model, all \( Y = 0 \) in equation (7.20) and the simplified line (transformer) can be represented by a twoport:

\[
\begin{bmatrix} U_a \\ I_a \end{bmatrix} = \begin{bmatrix} 1 & Z \\ 0 & 1 \end{bmatrix} \begin{bmatrix} U_b \\ I_b \end{bmatrix}
\]  

(7.21)

### 7.2.2 Connection to network

If power is taken from a certain connection in the network, the voltage at that connection usually drops in proportion to the power taken. For a connection called an “infinite bus”, the voltage is constant, independent on the power transfer.

According to the Thévenin-Helmholtz’ theorem, a linear network studied at one bus can be replaced with a constant voltage source behind an impedance, see section 5.5. Assume that by looking at one single bus, that part of the network can be replaced with a Thévenin-equivalent as shown in Figure 7.8. Assume that a solid short circuit (\( Z_k = 0 \)) is applied

![Figure 7.8. Thévenin-equivalent of a network at one bus](image-url)
7. Power transmission to impedance loads

between the buses a and b. This model implies that the short circuit current is

$$I_k = \frac{U_T}{Z_T}$$

(7.22)

expressed by the per-unit system. As given in Figure 7.8, the voltage between a and b is equal to $\overline{U}_T$ when no current flows.

The question is how well the linear model can be adapted to real conditions. Analyze the simplified network given in Figure 7.9. Assume that the transformer T2 is able to regulate the voltage at D to 10 kV automatically.

1: Assume that a short circuit is applied at bus D. The rest of the network can be replaced by a voltage source $\overline{U}_{TD}$ behind an impedance $\overline{Z}_{TD}$ where the voltage $\overline{U}_{TD} = U_T$ is expressed as the voltage in D prior to the short circuit and $\overline{Z}_{TD} = \overline{Z}_{T2} + \overline{Z}_{L1} + \overline{Z}_{T1} + \overline{Z}_{G1}$ (all values in per-unit). Of course, this will result in a voltage drop at D at a short circuit incident.

If a load is connected at D instead of a short circuit, the voltage will drop momentarily but the automatic regulation of the transformer will regulate the load voltage back to 10 kV. The Thévenin-equivalent is not appropriate in this case.

2: Instead, assume that a short circuit occur at bus E. The rest of the network can then be modeled as a voltage source $\overline{U}_{TE}$ behind an impedance $\overline{Z}_{TE}$ where the voltage $\overline{U}_{TE} = U_E$ is expressed as the voltage at E before the short circuit and $\overline{Z}_{TE} = \overline{Z}_{L2} + \overline{Z}_{T2} + \overline{Z}_{L1} + \overline{Z}_{T1} + \overline{Z}_{G1}$ (all values in per-unit). As in the same way as in case 1:, this implies that a voltage drop occur at bus E at a short circuit.

If a load is connected at bus E instead, the voltage will drop momentarily but recover to 10 kV at bus D due to the automatic voltage regulation of transformer T2. In this case, a permanent voltage drop will remain since the voltage is regulated at bus D and not at bus E. The remaining impedance $\overline{Z}_{L2}$ is the difference between the Thévenin impedance at bus D and E.

The conclusion is that for a bus located out in a distribution system, (i.e. a fairly voltage weak bus), the impedance to the closest bus with constant voltage can be calculated as the difference between the Thévenin-impedance for that bus and the Thévenin-impedance for the bus with voltage regulation. For the calculation of voltage drop at connection of load, this calculated difference in impedance should be used in the Thévenin-equivalent of the network.
In some cases, the term short circuit capacity \( S_k \) at a bus in the network, is used. It is defined as
\[
S_k = U_T T_k^* \tag{7.23}
\]
hence it follows that this is the power that is obtained in the Thévenin-impedance. Since this impedance often is mostly reactive, the angle of the power that is obtained is close to +90°. The short circuit capacity is of interest when the loadability of a certain bus is asked for. The short circuit capacity indicates how much the bus voltage will change for different magnitudes of bus loading. The voltage increase at generator buses can also be calculated.

**Example 7.1** At a bus with a short circuit capacity of 500 MVA and \( \cos \phi_k = 0 \), inductive, an impedance load of 4 MW, \( \cos \phi_{LD} = 0.8 \) at nominal voltage, is connected. Calculate the change in bus voltage when the load is connected.

**Solution**

Assume a voltage of 1 pu and a base power \( S_b = 500 \) MVA. The network can then be modeled as given in Figure 7.10.

![Figure 7.10. Single-phase model of system given in example](image)

The Thévenin-impedance can be calculated according to equation (7.22) and (7.23):
\[
Z_{T-pu} = \frac{U_T}{I_k} = \frac{U_T^2}{S_k} = \frac{1}{1/1 - 90°} = j1 \tag{7.24}
\]

The load impedance can be calculated as
\[
Z_{LD-pu} = (U_{LD-pu}^2/\sqrt{S_{LD-pu}}) = (1/[4/(500 \cdot 0.8)]) (0.8 + j0.6) = 80 + j60 \tag{7.25}
\]

By a simple voltage division, the voltage \( U'_{LD} \) at the load can be calculated as
\[
U'_{LD} = \frac{U_T Z_{LD-pu}}{Z_{T-pu} + Z_{LD-pu}} = \frac{1 \cdot (80 + j60)}{j1 + 80 + j60} = 0.9940 \angle -0.4556° \tag{7.26}
\]
i.e. the voltage drop is about 0.6%.

Conclusion: A load with an apparent power of 1% of the short circuit capacity at the bus connected, will cause a voltage drop at that bus of \( \approx 1\% \).
Example 7.2 A small industry is fed by a transformer (5 MVA, 70/10 kV, x = 4%) which is located at a distance of 5 km. The electric power demand of the industry is 400 kW at \(\cos \phi = 0.8\), lagging, at a voltage of 10 kV. The industry can be modeled as an impedance load. The 10 kV line has an series impedance of 0.9+j0.3 \(\Omega/\text{phase,km}\) and a shunt admittance of \(j3 \times 10^{-6}\) \(S/\text{phase,km}\). Assume that the line is modeled by the \(\Pi\)-equivalent. Calculate the voltage level at the industry as well as the power fed by the transformer into the line. When the industry is not connected, a short circuit current of 0.3 kA, pure inductive, can be obtained on the 70 kV side of the transformer when a three-phase short circuit is applied at nominal voltage.

Solution

Chose base values (MVA, kV, \(\Rightarrow\) kA, \(\Omega\)):
\[
S_b = 500 \text{ kVA} = 0.5 \text{ MVA}, \quad U_{b10} = 10 \text{ kV}
\]

\[
\Rightarrow I_{b10} = S_b/\sqrt{3}U_{b10} = 0.0289 \text{ kA}, \quad Z_{b10} = U_{b10}^2/S_b = 200 \Omega, \quad U_{b70} = 70 \text{ kV} \Rightarrow I_{b70} = S_b/\sqrt{3}U_{b70} = 0.0041 \text{ kA}
\]

Calculate the per-unit values of the Thévenin-equivalent of the system:

Equation (7.22) \(\Rightarrow Z_{Th} = U_{Th}/I_k \Rightarrow Z_{Thpu} = (U_{Th}/U_{b70}) \cdot (I_{b70}/I_k) = (70\angle0^\circ/70) \cdot (0.00412/0.3\angle-90^\circ) = j0.0137, \quad U_{Thpu} = U_{Th}/U_{b70} = 1.0\angle0^\circ
\]

Calculate the per-unit values of the transformer:
\[
Z_{trapu} = (Z_{tra\%}/100) \cdot Z_{b70} = (Z_{tra\%}/100) \cdot S_b/S_{tra} = (j4/100) \cdot 0.5/5 = j0.004
\]

Calculate the per-unit values of the line:
\[
\begin{align*}
\bar{A}_L &= 1 + \bar{Y}_{lpu} \cdot Z_{lpu} = 1.0000 + j0.0000 \\
\bar{B}_L &= Z_{lpu} = 0.0225 + j0.0075 \\
\bar{Y}_{lpu} &= 5 \cdot [j3 \times 10^{-6}/Y_{b10} = 5 \cdot [j3 \times 10^{-6}] \cdot Z_{b10} = j0.003 \\
\bar{C}_L &= \bar{Y}_{lpu}(1 + \frac{1}{4}\bar{Y}_{lpu} \cdot Z_{lpu}) = 0.0000 + j0.0030 \\
\bar{D}_L &= \bar{A}_L = 1.0000 + j0.0000
\end{align*}
\]

Calculate the per-unit values of the industry impedance:
\[
\bar{Z}_{indpu} = (U_{ind}^2/S_{ind})/Z_{b10} = (10^2/[0.4/0.8]) \cdot (0.8 + j0.6)/200 = 0.8 + j0.6
\]

The twoport of the whole system between the “infinite bus” and the industry, (power system diagrams)}
7.3 General method of calculations of symmetrical three-phase system with impedance loads

When analyzing large power systems, it is necessary to perform the analysis in a systematic manner. Below, a small system is analyzed with the same method as can be used for large systems. In Figure 7.12, a small system is shown where an impedance $Z_{LD1}$ is fed from an

![Impedance diagram of a symmetrical power system](image)

The impedance of the entire system (including the industry) can be calculated as

$$Z_{totpu} = \frac{U_{Th}}{I_1} = \frac{AU_3 + B I_3}{CU_3 + DI_3} = \frac{AZ_{indpu} + B}{CZ_{indpu} + D} = 0.8254 + j0.6244$$

$$\Rightarrow I_1 = \frac{U_{Th}}{Z_{totpu}} = 0.9662\angle -37.1035^\circ$$

The power fed by the transformer into the line can be calculated as

$$S_2 = U_2 I_2^* S_b = 0.3853 + j0.2832 \text{ MVA}$$

the voltage at the industry can be calculated as

$$U_3 (\text{kV}) = \frac{U_3}{U_{b10}} = 9.6796 \text{ kV}$$

7.3 General method of calculations of symmetrical three-phase system with impedance loads

When analyzing large power systems, it is necessary to perform the analysis in a systematic manner. Below, a small system is analyzed with the same method as can be used for large systems. In Figure 7.12, a small system is shown where an impedance $Z_{LD1}$ is fed from an

![Impedance diagram of a symmetrical power system](image)

infinite bus $U_3$ via a transformer $Z_T$ and a line $Z_L$. The Y-bus matrix for this system can be formulated as

$$[\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}] = [\begin{bmatrix} \frac{1}{Z_{LD1}} + \frac{1}{Z_L} & -\frac{1}{Z_L} & 0 \\ -\frac{1}{Z_L} & \frac{1}{Z_L} + \frac{1}{Z_T} & -\frac{1}{Z_T} \\ 0 & -\frac{1}{Z_T} & \frac{1}{Z_T} \end{bmatrix}] [\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}]$$
The Y-bus matrix can be inverted which results in the corresponding Z-bus matrix:

\[ \mathbf{U} = \mathbf{Y}^{-1} \mathbf{I} = \mathbf{ZI} \] (7.28)

Since \( \bar{I}_1 = \bar{I}_2 = 0 \), the third row in equation (7.28) can be written as

\[ \bar{U}_3 = Z(3,3) \cdot \bar{I}_3 \]
\[ \Rightarrow \bar{I}_3 = \bar{U}_3/Z(3,3) \] (7.29)

where \( Z(3,3) \) is an element in the Z-bus matrix. With that value of the current inserted into equation (7.28), all system voltages are obtained.

\[ \bar{U}_1 = Z(1,3) \cdot \bar{I}_3 \] (7.30)
\[ \bar{U}_2 = Z(2,3) \cdot \bar{I}_3 \]

Corresponding calculations can be performed for arbitrarily large systems containing impedance loads and one voltage source.

Assume that an impedance \( Z_{LD} \) is added to the system at bus 2, see Figure 7.13a. This will change the voltage magnitudes at the buses.

\[ \mathbf{U}' = \mathbf{U} + \mathbf{U}_\Delta \] (7.31)

where \( \mathbf{U}' \) are the voltages after the change and \( \mathbf{U}_\Delta \) is the applied change. This equation can be illustrated graphically as given in Figure 7.13, i.e. the total voltage can be calculated as a superposition of two systems with equal impedances but with different voltage sources. As indicated by the system in Figure 7.13c, the feeding voltage is \( -\bar{U}_2 \) while the voltage source \( \bar{U}_3 \) is shortened. The Y-bus matrix for this system can be obtained by canceling the third row as well as the third column in \( \mathbf{Y} \) (equation (7.27)) since bus 3 is grounded. If bus 3 was kept in the mathematical formulation, \( \mathbf{Y}(3,3) = \infty \) since the impedance to ground = 0.

\[ \begin{bmatrix} \bar{I}_{\Delta 1} \\ \bar{I}_{\Delta 2} \end{bmatrix} = \mathbf{I}_\Delta = \mathbf{Y}_\Delta \mathbf{U}_\Delta = \begin{bmatrix} \frac{1}{Z_{LD1}} + \frac{1}{Z_L} & -\frac{1}{Z_L} \\ -\frac{1}{Z_L} & \frac{1}{Z_L} + \frac{1}{Z_T} \end{bmatrix} \begin{bmatrix} \bar{U}_{\Delta 1} \\ \bar{U}_{\Delta 2} \end{bmatrix} \] (7.32)

The expression given above, can be inverted which gives the corresponding Z-bus matrix:

\[ \mathbf{U}_\Delta = \mathbf{Y}_\Delta^{-1} \mathbf{I}_\Delta = \mathbf{Z}_\Delta \mathbf{I}_\Delta \] (7.33)

In this equation, \( \bar{I}_{\Delta 1} = 0 \) which can be seen in Figure 7.13c. This gives that the second row can be written as

\[ \bar{U}_{\Delta 2} = Z_\Delta(2,2) \bar{I}_{\Delta 2} \] (7.34)

Figure 7.12 gives the same currents as Figure 7.13b because the voltage over \( \bar{Z}_{LD} \) in Figure 7.13b is = 0. This implies that the current through \( \bar{Z}_{LD} = 0 \). This gives that \( \bar{I}_{\Delta 2} = -\bar{I}_{LD2} \).

At bus 2 in Figure 7.13a the following is valid

\[ \bar{U}_2' = \bar{I}_{LD2} \cdot \bar{Z}_{LD2} = -\bar{I}_{\Delta 2} \cdot \bar{Z}_{LD2} \] (7.35)
7.3. General method of calculations of symmetrical three-phase system with impedance loads

By combining equation (7.31), (7.34) and (7.35), the following can be obtained

\[ I_{\Delta 2} = \frac{-U_2}{Z_{LD2} + Z_{\Delta}(2,2)} \]  

(7.36)

By inserting that value in the equations given above, all voltages after the system change can be calculated as:

\[ U_2' = \frac{Z_{LD2}}{Z_{LD2} + Z_{\Delta}(2,2)} U_2 \]  

(7.37)

\[ U_1' = U_1 - \frac{Z_{\Delta}(1,2)}{Z_{LD2} + Z_{\Delta}(2,2)} U_2 \]  

(7.38)

The procedure given above can be generalized to be used for an arbitrarily large system. Assume that the impedance \( Z_r \) is connected to bus \( r \) and an arbitrary bus is named \( i \). The current \( I_r (= -I_{\Delta r}) \) through \( Z_r \) can be calculated as well as the voltages after connection of
the impedance $Z_r$ at bus $r$. The equations are as follows.

$$I_r = \frac{U_r}{Z_r + Z(r,r)}$$  \hspace{1cm} (7.39)

$$U'_r = \frac{Z_r}{Z_r + Z(r,r)} U_r$$  \hspace{1cm} (7.40)

$$U'_i = U_i - \frac{Z(i,r)}{Z_r + Z(r,r)} U_r \quad i \neq r$$  \hspace{1cm} (7.41)

The Thévenin-equivalent at a bus in a symmetrical network can be calculated by using equations (7.31) and (7.33). At bus $r$ ($r=2$ in these equations), the equation will be as

$$U'(r) = U(r) + Z(r,r)I_{\Delta r}$$  \hspace{1cm} (7.42)

where

$U(r) =$ Thévenin-voltage at bus $r$, see Figure 7.14.

$Z(r,r) =$ Thévenin-impedance at bus $r$, see Figure 7.14.

$I_{\Delta r} =$ The current that is injected into bus $r$.

$U'(r) =$ the voltage at bus $r$.

![Figure 7.14. Thévenin-equivalent at bus $r$ in a symmetrical three-phase network](image)

As given by equation (7.42) and Figure 7.14, $U'(r) = U(r)$ when $I_{\Delta r} = 0$. This formulation shows that the Thévenin-voltage at bus $r$ can be calculated as the voltage at bus $r$ when the bus is not loaded, i.e. $I_{\Delta r} = 0$. The Thévenin-impedance can be calculated as the $r$-th diagonal element in the impedance matrix $Z_\Delta$. The matrix is calculated with all voltage sources shortened.

**Example 7.3** In Figure 7.15, an internal network of an industry is given. The energy is delivered by an infinite bus with a nominal voltage at bus 1. The energy is transmitted via transformer T1, line L2 and transformer T2 to the load LD2. There is also a high voltage load LD1 connected to T1 via the line L1. Ratings of the different components are:

- Transformer $T_1$: 800 kVA, 70/10 kV, $x = 7\%$
7.3. General method of calculations of symmetrical three-phase system with impedance loads

![Diagram](image)

**Figure 7.15.** Single-line diagram of an internal industry network

- **Transformer T2**: 300 kVA, 0.4/10 kV, \( x = 8 \% \)
- **Line L1**: \( r = 0.17 \Omega/km, \omega L = 0.3\Omega/km, \omega C = 3.2 \times 10^{-6} S/km, s = 2 \text{ km} \)
- **Line L2**: \( r = 0.17 \Omega/km, \omega L = 0.3\Omega/km, \omega C = 3.2 \times 10^{-6} S/km, s = 1 \text{ km} \)
- **Load LD1**: Impedance characteristic, 500 kW, \( \cos \phi = 0.80 \), inductive at 10 kV
- **Load LD2**: Impedance characteristic, 200 kW, \( \cos \phi = 0.95 \), inductive at 400 V

Assume that lines can be modeled by using the \( \Pi \)-equivalent. Calculate the efficiency of the internal network as well as the short circuit current that is obtained at a solid three-phase short circuit at bus 4.

**Solution**

Chose base values (MVA, kV, \( \Rightarrow \) kA, \( \Omega \)): \( S_b = 500 \text{ kVA} = 0.5 \text{ MVA}, U_{b10} = 10 \text{ kV} \)

\[ \Rightarrow I_{b10} = S_b / \sqrt{3}U_{b10} = 0.0289 \text{ kA}, Z_{b10} = U_{b10}^2 / S_b = 200 \Omega, U_{b04} = 0.4 \text{ kV} \Rightarrow I_{b04} = S_b / \sqrt{3}U_{b04} = 0.7217 \text{ kA}, Z_{b04} = U_{b04}^2 / S_b = 0.32 \Omega, \]

Calculate the per-unit values of the infinite bus:
\( \bar{U}_1 = 1 \)

Calculate the per-unit values of the transformer T1:
\( \bar{Z}_{T1pu} = (Z_{T1\%/100}) \cdot Z_{bT1-10}/Z_{b10} = (Z_{T1\%/100}) \cdot S_b/S_{T1} = (j7/100) \cdot 0.5/0.8 = j0.0438 \)
Calculate the per-unit values of the transformer \( T2 \):
\[
\overline{Z}_{T2pu} = (Z_{T2\%}/100) \cdot \frac{S_b}{S_{T2}} = (j0.1333)
\]

Calculate the per-unit values of the line \( L1 \):
\[
\overline{Z}_{L1pu} = 2 \cdot [0.17 + j0.3] / Z_{b10} = 0.0017 + j0.003
\]
\[
\overline{Y}_{lpu} = 2 \cdot [3.2 \times 10^{-6}] \cdot Z_{b10} = j0.0013
\]

Calculate the per-unit values of the line \( L2 \):
\[
\overline{Z}_{L2pu} = 1 \cdot [0.17 + j0.3] / Z_{b10} = 0.0009 + j0.0015
\]
\[
\overline{Y}_{lpu} = 1 \cdot [3.2 \times 10^{-6}] \cdot Z_{b10} = j0.00064
\]

Calculate the per-unit values of the impedance \( LD1 \):
\[
\overline{Z}_{LD1pu} = (U^2_{LD1}/S_{LD1})/Z_{b10} = (10^2/[0.5/0.8]) \cdot (0.8 + j0.6)/200 = 0.64 + j0.48
\]

Calculate the per-unit values of the impedance \( LD2 \):
\[
\overline{Z}_{LD2pu} = (U^2_{LD2}/S_{LD2})/Z_{b10} = (0.4^2/[0.2/0.95]) \cdot (0.95 + j0.95^2)/0.32 = 2.256 + j0.7416
\]

Calculate the Y-bus matrix of the network. The grounding point is not included in the Y-bus matrix since the system then is overdetermined.

\[
\begin{align*}
Y = \begin{bmatrix}
\overline{Y}_{22} & 0 & 0 & 0 \\
0 & \overline{Y}_{33} & 0 & 0 \\
0 & 0 & \overline{Y}_{44} & 0 \\
0 & 0 & 0 & \overline{Y}_{T2pu}
\end{bmatrix}
\end{align*}
\]

\[
\overline{Y}_{22} = \frac{1}{\overline{Z}_{T1pu}} + \frac{1}{\overline{Z}_{L1pu}} + \frac{\overline{Y}_{L1pu}}{2} + \frac{\overline{Y}_{L2pu}}{2}
\]

\[
\overline{Y}_{33} = \frac{1}{\overline{Z}_{L1pu}} + \frac{\overline{Y}_{L1pu}}{2} + \frac{\overline{Z}_{LD1pu}}{2}
\]

\[
\overline{Y}_{44} = \frac{1}{\overline{Z}_{L2pu}} + \frac{\overline{Y}_{L2pu}}{2} + \frac{\overline{Z}_{T2pu}}{2}
\]

The Y-bus matrix is defined as
\[
I = Yu
\]

this can be rewritten as
\[
\begin{bmatrix}
\overline{U}_1 \\
\overline{U}_2 \\
\overline{U}_3 \\
\overline{U}_4 \\
\overline{U}_5
\end{bmatrix}
= U = Y^{-1}I = ZI = Z
\]

\[
\begin{bmatrix}
\overline{I}_1 \\
\overline{I}_2 \\
\overline{I}_3 \\
\overline{I}_4 \\
\overline{I}_5
\end{bmatrix}
\]

The Z-bus matrix can be calculated by inverting the Y-bus matrix:
\[
Z = \begin{bmatrix}
0.510 + j0.375 & 0.510 + j0.331 & 0.508 + j0.329 & 0.510 + j0.331 & 0.516 + j0.298 \\
0.510 + j0.331 & 0.510 + j0.331 & 0.508 + j0.329 & 0.510 + j0.331 & 0.516 + j0.298 \\
0.508 + j0.329 & 0.508 + j0.329 & 0.509 + j0.330 & 0.508 + j0.329 & 0.515 + j0.296 \\
0.510 + j0.331 & 0.510 + j0.331 & 0.508 + j0.329 & 0.510 + j0.332 & 0.517 + j0.299 \\
0.516 + j0.298 & 0.516 + j0.298 & 0.515 + j0.296 & 0.517 + j0.299 & 0.529 + j0.397
\end{bmatrix}
\]

7. Power transmission to impedance loads
Since all injected currents except $I_1$ are $0$, then $I_1$ can be calculated using the first row in equation (7.45):

$$\bar{U}_1 = \bar{Z}_{11} \bar{I}_1 \Rightarrow \bar{I}_1 = \frac{\bar{U}_1}{\bar{Z}_{11}} = 1.0/\left((0.510 + j0.375) = 1.58\angle -36.33^\circ \right) \quad (7.47)$$

The voltages at the other buses can be easily solved by using equation (7.45):

$$\bar{U}_2 = \bar{Z}_{21} \bar{I}_1 = (0.510 + j0.331) \cdot (1.58\angle -36.33^\circ) = 0.9606\angle -3.324^\circ \quad (7.48)$$

$$\bar{U}_3 = \bar{Z}_{31} \bar{I}_1 = (0.508 + j0.329) \cdot (1.58\angle -36.33^\circ) = 0.9569\angle -3.423^\circ \quad (7.48)$$

$$\bar{U}_4 = \bar{Z}_{41} \bar{I}_1 = (0.510 + j0.331) \cdot (1.58\angle -36.33^\circ) = 0.9601\angle -3.350^\circ \quad (7.48)$$

$$\bar{U}_5 = \bar{Z}_{51} \bar{I}_1 = (0.516 + j0.298) \cdot (1.58\angle -36.33^\circ) = 0.9423\angle -6.351^\circ$$

The total amount of power delivered to the industry is

$$\bar{S}_1 = \bar{U}_1 \cdot \bar{I}_1 \cdot S_b = 0.6367 + j0.4682 \text{ MVA} \quad (7.49)$$

The power loss in line $L1$ can be calculated as

$$\bar{I}_{L1} = \frac{(\bar{U}_2 - \bar{U}_3)/\bar{Z}_{L1\text{pu}}} = 1.1957\angle -40.27^\circ \quad (7.50)$$

$$P_{fL1} = \text{Real}(\bar{Z}_{L1\text{pu}}) \bar{I}_{L1}^2 \cdot S_b = 0.0012 \text{ MW}$$

In the corresponding way, the power loss in line $L2$ can be calculated:

$$\bar{I}_{L2} = \frac{(\bar{U}_2 - \bar{U}_4)/\bar{Z}_{L2\text{pu}}} = 0.3966\angle -24.50^\circ \quad (7.51)$$

$$P_{fL2} = \text{Real}(\bar{Z}_{L2\text{pu}}) \bar{I}_{L2}^2 \cdot S_b = 0.0000669 \text{MW}$$

The efficiency for the network is then

$$\eta = \frac{\text{Real}(\bar{S}_1) - P_{fL1} - P_{fL2}}{\text{Real}(\bar{S}_1)} = 0.9980 \Rightarrow 99.80\% \quad (7.52)$$

A solid short circuit at bus 4 can be calculated by connecting an impedance with $\bar{Z}_4 = 0$ at bus 4. According to section 7.3, the current through the impedance $\bar{Z}_4$ can be determined by taking the row and the column in the $Y$-bus matrix that corresponds to the bus having the generator (in this example bus 1) $\Rightarrow$

$$\bar{Y} = \begin{bmatrix} Y_{22} & -\frac{1}{Z_{L1\text{pu}}} & -\frac{1}{Z_{L2\text{pu}}} & 0 \\ -\frac{1}{Z_{L1\text{pu}}} & Y_{33} & 0 & 0 \\ -\frac{1}{Z_{L2\text{pu}}} & 0 & Y_{44} & -\frac{1}{Z_{T2\text{pu}}} \\ 0 & 0 & -\frac{1}{Z_{T2\text{pu}}} & 0 \end{bmatrix} \quad (7.53)$$

The inverse of this matrix is

$$\bar{Z}_4 = \bar{Y}^{-1} = \begin{bmatrix} 0.0024+j0.0420 & 0.0025+j0.0418 & 0.0025+j0.0419 & 0.0046+j0.0410 \\ 0.0025+j0.0418 & 0.0043+j0.0446 & 0.0025+j0.0418 & 0.0046+j0.0408 \\ 0.0025+j0.0419 & 0.0025+j0.0418 & 0.0033+j0.0434 & 0.0055+j0.0424 \\ 0.0046+j0.0410 & 0.0046+j0.0408 & 0.0055+j0.0424 & 0.0144+j0.1719 \end{bmatrix} \quad (7.54)$$

The short circuit current at bus 4 can then be calculated according to equation (7.39).

$$\bar{I}_{k4} = \frac{\bar{U}_4}{\bar{Z}_4 + \bar{Z}_4(4,4)} \bar{I}_{b10} = \frac{0.9601\angle -3.350^\circ}{0 + (0.0033 + j0.0434)} \frac{0.0289 = \text{element (3,3)} \ in \ \bar{Z}_4}{0.6366\angle -88.97^\circ \text{ kA}} \quad (7.55)$$
7.4 Extended method to be used for power loads

The method described in section 7.3 is valid when all system load are modeled as impedance load, i.e. the power consumed is proportional to the voltage squared. In steady-state conditions, an often used load model is the constant power model. The method described in section 7.3 can be used in an iterative way, described in the following:

1. Calculate the per-unit values of all components that are of interest. Loads that are modeled with constant power (independent of the voltage) are replaced by an impedance under the assumption that nominal voltage apply. The impedance of a load at bus \( k \) can be calculated as \( Z_{LDk} = \frac{U_k^2}{S_{LDk}} \) where \( U_k = 1 \) pu, and \( S_{LDk} \) is the rated power of the load.

2. Calculate the Y-bus matrix and the corresponding Z-bus matrix of the network as well as the load impedances. By using the method described in section 7.3 (equation (7.29) and (7.30)), the voltage at all buses can be calculated.

3. Calculate the load demand at all loads. The power demand \( S_{LDk-b} \) at load \( LDk \) is obtained as \( S_{LDk-b} = U_k^2 / Z_{LDk}^* \) where \( U_k \) is the calculated voltage at bus \( k \).

4. Calculate the difference between calculated and specified load demand for all power loads:

\[
\Delta P_{LDk} = \left| Re(S_{LDk-b}) - Re(S_{LDk}) \right| \tag{7.56}
\]

\[
\Delta Q_{LDk} = \left| Im(S_{LDk-b}) - Im(S_{LDk}) \right| \tag{7.57}
\]

5. If \( \Delta P_{LDk} \) and/or \( \Delta Q_{LDk} \) are too large for a certain bus:

(a) Calculate new load impedances according to \( Z_{LDk} = \frac{U_k^2}{S_{LDk}} \) where \( U_k \) is the most recent estimation of the voltage at bus \( k \).

(b) Go back to item #2 and repeat the calculations

If \( \Delta P_{LDk} \) and \( \Delta Q_{LDk} \) are found to be acceptable for all power loads, the iteration process is finished.

A simple example will be given to clarify this method.

**Example 7.4** Assume a line operating with a voltage of \( U_1 = 225/0^\circ \) kV in the sending end, and with a load of \( P_{D2} = 80 \) MW and \( Q_{D2} = 60 \) MVAr in the receiving end. The line has a length of 100 km and has \( x = 0.4 \ \Omega/km \), \( r = 0.04 \ \Omega/km \) and \( b = 3 \times 10^{-6} \ S/km \). Calculate the receiving end voltage.

**Solution**

In Figure 7.17, the network modeled by impedance loads is given.

Assume \( S_b = 100 \) MVA and \( U_b = 225 \) kV which gives
Z_b = U_b^2 / S_b = 506.25 \Omega

This gives the following per-unit values of the line
U_1 = 225/U_b = 1.0 pu, P_D = P_{D2}/S_b = 0.8 pu, Q_D = Q_{D2}/S_b = 0.6 pu, R = 0.04 \cdot 100/Z_b = 0.0079 pu, X = 0.4 \cdot 100/Z_b = 0.0790 pu,

B = 3 \times 10^{-6} \cdot 100 \cdot Z_b/2 = 0.0759 pu

The iteration process can now be started:
1. U_2 = 1 pu, \bar{Z}_D = U_2^2 / (P_D - jQ_D) = 0.8 + j0.6 pu
2. I_{12} = U_1 / (\bar{Z} + \frac{1}{jB} \| \bar{Z}_D) = 0.7330 - j0.5415 pu \Rightarrow U_2 = |U_1 - I_{12} \bar{Z}| = 0.9529 pu
3. S_D = U_2^2 / \bar{Z}_D^* = 0.7265 + j0.5448 pu
4. \Delta P_D = 0.0735, \Delta Q_D = 0.0552
1. \bar{Z}_D = 0.7265 + j0.5448
2. U_2 = 0.9477
4. \Delta P_D = 0.0087, \Delta Q_D = 0.0066
1. \bar{Z}_D = 0.7185 + j0.5389
2. U_2 = 0.9471
4. \Delta P_D = 0.0011, \Delta Q_D = 0.00079
1. \bar{Z}_D = 0.7176 + 0.5382i
2. U_2 = 0.9470
4. \Delta P_D = 0.0001, \Delta Q_D = 0.0001

This is found to be acceptable, which gives a voltage magnitude in the sending end of U_2 = 0.9470 \cdot U_b = 213.08 kV. This simple example can be solved exactly by using a non-linear expression which will be shown in example 8.4.

**Example 7.5** A small industry is fed by a transformer (5 MVA, 70/10 kV, x = 4 %) which is located at a distance of 5 km. The industry has a power demand of 400 kW at cos\phi=0.8, inductive, independent of the voltage. The 10 kV-line has a series impedance of 0.9+j0.3 \Omega/phase.km. The shunt admittance of the load can be neglected. Calculate the voltage level at the industry. When the industry is not connected, a short circuit current of 0.3 kA (pure inductive) can be obtained at the 70 kV side of the transformer when a three-phase short circuit is applied at nominal voltage. This is the same case as in example 7.2 but assuming constant power load and neglected shunt admittance of the line.
Solution

1: Chose base values (MVA, kV, ⇒ kA, Ω): $S_b = 500$ kVA = 0.5 MVA, $U_{b10} = 10$ kV ⇒

$S_b = 500$ kVA = 0.5 MVA,

$U_{b10} = 10$ kV ⇒

$I_{b10} = \frac{S_b}{\sqrt{3}U_{b10}} = 0.0289$ kA,

$Z_{b10} = \frac{U_{b10}^2}{S_b} = 200$ Ω, $U_{b70} = 70$ kV ⇒

$I_{b70} = \frac{S_b}{\sqrt{3}U_{b70}} = 0.0041$ kA

According to example 7.2, the following is valid:

$U_{Thpu} = 1.0 \angle 0.0^\circ$ pu

$Z_{Thpu} = j0.0137$ pu

$Z_{trapu} = j0.004$ pu

$Z_{lpu} = 0.225 + j0.0075$ pu

The total impedance between $U_{Thpu}$ and the impedance of the industry can be calculated as:

$Z_{totpu} = Z_{Thpu} + Z_{trapu} + Z_{lpu} = 0.0225 + j0.0075$ pu

Calculate the per-unit values of the power demand of the industry as well as the corresponding impedance at nominal voltage:

$\overline{S}_{indpu} = (P_{ind} + j[P_{ind} / \cos \phi] \cdot \sin \phi) / S_b = 0.8000 + j0.6000$ pu

$\overline{Z}_{indpu} = (U_{ind}^2 / \overline{S}_{indpu}) / U_{b10}^2 = 0.8 + j0.6$ pu

2: The Y-bus matrix of the network can be calculated as:

$$Y = \begin{bmatrix} \frac{1}{Z_{totpu}} & -\frac{1}{Z_{totpu}} \\ -\frac{1}{Z_{totpu}} & \frac{1}{Z_{totpu}} \end{bmatrix} = \begin{bmatrix} 19.67 - j22.08 & -19.67 + j22.08 \\ -19.67 + j22.08 & 20.47 - j22.08 \end{bmatrix}$$ (7.58)

The Z-bus matrix is calculated as the inverse of the Y-bus matrix:

$$Z = Y^{-1} = \begin{bmatrix} 0.82 + j0.63 & 0.80 + j0.60 \\ 0.80 + j0.60 & 0.80 + j0.60 \end{bmatrix}$$ (7.59)

The voltage at the industry is now calculated according to equation (7.30):

$$\overline{U}_{indpu} = Z(2,1) \cdot \overline{U}_{Thpu} / Z(1,1) = 0.9679 \angle -0.3714^\circ$$ (7.60)

3: The power delivered to the industry can be calculated as:

$$\overline{S}_{indpu-b} = U_{ind}^2 / \overline{Z}_{indpu} = 0.7495 + j0.5621$$ (7.61)
4 : The difference between calculated and specified power can be calculated as:

\[
\Delta P_{\text{ind}} = |Re(\bar{S}_{\text{indpu}}) - Re(\bar{S}_{\text{indpu}})| = 0.0505 \\
\Delta Q_{\text{ind}} = |Im(\bar{S}_{\text{indpu}}) - Im(\bar{S}_{\text{indpu}})| = 0.0379
\] (7.62)

5 : These deviations are too large and the calculations are therefore repeated and a new industry impedance is calculated by using the new voltage magnitude:

\[
\bar{Z}_{\text{indpu}} = \left(\frac{U_{\text{indpu}}^2}{\bar{S}_{\text{indpu}}^*}\right) = 0.7495 + j0.5621
\] (7.64)

Repeat the calculations from item #2.

2, 3 : \(\bar{S}_{\text{indpu}} = 0.7965 + j0.5974\)

4 : \(\Delta P_{\text{ind}} = 0.0035, \Delta Q_{\text{ind}} = 0.0026\)

Unacceptable \(\Rightarrow\)

5 : \(\bar{Z}_{\text{indpu}} = 0.7462 + j0.5597\)

Continue from item #2.

2, 3 : \(\bar{S}_{\text{indpu}} = 0.7998 + j0.5998\)

4 : \(\Delta P_{\text{ind}} = 0.00024, \Delta Q_{\text{ind}} = 0.00018\)

Unacceptable \(\Rightarrow\)

5 : \(\bar{Z}_{\text{indpu}} = 0.7460 + j0.5595\)

Continue from item #2.

2, 3 : \(\bar{S}_{\text{indpu}} = 0.8000 + j0.6000\)

4 : \(\Delta P_{\text{ind}} = 0.000016, \Delta Q_{\text{ind}} = 0.000012\)

Acceptable \(\Rightarrow\)

\(\bar{U}_{\text{ind}} = \bar{U}_{\text{indpu}} \cdot U_{b10} = 9.6565 \angle -0.3974^\circ\)
7. Power transmission to impedance loads
Chapter 8
Non-linear static analysis

In chapter 6–7, two assumptions were made: firstly, the network has had only one generation bus, secondly, the loads have been modeled by impedances. The assumptions made gave a system that could be modeled by using a linear set of equations which can be solved quite easily.

In the following, these assumptions are not valid, i.e. the loads will be modeled as constant power loads and power can be generated at more than one location. First, the power transmission on an overhead line will be studied, followed by a more general system analysis.

8.1 Power flow on a line

Power lines can be modeled in several different ways depending on what parameters that can be considered as known and on the purpose of the system analysis. This leads to several different expressions on power flows on lines. The voltages are given by polar representation, i.e. voltage magnitude and angle as

\[ U_k = U_k e^{j\theta_k} \]  
\[ U_j = U_j e^{j\theta_j} \]  

where the phase angle difference between two buses has the following notation

\[ \theta_{kj} = \theta_k - \theta_j \]  

In the following, the shunt admittance of the line is neglected.

8.1.1 Line model with rectangular series impedance

The model presented here models a line with a \( \pi \)-equivalent with the series impedance given on rectangular form as given in Figure 8.1. The rectangular form means that the series impedance is given in one resistive part \( R \) and in one reactive part \( X \) as

\[ Z_{kj} = R + jX \]

The power \( S_{kj} \) in the sending end \( k \) can be calculated by using the per-unit system as

\[ S_{kj} = U_k \left( I_{kj0} + I_{kj} \right) = U_k \left( U_k^* Y^* + \frac{U_k^* - U_j^*}{Z_{kj}} \right) \]

\[ = U_k^2 (-jB) + \frac{U_k^2}{R - jX} - \frac{U_k U_j}{R - jX} e^{j(\theta_k - \theta_j)} \]

\[ = U_k^2 (-jB) + \frac{U_k^2}{Z^2} (R + jX) - \frac{U_k U_j}{Z^2} (R + jX) (\cos \theta_{kj} + j \sin \theta_{kj}) \]
By dividing equation (8.5) into a real and an imaginary part, expressions for the active and reactive power can be held, respectively, as

\[
P_{kj} = \frac{U_k^2}{Z^2} R + \frac{U_k U_j}{Z^2} (X \sin \theta_{kj} - R \cos \theta_{kj}) \quad (8.6)\\
Q_{kj} = -B U_k^2 + \frac{U_k U_j}{Z^2} X - \frac{U_k U_j}{Z^2} (R \sin \theta_{kj} + X \cos \theta_{kj}) \quad (8.7)
\]

One important conclusion that can be made from equation (8.6) and (8.7) is that if the voltage magnitude and the voltage angle at both ends of the line are known, the power flow can be uniquely determined. This implies that if all bus voltage magnitudes and angles in a system are known, the power flows in the whole system are known. The voltages can be said to define the system state.

**Example 8.1** Assume a line where the voltage in the sending end is \(U_1 = 225 \angle 0^\circ \) kV and in the receiving end \(U_2 = 213.08 \angle -3.572^\circ \) kV. The line has a length of 100 km and has \(x = 0.4 \Omega/\text{km}, r = 0.04 \Omega/\text{km} \) and \(b = 3 \times 10^{-6} \text{S/km} \). Calculate the amount of power transmitted from bus 1 to bus 2.

**Solution**

Assume \(S_b = 100 \text{ MVA} \) and \(U_b = 225 \text{ kV} \), this gives that \(Z_b = U_b^2 / S_b = 506.25 \Omega \)

The per-unit values for the line are

\[
U_1 = 225 / U_b = 1.0 \text{ pu}, \quad U_2 = 213.08 / U_b = 0.9470 \text{ pu}, \quad \theta_{12} = 0-(3.572) = 3.572^\circ \\
R = 0.04 \cdot 100 / Z_b = 0.0079 \text{ pu}, \quad X = 0.4 \cdot 100 / Z_b = 0.0790 \text{ pu}, \\
B = 3 \times 10^{-6} \cdot 100 \cdot Z_b / 2 = 0.0759 \text{ pu}, \quad Z = \sqrt{R^2 + X^2} = 0.0794 \text{ pu}
\]

The power flow in per-unit can be calculated by using equation (8.6) and 8.7:

\[
P_{12} = \frac{1.0^2}{0.0794^2} \cdot 0.0079 + \frac{1.0 \cdot 0.9470}{0.0794^2} (0.0790 \sin 3.572^\circ - 0.0079 \cos 3.572^\circ) = 0.8081 \text{ pu}\\
Q_{12} = -0.0759 \cdot 1.0^2 + \frac{1.0^2}{0.0794^2} \cdot 0.0790 - \frac{1.0 \cdot 0.9470}{0.0794^2} (0.0079 \sin 3.572^\circ + 0.0790 \cos 3.572^\circ) = 0.5373 \text{ pu}
\]
expressed in nominal values

\[ P_{12} = 0.8081 \cdot S_b = 80.81 \text{ MW} \]
\[ Q_{12} = 0.5373 \cdot S_b = 53.73 \text{ MVAr} \]

For this simple system, the calculations can be performed without using the per-unit system. By using equation (6.6), equation (8.6) can be rewritten as

\[
P_{kj}(\text{MW}) = P_{kj}(\text{pu}) S_b = \frac{U_k^2}{Z_b} \left\{ \frac{U_k^2}{Z^2} R + \frac{U_k U_j}{Z^2} (X \sin \theta_{kj} - R \cos \theta_{kj}) \right\} = \frac{U_k^2(\text{kV})}{Z^2(\Omega)} R(\Omega) + \frac{U_k(\text{kV}) \cdot U_j(\text{kV})}{Z^2(\Omega)} (X(\Omega) \sin \theta_{kj} - R(\Omega) \cos \theta_{kj})
\]
i.e. this equation is the same independent on if the values are given as nominal or per-unit values.

For a high voltage overhead line \((U > 70 \text{ kV})\), the line reactance is normally considerably higher than the resistance of the line, i.e. \(R \ll X\) in equation (8.6). An approximate form of that equation is

\[
P_{kj} \approx \frac{U_k U_j}{X} \sin \theta_{kj} \quad (8.8)
\]
i.e. the sign of \(\theta_{kj}\) determines the direction of the active power flow on the line. Normally, the active power will flow towards the bus with the lowest voltage angle. This holds also for lines having a pronounced resistivity.

Assume that the voltages \(U_k\) and \(U_j\) are in phase and that the reactance of the line is dominating the line resistance. This implies that the active power flow is very small. Equation (8.7) can be rewritten as

\[
Q_{kj} = -B U_k^2 + \frac{U_k(U_k - U_j)}{X} \quad (8.9)
\]

Equation (8.9) indicates that this type of line gives a reactive power flow towards the bus with the lowest voltage magnitude. The equation shows that if the difference in voltage magnitude between the ends of the line is small, the line will generate reactive power. This since the reactive power generated by the shunt admittances in that case dominates the reactive power consumed by the series reactance. The “rule of thumb” that reactive power flows towards the bus with lowest voltage is more vague than the rule that active power flows towards the bus with lowest angle. The fact that overhead lines and especially cables, generates reactive power when the active power flow is low, is important to be aware of.

**Example 8.2** Calculate the active and reactive power flows on the line in example 8.1, using the approximate expressions 8.8 and 8.9, respectively.

**Solution**

\[
P_{12} \approx \frac{1.0 \cdot 0.9470}{0.0790} \sin 3.572^\circ = 0.7468 \text{ pu} \Rightarrow 74.68 \text{ MW}
\]
\[
Q_{12} \approx -0.0759 \cdot 1.0^2 + \frac{1.0(1.0 - 0.9470)}{0.0790} = 0.5948 \text{ pu} \Rightarrow 59.48 \text{ MVAr}
\]
the answers are of right dimension and have correct direction of the power flow but the active power flow is about 8% too low and the reactive power flow is 11% too large.

8.1.2 Losses on a line

The active power losses on a three-phase line are dependent on the line resistance and one the actual line current. By using nominal values, the losses can be calculated as

\[ P_f = 3RI^2 \]  

(8.10)

The squared current dependence in equation (8.10) can be written as

\[ I^2 = Ie^{j\arg(T)}Ie^{-j\arg(T)} = \mathcal{M}^* = \frac{\mathcal{M}}{3U^*} \frac{\mathcal{M}}{3U} = \frac{S^2}{3U^2} = \frac{P^2 + Q^2}{3U^2} \]  

(8.11)

The active power losses for the line given in Figure 8.1 can be calculated as

\[ P_f = R \frac{P_{kj}^2 + (Q_{kj} + BU_k^2)^2}{U_k^2} \]  

(8.12)

where

\[ BU_k^2 = \text{the reactive power generated by the shunt capacitance at bus } k \]

The expression given by (8.12) is valid both for nominal and for per-unit values. This equation shows that a doubling of transmitted active power will increase the active power losses by a factor of four. If the voltage is doubled, the active power losses will decrease with a factor of four.

Assume that the active power injections at both ends of the line are known, i.e. both \( P_{kj} \) and \( P_{jk} \) have been calculated using equation (8.6). The active power losses can then be calculated as

\[ P_f = P_{kj} + P_{jk} \]  

(8.13)

The reactive power losses can be obtained in the corresponding manner

\[ Q_f = 3XI^2 = X \frac{P_{kj}^2 + (Q_{kj} + BU_k^2)^2}{U_k^2} \]  

(8.14)

Equations (8.11) and (8.12) shows that the losses are proportional to \( S^2 \) and that the losses will increase if reactive power is transmitted over the line. A natural solution to that is to generate the reactive power as close to the consumer as possible. Of course, active power is also generated as close to the consumer as possible, but the generation costs are of great importance.

**Example 8.3** Use the same line as in example 8.1 and calculate the active power losses.
8.1. Power flow on a line

Solution

The losses on the line can be calculated by using equation (8.12) and the conditions that apply at the sending end

\[ P_f(MW) = R \frac{P_{12}^2 + (Q_{12} + BU_{12}^2)^2}{U_{12}^2} S_b = \]
\[ = 0.0079 \frac{0.8081^2 + (0.5373 + 0.0759 \cdot 1.0)^2}{1.0^2} 100 = 0.81 \text{ MW} \]

The losses can also be calculated by using the receiving end conditions

\[ P_f(MW) = R \frac{P_{21}^2 + (Q_{21} + BU_{21}^2)^2}{U_{21}^2} S_b = \]
\[ = 0.0079 \frac{(-0.80)^2 + (-0.60 + 0.0759 \cdot 0.9470)^2}{0.9470^2} 100 = 0.81 \text{ MW} \]

or by using equation (8.13)

\[ P_f(MW) = [P_{12} + P_{21}] S_b = [0.8081 + (-0.80)] 100 = 0.81 \text{ MW} \]

8.1.3 Shunt capacitors and shunt reactors

As previously mentioned in subsection 8.1.2, transmission of reactive power will increase the line losses. An often used solution is to generate reactive power as close to the load as possible. This is done by switching in shunt capacitors. In Figure 8.2 it is shown how three capacitors are connected between phase and ground. Figure 8.2 also shows the single-

\[ Q_{sh} = B_{sh} U^2 = 2\pi f c U^2 \]  

\[ (8.15) \]

Figure 8.2: Y-connected shunt capacitors

An injection of reactive power into a certain bus will increase the bus voltage, see example 8.6. The insertion of shunt capacitors in the network is also called phase compensation. This because the phase displacement between voltage and current is reduced when the reactive power transmission on the line is reduced.
As previously mentioned in subsection 8.1.1, lines that are lightly loaded generates reactive power. The amount of reactive power generated is proportional to the length of the line. In such situations, the reactive power generation will be too large and it is necessary to consume the reactive power in order to avoid overvoltages. One possible countermeasure is to connect shunt reactors. They are connected and modeled in the same way as the shunt capacitors with the difference that the reactors consumes reactive power according to equation (8.15).

8.1.4 Series capacitors

By studying equation (8.8), an approximate expression of the maximum amount of power that can be transmitted over a line, at a certain voltage level, can be written as

\[ P_{kj\text{-max}} \approx \max_{\theta_{kj}} \frac{U_k U_j}{X} \sin \theta_{kj} = \frac{U_k U_j}{X} \]

(8.16)

i.e. the larger the reactance of the line is, the less amount of power can be transmitted. One possibility to increase the maximum loadability of a line is to compensate for the series reactance of the line by using series capacitors. In Figure 8.3, the way of connecting series capacitors is given as well as the single-phase equivalent of a series compensated line. The expression for the maximum loadability of a series compensated line is

\[ P_{kj\text{-max}} \approx \frac{U_k U_j}{X - X_c} \]

(8.17)

It is obvious that the series compensation increases the loadability of the line.

The use of series capacitors will also reduce the voltage drop along the line, see example 8.7.

8.2 Power flow calculations in a network - load flow

The technique of determining all bus voltages in a network is usually called load flow. When knowing the voltage magnitude and voltage angle at all buses, the system state is completely
determined and all system properties of interest can be calculated, e.g. line loadings and line losses.

In a network, power can be generated and consumed at many different locations. All locations where a line ends is often called bus (or node). In Figure 8.4, the representation of bus \( k \) is given under the assumption of symmetrical three-phase conditions.

![Figure 8.4. Notation of bus \( k \) in a network](image)

The generator generates the current \( T_{Gk} \), the load at the bus draws the current \( T_{Dk} \) and on the lines to the neighboring buses the currents \( T_{k1}, T_{k2} \ldots T_{kN} \) flows. According to Kirchoff’s first law, the sum of all currents flowing into bus \( k \) must equal zero, i.e.

\[
T_{Gk} - T_{Dk} = \sum_{j=1}^{N} T_{kj} \tag{8.18}
\]

By taking the conjugate of equation (8.18) and multiply the equation with the bus voltage, the following holds

\[
\overline{U}_k T_{Gk}^* - \overline{U}_k T_{Dk}^* = \sum_{j=1}^{N} \overline{U}_k T_{kj}^* \tag{8.19}
\]

This can be rewritten as an expression for complex power in the per-unit system as

\[
\overline{S}_{Gk} - \overline{S}_{Dk} = \sum_{j=1}^{N} \overline{S}_{kj} \tag{8.20}
\]

where

\[\overline{S}_{Gk} = P_{Gk} + jQ_{Gk} = \text{by the generator generated complex power}\]

\[\overline{S}_{Dk} = P_{Dk} + jQ_{Dk} = \text{by the load consumed complex power}\]

\[\overline{S}_{kj} = P_{kj} + jQ_{kj} = \text{transmitted power to bus } j\]

The power balance at the bus according to equation (8.20) must hold both for the active and for the reactive part of the expression. By using \( P_{GDK} \) and \( Q_{GDK} \) as notation for the net generation of active and reactive power at bus \( k \), respectively, the following expression holds

\[
P_{GDK} = P_{Gk} - P_{Dk} = \sum_{j=1}^{N} P_{kj} \tag{8.21}
\]

\[
Q_{GDK} = Q_{Gk} - Q_{Dk} = \sum_{j=1}^{N} Q_{kj} \tag{8.22}
\]
i.e. for every bus in the system, the power balance must hold for both active and reactive power.

Assume that a power system is operating under symmetrical conditions according to Figure 8.5. By formulating equation (8.21) and (8.22) at each bus in the system, a system of

\[
\begin{align*}
    P_{G1} & = P_{12} + P_{13} \\
    Q_{G1} & = Q_{12} + Q_{13} \\
    P_{G2} - P_{D2} & = P_{21} + P_{23} \\
    Q_{G2} - Q_{D2} & = Q_{21} + Q_{23} \\
    -P_{D3} & = P_{31} + P_{32} \\
    -Q_{D3} & = Q_{31} + Q_{32}
\end{align*}
\]

At each bus in Figure 8.5, four variables are of interest: net generation of active power \(P_{GDk}\), net generation of reactive power \(Q_{GDk}\), voltage magnitude \(U_k\) and voltage phase angle \(\theta_k\). This gives that the total number of variables for the system are \(3 \cdot 4 = 12\). The voltage phase angles must be given as an angle in relation to a reference angle. This since the phase angles are only relative to one another and not absolute. This reduces the number of system variables to \(12 - 1 = 11\). There are only six equations in the system of equations given in 8.23, this gives that five quantities must be known to be able to calculate the remaining six variables. Depending on what quantities that are known at a certain bus, the buses are mainly modeled in three different ways.

**PQ-bus, Load bus:** For this bus, the net generated power \(P_{GDk}\) and \(Q_{GDk}\) are assumed to be known. The name PQ-bus is based on that assumption. On the other hand, the voltage magnitude \(U_k\) and the voltage phase angle \(\theta_k\) are unknown. A PQ-bus is most often a bus

![Figure 8.5. A three bus power system](image)
with a pure load demand, as bus 3 in Figure 8.5. It represents a system bus where the power consumption can be considered to be independent of the voltage magnitude. This model is suitable for a load bus located on the low voltage side of a regulating transformer. The regulating transformer keeps the load voltage constant independent of the voltage fluctuations on the high voltage side of the transformer. Note that a PQ-bus can be a bus without generation as well as load, i.e. $P_{GD_k} = Q_{GD_k} = 0$. This holds e.g. at a bus where a line is connected to a transformer.

**PU-bus, Generator bus:** In a PU-bus, the net active power generation $P_{GD_k}$ as well as the voltage magnitude $U_k$ are assumed to be known. This gives that the net reactive power generation $Q_{GD_k}$ and the voltage angle $\theta_k$ are unknown. In a PU-bus some sort of voltage regulating device must be connected since the voltage magnitude is independent of the net reactive power generation. For example, in a synchronous machine, the terminal voltage can be regulated by changing the magnetizing current. In a system, voltage can be regulated by using controllable components as controllable shunt capacitors and controllable shunt reactors. A standard component is called SVC, Static VAR Compensator. This component change the reactive power flow in order to regulate the bus voltage. Assume that bus 2 in Figure 8.5 is modeled as a PU-bus. This gives that the active power generation of the generator as well as the active power consumption of the load are known. Also the reactive power consumption of the load is known. The bus voltage is constant due to the magnetization system of the generator. The generator may generate or consume reactive power in such a way that the relation in equation (8.22) holds.

**Uθ-nod, Slack bus:** At the slack bus (only one bus in each system), the voltage angle is chosen as a reference angle $\theta_k$, (often $0^\circ$) and the voltage magnitude is assumed to be known. Unknown quantities are the net generation of both active and reactive power. At this bus, (as for the PU-bus) a voltage regulating component must be present. Since the active power is allowed to vary, a generator or an active power in-feed into the system is assumed to exist at this bus. Since this bus also is the only bus where the active power is allowed to vary, the slack bus will take care of the system losses since they are unknown. If the loads have been modeled in the load flow as constant power loads and a line is tripped, the only bus which will change the active power generation is the slack bus. If bus 1 is chosen as slack bus in Figure 8.5, both $P_{G1}$ and $Q_{G1}$ are unknown but the voltage $U_1$ is given as well as the reference angle $\theta_1 = 0$.

Assume that a system contains $N$ buses and that $M$ of these are PU-buses. A summary of the different bus types is given in Table 8.1. As previously mentioned in subsection 8.1.1,

<table>
<thead>
<tr>
<th>Bus model</th>
<th>Number</th>
<th>Known quantities</th>
<th>Unknown quantities</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U\theta$-bus, Slack bus</td>
<td>1</td>
<td>$U, \theta$</td>
<td>$P_{GD}, Q_{GD}$</td>
</tr>
<tr>
<td>PU-bus, Generator bus</td>
<td>$M$</td>
<td>$P_{GD}, U$</td>
<td>$Q_{GD}, \theta$</td>
</tr>
<tr>
<td>PQ-bus, Load bus</td>
<td>$N-M-1$</td>
<td>$P_{GD}, Q_{GD}$</td>
<td>$U, \theta$</td>
</tr>
</tbody>
</table>

**Table 8.1.** Bus types for load flow calculations

equation (8.6)–(8.7), the active and reactive power transfer on a line can be expressed as a function of the voltage magnitude and voltage phase angle at both ends of the line. Assume that the power system in Figure 8.5 is modeled in such a way that bus 1 is a slack bus, bus
2 is a PU-bus and bus 3 is a PQ-bus. By using this bus type modeling, the equation system given in (8.23) can be written as

\[
\begin{align*}
P_{GD1} & \text{(unknown)} = P_{12}(U_1, \theta_1, U_2, \theta_2) + P_{13}(U_1, \theta_1, U_3, \theta_3) \\
Q_{G1} & \text{(unknown)} = Q_{12}(U_1, \theta_1, U_2, \theta_2) + Q_{13}(U_1, \theta_1, U_3, \theta_3) \\
P_{GD2} & = P_{21}(U_1, \theta_1, U_2, \theta_2) + P_{23}(U_2, \theta_2, U_3, \theta_3) \\
Q_{GD2} & \text{(unknown)} = Q_{21}(U_1, \theta_1, U_2, \theta_2) + Q_{23}(U_2, \theta_2, U_3, \theta_3) \\
P_{GD3} & = P_{31}(U_1, \theta_1, U_3, \theta_3) + P_{32}(U_2, \theta_2, U_3, \theta_3) \\
Q_{GD3} & = Q_{31}(U_1, \theta_1, U_3, \theta_3) + Q_{32}(U_2, \theta_2, U_3, \theta_3)
\end{align*}
\] (8.24)

where also \(\theta_2, U_3\) and \(\theta_3\) are unknown quantities whereas the others are known. As given by equation (8.24), unknown power quantities appear only on the left hand side for buses modeled as slack and PU-bus. These quantities can easily be calculated when voltage magnitudes and angles are known. These equations are not contributing to the system of equations since they only gives one extra equation, and one extra variable which easily can be calculated. The system of equations in (8.24) can therefore be simplified to system of equations containing unknown \(U\) and \(\theta\) as

\[
\begin{align*}
P_{GD2} & = P_{21}(U_1, \theta_1, U_2, \theta_2) + P_{23}(U_2, \theta_2, U_3, \theta_3) \\
P_{GD3} & = P_{31}(U_1, \theta_1, U_3, \theta_3) + P_{32}(U_2, \theta_2, U_3, \theta_3) \\
Q_{GD3} & = Q_{31}(U_1, \theta_1, U_3, \theta_3) + Q_{32}(U_2, \theta_2, U_3, \theta_3)
\end{align*}
\] (8.25)

The system of equations given by (8.25) is non-linear since the expressions for power flow on a line (equation (8.6)–(8.7)) includes squared voltages as well as trigonometric expressions. This system of equations can e.g. be solved by using the Newton-Raphson’s method.

The system of equations given by (8.25) can be generalized to a system containing \(N\) buses, of which \(M\) have a voltage regulating device in operation. A summary of this system is given in Table 8.2. As indicated in Table 8.2, the system of equations contains as many unknown quantities as the number of equations, and by that, the system is solvable.

<table>
<thead>
<tr>
<th>Bus model</th>
<th>Number</th>
<th>Balance equations</th>
<th>Unknown quantities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slack bus</td>
<td>1</td>
<td>(P_{GDk} = \sum P_{kj})</td>
<td>(U_k) \qquad 0 st</td>
</tr>
<tr>
<td>PU-bus</td>
<td>(M)</td>
<td>(Q_{GDk} = \sum Q_{kj})</td>
<td>(U_k) \qquad 0 st</td>
</tr>
<tr>
<td>PQ-bus</td>
<td>(N-M-1)</td>
<td>(N-M-1 st) \quad (N-M-1 st)</td>
<td>(N-M-1 st) \quad (N-M-1 st)</td>
</tr>
<tr>
<td>Total</td>
<td>(N)</td>
<td>2(N-M-2) \quad 2(N-M-2)</td>
<td></td>
</tr>
</tbody>
</table>

**Table 8.2.** Summary of equations and unknown quantities at load flow calculations

### 8.3 Load flow for a line

As given in section 8.2, constant power loads gives a non-linear system of equations. Generally, this will lead to more complicated calculations. When having only one line, the expressions are still suitable for calculations by hand. Below, expressions for two possible combinations of bus modeling when having a single line are given. Since a slack bus always must exist, the possible combinations are: slack bus + PU-bus and slack bus + PQ-bus.
8.3. Load flow for a line

8.3.1 Slack bus + PU-bus

This combination is of interest when the voltage magnitude is known at both ends of the line and the active power is known at one of the ends. This implies that the only unknown quantity is the voltage angle at the PU-bus, i.e. the bus having a known active power flow. Having the PU-bus as bus $k$, equation (8.6) can be rewritten in order to solve the phase angle of the voltage. Define

$$Z = Z e^{j(\frac{\pi}{2} - \delta)} \Rightarrow R = Z \sin \delta \quad X = Z \cos \delta \quad \delta = \arctan \frac{R}{X} \quad (8.26)$$

The phase angle of the voltage can be obtained by equation (8.6) as

$$\frac{Z}{U_k U_j} \left( P_{kj} - \frac{U_k^2 R}{Z^2} \right) = \cos \delta \sin \theta_{kj} - \sin \delta \cos \theta_{kj} = \sin(\theta_{kj} - \delta)$$

$$\Rightarrow \quad \theta_k = \theta_j + \delta + \arcsin \left( \frac{Z}{U_k U_j} \left( P_{kj} - \frac{U_k^2}{Z^2} R \right) \right) \quad (8.27)$$

**Example 8.4** Use the same line as in example 8.1 and assume that the voltage magnitude is known at both ends and that the first end phase angle is chosen as reference, 225° kV and $U_2 = 213.08$ kV. Assume also that the active load demand at bus 2 is $P_{D2} = 80$ MW. Calculate the voltage phase angle at bus 2 (the same as example 7.4).

**Solution**

The net active power generation at bus 2 is

$$P_{21} = -P_{D2} / S_b = -0.8 \text{ pu.}$$

The phase angle can now be calculated by using equation (8.27) where bus $k$ corresponds to bus 2:

$$\delta = \arctan \frac{0.0079}{0.0790} = 5.71^\circ$$

$$\theta_2 = \theta_1 + \delta + \arcsin \left( \frac{0.0794}{1.0 \cdot 0.9470} \left( -0.8 - \frac{0.9470^2}{0.0794^2} \cdot 0.0079 \right) \right) = -3.5724^\circ$$

8.3.2 Slack bus + PQ-bus

This combination is of interest when the voltage magnitude is known in one of the ends of the line and the net active and reactive power generation is known in the other end of the line. Assume that the powers are known at bus $k$ and the voltage magnitude is known at bus $j$. This implies that bus $j$ is modeled as slack bus and the reference angle at that bus is chosen as $\theta_j = 0^\circ$. By rewriting equations (8.6) and (8.7), the following equations can be obtained

$$\left\{ \begin{array}{l}
P_{kj} = \frac{U_k^2}{Z^2} R + \frac{U_k U_j}{Z^2} \left( X \sin \theta_k - R \cos \theta_k \right) \\
Q_{kj} = -BU_k^2 + \frac{U_k^2}{Z^2} X - \frac{U_k U_j}{Z^2} \left( R \sin \theta_k + X \cos \theta_k \right) \\
\end{array} \right. \quad (8.28)$$
Unknown quantities in equation (8.28) are the voltage magnitude $U_k$ and corresponding phase angle $\theta_k$. By trigonometric manipulations, an expression for $U_k^2$ can be obtained as

$$U_k^2 = \frac{a_4}{2a_3} + (-) \sqrt{\left( \frac{a_4}{2a_3} \right)^2 - \frac{1}{a_3} (a_1^2 + a_2^2)}$$  \hspace{1cm} (8.29)$$

where

$$a_1 = -RP_{kj} - XQ_{kj}$$
$$a_2 = -XP_{kj} + RQ_{kj}$$
$$a_3 = (1 - XB)^2 + R^2 B^2$$
$$a_4 = 2 \cdot a_1 (1 - XB) - U_j^2 + 2a_2 RB$$

The voltage $U_k$ can now be calculated as

$$U_k = + (-) \sqrt{U_k^2}$$  \hspace{1cm} (8.30)$$

**Example 8.5** Use the same line as in example 8.1 where the voltage at bus 1 is $225 \angle 0^\circ$ kV and bus 2 is loaded by $P_{D2} = 80$ MW and $Q_{D2} = 60$ MVAr. Calculate the voltage magnitude and phase angle at bus 2.

**Solution**

Use equation (8.29)–8.30 which gives that

$$a_1 = -0.0079(-0.8) - 0.0790(-0.6) = 0.0537$$
$$a_2 = -0.0790(-0.8) + 0.0079(-0.6) = 0.0585$$
$$a_3 = (1 - 0.0790 \cdot 0.0759)^2 + 0.0079^2 \cdot 0.0759^2 = 0.9880$$
$$a_4 = 2 \cdot 0.0537(1 - 0.0790 \cdot 0.0759) - 1.0^2 +$$
$$+ 2 \cdot 0.0585 \cdot 0.0079 \cdot 0.0759 = -0.8931$$
$$\Rightarrow$$
$$U_2^2 = 0.4520 + (-) 0.4449 = 0.8968$$
$$\Rightarrow$$
$$U_2 = + (-) \sqrt{0.8968} = 0.9470$$
$$\Rightarrow$$
$$U_2 \text{ kV} = 0.9470 \cdot S_b = 213.08 \text{ kV}$$

The voltage phase angle can now be calculated in exactly the same way as performed above in example 8.4, which result in the the same answer.

**Example 8.6** Use the same conditions as in example 8.5 that $P_{D2} = 80$ MW and $Q_{D2} = 60$ MVAr. Use these load levels as a base case and calculate the voltage $U_2$ when the active and reactive load demand are varying between 0–100 MW and 0–100 MVAr, respectively.
8.3. Load flow for a line

Solution

By using equation (8.29)–8.30, the voltage can be calculated. The result is shown in Figure 8.6. The base case, i.e. $P_{D2} = 80$ MW and $Q_{D2} = 60$ MVAr, is marked by circles on both curves. As given by the figure, the voltage drops at bus 2 as the load demand increases. The voltage at bus 2 is much more sensitive to a change in reactive load demand compared with a change in active demand. If a shunt capacitor generating 10 MVAr is connected at bus 2 when having a reactive load demand of 60 MVAr, the net demand of reactive power will decrease to 50 MVAr and the bus voltage will increase by two kV, from 213 kV to 215 kV. As discussed earlier in subsection 8.1.2, a reduced reactive power load demand will also reduce the losses on the line.

Example 8.7 Use the base case in example 8.5, i.e. $P_{D2} = 80$ MW and $Q_{D2} = 60$ MVAr. Calculate the voltage $U_2$ when the series compensation of the line is varied in the interval 0–100 %.

Solution

A series compensation of 0–100 % means that 0–100 % of the line reactance is compensated by series capacitors. 0 % means no series compensation at all and 100 % means that $X_c = X$. The voltage can be calculated by using equation (8.29)–(8.30). The result is given in Figure 8.7. As given in Figure 8.7, the voltage at bus 2 increases as the degree of series compensation increases. If the degree of compensation is 40 %, the voltage at bus 2 is increased by 4.5 kV (= 2 %) from 213.1 kV to 217.6 kV.

When having short lines or when only interested in approximate calculations, the shunt capacitance of a line can be neglected. In these conditions, the $B$-term in equation (8.29) disappears and the equation will look like

$$
U_k = \sqrt{U_j^2 - 2a_1} \pm \sqrt{\left(\frac{U_j^2 - 2a_1}{2}\right)^2 - (a_1^2 + a_2^2)}
$$

(8.31)
8. Non-linear static analysis

Figure 8.7. The voltage $U_2$ as a function of degree of compensation

where

\[
\begin{align*}
    a_1 &= -RP_{kj} - XQ_{kj} \\
    a_2 &= -XP_{kj} + RQ_{kj}
\end{align*}
\]

Example 8.8 Use the same line as in example 8.5. Calculate the magnitude of the voltage by using the approximate expression given by equation (8.31).

Solution

Equation (8.31) gives that

\[
\begin{align*}
    a_1 &= -0.0790(-0.8) + 0.0079(-0.6) = 0.0537 \\
    a_2 &= 0.0079(-0.6) - 0.0790(-0.8) = 0.0585 \\
    \Rightarrow & \\
    U_2 &= 0.9410 \\
    \Rightarrow & \\
    U_2 \text{ kV} &= 0.9410 \cdot S_b = 211.72 \text{ kV}
\end{align*}
\]

i.e. the voltage becomes 0.6 % too low compared with the more accurate result.

Another approximation often used, is to neglect $a_2$ in equation (8.31). That equation can then be rewritten as

\[
U_k \approx \frac{U_j}{2} + \sqrt{\frac{U_j^2}{4} + RP_{kj} + XQ_{kj}}
\] (8.32)

Example 8.9 Use the same line as in example 8.5. Calculate the voltage by using the approximate expression given by equation (8.32).
Solution
Equation (8.32) gives that
\[ U_2 = 0.9430 \]
\[ \Rightarrow U_2(\text{kV}) = 0.9439 \cdot S_b = 212.18 \text{kV} \]
i.e. the calculated voltage is 0.4 % too low. As indicated in this example, equation (8.32) gives a good approximation of the voltage drop on the line. In the equation, it is also clearly given that the active and reactive load demand have influence on the voltage drop. The reason why the voltage drop is more sensitive to a change in reactive power compared with a change in active power, is that the line reactance dominates the line resistance.

8.4 Newton-Raphson method

8.4.1 Theory

The Newton-Raphson method may be applied to solve for \( x_1, x_2, \cdots, x_n \) of the following non-linear equations,
\[
g_1(x_1, x_2, \cdots, x_n) = f_1(x_1, x_2, \cdots, x_n) - b_1 = 0 \\
g_2(x_1, x_2, \cdots, x_n) = f_2(x_1, x_2, \cdots, x_n) - b_2 = 0 \\
\vdots \\
g_n(x_1, x_2, \cdots, x_n) = f_n(x_1, x_2, \cdots, x_n) - b_n = 0
\]
(8.33)
or in the vector form
\[
g(x) = f(x) - b = 0
\]
(8.34)
where
\[
x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad g(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_n(x) \end{bmatrix}, \quad f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}
\]
x is an \( n \times 1 \) vector which contains variables, \( b \) is an \( n \times 1 \) vector which contains constants, and \( f(x) \) is an \( n \times 1 \) vector-valued function.

Taylor’s series expansion of (8.34) is the basis for the Newton-Raphson method of solving (8.34) in an iterative manner. From an initial estimate (or guess) \( x^{(0)} \), a sequence of gradually better estimates \( x^{(1)}, x^{(2)}, x^{(3)}, \cdots \) will be made that hopefully will converge to the solution \( x^* \).

Let \( x^* \) be the solution of (8.34), i.e. \( g(x^*) = 0 \), and \( x^{(i)} \) be an estimate of \( x^* \). Let also \( \Delta x^{(i)} = x^* - x^{(i)} \). Equation (8.34) can now be written as
\[
g(x^*) = g(x^{(i)} + \Delta x^{(i)}) = 0
\]
(8.35)
Taylor’s series expansion of (8.35) gives
\[
g(x^{(i)} + \Delta x^{(i)}) = g(x^{(i)}) + JAC(x^{(i)}) \Delta x^{(i)} = 0
\]  
where
\[
JAC(x^{(i)}) = \left[ \frac{\partial g(x)}{\partial x} \right]_{x=x^{(i)}} = \left[ \frac{\partial g_1(x)}{\partial x_1} \cdots \frac{\partial g_n(x)}{\partial x_n} \right]_{x=x^{(i)}}
\]
where, \( JAC \) is called the jacobian of \( g \).

From (8.36), \( \Delta x^{(i)} \) can be calculated as follows
\[
JAC(x^{(i)}) \Delta x^{(i)} = 0 - g(x^{(i)}) = \Delta g(x^{(i)}) \Rightarrow \Delta x^{(i)} = \left[ JAC(x^{(i)}) \right]^{-1} \Delta g(x^{(i)})
\]

Since \( g(x^{(i)}) = f(x^{(i)}) - b \), \( \Delta g(x^{(i)}) \) is given by
\[
\Delta g(x^{(i)}) = b - f(x^{(i)}) = -g(x^{(i)})
\]

Furthermore, since \( b \) is constant, \( JAC(x^{(i)}) \) is given by
\[
JAC(x^{(i)}) = \left[ \frac{\partial g(x)}{\partial x} \right]_{x=x^{(i)}} = \left[ \frac{\partial f_1(x)}{\partial x_1} \cdots \frac{\partial f_n(x)}{\partial x_n} \right]_{x=x^{(i)}}
\]

Therefore, \( \Delta x^{(i)} \) can be calculated as follows
\[
\Delta x^{(i)} = \begin{bmatrix} \Delta x_1^{(i)} \\ \vdots \\ \Delta x_n^{(i)} \end{bmatrix} = \left[ \frac{\partial f_1(x)}{\partial x_1} \cdots \frac{\partial f_n(x)}{\partial x_n} \right]^{-1} \begin{bmatrix} b_1 - f_1(x_1^{(i)}, \ldots, x_n^{(i)}) \\ \vdots \\ b_n - f_n(x_1^{(i)}, \ldots, x_n^{(i)}) \end{bmatrix}
\]

Finally, the following is obtained
\[
i = i + 1 \\
x^{(i)} = x^{(i-1)} + \Delta x^{(i-1)}
\]

The intention is that \( x^{(i)} \) will estimate the solution \( x^* \) better than what \( x^{(0)} \) does. In the same manner, \( x^{(2)}, x^{(3)}, \cdots \) can be determined until a specified condition is satisfied. Thus, we obtain an iterative method according to the flowchart in Figure 8.8.

**Example 8.10** Using the Newton-Raphson method, solve for \( x \) of the equation
\[
g(x) = k_1 x + k_2 \cos(x - k_3) - k_4 = 0
\]

Let \( k_1 = -0.2, k_2 = 1.2, k_3 = -0.07, k_4 = 0.4 \) and \( \epsilon = 10^{-4} \). (This equation is used in the assignment D1.)
8.4. Newton-Raphson method

Solution
This equation is of the form given by (8.34), with \( f(x) = k_1 x + k_2 \cos(x - k_3) \) and \( b = k_4 \).

Step 1
Set \( i = 0 \) and \( x^{(i)} = x^{(0)} = 0.0524 \) (radians), i.e. 3 (degrees).

Step 2
\( \Delta g(x^{(i)}) = b - f(x^{(i)}) = 0.4 - [(-0.2 \times 0.0524) + 1.2 \cos(0.0524 + 0.07)] = -0.7806 \)
Go to Step 4 since \( |\Delta g(x^{(i)})| > \epsilon \)

Step 4
\( JAC(x^{(i)}) = \left[ \frac{\partial f}{\partial x} \right]_{x=x^{(i)}} = -0.2 - 1.2 \sin(0.0524 + 0.07) = -0.3465 \)

Step 5
\( \Delta x^{(i)} = \left[ JAC(x^{(i)}) \right]^{-1} \Delta g(x^{(i)}) = \frac{-0.7806}{-0.3465} = 2.2529 \)

Step 6
\( i = i + 1 = 0 + 1 = 1 \)
\( x^{(i)} = x^{(i-1)} + \Delta x^{(i-1)} = 0.0524 + 2.2529 = 2.3053. \) Go to Step 2

After 5 iterations, i.e. \( i = 5 \), it was found that \( |\Delta g(x^{(i)})| < \epsilon \) for \( x^{(5)} = 0.9809 \) (rad.). Therefore, the solution becomes \( x = 0.9809 \) (rad.) or \( x = 56.2000 \) (deg.).

MATLAB-codes for this example can be found in appendix B.
Analysis of Example 8.10

Figure 8.9 shows variations of $g(x)$ versus $x$. The figure shows that the system (or equation) has only three solutions, i.e. the points at which $g(x) = 0$. Due to practical issues, $x^*$ indicated with (O) in the figure is the interesting solution.

![Figure 8.9. Variations of $g(x)$ vs. $x$.](image)

Figure 8.10 shows how the equation is solved by the Newton-Raphson method.

We first guess the initial estimate $x^{(0)}$. In this case $x^{(0)} = 0.0524$ (rad.), i.e 3 (deg.). The tangent to $g(x)$ through the point $(x^{(0)}, g(x^{(0)}))$, i.e. $g'(x^{(0)}) = \left[ \frac{dg(x)}{dx} \right]_{x=x^{(0)}} = JAC(x^{(0)})$, intersects the x-axis at point $x^{(1)}$. The equation for this tangent is given by

$$\mathcal{Y} - g(x^{(0)}) = g'(x^{(0)}) \times (x - x^{(0)})$$

The intersection point $x^{(1)}$ is obtained by setting $\mathcal{Y} = 0$, i.e.

$$x^{(1)} = x^{(0)} - \frac{g(x^{(0)})}{g'(x^{(0)})} = x^{(0)} - (g'(x^{(0)}))^{-1} g(x^{(0)})$$

$$\Delta x^{(1)} = x^{(1)} - x^{(0)} = - (g'(x^{(0)}))^{-1} g(x^{(0)}) = \left[ JAC(x^{(0)}) \right]^{-1} \Delta g(x^{(0)})$$

In a similar manner, $x^{(2)}$ can be obtained which is hopefully a better estimate than $x^{(1)}$. As shown in the figure, from $x^{(2)}$ we obtain $x^{(3)}$ which is a better estimate of $x^*$ than what $x^{(2)}$ does. This iterative method will be continued until $|\Delta g(x)| < \epsilon$.

**Example 8.11** Solve for $x$ in Example 8.10, but let $x^{(0)} = 0.0174$ (rad.), i.e. 1 (deg.).
8.4. Newton-Raphson method

Figure 8.10. Variations of $g(x)$ vs. $x$.

Solution

D.I.Y, (i.e., Do It Yourself)

8.4.2 Application to power systems

Consider a power system with $N$ buses. The aim is to determine the voltage at all buses in the system by applying the Newton-Raphson method. All variables are given in p.u.

Consider again Figure 8.1. Let

$$b_{kj0} = B$$
$$g_{kj} + j b_{kj} = \frac{1}{Z_{kj}} = \frac{1}{R + j X} = \frac{R}{Z^2} + j \frac{-X}{Z^2} \Rightarrow$$

$$g_{kj} = \frac{R}{Z^2}$$
$$b_{kj} = \frac{-X}{Z^2}$$

(8.43)

Based on (8.43), we rewrite (8.6) and (8.7) as follows

$$P_{kj} = g_{kj} U_k^2 - U_k U_j [g_{kj} \cos(\theta_{kj}) + b_{kj} \sin(\theta_{kj})]$$
$$Q_{kj} = U_k^2 (b_{kj0} - b_{kj}) - U_k U_j [g_{kj} \sin(\theta_{kj}) - b_{kj} \cos(\theta_{kj})]$$

(8.44)
(8.45)

The current through the line, and the loss in the line can be calculated by

$$I_{kj} = \frac{P_{kj} - j Q_{kj}}{U_k^*}$$
$$P_{lkj} = P_{kj} + P_{jk}$$
$$Q_{lkj} = Q_{kj} + Q_{jk}$$

(8.46)
(8.47)
(8.48)
Consider again Figure 8.4. Let \( Y = G + jB \) denote the admittance matrix of the system (or Y-matrix), where \( Y \) is an \( N \times N \) matrix, i.e. the system has \( N \) buses. The relation between the injected currents into the buses and the voltages at the buses is given by \( I = YU \), see section 5.1. Therefore, the injected current into bus \( k \) is given by \( I_k = \sum_{j=1}^{N} Y_{kj} U_j \).

The injected complex power into bus \( k \) can now be calculated by

\[
\bar{S}_k = U_k \bar{T}_k = U_k \sum_{j=1}^{N} Y_{kj}^* U_j^* = U_k \sum_{j=1}^{N} (G_{kj} - jB_{kj}) U_j (\cos(\theta_{kj}) + j \sin(\theta_{kj}))
\]

\[
= \left( U_k \sum_{j=1}^{N} U_j \left[ G_{kj} \cos(\theta_{kj}) + B_{kj} \sin(\theta_{kj}) \right] \right) + j \left( U_k \sum_{j=1}^{N} U_j \left[ G_{kj} \sin(\theta_{kj}) - B_{kj} \cos(\theta_{kj}) \right] \right)
\]

Let \( P_k \) denote the real part of \( \bar{S}_k \), i.e. the injected active power, and \( Q_k \) denote the imaginary part of \( \bar{S}_k \), i.e. the injected reactive power, as follows:

\[
P_k = U_k \sum_{j=1}^{N} U_j \left[ G_{kj} \cos(\theta_{kj}) + B_{kj} \sin(\theta_{kj}) \right]
\]

\[
Q_k = U_k \sum_{j=1}^{N} U_j \left[ G_{kj} \sin(\theta_{kj}) - B_{kj} \cos(\theta_{kj}) \right]
\]

Note that \( G_{kj} = -g_{kj} \) and \( B_{kj} = -b_{kj} \) for \( k \neq j \). Furthermore,

\[
P_k = \sum_{j=1}^{N} P_{kj}
\]

\[
Q_k = \sum_{j=1}^{N} Q_{kj}
\]

Equations (8.21) and (8.22) can now be rewritten as

\[
P_k - P_{GDk} = 0
\]

\[
Q_k - Q_{GDk} = 0
\]

which are of the form given in equation (8.34), where

\[
x = \begin{bmatrix}
\theta_1 \\
\vdots \\
\theta_N \\
U_1 \\
\vdots \\
U_N 
\end{bmatrix},
\]

\[
f(\theta, U) = \begin{bmatrix}
f_P(\theta, U) \\
f_Q(\theta, U)
\end{bmatrix},
\]

\[
b = \begin{bmatrix}
b_P \\
b_Q
\end{bmatrix}
\]

\[
\begin{bmatrix}
P_{GD1} \\
\vdots \\
P_GDN \end{bmatrix}
\]

\[
\begin{bmatrix}
Q_{GD1} \\
\vdots \\
Q_GDN
\end{bmatrix}
\]
the aim is to determine \( x = [\theta \ U]^T \) by applying the Newton-Raphson method.

Assume that there are 1 slack bus and \( M \) PU-buses in the system. Therefore, \( \theta \) becomes an \((N - 1) \times 1\) vector and \( U \) becomes an \((N - 1 - M) \times 1\) vector, why?

Based on (8.40), we define the following:
\[
\Delta P_k = P_{DK} - P_k \quad k \neq \text{slack bus}
\]
\[
\Delta Q_k = Q_{DK} - Q_k \quad k \neq \text{slack bus and PU-bus}
\]

Based on (8.41), the jacobian matrix is given by
\[
JAC = \begin{bmatrix}
\frac{\partial f_P(\theta,U)}{\partial \theta} & \frac{\partial f_P(\theta,U)}{\partial U} \\
\frac{\partial f_Q(\theta,U)}{\partial \theta} & \frac{\partial f_Q(\theta,U)}{\partial U}
\end{bmatrix} = \begin{bmatrix} H & N' \\ J & L' \end{bmatrix}
\]

where,
- \( H \) is an \((N - 1) \times (N - 1)\) matrix
- \( N' \) is an \((N - 1) \times (N - M - 1)\) matrix
- \( J \) is an \((N - M - 1) \times (N - 1)\) matrix
- \( L' \) is an \((N - M - 1) \times (N - M - 1)\) matrix

The entries of these matrices are given by:
\[
H_{kj} = \frac{\partial P_k}{\partial \theta_j} \quad k \neq \text{slack bus} \quad j \neq \text{slack bus}
\]
\[
N'_{kj} = \frac{\partial P_k}{\partial U_j} \quad k \neq \text{slack bus} \quad j \neq \text{slack bus and PU-bus}
\]
\[
J_{kj} = \frac{\partial Q_k}{\partial \theta_j} \quad k \neq \text{slack bus and PU-bus} \quad j \neq \text{slack bus}
\]
\[
L'_{kj} = \frac{\partial Q_k}{\partial U_j} \quad k \neq \text{slack bus and PU-bus} \quad j \neq \text{slack bus and PU-bus}
\]

Based on (8.38), (8.52) and (8.53), the following is obtained
\[
\begin{bmatrix} H & N' \\ J & L' \end{bmatrix} \begin{bmatrix} \Delta \theta \\ \Delta U \end{bmatrix} = \begin{bmatrix} \Delta P \\ \Delta Q \end{bmatrix}
\]

(8.54)

To simplify the entries of the matrices \( N' \) and \( L' \), these matrices are multiplied with \( U \). Then, (8.54) can be rewritten as
\[
\begin{bmatrix} H & N \\ J & L \end{bmatrix} \begin{bmatrix} \Delta \theta \\ \Delta U \end{bmatrix} = \begin{bmatrix} \Delta P \\ \Delta Q \end{bmatrix}
\]

(8.55)

where,

for \( k \neq j \)
\[
H_{kj} = \frac{\partial P_k}{\partial \theta_j} = U_k U_j [G_{kj} \sin(\theta_{kj}) - B_{kj} \cos(\theta_{kj})]
\]
\[
N_{kj} = U_j N'_{kj} = U_j \frac{\partial P_k}{\partial U_j} = U_k U_j [G_{kj} \cos(\theta_{kj}) + B_{kj} \sin(\theta_{kj})]
\]
\[
J_{kj} = \frac{\partial Q_k}{\partial \theta_j} = -U_k U_j [G_{kj} \cos(\theta_{kj}) + B_{kj} \sin(\theta_{kj})]
\]
\[
L_{kj} = U_j L'_{kj} = U_j \frac{\partial Q_k}{\partial U_j} = U_k U_j [G_{kj} \sin(\theta_{kj}) - B_{kj} \cos(\theta_{kj})]
\]

(8.56)
and for $k = j$

\[
H_{kk} = \frac{\partial P_k}{\partial U_k} = -Q_k - B_{kk} U_k^2
\]

\[
N_{kk} = U_k \frac{\partial P_k}{\partial \theta_k} = P_k + G_{kk} U_k^2
\]

\[
J_{kk} = \frac{\partial Q_k}{\partial \theta_k} = P_k - G_{kk} U_k^2
\]

\[
L_{kj} = U_k \frac{\partial Q_k}{\partial U_k} = Q_k - B_{kk} U_k^2
\]

(8.57)

Now based on (8.42), the following is obtained:

\[
\begin{bmatrix}
\Delta \theta \\
\Delta U
\end{bmatrix} =
\begin{bmatrix}
H & N \\
J & L
\end{bmatrix}^{-1}
\begin{bmatrix}
\Delta P \\
\Delta Q
\end{bmatrix}
\]

(8.58)

Finally, $U$ and $\theta$ will be updated as follows:

\[
\theta_k = \theta_k + \Delta \theta_k \quad k \neq \text{slack bus}
\]

\[
U_k = U_k \left(1 + \frac{\Delta U_k}{U_k}\right) \quad k \neq \text{slack bus and PU-bus}
\]

(8.59)

### 8.4.3 Newton-Raphson method for load flow solution

The Newton-Raphson load flow solution may be performed as follows:

- **Step 1**
  1a) Read bus and line data. Identify slack bus (i.e. Uθ-bus), PU-buses and PQ-buses.
  1b) Develop the Y-matrix and calculate the net productions, i.e. $P_{GD} = P_G - P_D$ and $Q_{GD} = Q_G - Q_D$.
  1c) Give the initial estimate of the unknown variables, i.e. $U$ for PQ-buses and $\theta$ for PU- and PQ-buses. It is very common to set $U = U_{U\theta-\text{nod}}$ and $\theta = \theta_{U\theta-\text{nod}}$. However, the flat initial estimate may also be applied, i.e. $U = 1$ and $\theta = 0$.
  1d) Go to Step 2.

- **Step 2**
  2a) Calculate the injected power into each bus by equation (8.49).
  2b) Calculate the difference between the net production and the injected power for each bus, i.e. $\Delta P$ and $\Delta Q$ by equation (8.52).
  2c) Is the magnitude of all entries of $[\Delta P \quad \Delta Q]^T$ less than a specified small positive constant $\epsilon$?
    - * If yes, go to Step 3.
    - * if no, go to Step 4.

- **Step 3**
3a) Calculate the power flows by equations (8.44) and (8.45).
3b) Calculate the generated powers, i.e. $P_G$ and $Q_G$ in the slack bus, and $Q_G$ in the PU-buses by equation (8.50).
3c) Print out the results.

• **Step 4**
  
  4a) Calculate the jacobian by equations (8.56) and (8.57).
  4b) Go to **Step 5**.

• **Step 5**
  
  5a) Calculate $[\Delta \theta \quad \Delta U]^T$ by equation (8.58).
  5b) Go to **Step 6**.

• **Step 6**
  
  6a) Update $U$ and $\theta$ by equation (8.59).
  6b) Go till **Step 2**.

**Example 8.12** Consider the power system shown in Figure 8.5. The following data (all in p.u.) are known:

- Line between Bus 1 and Bus 2: short line, $\bar{Z}_{12} = j0.2$
- Line between Bus 1 and Bus 3: short line, $\bar{Z}_{13} = j0.4$
- Line between Bus 2 and Bus 2: short line, $\bar{Z}_{23} = j0.5$
- Bus 1: $U_1 = 1$, $\theta_1 = 0$
- Bus 2: $U_2 = 1$, $P_{G2} = 2$, $P_{D2} = 1$, $Q_{D2} = 0.2$
- Bus 3: $P_{D3} = 2$, $Q_{D3} = -0.4$

By applying the Newton-Raphson method, calculate $P_{G1}$, $Q_{G1}$ and $Q_{G2}$ after 4 iterations.

**Solution**

**Step 1**

1a) 

$\bar{y}_{12} = \frac{1}{Z_{12}} = -j5$ , $\bar{y}_{13} = \frac{1}{Z_{13}} = -j2.5$ , $\bar{y}_{23} = \frac{1}{Z_{23}} = -j2$

Bus 1= slack bus , Bus 2= PU-bus , Bus 3= PQ-bus
8. Non-linear static analysis

1b)
\[ \ddot{y}_{11} = \ddot{y}_{12} + \ddot{y}_{13} \, , \, \ddot{y}_{22} = \ddot{y}_{12} + \ddot{y}_{23} \, , \, \ddot{y}_{33} = \ddot{y}_{13} + \ddot{y}_{23} \]
\[ Y = \begin{bmatrix} \dddot{y}_{11} & -\ddot{y}_{12} & -\ddot{y}_{13} \\ -\ddot{y}_{12} & \dddot{y}_{22} & -\ddot{y}_{23} \\ -\ddot{y}_{13} & -\ddot{y}_{23} & \dddot{y}_{33} \end{bmatrix} = \begin{bmatrix} jB_{11} & jB_{12} & jB_{13} \\ jB_{21} & jB_{22} & jB_{23} \\ jB_{31} & jB_{32} & jB_{33} \end{bmatrix} = \begin{bmatrix} -j7.5 & j5 & j2.5 \\ j5 & -j7 & j2 \\ j2.5 & j2 & -j4.5 \end{bmatrix} = jB \]

Note that \( Y = G + jB \). However, since the lines are lossless, i.e. \( R_{line} = 0 \), we have \( G = 0 \).

\[ P_{GD2} = P_{G2} - P_{D2} = 2 - 1 = 1 \]
\[ P_{GD3} = P_{G3} - P_{D3} = 0 - 2 = -2 \]
\[ Q_{GD3} = Q_{G3} - Q_{D3} = 0 - (-0.4) = 0.4 \]

1c)
\[ U_3 = 1 \, , \, \theta_2 = 0 \, , \, \theta_3 = 0 \]

Iteration 1

Step 2

2a)
\[
\begin{align*}
P_1 &= U_1 \left[ U_2 B_{12} \sin(\theta_1 - \theta_2) + U_3 B_{13} \sin(\theta_1 - \theta_3) \right] = \\
    &= 1 \ast [1 \ast 5 \ast \sin(0 - 0) + 1 \ast 2.5 \ast \sin(0 - 0)] = 0 \\
P_2 &= U_2 \left[ U_1 B_{21} \sin(\theta_2 - \theta_1) + U_3 B_{23} \sin(\theta_2 - \theta_3) \right] = \\
    &= 1 \ast [1 \ast 5 \ast \sin(0 - 0) + 1 \ast 2 \ast \sin(0 - 0)] = 0 \\
P_3 &= U_3 \left[ U_1 B_{31} \sin(\theta_3 - \theta_1) + U_2 B_{32} \sin(\theta_3 - \theta_2) \right] = \\
    &= 1 \ast [1 \ast 2.5 \ast \sin(0 - 0) + 1 \ast 2 \ast \sin(0 - 0)] = 0 \\
P_1 &= -U_1 \left[ U_2 B_{12} \cos(\theta_1 - \theta_2) + U_3 B_{13} \cos(\theta_1 - \theta_3) + U_1 B_{11} \right] = \\
    &= -1 \ast [1 \ast 5 \ast \cos(0 - 0) + 1 \ast 2.5 \ast \cos(0 - 0) + 1 \ast (-7.5)] = 0 \\
P_2 &= -U_2 \left[ U_1 B_{21} \cos(\theta_2 - \theta_1) + U_3 B_{23} \cos(\theta_2 - \theta_3) + U_2 B_{22} \right] = \\
    &= -1 \ast [1 \ast 5 \ast \cos(0 - 0) + 1 \ast 2 \ast \cos(0 - 0) + 1 \ast (-7)] = 0 \\
P_3 &= -U_3 \left[ U_1 B_{31} \cos(\theta_3 - \theta_1) + U_2 B_{32} \cos(\theta_3 - \theta_2) + U_3 B_{33} \right] = \\
    &= -1 \ast [1 \ast 2.5 \ast \cos(0 - 0) + 1 \ast 2 \ast \cos(0 - 0) + 1 \ast (-4.5)] = 0
\end{align*}
\]

2b)
\[
\begin{align*}
\Delta P &= \begin{bmatrix} \Delta P_1 \\ \Delta P_2 \\ \Delta P_3 \end{bmatrix} = \begin{bmatrix} P_{GD2} - P_2 \\ P_{GD3} - P_3 \end{bmatrix} = \begin{bmatrix} 1 - 0 \\ -2 - 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\
\Delta Q &= \begin{bmatrix} \Delta Q_1 \\ \Delta Q_2 \\ \Delta Q_3 \end{bmatrix} = \begin{bmatrix} Q_{GD3} - Q_3 \end{bmatrix} = [0.4 - 0] = [0.4]
\end{align*}
\]

2c)

Go directly to Step 4

Step 4

4a)
\[
H = \begin{bmatrix} \frac{\partial P_3}{\partial \theta_2} & \frac{\partial P_3}{\partial \theta_3} \\ \frac{\partial P_3}{\partial \theta_2} & \frac{\partial P_3}{\partial \theta_3} \end{bmatrix} , \quad N = \begin{bmatrix} U_3 \frac{\partial P_3}{\partial U_3} \end{bmatrix} , \quad L = \begin{bmatrix} \frac{\partial Q_3}{\partial \theta_2} & \frac{\partial Q_3}{\partial \theta_3} \end{bmatrix} , \quad J = \begin{bmatrix} U_3 \frac{\partial Q_3}{\partial U_3} \end{bmatrix}
\]
8.4. Newton-Raphson method

\[ JAC = \begin{bmatrix} H & N \\ L & J \end{bmatrix} = \begin{bmatrix} \frac{\partial P_2}{\partial \theta_2} & \frac{\partial P_2}{\partial \theta_3} & U_3 \frac{\partial P_2}{\partial U_3} \\ \frac{\partial P_3}{\partial \theta_2} & \frac{\partial P_3}{\partial \theta_3} & U_3 \frac{\partial P_3}{\partial U_3} \\ \frac{\partial Q_3}{\partial \theta_2} & \frac{\partial Q_3}{\partial \theta_3} & U_3 \frac{\partial Q_3}{\partial U_3} \end{bmatrix} \]

where,

**H:**

\[
\frac{\partial P_2}{\partial \theta_2} = -Q_2 - B_{22}U_2^2 = -0 - (-7*1^2) = 7 \\
\frac{\partial P_2}{\partial \theta_3} = -U_2U_3B_{23}\cos(\theta_2 - \theta_3) = -1*1*2*\cos(0 - 0) = -2 \\
\frac{\partial P_3}{\partial \theta_2} = -U_3U_2B_{32}\cos(\theta_3 - \theta_2) = -1*1*2*\cos(0 - 0) = -2 \\
\frac{\partial P_3}{\partial \theta_3} = -Q_3 - B_{33}U_3^2 = -0 - (-4.5*1^2) = 4.5
\]

**N:**

\[
U_3 \frac{\partial P_2}{\partial U_3} = U_2U_3B_{23}\sin(\theta_2 - \theta_3) = 1*1*2*\sin(0 - 0) = 0 \\
U_3 \frac{\partial P_3}{\partial U_3} = P_3 = 0
\]

**J:**

\[
\frac{\partial Q_3}{\partial \theta_2} = -U_3U_2B_{32}\sin(\theta_3 - \theta_2) = -1*1*2*\sin(0 - 0) = 0 \\
\frac{\partial Q_3}{\partial \theta_3} = P_3 = 0
\]

**L:**

\[
U_3 \frac{\partial Q_3}{\partial U_3} = Q_3 - B_{33}U_3^2 = 0 - (-4.5*1^2) = 4.5
\]

**Step 5**

\[
\begin{bmatrix} \Delta \theta_2 \\
\Delta \theta_3 \\
\Delta U_3 \\
\Delta Q_3 \\
\end{bmatrix} = \begin{bmatrix} \frac{\partial P_2}{\partial \theta_2} & \frac{\partial P_2}{\partial \theta_3} & U_3 \frac{\partial P_2}{\partial U_3} \\
\frac{\partial P_3}{\partial \theta_2} & \frac{\partial P_3}{\partial \theta_3} & U_3 \frac{\partial P_3}{\partial U_3} \\
\frac{\partial Q_3}{\partial \theta_2} & \frac{\partial Q_3}{\partial \theta_3} & U_3 \frac{\partial Q_3}{\partial U_3} \end{bmatrix}^{-1} \begin{bmatrix} \Delta P_2 \\
\Delta P_3 \\
\Delta Q_3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\
-2 4.5 0 \\
0 0 4.5 \end{bmatrix} \begin{bmatrix} 1 \\
-2 4.5 0 \\
0 0 4.5 \end{bmatrix} = \begin{bmatrix} 0.0182 \\
-0.4364 \\
0.0889 \end{bmatrix}
\]
8. Non-linear static analysis

Step 6

6a)

\[ \theta_2 = \theta_2 + \Delta \theta_2 = 0 + 0.0182 = 0.0182 \]

\[ \theta_3 = \theta_3 + \Delta \theta_3 = 0 + (-0.4364) = -0.4364 \]

\[ U_3 = U_3(1 + \frac{\Delta U_3}{U_3}) = 1 \times (1 + 0.0889) = 1.0889 \]

6b)  
Go to step 2

Iteration 2

Step 2

2a)

\[ P_1 = 1 \times [1 \times 5 \times \sin(0 - 0.0182) + 1.0889 \times 2.5 \times \sin(0 + 0.4364)] = 1.0596 \]

\[ P_2 = 1 \times [1 \times 5 \times \sin(0.0182 - 0) + 1.0889 \times 2 \times \sin(0.0182 + 0.4364)] = 1.0471 \]

\[ P_3 = 1.0889 \times [1 \times 2.5 \times \sin(-0.4364 - 0) + 1 \times 2 \times \sin(-0.4364 - 0.0182)] = -2.1067 \]

\[ Q_1 = -1 \times [1 \times 5 \times \cos(0 - 0.0182) + 1.0889 \times 2.5 \times \cos(0 + 0.4364) + 1 \times (-7.5)] = 0.0337 \]

\[ Q_2 = -1 \times [1 \times 5 \times \cos(0.0182 - 0) + 1.0889 \times 2 \times \cos(0.0182 + 0.4364) + 1 \times (-7)] = 0.0442 \]

\[ Q_3 = -1.0889 \times [1 \times 2.5 \times \cos(-0.4364 - 0) + 1 \times 2 \times \cos(-0.4364 - 0.0182) + 1.0889 \times (-4.5)] = 0.9118 \]

2b)

\[ \Delta P = \begin{bmatrix} \Delta P_2 \\ \Delta P_3 \end{bmatrix} = \begin{bmatrix} P_{GD2} - P_2 \\ P_{GD3} - P_3 \end{bmatrix} = \begin{bmatrix} 1 - 1.0471 \\ -2 + 2.1067 \end{bmatrix} = \begin{bmatrix} -0.0471 \\ 0.1067 \end{bmatrix} \]

\[ \Delta Q = \begin{bmatrix} \Delta Q_3 \end{bmatrix} = \begin{bmatrix} Q_{GD3} - Q_3 \end{bmatrix} = \begin{bmatrix} 0.4 - 0.9118 \end{bmatrix} = \begin{bmatrix} -0.5118 \end{bmatrix} \]

2c)  
Go to Step 4

Step 4

4a)  
H:

\[ \frac{\partial P_2}{\partial \theta_2} = -Q_2 - B_{22}U_2^2 = -0.0442 - (-7 \times 1^2) = 6.9558 \]

\[ \frac{\partial P_2}{\partial \theta_3} = -U_2U_3B_{23} \cos(\theta_2 - \theta_3) = -1 \times 1.0889 \times 2 \times \cos(0.0182 + 0.4364) = -1.9566 \]

\[ \frac{\partial P_3}{\partial \theta_2} = -U_3U_2B_{32} \cos(\theta_3 - \theta_2) = -1.0889 \times 1 \times 2 \times \cos(-0.0182 - 0.4364) = -1.9566 \]

\[ \frac{\partial P_3}{\partial \theta_3} = -Q_3 - B_{33}U_3^2 = -0.9118 - (-4.5 \times 1.0889^2) = 4.4238 \]
8.4. Newton-Raphson method

N:
\[ \frac{\partial P_2}{\partial U_3} = U_3 B_{23} \sin(\theta_2 - \theta_3) = 1 \times 1.0889 \times 2 \times \sin(0.0182 + 0.4364) = 0.9562 \]
\[ \frac{\partial P_3}{\partial U_3} = P_3 = -2.1067 \]

J:
\[ \frac{\partial Q_3}{\partial \theta_2} = -U_3 B_{32} \sin(\theta_3 - \theta_2) = -1.0889 \times 2 \times \sin(-0.0182 - 0.4364) = 0.9562 \]
\[ \frac{\partial Q_3}{\partial \theta_3} = P_3 = -2.1067 \]

L:
\[ \frac{\partial Q_3}{\partial U_3} = Q_3 - B_{33} U_3^2 = 0.9118 - (-4.5 \times 1.0889^2) = 6.2473 \]

Step 5

5a)
\[ \begin{bmatrix} \Delta \theta_2 \\ \Delta \theta_3 \\ \Delta U_3 / U_3 \end{bmatrix} = \begin{bmatrix} 6.9558 & -1.9566 & 0.9562 \\ -1.9566 & 4.4238 & -2.1067 \\ 0.9562 & -2.1067 & 6.2473 \end{bmatrix}^{-1} \begin{bmatrix} -0.0471 \\ 0.1067 \\ -0.5118 \end{bmatrix} = \begin{bmatrix} 0.0004 \\ -0.0175 \\ -0.0879 \end{bmatrix} \]

Step 6

6a)
\[ \theta_2 = \theta_2 + \Delta \theta_2 = 0.0182 + 0.0004 = 0.0186 \]
\[ \theta_3 = \theta_3 + \Delta \theta_3 = -0.4364 - 0.0175 = -0.4539 \]
\[ U_3 = U_3 (1 + \frac{\Delta U_3}{U_3}) = 1.0889 \times (1 - 0.0879) = 0.9932 \]

6b)
Go to step 2

Iteration 3

With these new values (in step 6) for \( \theta_2 \), \( \theta_3 \) and \( U_3 \), we run the third iteration which gives these updated values, i.e. \( \theta_2 = 0.0186 \), \( \theta_3 = -0.4611 \) and \( U_3 = 0.9824 \). Next we run the forth iteration with these updated values as follows:

Step 2 \rightarrow \text{Step 4} \rightarrow \text{Step 5}

5a)
\[ \begin{bmatrix} \Delta \theta_2 \\ \Delta \theta_3 \\ \Delta U_3 / U_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.0002 \\ -0.0002 \end{bmatrix} \]
8. Non-linear static analysis

Step 6

6a) \[ \theta_2 = \theta_2 + \Delta \theta_2 = 0.0186 + 0 = 0.0186 \]
\[ \theta_3 = \theta_3 + \Delta \theta_3 = -0.4611 - 0.0002 = -0.4613 \]
\[ U_3 = U_3 + (1 + \frac{\Delta U_3}{U_3}) = 0.9824(1 - 0.0002) = 0.9822 \]

Now go to step 3

Step 3

3b) \[ P_1 = U_1 [U_2 B_{12} \sin(\theta_1 - \theta_2) + U_3 B_{13} \sin(\theta_1 - \theta_3)] = \]
\[ = 1 \times [1 \times 5 \times \sin(0 - 0.0186) + 0.9822 \times 2.5 \times \sin(0 + 0.4613)] = 1.0000 \]
\[ Q_1 = -U_1 [U_2 B_{12} \cos(\theta_1 - \theta_2) + U_3 B_{13} \cos(\theta_1 - \theta_3) + U_1 B_{11}] = \]
\[ = -1 \times [1 \times 5 \times \cos(0 - 0.0186) + 0.9822 \times 2.5 \times \cos(0 + 0.4613) + 1 \times (-7.5)] = 0.3020 \]
\[ Q_2 = -U_2 [U_1 B_{21} \cos(\theta_2 - \theta_1) + U_3 B_{23} \cos(\theta_2 - \theta_3) + U_2 B_{22}] = \]
\[ = -1 \times [1 \times 5 \times \cos(0.0186 - 0) + 0.9822 \times 2 \times \cos(0.0186 + 0.4613) + 1 \times (-7)] = 0.2583 \]
\[ P_{G1} = P_1 + P_{D1} = 1 + 0 = 1 \]
\[ Q_{G1} = Q_1 + Q_{D1} = 0.3020 + 0 = 0.3020 \]
\[ Q_{G2} = Q_2 + Q_{D2} = 0.2583 + 0.2 = 0.4583 \]

Matlab-codes for this example can be found in appendix C.
Chapter 9

Analysis of three-phase systems using linear transformations

In this chapter, the possibilities of using linear transformations in order to simplify the analysis of three-phase systems, are briefly discussed. These transformations are general and are valid under both symmetrical and unsymmetrical conditions. By generalizing the expressions for a symmetric three-phase voltage given in equations (3.11) and (3.15), corresponding expressions for an arbitrary three-phase voltage at constant frequency can be obtained as

\[
\begin{align*}
    u_a(t) &= U_{Ma}\cos(\omega t + \gamma_a) & U_a &= U_a\angle \gamma_a^\circ \\
    u_b(t) &= U_{Mb}\cos(\omega t + \gamma_b) & U_b &= U_b\angle \gamma_b^\circ \\
    u_c(t) &= U_{Mc}\cos(\omega t + \gamma_c) & U_c &= U_c\angle \gamma_c^\circ
\end{align*}
\]  

(9.1)

where \( U_{Ma}, U_{Mb}, U_{Mc} \) are peak values, \( U_a, U_b, U_c \) are RMS-values and \( \gamma_a, \gamma_b, \gamma_c \) are phase angles of the three voltages. For the unsymmetrical currents, corresponding expressions hold as

\[
\begin{align*}
    i_a(t) &= I_{Ma}\cos(\omega t + \gamma_a - \phi_a) & I_a &= I_a\angle \gamma_a^\circ - \phi_a \\
    i_b(t) &= I_{Mb}\cos(\omega t + \gamma_b - \phi_b) & I_b &= I_b\angle \gamma_b^\circ - \phi_b \\
    i_c(t) &= I_{Mc}\cos(\omega t + \gamma_c - \phi_c) & I_c &= I_c\angle \gamma_c^\circ - \phi_c
\end{align*}
\]  

(9.2)

where \( I_{Ma}, I_{Mb}, I_{Mc} \) are peak values, \( I_a, I_b, I_c \) are RMS-values of the three phase currents whereas \( \phi_a, \phi_b, \phi_c \) are the phase of the currents in relation to the corresponding phase voltage.

The mean value of the total three-phase active power can be calculated as

\[
P_3 = \frac{U_{Ma}I_{Ma}}{\sqrt{2}}\cos\phi_a + \frac{U_{Mb}I_{Mb}}{\sqrt{2}}\cos\phi_b + \frac{U_{Mc}I_{Mc}}{\sqrt{2}}\cos\phi_c
\]  

(9.3)

whereas the total three-phase complex power is

\[
\begin{align*}
    \bar{S}_3 &= U_aI_a^* + U_bI_b^* + U_cI_c^*
    &= (U_aI_a\cos\phi_a + U_bI_b\cos\phi_b + U_cI_c\cos\phi_c) + \\
    &\quad + j(U_aI_a\sin\phi_a + U_bI_b\sin\phi_b + U_cI_c\sin\phi_c)
\end{align*}
\]  

(9.4)

This phase representation is in many cases sufficient for a three-phase system analysis. There are a number of important cases when the analysis can greatly be simplified by using linear transformations.

This chapter discuss the following items. First, the advantages of using linear transformations in three-phase system analysis are generally discussed. Later on, some specific transformations to be used in certain conditions are given. In order to really understand the subject of transformations, the reader is referred to text books on the subject, e.g. in electric machine theory or high power electronics. In chapter 10, one of the transformations of interest, symmetrical components, is discussed in more detail. The purpose of chapter 9 is to show that the idea and the mathematics behind the transformations are the same. It is only the choice of linear transformation, i.e. transformation matrix, that is different.
9. Analysis of three-phase systems using linear transformations

9.1 Linear transformations

By using transformations, components are mapped from an original space (the original space is here the instantaneous values or the complex representation of the phase quantities) to an image space. A linear transformation means that the components in the image space are a linear combination of the original space. The complex values of the phase voltages can be mapped with a linear transformation as

$$
U_A = w_{aa}U_a + w_{ab}U_b + w_{ac}U_c \\
U_B = w_{ba}U_a + w_{bb}U_b + w_{bc}U_c \\
U_C = w_{ca}U_a + w_{cb}U_b + w_{cc}U_c
$$

which in matrix form can be written as

$$
\begin{bmatrix}
  U_A \\
  U_B \\
  U_C 
\end{bmatrix} =
\begin{bmatrix}
w_{aa} & w_{ab} & w_{ac} \\
w_{ba} & w_{bb} & w_{bc} \\
w_{ca} & w_{cb} & w_{cc} 
\end{bmatrix}
\begin{bmatrix}
  U_a \\
  U_b \\
  U_c 
\end{bmatrix}
$$

or in a more compact notation

$$U_{ABC} = WU_{abc}$$

The elements in matrix $W$ are independent of the values of the original and image space components. In this example, the components in the original space $U_a$, $U_b$ and $U_c$ are mapped by using the linear transformation $W$ to the image space components $U_A$, $U_B$ and $U_C$. The original space components can be calculated from the image space components by using the inverse of matrix $W$ ($W^{-1} = T$), i.e.

$$U_{abc} = W^{-1}U_{ABC} = TU_{ABC}$$

The only mappings that are of interest, are those where $W^{-1}$ are existing. In the following, the matrix $T$ or its inverse $T^{-1}$ will represent the linear transformation.

9.1.1 Power invariants

A usual demand for the linear transformations in power system analysis is that it should be possible to calculate the electric power in the image space by using the same expressions as in the original space and that the two spaces should give the same result. A transformation that can meet that requirement is called power invariant. Using the complex representation, the electric power in the original space can be calculated by using equation (9.4), this gives

$$\bar{S}_{abc} = U_aI_a^* + U_bI_b^* + U_cI_c^* = U_{abc}^T I_{abc}^*$$

and in the image space, the corresponding expression is

$$\bar{S}_{ABC} = U_AI_A^* + U_BI_B^* + U_CI_C^* = U_{ABC}^T I_{ABC}^*$$

Power invariants implies that $\bar{S}_{ABC} = \bar{S}_{abc}$, i.e.

$$U_{ABC}^T I_{ABC} = U_{abc}^T I_{abc} = (TU_{ABC})^T (TI_{ABC})^* = U_{ABC}^T T^T T^* I_{ABC}$$
This gives that the transformation matrix \( T \) must fulfill the following condition:

\[
T^T T^* = E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]  \hspace{1cm} (9.12)

which can be rewritten as

\[
T^{-1} = (T^*)^T
\]  \hspace{1cm} (9.13)

i.e. the matrix \( T \) is unitary. If \( T \) is real, equation (9.13) implies that \( T \) is orthogonal.

### 9.1.2 The coefficient matrix in the original space

Consider a three-phase line between two buses. The voltage drop \( U_{abc} \) over the line depends on the current \( I_{abc} \) flowing in the different phases. The voltage drop can be expressed as

\[
U_{abc} = \begin{bmatrix} U_a \\ U_b \\ U_c \end{bmatrix} = \begin{bmatrix} Z_{aa} & Z_{ab} & Z_{ac} \\ Z_{ba} & Z_{bb} & Z_{bc} \\ Z_{ca} & Z_{cb} & Z_{cc} \end{bmatrix} \begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix} = Z_{abc} I_{abc}
\]  \hspace{1cm} (9.14)

where \( Z_{abc} \) is the coefficient matrix of the line. Note that each element in \( Z_{abc} \) is \( \neq 0 \) since a current in one phase has influence on the voltage drop in the other phases owing to the mutual inductance, see chapter 11.

### Symmetrical matrices

A matrix that is symmetrical around its diagonal is called a symmetrical matrix. For the \( Z \)-bus matrix in equation (9.14), this implies that \( Z_{ab} = Z_{ba}, Z_{ac} = Z_{ca} \) and \( Z_{bc} = Z_{cb} \), i.e.

\[
Z_{abc} = \begin{bmatrix} Z_{aa} & Z_{ab} & Z_{ac} \\ Z_{ab} & Z_{bb} & Z_{bc} \\ Z_{ac} & Z_{bc} & Z_{cc} \end{bmatrix} = Z^T_{abc}
\]  \hspace{1cm} (9.15)

An example of a symmetrical matrix is the one representing a line (or a cable) where the non-diagonal element are dependent on the mutual inductance, which is equal between the phases a–b and the phases b–a, see chapter 11.

### Cyclo-symmetrical matrices

The \( Z \)-bus matrix in equation (9.14) is cyclo-symmetric if \( Z_{ab} = Z_{bc} = Z_{ca}, Z_{ba} = Z_{ac} = Z_{cb} \) and \( Z_{aa} = Z_{bb} = Z_{cc} \), i.e.

\[
Z_{abc} = \begin{bmatrix} Z_{aa} & Z_{ab} & Z_{ba} \\ Z_{ab} & Z_{bb} & Z_{aa} \\ Z_{ba} & Z_{aa} & Z_{cc} \end{bmatrix}
\]  \hspace{1cm} (9.16)
All normal three-phase systems are cyclo-symmetrical, i.e. if \( i_a, i_b, i_c \) are permuted to \( i_b, i_c, i_a \), the voltages \( u_a, u_b, u_c \) will also be permuted to \( u_b, u_c, u_a \). This implies that ordinary overhead lines, cables, transformers and electrical machines can be represented by cyclo-symmetrical matrices.

9.1.3 The coefficient matrix in the image space

If both sides of equation (9.14) are multiplied with the matrix \( T^{-1} \), the following is obtained

\[
U_{ABC} = T^{-1}U_{abc} = T^{-1}Z_{abc}I_{abc} = (T^{-1}Z_{abc}T)I_{ABC} = Z_{ABC}I_{ABC} \quad (9.17)
\]

where

\[
Z_{ABC} = T^{-1}Z_{abc}T \quad (9.18)
\]

\( Z_{ABC} \) is the image space mapping of the coefficient matrix \( Z_{abc} \). This gives that if \( U_{ABC} \) represents the image space voltages, and \( I_{ABC} \) represents the image space currents then \( Z_{ABC} \) will represent the impedances in the image space.

One motive of introducing a linear transformation may be to obtain a diagonal coefficient matrix in the image space, i.e.

\[
Z_{ABC} = \begin{bmatrix}
Z_{AA} & 0 & 0 \\
0 & Z_{BB} & 0 \\
0 & 0 & Z_{CC}
\end{bmatrix} \quad (9.19)
\]

By having a diagonal coefficient matrix, equation (9.17) can be rewritten as

\[
\begin{align*}
\overline{U}_A &= Z_{AA}\overline{I}_A \\
\overline{U}_B &= Z_{BB}\overline{I}_B \\
\overline{U}_C &= Z_{CC}\overline{I}_C
\end{align*} \quad (9.20)
\]

i.e. the matrix equation (9.17) having mutual couplings between the phases is replaced by three un-coupled equations. If \( Z_{ABC} \) is diagonal as in equation (9.19), both sides in equation (9.18) can be multiplied with \( T \) and rewritten as

\[
TZ_{ABC} = \begin{bmatrix}
t_1 & t_2 & t_3
\end{bmatrix}
\begin{bmatrix}
Z_{AA} & 0 & 0 \\
0 & Z_{BB} & 0 \\
0 & 0 & Z_{CC}
\end{bmatrix}
= \begin{bmatrix}
Z_{AA}t_1 & Z_{BB}t_2 & Z_{CC}t_3
\end{bmatrix} = Z_{abc}T = Z_{abc} \begin{bmatrix}
t_1 & t_2 & t_3
\end{bmatrix} \quad (9.21)
\]

where \( t_1, t_2, t_3 \) are columns of \( T \). equation (9.21) can be rewritten as

\[
\begin{align*}
Z_{abc}t_1 - Z_{AA}t_1 &= 0 \\
Z_{abc}t_2 - Z_{BB}t_2 &= 0 \\
Z_{abc}t_3 - Z_{CC}t_3 &= 0
\end{align*} \quad (9.22)
\]
9.2 Examples of linear transformations that are used in analysis of three-phase systems

i.e. $Z_{AA}$, $Z_{BB}$ and $Z_{CC}$ are the eigenvalues of matrix $Z_{abc}$ and the vectors $t_1$, $t_2$ and $t_3$ are the corresponding eigenvectors. A transformation that maps the matrix $Z$ to a diagonal form should have a transformation matrix $T$ having columns that are the eigenvectors of the matrix $Z$. Note that eigenvectors can be scaled arbitrarily.

9.2 Examples of linear transformations that are used in analysis of three-phase systems

In the following, four commonly used linear transformations will briefly be introduced. In general, transformations can be presented in a little bit different ways in different textbook. It is therefore of importance to understand the definitions used by the author.

9.2.1 Symmetrical components

In the analysis of un-symmetrical conditions in a power system, symmetrical components are commonly used. This complex, linear transformation use the fact that all components (lines, machines, etc.) in normal systems are cyclo-symmetrical, i.e. their impedances can be modeled by equation (9.16). The power invariant transformation matrix and its inverse for the symmetrical components are

$$T_S = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \end{bmatrix}$$

$$T_S^{-1} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \end{bmatrix}$$

where $\alpha = e^{j120^\circ}$. As given by the definition $T_S^{-1} = (T_S^*)^T$ which correspond to the assumption of power invariance according to equation (9.13). By using this transformation, cyclo-symmetrical matrices are transformed into a diagonal form as given in equation (9.19), i.e. the column of matrix $T_S$ consist of the eigenvectors to a cyclo-symmetrical matrix. This will simplify the system analysis as indicated in equation (9.20). By using the given phase voltage $U_a$, $U_b$ and $U_c$, the power invariant symmetrical components can be calculated as

$$U_s = \begin{bmatrix} U_0 \\ U_1 \\ U_2 \end{bmatrix} = T_S^{-1}U_{abc} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \end{bmatrix} \begin{bmatrix} U_a \\ U_b \\ U_c \end{bmatrix}$$

The three components $U_0$, $U_1$ and $U_2$ are called zero-sequence, positive-sequence and negative-sequence, respectively. A cyclo-symmetrical impedance matrix according to equation (9.16) can be diagonalized by using symmetrical components according to equation (9.18) as

$$Z_S = T_S^{-1} \begin{bmatrix} Z_{aa} & Z_{ab} & Z_{ba} \\ Z_{ba} & Z_{aa} & Z_{ab} \\ Z_{ab} & Z_{ba} & Z_{aa} \end{bmatrix} T_S = \begin{bmatrix} Z_0 & 0 & 0 \\ 0 & Z_1 & 0 \\ 0 & 0 & Z_2 \end{bmatrix}$$

(9.25)
where
\[ Z_0 = Z_{aa} + Z_{ab} + Z_{ba} = \text{zero-sequence impedance} \]
\[ Z_1 = Z_{aa} + \alpha^2 Z_{ab} + \alpha Z_{ba} = \text{positive-sequence impedance} \] \hspace{1cm} (9.26)
\[ Z_2 = Z_{aa} + \alpha Z_{ab} + \alpha^2 Z_{ba} = \text{negative-sequence impedance} \]

The three impedances \( Z_0, Z_1 \) and \( Z_2 \) are the eigenvalues of the cyclo-symmetrical impedance matrix. For an impedance matrix that is both cyclo-symmetric and symmetric, i.e. \( Z_{ba} = Z_{ab} \), the result after a diagonalization will be that
\[ Z_0 = Z_{aa} + 2Z_{ab} \]
\[ Z_1 = Z_{aa} - Z_{ab} \]
\[ Z_2 = Z_{aa} - Z_{ab} \] \hspace{1cm} (9.27)

Transformers, overhead lines, cables and symmetrical loads (not electrical machines) can normally be represented by impedance matrices that are both symmetrical and cyclo-symmetrical, i.e. all diagonal elements are equal and all non-diagonal elements are equal. This gives that the positive-sequence impedance and the negative-sequence impedance are equal.

In order to make the positive-sequence phase voltage equal to the phase voltage, a reference invariant form of transformation for the symmetrical components is normally used. The reference invariant transformation matrix and its inverse are
\[ T_{S'} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha^2 & \alpha \\ 1 & \alpha & \alpha^2 \end{bmatrix} = \sqrt{3} \cdot T_S \quad T_{S'}^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \end{bmatrix} = \frac{1}{\sqrt{3}} \cdot T_S^{-1} \] \hspace{1cm} (9.28)

The reference invariant transformation is not power invariant since \( T_{S'}^{-1} = \frac{1}{3}(T_{S'}^*)^T \). The name reference invariant means that in symmetrical conditions \( U_1 = U_a \). Note that transformations of coefficient matrices, according to equation (9.18), are not influenced whether the power invariant or the reference invariant matrix is used since
\[ Z_{ABC}(eff-inv) = T_{S'}^{-1}Z_{abc}T_S = \left( \frac{1}{\sqrt{3}} T_S^{-1} \right) Z_{abc} \left( \sqrt{3} T_S \right) = T_{S'}^{-1}Z_{abc}T_{S'} = Z_{ABC}(ref-inv) \] \hspace{1cm} (9.29)

A third variation of the transformation matrix for the symmetrical components arise when the ordering of the sequences is changed. If the positive-sequence is given first and the zero-sequence last, the columns of the \( T \)-matrix and the rows in the \( T^{-1} \) are permuted, respectively. This results in the following reference invariant transformations matrices :
\[ T_{S''} = \begin{bmatrix} 1 & 1 & 1 \\ \alpha^2 & \alpha & 1 \\ \alpha & \alpha^2 & 1 \end{bmatrix} \quad T_{S''}^{-1} = \frac{1}{3} \begin{bmatrix} 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \\ 1 & 1 & 1 \end{bmatrix} \] \hspace{1cm} (9.30)

This form of the transformation matrices will be used in chapter 10 where symmetrical components are discussed in more detail. The only thing that happens with the coefficient matrix in the image space is that the diagonal elements change places.

As described above, a number of different variations of the symmetrical components can be used, all having the same fundamental purpose, to diagonalize the cyclo-symmetrical impedance matrices.
9.2.2 Clarke’s components

Clarke’s components, also called $\alpha - \beta$-components or orthogonal components, divides those phase-quantities not having any zero-sequence into two orthogonal components. The word zero-sequence means the same as when discussing symmetrical components, the sum of the phase components. Components not having any zero-sequence are those where the sum of all phase components are $= 0$. The power invariant ($T^{-1} = (T^*)^T$, see equation (9.13)) transformation matrix and its inverse for the Clarke’s components are

$$T_C = \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \quad T_C^{-1} = \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ 1 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & 0 \end{bmatrix}$$

(9.31)

Clarke’s components are a real orthogonal transformation that is mainly used in transformations of time quantities, e.g.

$$\begin{bmatrix} i_0(t) \\ i_\alpha(t) \\ i_\beta(t) \end{bmatrix} = i_{0\alpha\beta}(t) = T_C^{-1} i_{abc}(t) = \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ 1 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & 0 \end{bmatrix} \begin{bmatrix} i_a(t) \\ i_b(t) \\ i_c(t) \end{bmatrix}$$

(9.32)

where $i_0(t)$ is the zero-sequence component, $i_\alpha(t)$ is the $\alpha$-component and $i_\beta$ is the $\beta$-component for Clarke’s transformation of the phase currents $i_a(t), i_b(t)$ and $i_c(t)$. The reason why the transformation is orthogonal is given by column two and three of $T_C$ (corresponds to the $\alpha$- and $\beta$-components) since the columns are orthogonal.

For a symmetrical three-phase current given by equation (3.13)

$$i_a(t) = I_M \cos(\omega t - \phi) \quad i_b(t) = I_M \cos(\omega t - 120^\circ - \phi) \quad i_c(t) = I_M \cos(\omega t + 120^\circ - \phi)$$

(9.33)

the Clarke’s components are given by equation (9.32)

$$i_0(t) = \frac{1}{\sqrt{3}} (i_a(t) + i_b(t) + i_c(t)) = 0$$

$$i_\alpha(t) = \sqrt{\frac{2}{3}} \left( i_a(t) - \frac{1}{2} i_b(t) - \frac{1}{2} i_c(t) \right) = \sqrt{\frac{3}{2}} I_M \cos(\omega t - \phi)$$

$$i_\beta(t) = \sqrt{\frac{2}{3}} \left( \frac{\sqrt{3}}{2} i_b(t) - \frac{\sqrt{3}}{2} i_c(t) \right) = \sqrt{\frac{3}{2}} I_M \cos(\omega t - \phi - 90^\circ)$$

(9.34)

As given above, conditions not having any zero-sequence can be fully represented by Clarke’s $\alpha$- and $\beta$-components. Conditions not having any zero-sequence are quite common and depends, among other things, on the type of transformer connection used.

Matrices that are both symmetrical and cyclo-symmetrical can be diagonalized by using
Clarke’s transform as

\[
\mathbf{Z}_C = T_C^{-1} \begin{bmatrix}
Z_{aa} & Z_{ab} & Z_{ab} \\
Z_{ab} & Z_{aa} & Z_{ab} \\
Z_{ab} & Z_{ab} & Z_{aa}
\end{bmatrix} T_C = \begin{bmatrix}
Z_{aa} + 2Z_{ab} & 0 & 0 \\
0 & Z_{aa} - Z_{ab} & 0 \\
0 & 0 & Z_{aa} - Z_{ab}
\end{bmatrix}
\] (9.35)

For this type of matrices, the diagonalizing using Clarke’s components gives exactly the same answer as a diagonalizing using symmetrical components, see equations (9.25) and (9.27).

This gives that the matrix representation of transformers, overhead lines, cables and symmetrical loads (not electrical machines) can be diagonalized. The advantage of using Clarke’s components is that the transformation is real which gives that the mapping of real instantaneous value mappings also are real. The disadvantage is that electrical machines cannot be represented by three independent variables by using Clarke’s components.

Clarke’s components are used in order to simplify the analysis of e.g. multi-phase short circuits, transient system behavior, converter operation, etc.

### 9.2.3 Park’s transformation

Park’s transformation (also called dq-transformation or Blondell’s transformation) is a linear transformation between the three physical phases and three new components. This transformation is often used when analyzing synchronous machines.

In Figure 9.1, a simplified description of the internal conditions of a synchronous machine having salient poles, is given. Two orthogonal axis are defined. One is directed along the magnetic flux induced in the rotor. The second axis is orthogonal to the first axis. The first axis is called the direct-axis (d-axis) and the second axis is called the quadrature-axis (q-axis). Note that this system of coordinates follows the rotation of the rotor. The machine given in Figure 9.1 is a two-pole machine, but Park’s transformation can be used for machines having an arbitrary number of poles.

As indicated above, Park’s transformation is time independent since the displacement between the dq-axes and the abc-axes is changed when the rotor revolves. The Park’s transformation includes not only the d- and q-components, but also the zero-sequence in order to achieve a complete representation. The connection between phase currents \(i_a\), \(i_b\), and \(i_c\) and the dq0-components is given by the notation given in Figure 9.1

\[
\begin{align*}
i_0 &= \frac{1}{\sqrt{3}} (i_a + i_b + i_c) \\
i_d &= \sqrt{\frac{2}{3}} (i_a \cos \theta + i_b \cos (\theta - 120^\circ) + i_c \cos (\theta + 120^\circ)) \\
i_q &= \sqrt{\frac{2}{3}} (-i_a \sin \theta - i_b \sin (\theta - 120^\circ) - i_c \sin (\theta + 120^\circ))
\end{align*}
\] (9.36)
Examples of linear transformations that are used in analysis of three-phase systems

This equation can be written on matrix form as

\[
\begin{bmatrix}
i_0 \\
i_d \\
i_q
\end{bmatrix} = \sqrt{\frac{2}{3}} \begin{bmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\cos \theta & \cos (\theta - 120^\circ) & \cos (\theta + 120^\circ) \\
-\sin \theta & -\sin (\theta - 120^\circ) & -\sin (\theta + 120^\circ)
\end{bmatrix}
\begin{bmatrix}
i_a \\
i_b \\
i_c
\end{bmatrix} = T_P^{-1}i_{abc}
\] (9.37)

The transformation is hence power invariant according to equation (9.13). Park’s transformation is real and usable when transforming time quantities. Note that the Park’s transformation is linear but the transformation matrix is time dependent. At constant frequency \( \theta = \omega t + \theta_0 \).

The Park’s transformation is a frequency transformed version of Clarke’s transformation. When \( \theta = 0 \) (Park’s transformation), the transformation matrices are identical, i.e. \( T_C = T_P(\theta = 0) \).

9.2.4 Phasor components

Phasor components are mainly used at instantaneous value analysis when a single machine or when several machines are connected together. The power invariant \( (T^{-1} = (T^*)^T \), see
9. Analysis of three-phase systems using linear transformations

equation (9.13) transformation matrix and its inverse for these components are

\[ T_R = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & \frac{e^{j\theta}}{\alpha^2} & e^{-j\theta} \\ 1 & \alpha e^{j\theta} & \alpha^2 e^{-j\theta} \\ 1 & \alpha^2 e^{j\theta} & \alpha e^{-j\theta} \end{bmatrix} \]  

(9.39)

\[ T_R^{-1} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & \frac{e^{-j\theta}}{\alpha} & \frac{1}{\alpha^2} \\ e^{j\theta} & \frac{e^{-j\theta}}{\alpha} & \frac{1}{\alpha^2} \\ e^{-j\theta} & \frac{e^{-j\theta}}{\alpha^2} & \frac{e^{j\theta}}{\alpha} \end{bmatrix} \]

where \( \alpha = e^{j120^\circ} \). The phasor components of the three-phase currents \( i_a(t) \), \( i_b(t) \) and \( i_c(t) \) can be obtained as

\[ \begin{bmatrix} i_0(t) \\ \tilde{i}_s(t) \\ \tilde{i}_z(t) \end{bmatrix} = i_{0sz}(t) = T_R^{-1} \begin{bmatrix} i_a(t) \\ i_b(t) \\ i_c(t) \end{bmatrix} \]

(9.40)

where \( i_0(t) \) is the zero-sequence component and \( \tilde{i}_s(t) \) is called the field vector current, the complex phasor of the current. The current \( \tilde{i}_z(t) \) is complex since the transformation matrix is complex. By assuming that \( i_a(t) \), \( i_b(t) \) and \( i_c(t) \) are real, the expression for \( \tilde{i}_z(t) \) can be written as

\[ \tilde{i}_z(t) = \frac{1}{\sqrt{3}} e^{j\theta} \left( i_a(t) + \alpha^2 i_b(t) + \alpha i_c(t) \right) = \left[ \frac{1}{\sqrt{3}} e^{-j\theta} \left( i_a(t) + \alpha i_b(t) + \alpha^2 i_c(t) \right) \right]^* = \tilde{i}_s(t) \]

(9.41)

i.e. \( \tilde{i}_z(t) \) is known if the field vector \( \tilde{i}_s(t) \) is known. Under conditions of no zero-sequence components, the field vector is fully describing an arbitrary real three-phase quantity.

For a symmetrical three-phase current as given in equation (9.33), the phasor components can be obtained according to equation (9.40)

\[ i_0(t) = \frac{1}{\sqrt{3}} (i_a(t) + i_b(t) + i_c(t)) = 0 \]

\[ \tilde{i}_s(t) = \frac{e^{-j\theta}}{\sqrt{3}} \left( i_a(t) + \alpha i_b(t) + \alpha^2 i_c(t) \right) = \sqrt{3} I_M e^{j(\omega t - \phi - \theta)} \]

(9.42)

\[ \tilde{i}_z(t) = \frac{e^{j\theta}}{\sqrt{3}} \left( i_a(t) + \alpha^2 i_b(t) + \alpha i_c(t) \right) = \sqrt{3} I_M e^{-j(\omega t - \phi - \theta)} = \tilde{i}_s(t) \]

Finally, for \( \theta = \omega t \) the following is obtained

\[ i_0(t) = 0 \]

\[ \tilde{i}_s(t) = \frac{\sqrt{3}}{2} I_M e^{-j\phi} \]

(9.43)

\[ \tilde{i}_z(t) = \frac{\sqrt{3}}{2} I_M e^{j\phi} = \tilde{i}_s(t) \]

i.e. the field vector current \( \tilde{i}_s(t) \) has a constant magnitude, independent of time.
By assuming real phase currents having no zero-sequence, they can be calculated by using the field vector current as

\[
\begin{bmatrix}
  i_a(t) \\
  i_b(t) \\
  i_c(t)
\end{bmatrix} = T_R \text{ios}z(t) = \frac{1}{\sqrt{3}} \begin{bmatrix}
  1 & e^{j\theta} & e^{-j\theta} \\
  1 & \alpha^2 e^{j\theta} & \alpha e^{-j\theta} \\
  1 & \alpha e^{j\theta} & \alpha^2 e^{-j\theta}
\end{bmatrix} \begin{bmatrix}
  0 \\
  \tilde{i}_s(t) \\
  \tilde{i}_s^*(t)
\end{bmatrix} = (9.44)
\]

\[
= \frac{1}{\sqrt{3}} \begin{bmatrix}
  e^{j\theta} \tilde{i}_s(t) + e^{j\theta} \tilde{i}_s(t)^* \\
  \alpha^2 e^{j\theta} \tilde{i}_s(t) + \alpha^2 e^{j\theta} \tilde{i}_s(t)^* \\
  \alpha e^{j\theta} \tilde{i}_s(t) + \alpha e^{j\theta} \tilde{i}_s(t)^*
\end{bmatrix} = \frac{2}{\sqrt{3}} \begin{bmatrix}
  \text{Re} \left[ e^{j\theta} \tilde{i}_s(t) \right] \\
  \text{Re} \left[ \alpha e^{j\theta} \tilde{i}_s(t) \right] \\
  \text{Re} \left[ \alpha^2 e^{j\theta} \tilde{i}_s(t) \right]
\end{bmatrix}
\]

Phasor components are a frequency transformed form of the symmetrical components. For \( \theta = 0 \) (phasor components), the transformation matrix for the phasor components and the symmetrical components are identical, i.e. \( T_R(\theta = 0) = T_S \).
9. Analysis of three-phase systems using linear transformations
Chapter 10
Symmetrical components

10.1 Definitions

Assume an arbitrary un-symmetric combination of three phases, exemplified by the currents \( I_a, I_b \) and \( I_c \), given in Figure 10.1a. Fortesque has shown that it is possible to replace three phase-components with three symmetrical components:

A. The positive-sequence is a sequence of three components that all have the same amplitude but have a phase displacement of 120 and 240°, respectively, and the phase sequence is \( abc \), see Figure 10.1b.

B. The negative-sequence is a sequence of three components that all have the same amplitude but have a phase displacement of 240 and 120°, respectively, and the phase sequence is \( acb \), see Figure 10.1c.

C. The zero-sequence is a sequence of three components that all have the same amplitude and phase, see Figure 10.1d.

The three system of components can be symbolized with 1 (positive-sequence), 2 (negative-sequence) and 0 (zero-sequence).

\[ \text{Figure 10.1. Un-symmetric three-phase current expressed as the sum of positive-, negative-, and zero-sequences} \]
The result shown in Figure 10.1 can mathematically be expressed as:

\[
\begin{align*}
I_a &= I_{a1} + I_{a2} + I_{a0} \\
I_b &= I_{b1} + I_{b2} + I_{b0} \\
I_c &= I_{c1} + I_{c2} + I_{c0}
\end{align*}
\] (10.1)

The three positive-sequence components can be denoted as:

\[
\begin{align*}
I_{b1} &= I_{a1}e^{-j120^\circ} \\
I_{c1} &= I_{a1}e^{j120^\circ}
\end{align*}
\] (10.2)

The corresponding expressions for the negative- and zero-sequence are as:

\[
\begin{align*}
I_{b2} &= I_{a2}e^{j120^\circ} \\
I_{c2} &= I_{a2}e^{-j120^\circ} \\
I_{a0} &= I_{b0} = I_{c0}
\end{align*}
\] (10.3)

By inserting equation (10.2) and (10.3) into equation (10.1), the following is obtained:

\[
\begin{align*}
I_a &= I_{a1} + I_{a2} + I_{a0} \\
I_b &= \alpha^2I_{a1} + \alpha I_{a2} + I_{a0} \\
I_c &= \alpha I_{a1} + \alpha^2I_{a2} + I_{a0}
\end{align*}
\] (10.4)

In order to simplify the expressions, the symbol \( \alpha \) has been introduced:

\[
\alpha = e^{j120^\circ} = \cos 120^\circ + j \sin 120^\circ = -\frac{1}{2} + j\frac{\sqrt{3}}{2}
\] (10.5)

The following expressions of the symbol \( \alpha \) are valid:

\[
\alpha^2 = e^{j240^\circ} = e^{-j120^\circ} = -\frac{1}{2} - j\frac{\sqrt{3}}{2} \\
\alpha^3 = 1 \\
1 + \alpha + \alpha^2 = 0 \\
\alpha^* = \alpha^2 \\
(\alpha^2)^* = \alpha
\] (10.6)

Equation (10.4) can, by using matrix form, be written as:

\[
I_f = TI_s
\] (10.7)

where the matrix

\[
T = \begin{bmatrix}
1 & 1 & 1 \\
\alpha^2 & \alpha & 1 \\
\alpha & \alpha^2 & 1
\end{bmatrix}
\] (10.8)
which is called the transformation matrix for the symmetrical components. This matrix is equal to the reference invariant matrix $T_{S''}^{-1}$ according to equation (9.30). The current vector

$$I_f = \begin{bmatrix} \bar{I}_a \\ \bar{I}_b \\ \bar{I}_c \end{bmatrix}$$

(10.9)

represents the phase components of the current whereas

$$I_s = \begin{bmatrix} \bar{I}_{a1} \\ \bar{I}_{a2} \\ \bar{I}_{a0} \end{bmatrix} \text{ or shorter } I_s = \begin{bmatrix} I_1 \\ I_2 \\ I_0 \end{bmatrix}$$

(10.10)

represents the symmetrical components of the current.

By using equation (10.7), the symmetrical components as a function of phase components can be obtained:

$$I_s = T^{-1} I_f$$

(10.11)

where

$$T^{-1} = \frac{1}{3} \begin{bmatrix} 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \\ 1 & 1 & 1 \end{bmatrix}$$

(10.12)

which is equal to $T_{S''}^{-1}$ according to equation (9.30). Of course, the symmetrical components can also be applied to voltages. Using the vectors

$$U_f = \begin{bmatrix} \bar{U}_a \\ \bar{U}_b \\ \bar{U}_c \end{bmatrix} \text{ and } U_s = \begin{bmatrix} \bar{U}_{a1} \\ \bar{U}_{a2} \\ \bar{U}_{a0} \end{bmatrix} \text{ or shorter } U_s = \begin{bmatrix} U_1 \\ U_2 \\ U_0 \end{bmatrix}$$

(10.13)

for representing phase-quantities and symmetrical components, respectively, the relation between them can be written as

$$U_f = TU_s$$

(10.14)

$$U_s = T^{-1} U_f$$

(10.15)

Example 10.1 Calculate the symmetrical components for the following symmetrical voltages

$$U_f = \begin{bmatrix} \bar{U}_a \\ \bar{U}_b \\ \bar{U}_c \end{bmatrix} = \begin{bmatrix} 277 \angle 0^\circ \\ 277 \angle -120^\circ \\ 277 \angle +120^\circ \end{bmatrix} V$$

(10.16)

Solution

By using equation (10.12) and (10.15), the symmetrical components of the voltage $U_f$ can
be calculated as
\[
\begin{bmatrix}
U_1 \\
U_2 \\
U_0
\end{bmatrix} = U_s = T^{-1} U_f = \frac{1}{3} \begin{bmatrix}
1 & \alpha & \alpha^2 \\
1 & \alpha^2 & \alpha \\
1 & 1 & 1
\end{bmatrix} \begin{bmatrix}
U_a \\
U_b \\
U_c
\end{bmatrix} = \begin{bmatrix}
1 \cdot 277 + 1 \angle 120^\circ \cdot 277 \angle -120^\circ + 1 \angle 120^\circ \cdot 277 \angle +120^\circ \\
1 \cdot 277 + 1 \angle 240^\circ \cdot 277 \angle -120^\circ + 1 \angle 120^\circ \cdot 277 \angle +120^\circ \\
1 \cdot 277 + 1 \angle 277 \angle -120^\circ + 1 \cdot 277 \angle +120^\circ \\
\end{bmatrix} = \begin{bmatrix}
277 \angle 0^\circ \\
0 \\
0
\end{bmatrix}
\]

As given in the example, a symmetric three-phase system with a phase sequence of \(abc\) gives rise to a positive-sequence voltage only, having the same amplitude and angle as the voltage in phase \(a\).

**Example 10.2** For a Y-connected three-phase load with zero conductor, phase \(b\) is at one occasion disconnected. The load currents at that occasion is:
\[
I_f = \begin{bmatrix}
I_a \\
I_b \\
I_c
\end{bmatrix} = \begin{bmatrix}
10 \angle 0^\circ \\
0 \\
10 \angle +120^\circ
\end{bmatrix} A
\]

Calculate the symmetrical components of the load current as well as the current in the zero conductor, \(I_n\).

**Solution**
\[
\begin{bmatrix}
I_1 \\
I_2 \\
I_0
\end{bmatrix} = \frac{1}{3} \begin{bmatrix}
1 \cdot 10 \angle 0^\circ + 1 \angle 120^\circ \cdot 0 + 1 \angle 240^\circ \cdot 10 \angle +120^\circ \\
1 \cdot 10 \angle 0^\circ + 1 \angle 240^\circ \cdot 0 + 1 \angle 120^\circ \cdot 10 \angle +120^\circ \\
1 \cdot 10 \angle 0^\circ + 1 \cdot 0 + 1 \angle 10 \angle +120^\circ \\
\end{bmatrix} = \begin{bmatrix}
6.667 \angle 0^\circ \\
3.333 \angle -60^\circ \\
3.333 \angle 60^\circ \\
\end{bmatrix}
\]
\[
I_n = I_a + I_b + I_c = 10 \angle 0^\circ + 0 + 10 \angle +120^\circ = 10 \angle 60^\circ = 3I_0
\]

As given in the example, the current in the zero conductor is three times as large as the zero-sequence current.

### 10.2 Power calculations in un-symmetrical conditions

The three-phase complex power of the phase-components can be calculated as
\[
\mathcal{S} = P + jQ = U_a I_a^* + U_b I_b^* + U_c I_c^* = U_f^T I_f^*
\]
By introducing symmetrical components, the expression above can be converted to

$$\bar{S} = U_T^T I_f^* = (TU_s)^T (TI_s)^* = U_s^T T^T T^* I_s^*$$

(10.22)

The expression $T^T T^*$ can be written as

$$T^T T^* = \begin{bmatrix} 1 & \alpha^2 & \alpha \\ 1 & \alpha & \alpha^2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ \alpha & \alpha^2 & 1 \\ \alpha^2 & \alpha & 1 \end{bmatrix} = 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(10.23)

i.e. the transformation is not power invariant, see subsection 9.1.1, equation (9.12). Equation (10.22) can be rewritten as

$$\bar{S} = 3 U_s^T I_s^* = 3 \bar{U}_{s1} \bar{T}_{s1}^* + 3 \bar{U}_{s2} \bar{T}_{s2}^* + 3 \bar{U}_{s0} \bar{T}_{s0}^*$$

(10.24)

Since the magnitude of the phase-to-phase voltages are $\sqrt{3}$ times larger than the phase-to-ground voltages and $S_b = \sqrt{3} \cdot U_b \cdot I_b$, the introduction of the per-unit system gives that equation (10.24) can be rewritten as

$$\bar{S}_{pu} = \frac{\sqrt{3} (\sqrt{3} U_s)^T \cdot I_s^*}{\sqrt{3} \cdot U_b \cdot I_b} = \bar{U}_{s1-pu} \bar{T}_{s1-pu}^* + \bar{U}_{s2-pu} \bar{T}_{s2-pu}^* + \bar{U}_{s0-pu} \bar{T}_{s0-pu}^*$$

(10.25)

This implies that the total power in an un-symmetric system can be calculated as the sum of the powers in the positive-, negative-, and zero-sequences.
10. Symmetrical components
Chapter 11

Line model for un-symmetric three-phase conditions

11.1 Series impedance of single-phase overhead line

The theory of having an overhead line using the ground as a return conductor was discussed by Carson in 1923. Carson considered a single conductor $a$ of unity length (e.g. one meter) running in parallel with the ground, see Figure 11.1.

![Figure 11.1. Carson’s single-phase overhead line using the ground as return path](image)

The current $I_a$ flows in the conductor using the ground between $d - d'$ as return path. The ground is assumed to have an uniform resistance and an infinite extension. The current $I_d$ ($=-I_a$) is distributed over a large area, flowing along the ways of least resistance. Kirchhoff’s law about the same voltage drop along each path is fulfilled. It has been shown that these distributed return paths may, in the analysis, be replaced by a single return conductor having a radius $r_d$ located at a distance $D_{ad}$ from the overhead line according to Figure 11.1. The distance $D_{ad}$ is a function of the resistivity of the ground $\rho$. The distance $D_{ad}$ increases as the resistivity $\rho$ increases.

The inductance of this circuit can be calculated as

$$L_a = \frac{\mu}{2\pi} \ln \frac{1}{D_a} + \frac{\mu}{2\pi} \ln \frac{1}{D_d} - 2 \frac{\mu}{2\pi} \ln \frac{1}{D_{ad}} = \frac{\mu}{2\pi} \left( \ln \frac{D_{ad}}{D_a} + \ln \frac{D_{ad}}{D_d} \right)$$  \hspace{1cm} (11.1)

where

- $\mu$ = the permeability of the conductor
- $D_a$ = $e^{-1/4}r_a$ for a single conductor with radius $r_a$
- $D_d$ = $e^{-1/4}r_d$ for a return conductor in ground with radius $r_d$

The inductance can according to equation (11.1) be divided into three parts, two apparent self inductances ($L_{aa}, L_{dd}$) and one apparent mutual inductance ($L_{ad}$). Note that these quantities
are only mathematical quantities without any physical meaning. For instance, they have wrong dimension inside the ln-sign. It is only after the summation they achieve a physical meaning. Hopefully, the different part expressions will simplify the understanding of the behavior of a three-phase line. The total series reactance of this single-phase conductor is

\[ X_a = \omega L_a = \omega (L_{aa} + L_{dd} - 2L_{ad}) \]  

(11.2)

By using this line model, having apparent inductances, the voltage drop for a single-phase line can be calculated as

\[
\begin{bmatrix}
U_{aa}' \\
U_{bb}' \\
U_{cc}' \\
U_{dd}'
\end{bmatrix}
= \begin{bmatrix}
\Delta U_a \\
\Delta U_b \\
\Delta U_c \\
\Delta U_d
\end{bmatrix}
= \begin{bmatrix}
z_{aa} & z_{ab} & z_{ac} & z_{ad} \\
z_{ba} & z_{bb} & z_{bc} & z_{bd} \\
z_{ca} & z_{cb} & z_{cc} & z_{cd} \\
z_{da} & z_{db} & z_{dc} & z_{dd}
\end{bmatrix}
\begin{bmatrix}
I_a \\
I_b \\
I_c \\
I_d
\end{bmatrix}
\text{volt/length unit}
\]  

(11.3)

where \( U_a, U_{a'}, U_d \) and \( U_{d'} \) are given in proportion to the same reference. Since \( U_d = 0 \) and \( U_{a'} - U_{d'} = 0 \), \( U_a \) can be obtained by subtracting the two equations from each other:

\[
U_a = (z_{aa} + z_{dd} - 2z_{ad})I_a = Z_a I_a
\]  

(11.4)

By definition

\[
\overline{Z}_a \equiv z_{aa} + z_{dd} - 2z_{ad} \quad \Omega/\text{length unit}
\]  

(11.5)

The impedances in this equation can be calculated as

\[
\begin{align*}
z_{aa} &= r_a + jx_{aa} = r_a + j\omega L_{aa} \quad \Omega/\text{length unit} \\
z_{dd} &= r_d + jx_{dd} = r_d + j\omega L_{dd} \quad \Omega/\text{length unit} \\
z_{ad} &= jx_{ad} = j\omega L_{ad} \quad \Omega/\text{length unit} \\
Z_a &= r_a + r_d + jX_a \quad \Omega/\text{length unit}
\end{align*}
\]  

(11.6)

where

\[
\begin{align*}
r_a &= \text{conductor resistance per length unit} \\
r_d &= \text{ground resistance per length unit}
\end{align*}
\]

### 11.2 Series impedance of a three-phase overhead line

In order to obtain the series impedance of a three-phase line, the calculations are performed in the same way as for the single-phase line. In Figure 11.2, the impedances, voltages and currents of the line are given.

Since all conductors are grounded at \( a', b', c' \), the following are valid

\[
\begin{align*}
\overline{I}_d &= -(\overline{I}_a + \overline{I}_b + \overline{I}_c) \\
\overline{U}_{a'} - \overline{U}_{d'} &= 0 \\
\overline{U}_{b'} - \overline{U}_{d'} &= 0 \\
\overline{U}_{c'} - \overline{U}_{d'} &= 0
\end{align*}
\]  

(11.7)

The voltage drop over the conductors can be calculated as

\[
\begin{bmatrix}
\Delta U_{aa'} \\
\Delta U_{bb'} \\
\Delta U_{cc'} \\
\Delta U_{dd'}
\end{bmatrix}
= \begin{bmatrix}
\Delta U_a \\
\Delta U_b \\
\Delta U_c \\
\Delta U_d
\end{bmatrix}
= \begin{bmatrix}
z_{aa} & z_{ab} & z_{ac} & z_{ad} \\
z_{ba} & z_{bb} & z_{bc} & z_{bd} \\
z_{ca} & z_{cb} & z_{cc} & z_{cd} \\
z_{da} & z_{db} & z_{dc} & z_{dd}
\end{bmatrix}
\begin{bmatrix}
\overline{I}_a \\
\overline{I}_b \\
\overline{I}_c \\
\overline{I}_d
\end{bmatrix}
\text{volt/length unit}
\]  

(11.8)
11.2. Series impedance of a three-phase overhead line

In the same way as for the single-phase conductor, the impedances in equation (11.8) are apparent without any physical relevance. With $U_d = 0$ and by using equation (11.7), row four can be subtracted from row one in equation (11.8) which gives

$$U_a - (U_a' - U_d') = (z_{aa} - 2z_{ad} + z_{dd})T_a + (z_{ab} - z_{ad} - z_{bd} + z_{dd})T_b +$$

$$+ (z_{ac} - z_{ad} - z_{cd} + z_{dd})T_c$$

(11.9)

This can be simplified to

$$\begin{bmatrix} U_a \\ U_b \\ U_c \end{bmatrix} = \begin{bmatrix} Z_{aa} & Z_{ab} & Z_{ac} \\ Z_{ab} & Z_{bb} & Z_{bc} \\ Z_{ac} & Z_{bc} & Z_{cc} \end{bmatrix} \begin{bmatrix} T_a \\ T_b \\ T_c \end{bmatrix}$$

volt/length unit

(11.10)

where

$$Z_{aa} = z_{aa} - 2z_{ad} + z_{dd} \ \Omega/\text{length unit}$$

$$Z_{bb} = z_{bb} - 2z_{bd} + z_{dd} \ \Omega/\text{length unit}$$

$$Z_{cc} = z_{cc} - 2z_{cd} + z_{dd} \ \Omega/\text{length unit}$$

$$Z_{ab} = Z_{ba} = z_{ab} - z_{ad} - z_{bd} + z_{dd} \ \Omega/\text{length unit}$$

$$Z_{bc} = Z_{cb} = z_{bc} - z_{bd} - z_{cd} + z_{dd} \ \Omega/\text{length unit}$$

$$Z_{ac} = Z_{ca} = z_{ac} - z_{ad} - z_{cd} + z_{dd} \ \Omega/\text{length unit}$$

(11.11)

The magnitude of the impedances can be calculated as the impedances in section 11.1, equation (11.1) and 11.6. It is important to note the coupling between the phases. A current flowing in one phase will influence the voltage drop in other phases. The replacing of a three-phase line with three parallel impedances, is an approximation which gives that all
11. Line model for un-symmetric three-phase conditions

The non-diagonal element of the $Z$-bus matrix in equation (11.10) are neglected. In other words, the mutual inductance between the conductors are neglected. The error this simplification gives is dependent on several things, e.g. the distance between the conductors, the length of the conductors and the magnitude of the currents in the conductors.

### 11.2.1 Symmetrical components of the series impedance of a three-phase line

Symmetrical components are often used in the analysis of power systems having three-phase lines, in order to simplify the complicated cross-couplings that exist between the phases. The quantities in equation (11.10) can be defined as:

$$
\begin{bmatrix}
U_a \\
U_b \\
U_c
\end{bmatrix} = U_f = \begin{bmatrix}
Z_{aa} & Z_{ab} & Z_{ac} \\
Z_{ab} & Z_{bb} & Z_{bc} \\
Z_{ac} & Z_{bc} & Z_{cc}
\end{bmatrix} \begin{bmatrix}
I_a \\
I_b \\
I_c
\end{bmatrix}
$$

(11.12)

The voltage phase vectors ($U_f$) and current vectors ($I_f$) can be replaced by the corresponding symmetrical component multiplied with matrix $T$ according to the section on symmetrical components:

$$
U_f = TU_s = Z_f TI_s = Z_f I_f
$$

(11.13)

This equation can be rewritten as

$$
U_s = T^{-1}Z_f TI_s = Z_s I_s
$$

(11.14)

If a symmetrical overhead line (or cable) is assumed, i.e. $Z_{aa} = Z_{bb} = Z_{cc}$ and $Z_{ab} = Z_{bc} = Z_{ac}$, the following is obtained:

$$
Z_s = T^{-1}Z_f T = \frac{1}{3} \begin{bmatrix}
1 & \alpha & \alpha^2 \\
1 & \alpha^2 & \alpha \\
1 & 1 & 1
\end{bmatrix} \begin{bmatrix}
Z_{aa} & Z_{ab} & Z_{ac} \\
Z_{ab} & Z_{bb} & Z_{bc} \\
Z_{ac} & Z_{bc} & Z_{cc}
\end{bmatrix} \begin{bmatrix}
1 & 1 & 1 \\
1 & \alpha & 1 \\
\alpha^2 & \alpha & 1
\end{bmatrix} =
\begin{bmatrix}
Z_{aa} - Z_{ab} & 0 & 0 \\
0 & Z_{aa} - Z_{ab} & 0 \\
0 & 0 & Z_{aa} + 2Z_{ab}
\end{bmatrix}
$$

(11.15)

Equation (11.14) can be rewritten as

$$
\begin{bmatrix}
\bar{U}_1 \\
\bar{U}_2 \\
\bar{U}_0
\end{bmatrix} = U_s = Z_s I_s
$$

(11.16)

$$
\begin{bmatrix}
\bar{Z}_{aa} - \bar{Z}_{ab} & 0 & 0 \\
0 & \bar{Z}_{aa} - \bar{Z}_{ab} & 0 \\
0 & 0 & \bar{Z}_{aa} + 2\bar{Z}_{ab}
\end{bmatrix} \begin{bmatrix}
\bar{I}_1 \\
\bar{I}_2 \\
\bar{I}_0
\end{bmatrix} = \begin{bmatrix}
\bar{Z}_1 \bar{I}_1 \\
\bar{Z}_2 \bar{I}_2 \\
\bar{Z}_0 \bar{I}_0
\end{bmatrix}
$$

where

$$
\bar{Z}_1 = \bar{Z}_{aa} - \bar{Z}_{ab} = \text{positive-sequence impedance}
$$

$$
\bar{Z}_2 = \bar{Z}_{aa} - \bar{Z}_{ab} = \text{negative-sequence impedance}
$$

(11.17)

$$
\bar{Z}_0 = \bar{Z}_{aa} + 2\bar{Z}_{ab} = \text{zero-sequence impedance}$$
By inserting the expressions used in equation (11.11) into equation (11.17), the following can be obtained

\[ Z_1 = Z_2 = z_{aa} - z_{ab} \]

\[ Z_0 = z_{aa} + 2z_{ab} - 6z_{ad} + 3z_{dd} \]  

(11.18)

Note that the coupling to ground are not present in the expressions for the positive- and negative-sequence impedances, i.e. the elements having index d in the Z-bus matrix in equation (11.8) are not included. This means that the zero-sequence current is zero in the positive- and negative-sequence reference frame, which is quite logical. All couplings to ground are represented in the zero-sequence impedance. As indicated above, a line by using this model, can be represented as three non-coupled components: positive-, negative-, and zero-sequence components. It should be pointed out that some loss of information will occur when using this model. For example, if only positive-, negative-, and zero-sequence data are given, the potential of the ground, \( U_{d'} \) in Figure 11.2, cannot be calculated. To calculate that potential, more detailed data are needed. The line model introduced in section 7.2, is based on positive-sequence data only, since symmetrical conditions are assumed.

### 11.2.2 Equivalent diagram of the series impedance of a line

As given above, for a symmetrical line \( Z_1 = Z_2 \). Assume that this line can be replaced by an equivalent circuit according to Figure 11.3, i.e. three phase impedances \( Z_a \) and one return impedance \( Z_\beta \) where the mutual inductance between the phases = 0. With three phases and one return path, as given by the equivalent in Figure 11.3, the following is valid

\[ I_0 = I_a + I_b + I_c \]  

(11.19)

By using equation (11.19), the voltage drop between the phases and the return conductor can be calculated as

\[ U'_a - U'_0 = U_a - U_0 - I_a \cdot Z_a - (I_a + I_b + I_c)Z_\beta \]

\[ U'_b - U'_0 = U_b - U_0 - I_b \cdot Z_a - (I_a + I_b + I_c)Z_\beta \]  

(11.20)

\[ U'_c - U'_0 = U_c - U_0 - I_c \cdot Z_a - (I_a + I_b + I_c)Z_\beta \]
which can be rewritten to matrix form

\[
\begin{bmatrix}
U'_a - U'_0 \\
U'_b - U'_0 \\
U'_c - U'_0
\end{bmatrix} = \begin{bmatrix}
U_a - U_0 \\
U_b - U_0 \\
U_c - U_0
\end{bmatrix} = \begin{bmatrix}
Z_\alpha + Z_\beta & Z_\beta & Z_\beta \\
Z_\beta & Z_\alpha + Z_\beta & Z_\beta \\
Z_\beta & Z_\beta & Z_\alpha + Z_\beta
\end{bmatrix} \begin{bmatrix}
I_a \\
I_b \\
I_c
\end{bmatrix}
\]

(11.21)

or

\[
U'_f = U_f - Z_{\alpha\beta}I_f
\]

(11.22)

Since the matrix \( Z_{\alpha\beta} \) is both symmetric and cyclo-symmetric, it can represent a line according to the assumption made. The matrix \( Z_{\alpha\beta} \) can be converted to symmetrical components by using equation (11.15):

\[
Z_{s\alpha\beta} = T^{-1}Z_{\alpha\beta}T = \begin{bmatrix}
Z_\alpha & 0 & 0 \\
0 & Z_\alpha & 0 \\
0 & 0 & Z_\alpha + 3Z_\beta
\end{bmatrix}
\]

(11.23)

When the symmetrical components \( Z_1 = Z_2 \) and \( Z_0 \) for the line are known, the following is obtained

\[
\begin{align*}
Z_\alpha &= \frac{Z_1}{3} \\
Z_\beta &= \frac{Z_0 - Z_1}{3}
\end{align*}
\]

(11.24)

With these values of \( Z_\alpha \) and \( Z_\beta \), the equivalent in Figure 11.3 can be used, together with equation (11.21), to calculate the voltage drop between the phases and the return conductor \((= U_f - U'_f)\) as a function of the phase currents \((= I_f)\).

Note that the equivalent cannot be used to calculate e.g. \( U'_0 - U_0 \) or \( U'_a - U_a \) but only e.g. \((U'_a - U'_0) - (U_a - U_0)\).

Example 11.1 Solve example 3.5 by using symmetrical components.

![Network diagram for example 11.1](image)

Figure 11.4. Network diagram for example 11.1

Solution

According to the solutions in example 3.5, the impedances of interest are \( Z_L = 2.3 + j0.16 \) \( \Omega \), \( Z_{L0} = 2.3 + j0.03 \) \( \Omega \), \( Z_a = 47.9 + j4.81 \) \( \Omega \), \( Z_b = 15.97 + j1.60 \) \( \Omega \), \( Z_c = 23.96 + j2.40 \) \( \Omega \).
The symmetrical components of the line will first be calculated. Note that the line in the example is given in the same way as the equivalent. The symmetrical components can be calculated by using equation (11.23):

\[
Z_1 = Z_2 = Z_L = 2.3 + j0.16 \, \Omega \\
Z_0 = Z_L + 3Z_{L0} = 9.2 + j0.25 \, \Omega
\]

(11.25)

which gives that

\[
Z_s = \begin{bmatrix}
Z_1 & 0 & 0 \\
0 & Z_2 & 0 \\
0 & 0 & Z_0
\end{bmatrix} = \begin{bmatrix}
2.3 + j0.16 & 0 & 0 \\
0 & 2.3 + j0.16 & 0 \\
0 & 0 & 9.2 + j0.25
\end{bmatrix}
\]

(11.26)

The symmetrical components for the load can be calculated by using equation (11.15)

\[
Z_{LDS} = \frac{1}{3} \begin{bmatrix}
1 & \alpha & \alpha^2 \\
1 & \alpha^2 & \alpha \\
\alpha & 1 & 1
\end{bmatrix} \begin{bmatrix}
Z_a & 0 & 0 \\
0 & Z_b & 0 \\
0 & 0 & Z_c
\end{bmatrix} \begin{bmatrix}
1 & \alpha & \alpha^2 \\
\alpha & 1 & 1 \\
\alpha & \alpha^2 & 1
\end{bmatrix} = \frac{1}{3} \begin{bmatrix}
29.28 + j2.94 & 9.09 + j3.24 & 9.55 - j1.37 \\
9.55 - j1.37 & 29.28 + j2.94 & 9.09 + j3.24 \\
9.09 + j3.24 & 9.55 - j1.37 & 29.28 + j2.94
\end{bmatrix} \Omega
\]

(11.27)

The applied voltage is symmetric, i.e. it has only one sequence, the positive one:

\[
U_S = T^{-1}U_f = \begin{bmatrix}
220^\circ & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix} \, V
\]

(11.28)

The equation for this un-symmetric three-phase network can be described as

\[
U_S = (Z_s + Z_{LDS})I_S
\]

(11.29)

which can be rewritten as

\[
I_S = (Z_s + Z_{LDS})^{-1}U_S = \begin{bmatrix}
8.11^\circ - 5.51^\circ \\
2.22^\circ 149.09^\circ \\
1.75^\circ - 155.89^\circ
\end{bmatrix} \, A
\]

(11.30)

The symmetrical components for the voltage at the load can be calculated as

\[
U_{LDS} = Z_{LDS}I_S = \begin{bmatrix}
201.32^\circ - 0.14^\circ \\
5.13^\circ - 26.93^\circ \\
16.10^\circ 25.67^\circ
\end{bmatrix} \, V
\]

(11.31)

The power obtained in the radiators can be calculated by using equation (10.24)

\[
\overline{S} = 3U_{LDS}^T I_S^* = 4754 + j477 \, VA
\]

(11.32)

i.e. the thermal power is 4754 W.
As given above, only the voltage drop at the load and the load currents can be calculated by using the symmetrical components. The ground potential at the load cannot be calculated, but that is usually of no interest.

Previously, in example 3.5, 5.1 and 5.2, the ground potential at the load has been calculated by using other types of circuit analyses. It should be pointed out that the value of the ground potential has no physical interpretation if the value of $Z_L$ and $Z_{L0}$ has been obtained by using the symmetrical components of the line according to equation (11.24). As given by the solutions, the load demand, phase voltages at the load and the currents at the load are physically correct by using either one of the four methods of solution.

### 11.3 Shunt capacitance of a three-phase line

The line resistance and inductance are components that together form the *series* impedance of the line. The capacitance that is of interest in this section, forms the *shunt* component.

The series component, usually the inductance, gives a limit on the maximum amount of the current that can be transmitted over the line, and by that also the maximum power limit. The capacitive *shunt* component behaves as a reactive power source. The reactive power generated, is proportional to the voltage squared, which implies that the importance of the shunt capacitance increases with the voltage level. For lines having a nominal voltage of 300–500 kV and a length of more than 200 km, these capacitances are of great importance. In high voltage *cables* where the conductors are more close to one another, the capacitance is up to 20–40 times larger than for overhead lines. The reactive power generation can be a problem in cables having a length of only 10 km.

There is a fundamental law about electric fields saying that the electric potential $v$ at a certain point on the distance $r$ from a point charge $q$, can be calculated as:

$$v = \frac{q}{4\pi\epsilon_0 r} \text{ V}$$  \hspace{1cm} (11.33)

where $\epsilon_0 = 8.854 \times 10^{-12}$ F/m, permittivity of vacuum. This law gives that there is a direct relationship between the difference in potential and accumulation of charges. If two long, parallel conductors are of interest, and if there is a voltage difference, $v_1 - v_2$, between the lines, an accumulation of charges with different sign, $+Q$ and $-Q$, will take place. The magnitude of the total charge $Q$ depends mainly on the distance between the lines but also on the design of the lines. For cables, the material between the conductors will also have an influence on the charge accumulation. The capacitance between the two conductors is equal to the quotient between the charge $Q$ and the difference in potential:

$$C \equiv \frac{Q}{v_1 - v_2}$$  \hspace{1cm} (11.34)

For a three-phase line, the corresponding capacitance is located between all conductors. When having a difference in potential between a conductor and ground, an accumulation of charges will also occur in relation to the magnitude of the capacitance. In Figure 11.5, the different capacitances of a three-phase overhead line are given. A line is normally constructed
11.3. Shunt capacitance of a three-phase line

In a symmetrical way, i.e. the mean distance between the phases are equal. Also, the mean distance between a phase and ground is the same for all phases. In Figure 11.5, this corresponds to the case that \( c_{ab} = c_{bc} = c_{ac} \) and \( c_{ag} = c_{bg} = c_{cg} \) when the entire line is of interest.

In the same way as given earlier for the series impedances, the positive-, negative- and zero-sequence capacitances can be calculated. Only the results from the calculations will be presented here.

\[
\begin{align*}
C_1 &= C_2 = \frac{2\pi \varepsilon_0}{\ln \left( \frac{2H_a}{Ar} \right)} \text{ F/m} \quad (11.35) \\
C_0 &= \frac{2\pi \varepsilon_0}{\ln \left( \frac{2H A^2}{ra^2} \right)} \text{ F/m} \quad (11.36)
\end{align*}
\]

where, according to Figure 11.6

- \( C_1 \) = positive-sequence capacitance
- \( C_2 \) = negative-sequence capacitance
- \( C_0 \) = zero-sequence capacitance
- \( H = \sqrt[3]{H_1 H_2 H_3} \)
- \( A = \sqrt[3]{A_1 A_2 A_3} \)
- \( a = \sqrt[3]{a_{12} a_{13} a_{23}} \)
- \( r = \) the equivalent radius of the line = \( e^{-1/4} \cdot \) real radius of the line

Note that \( C_1 \) is equal to \( C_2 \), but \( C_0 \) has a different value. When having a closer look at the equations for \( C_1 \), it can be seen that \( 2H/A \approx 1 \) according to Figure 11.6, which means that the distance to ground has a relatively small influence. If the conductors are located close to one another, then \( 2H = A \). The line model described in section 7.2, uses only the positive-sequence capacitance \( C_1 \) for the line. In principle, this can be regarded as a \( \Delta-Y \)-transformation of the capacitances between the phases since they are the main contributors to the positive- and negative-sequence capacitances. The coupling to ground is of less importance. In cables, the positive- and negative-sequence capacitances are usually higher owing to the short distance between the phases.
For $C_0$, the coupling to ground is very important. When calculating $C_0$, all phases have the same potential by the definition of zero-sequence. This implies that the capacitances between the phases $c_{ab}, c_{bc}, c_{ac}$ are not of interest. However, the electric field is changed since all three conductors have the same potential. As given in the equation, the distance to ground is very important (power of three inside the ln-sign) in the calculations of the zero-sequence capacitance.
Chapter 12

Transformer model in un-symmetric three-phase conditions

In the analysis of un-symmetric conditions in three-phase circuits, the transformer is represented by its positive-, negative- and zero-sequence impedances. These can be determined by analyzing the three-phase transformer, e.g. the Y-Δ-coupling shown in Figure 12.1. Three single-phase transformers are used in this coupling and the zero connection point is grounded via an impedance $Z_n$. The impedance $Z_e$ represents the equivalent impedance of each single-phase transformer and consists of both leakage reactance of the primary and secondary winding as well as the resistance of the windings. The magnetizing current of the transformer can be neglected, i.e. the magnetizing impedance is assumed to be infinitely large.

By using the direction of currents as given in Figure 12.1, the following expressions for the three phases of the transformer can be held

\[
\begin{align*}
\Delta U_a &= I_a Z_e + I_n Z_n \\
\Delta U_b &= I_b Z_e + I_n Z_n \\
\Delta U_c &= I_c Z_e + I_n Z_n
\end{align*}
\]

(12.1)

Figure 12.1. Y-Δ-coupled transformer to determine the positive-, negative- and zero-sequence impedances
Since $I_n = I_a + I_b + I_c$, this can be rewritten as

\[
\begin{align*}
\Delta U_a &= I_a (Z_e + Z_n) + I_b Z_n + I_c Z_n \\
\Delta U_b &= I_a Z_n + I_b (Z_e + Z_n) + I_c Z_n \\
\Delta U_c &= I_a Z_n + I_b Z_n + I_c (Z_e + Z_n)
\end{align*}
\] (12.2)

which can be written on matrix form

\[
\Delta U = \begin{bmatrix}
\Delta U_a \\
\Delta U_b \\
\Delta U_c
\end{bmatrix} = \begin{bmatrix}
Z_e + Z_n & Z_n & Z_n \\
Z_n & Z_e + Z_n & Z_n \\
Z_n & Z_n & Z_e + Z_n
\end{bmatrix} \begin{bmatrix}
I_a \\
I_b \\
I_c
\end{bmatrix} = Z_t I_t
\] (12.3)

By diagonalizing the matrix $Z_t$, the symmetrical components of the transformer can be obtained as the eigenvalues of $Z_t$

\[
Z_{ts} = T^{-1}Z_tT = \begin{bmatrix}
Z_e & 0 & 0 \\
0 & Z_e & 0 \\
0 & 0 & Z_e + 3Z_n
\end{bmatrix}
\] (12.4)

i.e.

\[
\begin{align*}
Z_1 &= Z_e = \text{positive-sequence impedance} \\
Z_2 &= Z_e = \text{negative-sequence impedance} \\
Z_0 &= Z_e + 3Z_n = \text{zero-sequence impedance}
\end{align*}
\] (12.5)

As given above, the positive- and negative-sequence impedances are the same and equal to the leakage reactance of each phase. That $Z_1 = Z_2$ is not surprising since the transformer impedance does not change if the phase ordering is changed from $abc$ (positive-sequence) to $acb$ (negative-sequence).

The zero-sequence impedance includes the leakage reactance but a factor of $3Z_n$ is added where $Z_n$ is the impedance connected between the transformer zero connection point and the reference ground point. If $Z_n = 0$, the zero-sequence impedance will be equal to the leakage reactance of the transformer.

It should be pointed out that this analysis of the zero-sequence impedance is dependent on the Y-Δ-coupling according to Figure 12.1. There must be a connection from the transformer zero connection point to ground to obtain a zero-sequence current. On the secondary side, the Δ-coupling side in Figure 12.1, there is no such connection, i.e. seen from the secondary side, $Z_0 = \infty$ and a zero-sequence current cannot flow. In Figure 12.2, the zero-sequence impedance diagram for some different transformer couplings is shown. Whereas the positive- and negative-sequence impedances of the transformer are independent on from which side of the transformer the analysis is performed, the zero-sequence impedance can vary with a large amount. In Figure 12.2a, a Y0-Y0-coupling is shown, which gives that a zero-sequence current can flow through the transformer. This gives that the zero-sequence impedance is equal to the leakage reactance as discussed above.

In Figure 12.2b a Y0-Δ-coupling is given which only allows for a zero-sequence current on the Y0-side since a return conductor exist and mmf-balance can be obtained owing to the Δ-winding. On the Δ-side, no zero-sequence current can flow.
In Figure 12.2c–e, no zero-sequence current can flow on either side due to the connection types.
12. Transformer model in un-symmetric three-phase conditions
Chapter 13
Analysis of un-symmetric three-phase systems

As discussed in chapters 11 and 12, lines and transformers can be represented by their positive-, negative- and zero-sequence impedances. These are un-coupled which, e.g. gives that a certain zero-sequence current will only cause a zero-sequence voltage drop whereas positive- and negative-sequence voltages are unchanged. Also three-phase generators can in an equivalent way be described by un-coupled positive-, negative- and zero-sequence systems.

The consequences of this, is that the whole system of generators, lines and transformers, can be represented by three decoupled systems.

13.1 Load modeling in the analysis of un-symmetric conditions

It is assumed that a load can be modeled as an impedance. A three-phase load is normally Y- or ∆-connected according to Figure 13.1. As given in Figure 13.1a, a Y-connected load may have an impedance between the zero connection point and the grounding point. For a Y-connected load, the following equation apply for the phase components :

\[
\begin{pmatrix}
U_f \\
U_f \\
U_f \\
\end{pmatrix} =
\begin{bmatrix}
Z_a + Z_n & Z_n & Z_n \\
Z_n & Z_b + Z_n & Z_n \\
Z_n & Z_n & Z_c + Z_n \\
\end{bmatrix}
\begin{pmatrix}
I_a \\
I_b \\
I_c \\
\end{pmatrix} = Z_f I_f
\]

This equation can be transformed to symmetrical components :

\[
U_f = T U_s = Z_f I_f = Z_f T I_s
\]

⇒

\[
U_s = T^{-1} Z_f T I_s = Z_s I_s
\]
where

\[
Z_s \equiv T^{-1}Z_t T = \begin{bmatrix}
\frac{Z_a + Z_b + Z_c}{3} & \frac{Z_a + \alpha^2 Z_b + \alpha Z_c}{3} & \frac{Z_a + \alpha Z_b + \alpha^2 Z_c}{3} \\
\frac{Z_a + \alpha Z_b + \alpha^2 Z_c}{3} & \frac{Z_a + Z_b + Z_c}{3} & \frac{Z_a + \alpha^2 Z_b + \alpha Z_c}{3} \\
\frac{Z_a + \alpha^2 Z_b + \alpha Z_c}{3} & \frac{Z_a + \alpha Z_b + \alpha^2 Z_c}{3} & \frac{Z_a + Z_b + Z_c + 9Z_n}{9}
\end{bmatrix}
\] (13.3)

As indicated in this matrix, there are non-diagonal elements that are \( \neq 0 \), i.e. there exists couplings between the positive-, negative- and zero-sequences. A special case is when \( Z_a = Z_b = Z_c \). In this special case, \( Z_s \) can be written as

\[
Z_s = \begin{bmatrix}
Z_a & 0 & 0 \\
0 & Z_a & 0 \\
0 & 0 & Z_a + 3Z_n
\end{bmatrix}
\] (13.4)

For a \( Y \)-connected symmetric load with an impedance between zero connection and grounding point, the three symmetrical components are un-coupled. If there is no connection between the zero connection and grounding point (\( Z_n = \infty \)), the zero-sequence impedance is = \( \infty \), i.e. no zero-sequence current can flow.

For a \( \Delta \)-connected load given in Figure 13.1b, the impedance can first be \( \Delta \)-\( Y \) transformed which results in a \( Y \)-connection without impedance to ground :

\[
\begin{align*}
\overline{Z}_a &= \frac{\overline{Z}_{ab}\overline{Z}_{ac}}{\overline{Z}_{ab} + \overline{Z}_{ac} + \overline{Z}_{bc}} \\
\overline{Z}_b &= \frac{\overline{Z}_{ab}\overline{Z}_{bc}}{\overline{Z}_{ab} + \overline{Z}_{ac} + \overline{Z}_{bc}} \\
\overline{Z}_c &= \frac{\overline{Z}_{ac}\overline{Z}_{bc}}{\overline{Z}_{ab} + \overline{Z}_{ac} + \overline{Z}_{bc}} \\
\overline{Z}_n &= \infty
\end{align*}
\] (13.5) 13.6) 13.7)

For a symmetric \( \Delta \)-connected load, i.e. \( \overline{Z}_{ab} = \overline{Z}_{bc} = \overline{Z}_{ac} \), the symmetrical components can be calculated by using equations (13.4) to 13.8 :

\[
\begin{align*}
\overline{Z}_1 &= \overline{Z}_{ab}/3 \\
\overline{Z}_2 &= \overline{Z}_{ab}/3 \\
\overline{Z}_0 &= \infty
\end{align*}
\] (13.9) 13.10) 13.11)

### 13.2 Connection to a system in un-symmetric conditions

In subsection 7.2.2, the connection to a network in symmetrical conditions was discussed. It was given that a system studied at one bus can be modeled by using the Thévenin-equivalent, i.e. a voltage source behind an impedance. The value of the impedance could be calculated when knowing the three-phase short circuit current at the bus.
13.3. Single-phase short circuit to ground

Assume that a system having components that can be modeled as un-coupled symmetrical components is of interest. That is the case if all system components are symmetrical. This implies that power system can be described with three systems: a positive-, a negative and a zero-sequence system. The model of the positive-sequence system is the one valid in symmetrical conditions as described in subsection 7.2.2.

According to Thévenin’s theorem, a linear network analyzed at one bus be replaced with a voltage source behind an impedance. There are no voltage sources in the network for the negative- and zero-sequence systems. This gives that at a bus in the network, the negative- and zero-sequence systems consists only of impedances, as given in Figure 13.2. The conditions in Figure 13.2, can be given as

\[
\begin{align*}
\bar{U}_1 &= \bar{U}_T - Z_{T1} \bar{I}_1 \\
\bar{U}_2 &= 0 - Z_{T2} \bar{I}_2 \\
\bar{U}_0 &= 0 - Z_{T0} \bar{I}_0 
\end{align*}
\]  \hspace{1cm} (13.12)

**Figure 13.2.** Thévenin-equivalent at a bus in the network

\[
\begin{align*}
\bar{U}_1 &= \bar{U}_T - Z_{T1} \bar{I}_1 \\
\bar{U}_2 &= 0 - Z_{T2} \bar{I}_2 \\
\bar{U}_0 &= 0 - Z_{T0} \bar{I}_0 
\end{align*}
\]

13.3 Single-phase short circuit to ground

Assume that a single-phase (phase \(a\)) short circuit with an impedance of \(Z_f\), occur at a bus in the network, as given in Figure 13.3. According to Figure 13.3, \(\bar{I}_b = \bar{I}_c = 0\) which

\[
\begin{align*}
\bar{U}_a &= - Z_f \bar{I}_a \\
\bar{I}_b &= \\
\bar{I}_c &= 
\end{align*}
\]

**Figure 13.3.** Single-phase short circuit in phase \(a\)
13. Analysis of un-symmetric three-phase systems

together with equation (10.11) gives that

\[
I_s = \begin{bmatrix}
I_1 \\
I_2 \\
I_0
\end{bmatrix} = \frac{1}{3} \begin{bmatrix}
1 & 0 & \alpha^2 \\
1 & \alpha & 0 \\
1 & 1 & 1
\end{bmatrix} \begin{bmatrix}
I_a \\
I_a \\
I_a
\end{bmatrix} = \frac{1}{3} \begin{bmatrix}
I_a \\
I_a \\
I_a
\end{bmatrix}
\]

\[\Rightarrow I_1 = I_2 = I_0 = \frac{1}{3} I_a \quad (13.13)\]

By using the notation in Figure 13.3, \(U_a = Z_f I_a\). By using the first row in equation (10.14) as well as equation (13.13), the following is obtained

\[
U_a = U_1 + U_2 + U_0 = Z_f I_a = 3Z_f I_1 \quad (13.14)
\]

By combining equations (13.12), (13.13) and (13.14) gives that

\[
U_T - Z_1 I_1 - Z_2 I_1 - Z_0 I_1 = 3Z_f I_1
\]

\[
I_a = 3I_1 = \frac{3U_T}{Z_1 + Z_2 + Z_0 + 3Z_f} \quad (13.15)
\]

This equation presuppose that the per-unit system is used. If the equivalent diagram of a line, Figure 11.3, is used, and if \(U'_a\) in that figure is connected to \(U'_0\) over an impedance \(Z_f\), and in combination with equation (11.24) the current \(I_a\) can be obtained

\[
I_a = \frac{U_a - U_0}{Z_a + Z_\beta + Z_f} = \frac{U_a - U_0}{Z_1 + \frac{2a-\frac{Z_1}{3}}{Z_0 + 3Z_f}} = \frac{3(U_a - U_0)}{2Z_1 + Z_0 + 3Z_f} \quad (13.16)
\]

Which corresponds to equation (13.15). To calculate the current at a single-phase fault, the equivalent in Figure 11.3 can be used if \(Z_1 = Z_2\).

Example 13.1 At a 400 kV bus, a solid three-phase short circuit occur, giving a fault current of 20 kA per phase. If a single-phase solid short circuit occur at the same bus, the fault current will be 15 kA in the faulted phase. The Thévenin-impedances in the positive- and negative-sequence systems at the bus can be assumed to be purely reactive and equal. (This is normal for high voltage systems since the dominating impedances origin from lines and transformers which have dominating reactive characteristics, equal for positive- and negative-sequences). Also the zero-sequence impedance can be assumed to be purely reactive. Calculate the Thévenin-equivalents for the positive-, negative- and zero-sequences at the fault.

Solution

Solid short circuits means that \(Z_f = 0\). Since all impedances are purely inductive, the fault currents will also be inductive, i.e.

\[
I_{3k} = -j20 \text{ kA} \quad I_{1k} = -j15 \text{ kA} \quad (13.17)
\]

Three-phase fault:

According to equation (7.22), (observe that this equation is given in the per-unit system),
the positive-sequence impedance (= negative-sequence impedance in this example) can be calculated as

$$Z_1 = Z_2 = \frac{U_T}{\sqrt{3}I_{sk}} = \frac{400}{\sqrt{3} \cdot 20} = 11.55 \ \Omega$$

$$\Rightarrow Z_1 = Z_2 = j11.55 \ \Omega$$ \hspace{1cm} (13.18)

Single-phase fault:

From equation (13.15), (converted to nominal quantities), the zero-sequence impedance can be calculated as

$$Z_0 = \frac{3U_T}{\sqrt{3}I_{sk}} - Z_1 - Z_2 - 3Z_f = \frac{3 \cdot 400}{\sqrt{3} \cdot (-j15)} - 2 \cdot (j11.55) = j23.09 \ \Omega$$ \hspace{1cm} (13.19)

13.4 Analysis of a network with one un-symmetrical load

As given in section 13.1, the symmetrical components are un-coupled when having symmetrical loads. However, in the case having un-symmetrical loads, these components will be coupled. Assume that a network consists of lines, transformers, a connection to a system as well as loads of which one is un-symmetrical. This system can be analyzed as follows:

1. Build an impedance diagram each for the positive-, negative and zero-sequence systems, for the entire network except for the un-symmetrical load.
2. Calculate the Thévenin-equivalent for the positive-, negative- and zero-sequences at the bus where the un-symmetrical load is located.
3. Calculate the positive-, negative- and zero-sequence currents through the load.
4. Calculate the positive-, negative- and zero-sequence voltages at the locations that are of interest.
5. Calculate the positive-, negative- and zero-sequence currents through components that are of interest.
6. Transform those symmetrical components to phase-quantities that are asked for.

The items above can be treated in different ways which will be shown in the following example.

**Example 13.2** A small industry is fed by a transformer (5 MVA, 70/10 kV, \(x = 4\) \%, \(\Delta-Y0\)-connected with \(Y0\) on the 10 kV-side) located at a distance of 5 km. The electric power demand of the industry is 400 kW at \(\cos \phi = 0.8\), lagging, at a voltage of 10 kV. The industry can be modeled as an impedance load. The 10 kV-line has a series impedance of
Analysis of un-symmetric three-phase systems

$Z_{L1} = 0.9 + j0.3 \ \Omega/\text{phase, km}$ and a shunt admittance of $Y_{L1} = j 3 \times 10^{-6} \ \text{S/phase, km}$. The zero-sequence data of the line is $Z_{L0} = 3Z_{L1}$ and $Y_{L0} = 0.5 Y_{L1}$. The industry load can be assumed to be $Y$-connected with the zero connection point connected to ground. At the point in time of interest, half of the normal load connected to phase $a$ is disconnected while the other phases are loaded as normal. Use the $\Pi$-equivalent in the line modeling and calculate the voltage level at the industry as well as the power fed into the line at the transformer. Before the industry was connected to the transformer, a short circuit current of 0.3 kA on the 70 kV-side when a solid three-phase short circuit was applied at nominal voltage. If a single-phase short circuit was applied in the same way, a current of 0.2 kA was obtained. The three Thévenin-impedances of the system can be assumed to be reactive and $Z_{Th1} = Z_{Th2}$.

(Same example as 7.2 but with half load in phase $a$ at the industry, i.e. un-symmetrical).

Solution

1: Start with the building of the impedance diagram of the positive-, negative- and zero-sequence for the whole system except for the un-symmetrical load, see Figure 13.4. Chose

![Diagram](image_url)

**Figure 13.4.** System in example 13.2

base-quantities (MVA, kV, $\Rightarrow$ kA, $\Omega$) : $S_b = 500 \ \text{kVA} = 0.5 \ \text{MVA}$, $U_{b10} = 10 \ \text{kV} \Rightarrow I_{b10} =
\[ S_b / \sqrt{3} U_{b10} = 0.0289 \text{ kA}, \quad Z_{b10} = U_{b10}^2 / S_b = 200 \Omega, \quad U_{b70} = 70 \text{ kV} \Rightarrow I_{b70} = S_b / \sqrt{3} U_{b70} = 0.0041 \text{ kA}, \quad Z_{b70} = U_{b70}^2 / S_b = 9800 \Omega \]

Calculate the per-unit values of the Thévenin-equivalents of the system:

1. **Solid short circuit:** Replace the networks with Thévenin-equivalents (\( U_{\text{Th}} \)).
2. **Single-phase fault:** Equation (13.15)

\[ U_{\text{Th}0} = U_{\text{Th}} / U_{b70} = 1.0 \angle 0^\circ \]

Calculate the per-unit values of the Thévenin-equivalents of the system:

\[ Z_{\text{Thpu}} = Z_{\text{Thpu1}} = Z_{\text{Thpu2}} = (U_{\text{Th}} / U_{b70}) \cdot (I_{b70} / I_{3k}) = (70 / 70) \cdot (0.00412 / 0.3) \Rightarrow Z_{\text{Thpu1}} = Z_{\text{Thpu2}} = j0.0137 \]

Next step is to replace the networks with Thévenin-equivalents (\( U_{\text{Th}C} \)), see Figure 13.2) at the industry connection point.

The twoport of the whole positive-sequence network between the “infinite bus” (\( U_{\text{Thpu}} \)) and the industry, (network + transformer + line) can be formulated as

\[
\begin{bmatrix}
U_{\text{Thpu}} \\
T_{A1}
\end{bmatrix} = \begin{bmatrix}
1 & Z_{\text{Thpu1}} + Z_{\text{trapu1}} \\
0 & 1
\end{bmatrix} \begin{bmatrix}
\overline{A}_L1 \\
\overline{C}_L1
\end{bmatrix} \begin{bmatrix}
\overline{B}_L1 \\
\overline{D}_L1
\end{bmatrix} \begin{bmatrix}
U_{C1} \\
T_{C1}
\end{bmatrix} = \begin{bmatrix}
\overline{A}_1 \\
\overline{C}_1
\end{bmatrix} \begin{bmatrix}
\overline{B}_1 \\
\overline{D}_1
\end{bmatrix} \begin{bmatrix}
U_{C1} \\
T_{C1}
\end{bmatrix} = \begin{bmatrix}
0.9999 + j0.0000 & 0.0225 + j0.0252 \\
0.0000 + j0.0030 & 1.0000 + j0.0000
\end{bmatrix} \begin{bmatrix}
\overline{U}_{C1} \\
\overline{T}_{C1}
\end{bmatrix} (13.20)
\]

By comparing with Figure 13.2a \( U_{C1} = U_{\text{ThC}} \) when \( T_{C1} = 0 \):

\[ U_{\text{Thpu}} = \overline{A}_1 \overline{U}_{C1} + \overline{B}_1 \cdot 0 \]

\[ U_{\text{ThC}} = \overline{U}_{C1} = U_{\text{Thpu}} / \overline{A}_1 = 1.0001 \angle -0.0019^\circ (13.21) \]
The Thévenin-impedance $Z_{ThC1}$ is calculated as the quotient $-U_{C1}/I_{C1}$ when $U_{Thpu} = 0$:

$$0 = \frac{A_1U_{C1} + B_1I_{C1}}{A_1}$$

$$\Rightarrow Z_{ThC1} = -\frac{U_{C1}}{I_{C1}} = \frac{B_1}{A_1} = 0.0225 + j0.0252 \quad (13.22)$$

The Thévenin-impedance of the negative-sequence at the industry connection point is the same as for the positive-sequence since the only difference between the impedance networks is the voltage source $U_{Thpu}$:

$$Z_{ThC2} = Z_{ThC1} = 0.0225 + j0.0252$$

According to Figure 12.2c, the zero-sequence of a $\Delta$-Y0-transformer should be modeled as an impedance to ground on the Y0-side, as shown in Figure 13.4c. As seen in the figure, the feeding network is not connected to the industry load from a zero-sequence point of view, owing to this transformer. The twoport of the network from the transformer to the connection point of the industry (transformer + line) can be given as

$$\begin{bmatrix} U_{T0} \\ I_{T0} \end{bmatrix} = \begin{bmatrix} 0 \\ I_{T0} \end{bmatrix} = \begin{bmatrix} 1 & Z_{tr} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A_{L0} & B_{L0} \\ C_{L0} & D_{L0} \end{bmatrix} \begin{bmatrix} U_{C0} \\ I_{C0} \end{bmatrix} =$$

$$= \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} \begin{bmatrix} U_{C0} \\ I_{C0} \end{bmatrix} =$$

$$= \begin{bmatrix} 1.0000 + j0.0001 & 0.0675 + j0.0265 \\ 0.0000 + j0.0015 & 1.0000 + j0.0001 \end{bmatrix} \begin{bmatrix} U_{C0} \\ I_{C0} \end{bmatrix} \quad (13.23)$$

The Thévenin-impedance $Z_{ThC0}$ can be calculated as the quotient $-U_{C0}/I_{C0}$ since $U_{T0} = 0$:

$$0 = \frac{A_0U_{C0} + B_0I_{C0}}{A_0}$$

$$\Rightarrow Z_{ThC0} = -\frac{U_{C0}}{I_{C0}} = \frac{B_0}{A_0} = 0.0675 + j0.0265 \quad (13.24)$$

3: Calculate the per-unit values of the industry impedance in normal conditions:

$$Z_{indpu} = (\frac{U_{ind}^2}{S_{ind}^*})/Z_{b10} = (10^2/[0.4/0.8]) \cdot (0.8 + j0.6)/200 = 0.8 + j0.6 \quad (13.25)$$

At half load in phase $a$ ($Z_a = 2Z_{indpu}$), the impedance matrix of the symmetrical components can be calculated according to equation (13.3):

$$Z_{lds} = T^{-1}Z_T T = T^{-1} \begin{bmatrix} 2Z_{indpu} & 0 & 0 \\ 0 & Z_{indpu} & 0 \\ 0 & 0 & Z_{indpu} \end{bmatrix} T$$

$$= \begin{bmatrix} 1.0667 + j0.8000 & 0.2667 + j0.2000 & 0.2667 + j0.2000 \\ 0.2667 + j0.2000 & 1.0667 + j0.8000 & 0.2667 + j0.2000 \\ 0.2667 + j0.2000 & 0.2667 + j0.2000 & 1.0667 + j0.8000 \end{bmatrix} \quad (13.26)$$

The equation of the whole system can be formed as:

$$U_{Th} = \begin{bmatrix} U_{ThC} \\ 0 \\ 0 \end{bmatrix} = (Z_a + Z_{lds})I_s \quad (13.27)$$
13.4. Analysis of a network with one un-symmetrical load

where

\[
Z_u = \begin{bmatrix}
Z_{ThC1} & 0 & 0 \\
0 & Z_{ThC2} & 0 \\
0 & 0 & Z_{ThC0}
\end{bmatrix} \quad \text{and} \quad I_s = \begin{bmatrix}
I_{C1} \\
I_{C2} \\
I_{C0}
\end{bmatrix}
\]

The symmetrical components of the currents through the load can be calculated as:

\[
I_s = (Z_u + Z_{lds})^{-1} U_{Th} = \begin{bmatrix}
0.8084\angle 37.1616^\circ \\
0.1596\angle 142.3433^\circ \\
0.1541\angle 143.7484^\circ
\end{bmatrix}
\] (13.28)

4-5: The symmetrical components of the voltage level at the industry can be calculated as:

\[
U_C = \begin{bmatrix}
\bar{U}_{C1} \\
\bar{U}_{C2} \\
\bar{U}_{C0}
\end{bmatrix} = Z_{lds} I_s = \begin{bmatrix}
0.9733\angle -0.3126^\circ \\
0.0054\angle 10.6340^\circ \\
0.0112\angle -14.8199^\circ
\end{bmatrix}
\] (13.29)

The positive-sequence of the voltage and the current at the transformer connection to the line can be calculated as:

\[
\begin{bmatrix}
\bar{U}_{B1} \\
\bar{I}_{B1}
\end{bmatrix} = \begin{bmatrix}
\bar{A}_{L1} & \bar{B}_{L1} \\
\bar{C}_{L1} & \bar{D}_{L1}
\end{bmatrix} \begin{bmatrix}
\bar{U}_{C1} \\
\bar{I}_{C1}
\end{bmatrix} = \begin{bmatrix}
0.9915\angle -0.6607^\circ \\
0.8066\angle -36.9937^\circ
\end{bmatrix}
\] (13.30)

\[
\Rightarrow \bar{S}_{B1} = \bar{U}_{B1}\bar{I}_{B1}S_b = 0.3221 + j0.2369 \text{ MVA}
\] (13.31)

Note that the power \(\bar{S}_{B1}\) is the three-phase power.

The negative-sequence of the voltage and the current at the transformer connection to the line as well as the negative-sequence of the three-phase power can be calculated as:

\[
\begin{bmatrix}
\bar{U}_{B2} \\
\bar{I}_{B2}
\end{bmatrix} = \begin{bmatrix}
\bar{A}_{L2} & \bar{B}_{L2} \\
\bar{C}_{L2} & \bar{D}_{L2}
\end{bmatrix} \begin{bmatrix}
\bar{U}_{C2} \\
\bar{I}_{D2}
\end{bmatrix} = \begin{bmatrix}
0.0028\angle 52.3414^\circ \\
0.1596\angle 142.3414^\circ
\end{bmatrix}
\] (13.32)

\[
\Rightarrow \bar{S}_{B2} = \bar{U}_{B2}\bar{I}_{B2}S_b = 0 - j0.0002 \text{ MVA}
\] (13.33)

Finally, the zero-sequence of the voltage and the current at the transformer connection to the line as well as the zero-sequence of the three-phase power can be calculated as:

\[
\begin{bmatrix}
\bar{U}_{B0} \\
\bar{I}_{B0}
\end{bmatrix} = \begin{bmatrix}
\bar{A}_{L0} & \bar{B}_{L0} \\
\bar{C}_{L0} & \bar{D}_{L0}
\end{bmatrix} \begin{bmatrix}
\bar{U}_{C0} \\
\bar{I}_{D0}
\end{bmatrix} = \begin{bmatrix}
0.0004\angle 0.0005^\circ \\
0.1541\angle 143.7455^\circ
\end{bmatrix}
\] (13.34)

\[
\Rightarrow \bar{S}_{B0} = \bar{U}_{B0}\bar{I}_{B0}S_b = 0 - j0.0005 \text{ MVA}
\] (13.35)

6: Phase voltages must be multiplied with a base voltage that is actual base value of the phase-to-phase voltage/\(\sqrt{3}\) to achieve nominal values. The phase components of the voltage at the industry can be calculated as:

\[
\begin{bmatrix}
\bar{U}_{Ca} \\
\bar{U}_{Cb} \\
\bar{U}_{Cc}
\end{bmatrix} = TU_C \cdot \frac{U_{b10}}{\sqrt{3}} = \begin{bmatrix}
5.7122\angle -0.4154^\circ \\
5.5918\angle -119.9774^\circ \\
5.5535\angle 119.4555^\circ
\end{bmatrix}
\] (13.36)

According to section 10.2 equation (10.25), the total power delivered by the transformer can be calculated as the sum of the positive-, negative- and zero-sequence powers:

\[
\bar{S} = \bar{S}_{B1} + \bar{S}_{B2} + \bar{S}_{B0} = 0.3221 + j0.2366 \text{ MVA}
\] (13.37)
13. Analysis of un-symmetric three-phase systems

13.5 General method of analysis in systems having impedance loads of which one is un-symmetrical

In larger un-symmetrical systems, it is necessary to use a systematic approach to analyze system voltages and currents. In this section, all system components except one load, will be symmetrical. In the demonstration below, a small system is analyzed in the same way as can be performed for a large system. The following notation for variable index is used, $-1$ indicate positive-sequence, $-2$ indicate negative-sequence and $-0$ zero-sequence. For vectors, the corresponding index are 1, 2 and 0. The example given below is identical to the one in section 7.3 but with the difference that the load connected at bus 2 is un-symmetric. In the initial stage, the only difference compared with the symmetrical case is that a new notation is introduced. In Figure 13.5, an example of a simple symmetric system is given. Since symmetrical conditions apply, all components can be represented by the positive-sequence diagram only. In the system, a load $Z_{LD1}$ is fed by an infinite bus $U_3$ via a transformer $Z_T$ and a line $Z_L$. The Y-bus matrix of the positive-sequence of the system can be formed as

$$\begin{bmatrix} \bar{I}_{1-1} \\ \bar{I}_{2-1} \\ \bar{I}_{3-1} \end{bmatrix} = \mathbf{I}_1 = \mathbf{Y}_1 \mathbf{U}_1 = \begin{bmatrix} \frac{1}{Z_{LD1}} + \frac{1}{Z_{LD2}} & \frac{1}{Z_{L1}} & 0 \\ \frac{1}{Z_{L2}} & \frac{1}{Z_{L1}} + \frac{1}{Z_{T1}} & \frac{1}{Z_{T1}} \\ 0 & \frac{1}{Z_{T1}} & \frac{1}{Z_{T1}} \end{bmatrix} \begin{bmatrix} \bar{U}_{1-1} \\ \bar{U}_{2-1} \\ \bar{U}_{3-1} \end{bmatrix} \quad (13.38)$$

This Y-bus matrix can be inverted which results in the corresponding Z-bus matrix:

$$\mathbf{U}_1 = \mathbf{Y}_1^{-1} \mathbf{I}_1 = \mathbf{Z}_1 \mathbf{I}_1 \quad (13.39)$$

Since $\bar{I}_{1-1} = \bar{I}_{2-1} = 0$, the third row in equation (13.39) can be written as

$$\bar{U}_{3-1} = \mathbf{Z}_1(3,3) \cdot \bar{I}_{3-1}$$

$$\Rightarrow \quad \bar{I}_{3-1} = \bar{U}_{3-1}/\mathbf{Z}_1(3,3) \quad (13.40)$$

where $\mathbf{Z}_1(3,3)$ is an element in the Z-bus matrix. With this value of the current inserted into equation (13.39) all voltages are obtained.

$$\bar{U}_{1-1} = \mathbf{Z}_1(1,3) \cdot \bar{I}_{3-1} \quad (13.41)$$

$$\bar{U}_{2-1} = \mathbf{Z}_1(2,3) \cdot \bar{I}_{3-1} \quad (13.42)$$
13.5. General method of analysis in systems having impedance loads of which one is un-symmetrical

So far, all calculations are identical to those in section 7.3. Corresponding calculations can be performed for an arbitrary large system having impedance loads and one voltage source. Assume a system with a voltage source at bus $i$. The current at bus $i$ and the voltage at another bus $r$ can then be calculated as

$$I_{i-1} = \frac{U_{i-1}}{Z_1(i,i)} \quad (13.43)$$

$$U_{r-1} = Z_1(r,i) \cdot I_{i-1} \quad (13.44)$$

Assume that an un-symmetrical impedance load is added to the system at bus 2. The system voltages and currents will then be changed in the whole system. The new voltages can be obtained by using the theorem of superposition, i.e. as the sum of the voltages before the connection of the load and with the voltage change obtained by the load connection. This can be expressed by using symmetrical components as:

$$U'_1 = U_1 + U_{A1}$$

$$U'_2 = U_2 + U_{A2} \quad (13.45)$$

$$U'_0 = U_0 + U_{A0}$$

where

$U'_1 =$ vector of positive-sequence voltages at all buses after the connection of the un-symmetrical load

$U'_2 =$ vector of negative-sequence voltages at all buses after the connection of the un-symmetrical load

$U'_0 =$ vector of zero-sequence voltages at all buses after the connection of the un-symmetrical load

$U_1 =$ vector of positive-sequence voltages at all buses prior to the connection of the un-symmetrical load, same vector as above

$U_2 =$ vector of negative-sequence voltages at all buses prior to the connection of the un-symmetrical load. This vector is here $\equiv 0$ since the conditions are symmetric prior to the connection of the un-symmetrical load

$U_0 =$ vector of zero-sequence voltages at all buses prior to the connection of the un-symmetrical load. This vector is here $\equiv 0$ since the conditions are symmetric prior to the connection of the un-symmetrical load

$U_{A1} =$ vector of changes in the positive-sequence voltages at all buses caused by the connection of the un-symmetrical load

$U_{A2} =$ vector of changes in the negative-sequence voltages at all buses caused by the connection of the un-symmetrical load

$U_{A0} =$ vector of changes in the zero-sequence voltages caused by the connection of the un-symmetrical load
Equation (13.45) can be rewritten by expressing the voltage changes by a Z-bus matrix multiplied with the current changes injected at the buses.

\[
\begin{align*}
U'_1 &= U_1 + Z_{\Delta 1}I_{\Delta 1} \\
U'_2 &= 0 + Z_{\Delta 2}I_{\Delta 2} \\
U'_0 &= 0 + Z_{\Delta 0}I_{\Delta 0}
\end{align*}
\] (13.46)

where

\[
\begin{align*}
Z_{\Delta 1} &= \text{Z-bus matrix of the positive-sequence data with voltage sources shortened} \\
Z_{\Delta 2} &= \text{Z-bus matrix of the negative-sequence data} \\
Z_{\Delta 0} &= \text{Z-bus matrix of the zero-sequence data}
\end{align*}
\]

\[
\begin{align*}
I_{\Delta 1} &= \text{vector of the injected positive-sequence current changes at the buses, only } I_{\Delta 1}(2) \neq 0, \text{ since the load is connected at that bus} \\
I_{\Delta 2} &= \text{vector of the injected negative-sequence current changes at the buses, only } I_{\Delta 2}(2) \neq 0 \\
I_{\Delta 0} &= \text{vector of the injected zero-sequence current changes at the buses, only } I_{\Delta 0}(2) \neq 0
\end{align*}
\]

In Figure 13.6, the positive-, negative- and zero-sequence systems used in the calculations of the voltage changes when connecting an un-symmetrical load at any bus are given. The difference between the positive-sequence system in Figure 13.6 and the \(\Delta\)-system used in section 7.3 is that the load is now represented by the currents injected into the buses. The infinite bus is assumed to be directly connected to ground and the transformer is \(Y_oY_o\)-connected. The Y-bus matrices of the network given in Figure 13.6 can be formed as

\[
\begin{align*}
Y_{\Delta 1} &= \left[\begin{array}{ccc}
\frac{1}{Z_{L1}} & \frac{1}{Z_{L1}} & \frac{-1}{Z_{L1}} \\
\frac{-1}{Z_{L1}} & \frac{1}{Z_{L1}} & \frac{-1}{Z_{T1}} \\
\frac{1}{Z_{L1}} & \frac{-1}{Z_{L1}} & \frac{1}{Z_{T1}}
\end{array}\right] \\
Y_{\Delta 2} &= \left[\begin{array}{ccc}
\frac{1}{Z_{L2}} & \frac{1}{Z_{L2}} & \frac{-1}{Z_{L2}} \\
\frac{-1}{Z_{L2}} & \frac{1}{Z_{L2}} & \frac{-1}{Z_{T2}} \\
\frac{1}{Z_{L2}} & \frac{-1}{Z_{L2}} & \frac{1}{Z_{T2}}
\end{array}\right] \\
Y_{\Delta 0} &= \left[\begin{array}{ccc}
\frac{1}{Z_{L0}} & \frac{1}{Z_{L0}} & \frac{-1}{Z_{L0}} \\
\frac{-1}{Z_{L0}} & \frac{1}{Z_{L0}} & \frac{-1}{Z_{T0}} \\
\frac{1}{Z_{L0}} & \frac{-1}{Z_{L0}} & \frac{1}{Z_{T0}}
\end{array}\right]
\end{align*}
\] (13.47) (13.48) (13.49)

The matrix \(Y_{\Delta 1}\) is the matrix \(Y_1\) but the row and column that corresponds to the generator bus (= bus 1) are cancelled, compare section 7.3. When only having lines, transformers and symmetrical loads in the system, \(Y_{\Delta 2} = Y_{\Delta 1}\) since all components have positive-sequence data that are equal to the negative-sequence data. From these Y-bus matrices the corresponding Z-bus matrices can be calculated as

\[
\begin{align*}
Z_{\Delta 1} &= Y^{-1}_{\Delta 1} \\
Z_{\Delta 2} &= Y^{-1}_{\Delta 2} \\
Z_{\Delta 0} &= Y^{-1}_{\Delta 0}
\end{align*}
\] (13.50) (13.51) (13.52)
13.5. General method of analysis in systems having impedance loads of which one is un-symmetrical

It is only at the bus having the un-symmetrical load that the injected currents $\neq 0$. This means that the rows that corresponds to the bus with the un-symmetrical load connected (bus 2) in equation (13.46) can be rewritten as

$$
\begin{align*}
U'_{2-1} &= U'_1(2) = U_{2-1}' + Z_{\Delta 1}(2, 2)I_{\Delta 1}(2) \\
\text{from eq. 13.44} \\
U'_{2-2} &= U'_2(2) = Z_{\Delta 2}(2, 2)I_{\Delta 2}(2) \\
U'_{2-0} &= U'_0(2) = Z_{\Delta 0}(2, 2)I_{\Delta 0}(2)
\end{align*}
$$

(13.53)

By assuming that the un-symmetrical load is connected at bus $r$ (here $r = 2$), the equations can be summarized as

$$
U'(r) = \begin{bmatrix}
U'_{r-1} \\
U'_{r-2} \\
U'_{r-0}
\end{bmatrix} = U(r) + Z_{\Delta}(r, r)I_{\Delta}(r) \equiv 
$$

$$
\equiv \begin{bmatrix}
0 \\
0 \\
U_r
\end{bmatrix} + \begin{bmatrix}
Z_{\Delta 1}(r, r) & 0 & 0 \\
0 & Z_{\Delta 2}(r, r) & 0 \\
0 & 0 & Z_{\Delta 0}(r, r)
\end{bmatrix} \begin{bmatrix}
I_{\Delta 1}(r) \\
I_{\Delta 2}(r) \\
I_{\Delta 0}(r)
\end{bmatrix}
$$

(13.54)

It should be pointed out that equation (13.54) describes the Thévenin-equivalent at bus $r$, where the voltage behind the positive-sequence impedance is $U_r(2)$ and the three Thévenin-impedances are $Z_{\Delta 1}(r, r), Z_{\Delta 2}(r, r)$ and $Z_{\Delta 0}(r, r)$. 

**Figure 13.6.** Positive-, negative- and zero-sequence diagram of the calculations of voltage changes
Assume that the un-symmetrical load is Y-connected and connected to ground and consists of three different impedances in the three phases: $Z_{LD2-a}$, $Z_{LD2-b}$ and $Z_{LD2-c}$. The voltage drop (in the phase components) over the load is

$$U_{LD2-phase} = \begin{bmatrix} \hat{U}_{LD2-a} \\ \hat{U}_{LD2-b} \\ \hat{U}_{LD2-c} \end{bmatrix} = \begin{bmatrix} Z_{LD2-a} & 0 & 0 \\ 0 & Z_{LD2-b} & 0 \\ 0 & 0 & Z_{LD2-c} \end{bmatrix} \begin{bmatrix} \hat{I}_{LD2-a} \\ \hat{I}_{LD2-b} \\ \hat{I}_{LD2-c} \end{bmatrix} = Z_{LD2-phase} I_{LD2-phase}$$

(13.55)

By introducing symmetrical components, this can be converted to

$$U'(2) = \begin{bmatrix} U'_{2-1} \\ U'_{2-2} \\ U'_{2-0} \end{bmatrix} = T^{-1} U_{LD2-phase} = T^{-1} Z_{LD2-phase} I_{LD2-phase} =$$

$$= \begin{bmatrix} -Z_{LD2-phase} I_{LD2-sym} \end{bmatrix} = Z_{LD2-phase} I_{LD2-phase}$$

(13.56)

since the direction of definitions for currents in equation (13.55) is down into the load (=out from the bus) whereas $I_\Delta(2)$ has the direction of definitions into the bus.

By assuming that the un-symmetrical load is connected to bus $r$ (here $r = 2$) the current $I_\Delta(r)$ can be calculated by summarizing equation (13.54) and (13.56).

$$I_\Delta(r) = -[Z_\Delta(r, r) + Z_{LDr-sym}]^{-1} U(r)$$

(13.57)

These values of the symmetrical components of the current at bus $r$ can then be inserted into equation (13.46) where the symmetrical components of all voltages can be calculated. The voltage at bus $k$ can then be calculated as:

$$U_1'(k) = U_1(k) + Z_{\Delta_1}(k, r) I_{\Delta_1}(r)$$

$$U_2'(k) = Z_{\Delta_2}(k, r) I_{\Delta_2}(r)$$

(13.58)

$$U_0'(k) = Z_{\Delta_0}(k, r) I_{\Delta_0}(r)$$

**Example 13.3** In Figure 13.7, an internal industry network is shown. The electric energy is delivered by an infinite bus 1 having nominal voltage and a grounded zero connection point. The energy is distributed via transformer $T1$, the line $L2$ and the transformer $T2$ to the load $LD2$. There is also a high voltage load $LD1$ connected to $T1$ via the line $L1$. Component data are:

- **Transformer $T1$**: 800 kVA, 70/10 kV, $x = 7\%$, $Y0-\Delta$ with $Y0$ on the 10-kV side
- **Transformer $T2$**: 300 kVA, 0.4/10 kV, $x = 8\%$, $Y0 - Y0$
- **Line $L1$**: $r_{L1-1} = 0.17$ $\Omega/km$, $\omega L_{L1-1} = 0.3$ $\Omega/km$, $\omega C_{L1-1} = 3.2 \times 10^{-6}$ $S/km$, $\ell = 2$ km, $Z_{L1-0} = 3Z_{L1-1}$, $\bar{Y}_{L1-0} = 0.5\bar{Y}_{L1-1}$,
- **Line $L2$**: $r_{L2-1} = 0.17$ $\Omega/km$, $\omega L_{L2-1} = 0.3$ $\Omega/km$, $\omega C_{L2-1} = 3.2 \times 10^{-6}$ $S/km$, $\ell = 1$ km, $Z_{L2-0} = 3Z_{L2-1}$, $\bar{Y}_{L2-0} = 0.5\bar{Y}_{L2-1}$,
13.5. **General method of analysis in systems having impedance loads of which one is un-symmetrical**

![Figure 13.7. Single-line diagram of internal industry network](image)

- **Load LD1**: Impedance load, 500 kW, \( \cos \phi = 0.80 \), inductive at 10 kV, \( \Delta \)-connected.
- **Load LD2**: Impedance load, 200 kW, \( \cos \phi = 0.95 \), inductive at 400 V, \( Y \)-connected with grounded zero connection point.

Assume that the lines are modeled by using the \( \Pi \)-equivalent. Assume that half of the load in phase \( a \) at LD2 is disconnected. Calculate the efficiency of the internal network operating in this condition. This is the same example as example 7.3 but with the difference that the load LD2 is un-symmetric.

**Solution**

In the following, index \( -1 \) means positive-sequence, \( -2 \) means negative-sequence and \( -0 \) means zero-sequence. Example: \( \overline{Z}_{L1-2} \) is the negative-sequence impedance of line L1 and \( \overline{U}_{5-0} \) is the zero-sequence voltage at bus 5.

1: Start with the calculations of per-unit values of the positive-, negative- and zero-sequence systems of the whole network except for the un-symmetrical load LD2.

Chose base values (MVA, kV, \( \Rightarrow \) kA, \( \Omega \)):

\[
S_b = 500 \text{ kVA} = 0.5 \text{ MVA}, \quad U_{b10} = 10 \text{ kV} \Rightarrow I_{b10} = S_b/\sqrt{3}U_{b10} = 0.0289 \text{ kA}, \quad Z_{b10} = U_{b10}^2/S_b = 200 \Omega, \quad U_{b04} = 0.4 \text{ kV} \Rightarrow I_{b04} = S_b/\sqrt{3}U_{b04} = 0.7217 \text{ kA}, \quad Z_{b04} = U_{b04}^2/S_b = 0.32 \Omega
\]

Calculate the per-unit values of the infinite bus:

Infinite bus operating at nominal voltage and having a grounded zero connection point gives that \( \overline{U}_1 = 1 \text{ pu} \) and that the Thévenin-impedances of the positive-, negative- and zero-sequence = 0, i.e. no impedance between bus 1 and ground in Figure 13.8a, b and c.

Calculate the per-unit values of the transformer \( T1 \):

\[
\overline{Z}_{T1-1} = (\overline{Z}_{T1\%}/100) \cdot Z_{bT1-10}/Z_{b010} = (\overline{Z}_{T1\%}/100) \cdot S_b/S_{T1} = (j7/100) \cdot 0.5/0.8 = j0.0438, \quad \overline{Z}_{T1-2} = \overline{Z}_{T1-0} = \overline{Z}_{T1-1} = j0.0438, \quad \text{zero-sequence diagram is given in Figure 13.8c.}
\]

Calculate the per-unit values of the transformer \( T2 \):

\[
\overline{Z}_{T2-1} = (\overline{Z}_{T2\%}/100) \cdot S_b/S_{T2} = (j8/100) \cdot 0.5/0.3 = j0.1333, \quad \overline{Z}_{T2-2} = \overline{Z}_{T2-0} = \overline{Z}_{T2-1} = j0.1333, \quad \text{zero-sequence diagram is given in Figure 13.8c.}
\]

Calculate the per-unit values of the line L1:

\[
\overline{Z}_{L1-1} = 2 \cdot [0.17 + j0.3]/Z_{b10} = 0.0017 + j0.003, \quad \overline{Z}_{L1-2} = \overline{Z}_{L1-1} = 0.0017 + j0.003, \quad \overline{Z}_{L1-0} = 3\overline{Z}_{L1-1} = 0.0051 + j0.009
\]
Figure 13.8. Internal industry network in example 13.3

\[
\begin{align*}
\bar{Y}_{L1-1} &= 2 \cdot [3.2 \times 10^{-6}] \cdot Z_{b10} = j0.0013, \\
\bar{Y}_{L1-2} &= \bar{Y}_{L1-1} = j0.0013, \\
\bar{Y}_{L1-0} &= 0.5\bar{Y}_{L1-1} = j0.00064
\end{align*}
\]

Calculate the per-unit values of the line \(L2\) :
\[
\begin{align*}
\bar{Z}_{L2-1} &= 1 \cdot [0.17 + j0.3] / Z_{b10} = 0.0009 + j0.0015, \\
\bar{Z}_{L2-2} &= \bar{Z}_{L2-1} = 0.0009 + j0.0015, \\
\bar{Z}_{L2-0} &= 3\bar{Z}_{L2-1} = 0.0026 + j0.0045 \\
\bar{Y}_{L2-1} &= 1 \cdot [3.2 \times 10^{-6}] \cdot Z_{b10} = j0.00064, \\
\bar{Y}_{L2-2} &= \bar{Y}_{L2-1} = j0.00064, \\
\bar{Y}_{L2-0} &= 0.5\bar{Y}_{L2-1} = j0.00032
\end{align*}
\]

Calculate the per-unit values of the impedance of the load \(LD1\) :
Calculate as \(Y\)-connected load with an isolated zero connection point, which gives the same result as in section 13.1 : \(\bar{Z}_{LD1-1} = (\bar{U}_{LD1}^2 / \bar{I}_{LD1}) / Z_{b10} = (10^2 / [0.5/0.8]) \cdot (0.8 + j0.6) / 200 = 0.64 + j0.48, \bar{Z}_{LD1-2} = \bar{Z}_{LD1-1} = 0.64 + j0.48\). No zero-sequence since the zero connection point is isolated, see Figure 13.8c.

It is now possible to form the \(Y\)-bus matrix of the positive-sequence in the same way as in
section 13.5. Bus 1 is included in order to determine the voltage at all buses prior to the connection of the un-symmetrical load. The zero connection point is not included in the Y-bus matrix since the system of equations in that case will be overdetermined. Compared with example 7.3, the load \( LD2 \) is not included in the Y-bus matrix.

\[
Y_1 = \begin{bmatrix}
\frac{1}{Z_{T1-1}} & -\frac{1}{Z_{L1-1}} & 0 & 0 & 0 \\
-\frac{1}{Z_{T1-1}} & \frac{1}{Z_{L1-1}} & \frac{1}{Z_{T2-1}} & -\frac{1}{Z_{L1-1}} & 0 \\
0 & -\frac{1}{Z_{L1-1}} & \frac{1}{Z_{L1-1}} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{Z_{T2-1}} & -\frac{1}{Z_{T2-1}}
\end{bmatrix}
\]  

(13.59)

where

\[
\begin{align*}
\gamma_{22-1} &= \frac{1}{Z_{T1-1}} + \frac{1}{Z_{L1-1}} + \frac{\gamma_{L1-1}}{2} + \frac{1}{Z_{L2-1}} \frac{\gamma_{L2-1}}{2} \\
\gamma_{33-1} &= \frac{1}{Z_{L1-1}} + \frac{\gamma_{L1-1}}{2} + \frac{1}{Z_{LD1-1}} \\
\gamma_{44-1} &= \frac{1}{Z_{L2-1}} + \frac{\gamma_{L2-1}}{2} + \frac{1}{Z_{T2-1}}
\end{align*}
\]

2: Next step is to replace the entire network with Thévenin-equivalents \((U_{Th5}, Z_{Th5-1}, Z_{Th5-2}\) and \(Z_{Th5-0}\)) at the bus having un-symmetrical condition, i.e. bus 5, according to equation (13.54).

2a: The Thévenin-voltage \( U_{Th5} \) can be calculated according to equation (13.44). The first task is to calculate the Z-bus matrix which can be obtained as the inverse of the Y-bus matrix:

\[
Z_1 = Y^{-1} = \begin{bmatrix}
0.6429+j0.5264 & 0.6429+j0.4827 & 0.6412+j0.4797 & 0.6429+j0.4827 & 0.6429+j0.4827 \\
0.6429+j0.4827 & 0.6429+j0.4827 & 0.6412+j0.4797 & 0.6429+j0.4827 & 0.6429+j0.4827 \\
0.6412+j0.4797 & 0.6412+j0.4797 & 0.6412+j0.4797 & 0.6412+j0.4797 & 0.6412+j0.4797 \\
0.6429+j0.4827 & 0.6429+j0.4827 & 0.6412+j0.4797 & 0.6437+j0.4842 & 0.6437+j0.4842 \\
0.6429+j0.4827 & 0.6429+j0.4827 & 0.6412+j0.4797 & 0.6437+j0.4842 & 0.6437+j0.6175
\end{bmatrix}
\]

(13.60)

By using equation (13.44), the Thévenin-voltage \( U_{Th5} \) can be obtained as

\[
U_{Th5} = Z_1(5,1)I_1-1 = 0.968\angle -2.413^\circ
\]

(13.61)

2b: According to equation (13.54), the positive-sequence impedance of the Thévenin-equivalent can be obtained from the Z-bus matrix of the positive-sequence system with all voltage sources shortened. The Y-bus matrix of the system can be formed by cancellation of the row and column in matrix \( Y_1 \) given above, that corresponds to the generator bus, i.e. row and column 1.

\[
Y_{\Delta1} = \begin{bmatrix}
\gamma_{22-1} & -\frac{1}{Z_{L1-1}} & -\frac{1}{Z_{L2-1}} & 0 \\
-\frac{1}{Z_{L1-1}} & \gamma_{33-1} & 0 & 0 \\
-\frac{1}{Z_{L2-1}} & 0 & \gamma_{44-1} & -\frac{1}{Z_{T2-1}} \\
0 & 0 & -\frac{1}{Z_{T2-1}} & \frac{1}{Z_{T2-1}}
\end{bmatrix}
\]

(13.62)
The matrix $Z_{\Delta 1}$ is obtained as the inverse:

$$Z_{\Delta -1} = Y_{\Delta -1} = \begin{bmatrix}
0.0018+j0.0423 & 0.0018+j0.0423 & 0.0018+j0.0423 & 0.0018+j0.0423 \\
0.0018+j0.0423 & 0.0036+j0.0449 & 0.0018+j0.0423 & 0.0018+j0.0423 \\
0.0018+j0.0423 & 0.0018+j0.0423 & 0.0026+j0.0438 & 0.0026+j0.0438 \\
0.0018+j0.0423 & 0.0018+j0.0423 & 0.0026+j0.0438 & 0.0026+j0.1771
\end{bmatrix}$$  \hspace{1cm} (13.63)

Note that element (4,4) corresponds to bus 5 since bus 1 is cancelled. This implies that

$$Z_{Th5-1} = 0.0026 + j0.1771 $$  \hspace{1cm} (13.64)

2c : The Thévenin-impedance of the negative-sequences $Z_{Th5-2}$ can be calculated using the corresponding matrix of the negative-sequence. The only difference between the positive- and negative-sequence is that there is no voltage source in the negative-sequence system. To calculate the $Y_{\Delta 2}$-matrix, this difference disappears since the $Y_{\Delta 1}$-matrix has shortened voltage sources. Since the positive- and negative-sequence components are equal for all system components, the corresponding Y-bus matrix:

$$Y_{\Delta 2} = Y_{\Delta 1} $$  \hspace{1cm} (13.65)

This gives that the corresponding inverses are equal also, which gives that

$$Z_{Th5-2} = Z_{Th5-1} = 0.0026 + j0.1771 $$  \hspace{1cm} (13.66)

2d : The Y-bus matrix of the zero-sequence is different compared with the other sequences, both owing to different numerical values but also because of the zero-sequence connections in transformers and loads.

$$Y_{\Delta 0} = \begin{bmatrix}
\bar{Y}_{22-0} & \frac{1}{\bar{Z}_{L1-0}} & \frac{1}{\bar{Z}_{L2-0}} & 0 \\
\frac{1}{\bar{Z}_{L1-0}} & \bar{Y}_{33-0} & 0 & 0 \\
\frac{1}{\bar{Z}_{L2-0}} & 0 & \bar{Y}_{44-0} & \frac{1}{\bar{Z}_{T2-0}} \\
0 & 0 & \frac{1}{\bar{Z}_{T2-0}} & \frac{1}{\bar{Z}_{T2-0}}
\end{bmatrix}$$  \hspace{1cm} (13.67)

where

$$\bar{Y}_{22-0} = \frac{1}{\bar{Z}_{T1-0}} + \frac{1}{\bar{Z}_{L1-0}} + \frac{\bar{Y}_{L1-0}}{2} + \frac{1}{\bar{Z}_{L2-0}} + \frac{\bar{Y}_{L2-0}}{2}$$

$$\bar{Y}_{33-0} = \frac{1}{\bar{Z}_{L1-0}} + \frac{\bar{Y}_{L1-0}}{2}$$

$$\bar{Y}_{44-0} = \frac{1}{\bar{Z}_{L2-0}} + \frac{\bar{Y}_{L2-0}}{2} + \frac{1}{\bar{Z}_{T2-0}}$$

Corresponding Z-bus matrix is obtained as the inverse

$$Z_{\Delta 0} = Y_{\Delta 0}^{-1} = \begin{bmatrix}
0.0000+j0.0438 & 0.0000+j0.0438 & 0.0000+j0.0438 & 0.0000+j0.0438 \\
0.0000+j0.0438 & 0.0051+j0.0528 & 0.0000+j0.0438 & 0.0000+j0.0438 \\
0.0000+j0.0438 & 0.0000+j0.0438 & 0.0026+j0.0483 & 0.0026+j0.0483 \\
0.0000+j0.0438 & 0.0000+j0.0438 & 0.0026+j0.0483 & 0.0026+j0.1816
\end{bmatrix}$$  \hspace{1cm} (13.68)
13.5. General method of analysis in systems having impedance loads of which one is un-symmetrical

Note that element $(4,4)$ corresponds to bus 5 since bus 1 is cancelled. This gives that

$$Z_{Th5-0} = 0.0026 + j0.1816 \quad (13.69)$$

By that, all Thévenin-quantities of bus 5 are calculated and the voltage vector and impedance matrix according to equation (13.54) are calculated as

$$U(5) = \begin{bmatrix} U_{Th5} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.968\angle -2.413^\circ \\ 0 \\ 0 \end{bmatrix}$$

$$Z_{\Delta(5,5)} = \begin{bmatrix} Z_{Th5-1} \\ 0 \\ 0 \\ 0 \\ Z_{Th5-2} \end{bmatrix} = \begin{bmatrix} 0.0026 + j0.1771 \\ 0 \\ 0.0026 + j0.1771 \\ 0 \\ 0.0026 + j0.1816 \end{bmatrix} \quad (13.70)$$

3: Calculate the per-unit values of the impedance of LD2 in normal conditions:

$$Z_{LD2_{pu}} = \left(\frac{U_{LD2}^2}{S_{LD2}}\right)/Z_{b04} = \quad (13.71)$$

$$= (0.4^2/[0.2/0.95]) \cdot (0.95 + j\sqrt{1 - 0.95^2})/0.32 = 2.2562 + j0.7416$$

At half load in phase $a$ ($Z_a = 2Z_{LD2}$), the impedance matrix of the symmetrical components can be calculated by using equation (13.3):

$$Z_{LD2\text{-sym}} = T^{-1}Z_{LD2\text{-phase}}T = T^{-1} \begin{bmatrix} 2Z_{LD2_{pu}} & 0 & 0 \\ 0 & Z_{LD2_{pu}} & 0 \\ 0 & 0 & Z_{LD2_{pu}} \end{bmatrix} T =$$

$$= \begin{bmatrix} 3.0083 + j0.9888 & 0.7521 + j0.2472 & 0.7521 + j0.2472 \\ 0.7521 + j0.2472 & 3.0083 + j0.9888 & 0.7521 + j0.2472 \\ 0.7521 + j0.2472 & 0.7521 + j0.2472 & 3.0083 + j0.9888 \end{bmatrix} \quad (13.72)$$

The symmetrical components of the currents through the load can be calculated by using equation (13.57) together with data from equation (13.70)

$$I_{\Delta}(5) = -[Z_{\Delta(5,5)} + Z_{LD2\text{-sym}}]^{-1}U(5) = \begin{bmatrix} 0.3315\angle 155.8442^\circ \\ 0.0653\angle -26.5244^\circ \\ 0.0653\angle -26.6221^\circ \end{bmatrix} \quad (13.73)$$

4: The symmetrical components of all voltages can be calculated by using equation (13.58)
By using equation (10.25), the total power flowing into the line as 

\[ \bar{S}_{L1-bus2-1} = U_1'(2)\bar{T}_{L1-1} = 0.9203 + j0.6921 \]

\[ \bar{S}_{L1-bus2-2} = U_0'(2)\bar{T}_{L1-2} = (7.60 + j5.72) \times 10^{-6} \]

\[ \bar{S}_{L1-bus2-0} = U_0'(2)\bar{T}_{L1-0} = 4.26 \times 10^{-15} + j2.61 \times 10^{-9} \]

By using equation (10.25), the total power flowing into the line \( L1 \) at bus 2 can be calculated as

\[ \bar{S}_{L1-bus2} = \bar{S}_{L1-bus2-1} + \bar{S}_{L1-bus2-2} + \bar{S}_{L1-bus2-0} = 0.4602 + j0.3461 \]
13.5. General method of analysis in systems having impedance loads of which one is un-symmetrical

In the corresponding manner, the total power out from the line $L_1$ at bus 3, as well as the power in and out from the line $L_2$ can be calculated as

\[ S_{L_1-bus3} = 0.4589 + j0.3461 \]
\[ S_{L_2-bus2} = 0.1489 + j0.0567 \]
\[ S_{L_2-bus4} = 0.1488 + j0.0567 \]  

(13.78)

The only system losses are the line losses. These losses can be calculated as the difference between the sending and the receiving end power. The efficiency can be calculated as:

\[ \eta = 100 \cdot \frac{\text{Real}(S_{L_1-bus3}) + \text{Real}(S_{L_2-bus4})}{\text{Real}(S_{L_1-bus2}) + \text{Real}(S_{L_2-bus2})} = 99.7911\% \]  

(13.79)
13. Analysis of un-symmetric three-phase systems
Chapter 14
Power system harmonics

Currents and voltages that are not pure sinusoidal occur when non-linear components are connected to the power system. Non-linear components gives for example that a complex notation of the voltage drop

\[ \overline{U} = \overline{ZI} \]  

(14.1)

over a component cannot be used directly since the component itself cannot be described as an impedance. Examples of non-linear components are power electronic devices and transformer operating with saturation.

The influence non-linear components may have on a power system will be illustrated by an example. Assume that a symmetric three-phase voltage, \( U_M = 400 \text{ V}, f = 50 \text{ Hz} \), feeds a resistive, Y-connected three-phase load, \( R = 400 \text{ Ω} \), via anti-parallel connected thyristors as given in Figure 14.1.

![Figure 14.1. Three-phase load fed via thyristors all having the same firing angle \( \alpha \)](image)

Thyristors are used in power systems to control the power flow. A thyristor is a controllable diode where the firing angle \( \alpha \) denotes the delay after the voltage zero crossing that the thyristor starts to behave as a conductor. Anti-parallel connected thyristors are used in e.g. dimmers that are used to control the light from a lamp. In Figure 14.2 the voltage in phase \( a \) in Figure 14.1 is shown as well as how the current changes when the firing angle of the thyristors T1 and T2 is changed. As indicated in Figure 14.2, \( \int i_a \, dt \) decreases as the firing angle \( \alpha \) increases. The firing angle \( \alpha \) can be increased to a maximum of \( 180^\circ \), i.e. half cycle. The firing angle \( \alpha = 45^\circ \) means that the thyristor is ignited and behaves as a conductor after \( \frac{45}{360} = \frac{1}{8} \) cycle, i.e. after 0.0025 s. When \( \int i_a^2 \, dt \) decreases, the mean power delivered is also decreased, i.e. the thyristors controls the power flow. The figure also shows that when the firing angle \( \alpha \neq 0^\circ \), the current is not purely sinusoidal.

If all thyristors in Figure 14.1 have the firing angle \( \alpha = 0^\circ \), the phase currents \( i_a, i_b \) and \( i_c \) will be sinusoidal and thereby the load will draw a sinusoidal current. The current \( i_0 \) is equal to zero.
Assume that all thyristors instead have a firing angle of \(45^\circ\). The three phase currents as well as the current in the neutral will have the shape shown in Figure 14.3.

As given in Figure 14.3, the occurrence of thyristors gives a current in the neutral conductor. If the firing angle increases in the circuit, Figure 14.1, the shape of the currents will also change. In Figure 14.4 the currents when having a firing angle of \(135^\circ\) at all thyristors are shown.

\[
\begin{align*}
&\text{Figure 14.2. The voltage } u_a(t) \text{ and the current } i_a \text{ at a firing angle of } 0^\circ, 45^\circ \text{ and } 135^\circ, \text{ respectively, of the thyristors T1 and T2} \\
&\text{Figure 14.3. Currents in the circuit given in figure 14.1 when the firing angle is } 45^\circ
\end{align*}
\]

In the analysis of not purely sinusoidal quantities in the power system, Fourier analysis is often used. The quantity of interest is then studied as the sum of sinusoidal quantities with
a fundamental frequency and multiples of it, so called harmonics. The current in e.g. phase $a$ can then be calculated as

$$i_a(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t + \gamma_n)$$  \hspace{1cm} (14.2)

The amplitudes $a_n$ and phase angles $\gamma_n$ of the fundamental frequency as well as harmonics of the current, can be calculated by using Fast Fourier Transform, FFT, using samples of the waveform of the current as given in Figure 14.3.

![Graphs showing waveforms for different phases and frequencies.](image)

**Figure 14.4.** Currents in the circuit according to figure 14.1 when the firing angle is 135°

In Figure 14.5, the result of a Fourier analysis of the currents given in Figure 14.3 is shown. As shown in the figure, the harmonics in each phase are the same since the waveform of
the currents are the same in the three phases. For the current $i_0$, the rate of harmonics is different. The rate of harmonics is also given in Table 14.1 which also includes an analysis of the system condition at a firing angle of 135°.

<table>
<thead>
<tr>
<th>Order</th>
<th>Hz</th>
<th>$i_a$</th>
<th>% of 50 Hz</th>
<th>$i_0$</th>
<th>% of 50 Hz</th>
<th>$i_a$</th>
<th>% of 50 Hz</th>
<th>$i_0$</th>
<th>% of 50 Hz</th>
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<td>100.0</td>
<td>0.004</td>
<td>0.9</td>
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<td>0.000</td>
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<td>0.480</td>
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<td>0.000</td>
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<tr>
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<td>0.120</td>
<td>64.2</td>
<td>0.004</td>
<td>0.8</td>
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<td>0.075</td>
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<td>3.5</td>
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<td>20.1</td>
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<td>17.3</td>
<td>0.096</td>
<td>20.1</td>
</tr>
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<td>0.025</td>
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<tr>
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<tr>
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<table>
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<th>THD</th>
<th>RMS</th>
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<tr>
<td>50-∞</td>
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</table>

Table 14.1. Harmonics in the phase currents in the circuit given in Figure 14.1 at $\alpha = 45^\circ$ and $\alpha = 135^\circ$

As given in table 14.1, no even harmonics exist. This owing to the anti-parallel connected thyristors in the three phases that all have the same firing angle giving that the positive and the negative part of the current have the same waveform, i.e. a mirror image along the time-axis, see Figure 14.3 and 14.4. In Table 14.1, the Total Harmonic Distortion, THD, is given which is an RMS-value of the total harmonics.

$$THD = \sqrt{\frac{1}{2} \sum_{n=2}^{\infty} a_n^2}$$  \hspace{1cm} (14.3)

As given in Table 14.1, the harmonics in the neutral conductor is (mainly) the odd multiples of three times the fundamental frequency, i.e. 3, 9, 15, etc. (The low percentage levels given for other harmonics in the table are due to limitations in the calculation routine used.) Also the amplitude of the harmonics are higher in the neutral conductor compared with the phase conductors. The reason for this is that the harmonics in the three phases have the same phase position since the phase displacement of 120° is an even number of cycles of the
frequency, because

\[ i_{0-3n} = i_{a-3n} + i_{b-3n} + i_{c-3n} = \]
\[ = a_n \cos(3n\omega t + \gamma_n) + a_n \cos(3n\omega t + \gamma_n + 120^\circ) + \]
\[ + a_n \cos(3n\omega t + \gamma_n - 120^\circ) = 3a_n \cos(3n\omega t + \gamma_n) \]  \hspace{1cm} (14.4)

For the other harmonics, the current in the different phases are cancelling one another in the neutral conductor since each of them represents a symmetrical phase sequence

![Figure 14.6. RMS\(i_a\) ('—'), RMS\(i_0\) ('- - -'), and the quotient RMS\(i_0\)/RMS\(i_a\) ('...')](image)

\[ i_{0-(3n\pm1)} = i_{a-(3n\pm1)} + i_{b-(3n\pm1)} + i_{c-(3n\pm1)} = \]
\[ = a_n \cos(3n\omega t + \gamma_n) + a_n \cos(3n\omega t + \gamma_n \pm 120^\circ) + \]
\[ + a_n \cos(3n\omega t + \gamma_n \mp 120^\circ) = 0 \]  \hspace{1cm} (14.5)

Note that if no neutral conductor existed, no harmonics having a multiple of three times the fundamental frequency could occur in the phase conductors either. Since the harmonics of multiples of three times the fundamental frequency has the same phase position, they will appear as zero-sequence components which implies that they cannot pass \(\Delta\)-\(Y\)-connected transformers.

The firing angle of the thyristors will of course influence the currents in the phases as well as in the neutral conductor. In Figure 14.6, the change in the RMS-value of the current when the firing angle is increased is shown.

As shown in Figure 14.6, the \(i_0\)-current is relatively small for firing angles up to approximately 20°. When the firing angle has increased to about 70°, the RMS-values of the current in the neutral conductor is as large as the RMS-value of the phase current. The RMS-value of the current in the neutral conductor reach a maximum value when the firing angle is \(\alpha = 90^\circ\) and for even higher firing angles, the current in the neutral conductor is \(\sqrt{3}\) times the phase current.
14. Power system harmonics
Appendix A
MATLAB-codes for Examples 7.2, 7.3, 13.2 and 13.3

%--- Example 7.2
l=5 ;
zled=0.9+j*0.3 ;
yled=j*3e-6 ;
%
pind=0.400 ;
cosind=0.8 ;
uind=10 ;
%
uk=70 ;
%
str=5 ;
xtraproc=4 ;
%--- Select base values
Sb=0.5 ;
Ub10=10 ;
Ub70=70 ;
Ib10=Sb/sqrt(3)/Ub10 ;
Ib70=Sb/sqrt(3)/Ub70 ;
Zb10=Ub10^2/Sb ;
%--- Calculate per-unit values for the Thevenin equivalent
Uthpu=uk/Ub70 ;
Zthpu=uk/Ub70*Ib70/ik ;
%
%f--- Calculate per-unit values for the transformer
ztra=j*xtraproc/100*Sb/stra ;
%
%f--- Calculate per-unit values for the line
LZ = zled*l/Zb10 ;
LY = yled*l*Zb10 ;
LA = 1+0.5*LY*LZ ;
LB = LZ ;
LC = LY*(1+0.25*LY*LZ) ;
LD = LA ;
%--- Calculate per-unit values for the industry impedance
Zindpu=uind^2/(pind/cosind)*(cosind+j*sqrt(1-cosind^2))/Zb10 ;
%f--- Calculate the two port matrix for grid+transformer+line
fyrpol = [1 Zthpu ; 0 1]*[1 ztra ; 0 1]*[LA LB ; LC LD] ;
A= fyrpol(1,1) ;
B= fyrpol(1,2) ;
C= fyrpol(2,1) ;
D= fyrpol(2,2) ;
\%--- Calculate the total impedance of the grid
Ztot = (A*Zindpu+B)/(C*Zindpu+D) ;
i1 = Uthpu/Ztot ;
\%--- Calculate u2 and i2
trakramat=[1 Zthpu+ztra ; 0 1] ;
u2i2=inv(trakramat)*[Uthpu ;i1] ;
s2=u2i2(1)*conj(u2i2(2))*Sb ;
\%--- Calculate the voltage at the industry
matmat = [A B ; C D] ;
u3i3=inv(matmat)*[Uthpu ;i1] ;
u3=u3i3(1)*Ub10 ;

\%--- Example 7.3
l1=2 ;
l2=1 ;
zled=0.17+j*0.3 ;
yled=j*3.2e-6 ;
\%
pld1=0.500 ;
cosl1d=0.8 ;
u1d1=10 ;
pld2=0.200 ;
cosl2d=0.95 ;
u1d2=0.4 ;
\%
st1=0.8 ;
xt1proc=7 ;
st2=0.3 ;
xt2proc=8 ;
\%--- Select base values
Sb=0.5 ;
Ub10=10 ;
Ub04=0.4 ;
Ib10=Sb/sqrt(3)/Ub10 ;
Ib04=Sb/sqrt(3)/Ub04 ;
Zb10=Ub10^2/Sb ;
Zb04=Ub04^2/Sb ;
\%--- Calculate per-unit values for the strong grid
u1=1 ;
\%--- Calculate per-unit values for transformer T1
zT1pu=j*xt1proc/100*Sb/st1 ;
\%--- Calculate per-unit values for transformen T2
zT2pu=j*xt2proc/100*Sb/st2 ;
%--- Calculate per-unit values for line L1
zl1pu = zled*11/Zb10;
yl1pu = yled*11*Zb10;
%--- Calculate per-unit values for line L2
zl2pu = zled*12/Zb10;
yl2pu = yled*12*Zb10;
%--- Calculate per-unit values for the LD1-impedance
zld1pu=uld1^2/(pld1/cosld1)*(cosld1+j*sqrt(1-cosld1^2))/Zb10;
%--- Calculate per-unit values for the LD2-impedance
zld2pu=uld2^2/(pld2/cosld2)*(cosld2+j*sqrt(1-cosld2^2))/Zb04;
%--- Formulate the Y-bus matrix for the grid
y22=1/zT1pu+1/zL1pu+(yL1pu/2)+1/zL2pu+(yL2pu/2);
y33=1/zL1pu+(yL1pu/2)+1/zLD1pu;
y44=1/zL2pu+(yL2pu/2)+1/zT2pu;
Ybus=[1/zT1pu -1/zT1pu 0 0 0; -1/zT1pu y22 -1/zL1pu -1/zL2pu 0; 0 -1/zL1pu y33 0 0; 0 -1/zL2pu 0 y44 -1/zT2pu; 0 0 0 -1/zT2pu 1/zT2pu+1/zLD2pu];
Zbus=inv(Ybus);
%--- Calculate the efficiency
i1=u1/Zbus(1,1);
u2=Zbus(1,2)*i1;
u3=Zbus(1,3)*i1;
u4=Zbus(1,4)*i1;
u5=Zbus(1,5)*i1;
S1=u1*conj(i1)*Sb;
iL1=(u2-u3)/zL1pu;
PfL1=real(zL1pu)*abs(iL1)^2*Sb;
iL2=(u2-u4)/zL2pu;
PfL2=real(zL2pu)*abs(iL2)^2*Sb;
verkn=100*(real(S1)-PfL1-PfL2)/real(S1);
%--- Calculation of the Z-bus matrix for fault current calculations
Ybusk=Ybus(2:5,2:5);
Zbusk=inv(Ybusk);
ike= u4/Zbusk(3,3)*Ib10;

%--- Example 13.2
alfa = exp(j*120*pi/180);
tmat = [1 1 1 ; alfa*alfa alfa 1 ; alfa alfa*alfa 1 ];
l=5;
zled1=0.9+j*0.3;
zledz=3*zled1;
yled1=j*3e-6;
yledz=0.5*yled1;
% pind=0.400;
cosind=0.8 ;
uin=10 ;
fasafaktor=2 ;
%
iki=-j*0.3 ;
lok=-j*0.2 ;
uk=70 ;
%
stra=5 ;
xtraproc=4 ;
%--- Select base values
Sb=0.5 ;
Ub10=10 ;
Ub70=70 ;
Ib10=Sb/sqrt(3)/Ub10 ;
Ib70=Sb/sqrt(3)/Ub70 ;
Zb10=Ub10^2/Sb ;
Zb70=Ub70^2/Sb ;
%--- Calculate per-unit values for the Thevenin equivalent
Uthpu=uk/Ub70 ;
Zthpu1=uk/Ub70*Ib70/iki3 ;
Zthpu2=3*uk/Ub70*Ib70/loki-2*Zthpu1 ;
%--- Calculate per-unit values for the transformer
Ztrapu1=j*xtraproc/100*Sb/stra ;
Ztrapu2=Ztrapu1 ;
%--- Calculate per-unit values for the line
Zlpu1 = zled1*1/Zb10 ;
Zlpu2 = zledz*1/Zb10 ;
Ylpu1 = yled1*1*Zb10 ;
Ylpu2 = yledz*1*Zb10 ;
AL1 = 1+0.5*Ylpu1*Zlpu1 ;
BL1 = Zlpu1 ;
CL1 = Ylpu1*(1+0.25*Ylpu1*Zlpu1) ;
DL1 = AL1 ;
ALz = 1+0.5*Ylpu2*Zlpu2 ;
BLz = Zlpu2 ;
CLz = Ylpu2*(1+0.25*Ylpu2*Zlpu2) ;
DLz = ALz ;
%--- Calculate the positive sequence two-port matrix for grid+traf+line
plusmat = [1 Zthpu1 ; 0 1]*[1 Ztrapu1 ; 0 1]*[AL1 BL1 ; CL1 DL1] ;
A1= plusmat(1,1) ;
B1= plusmat(1,2) ;
C1= plusmat(2,1) ;
D1= plusmat(2,2) ;
UThc=Uthpu/A1 ;
ZThC1=B1/A1 ;
ZThC2=ZThC1 ;

%--- Calculate the two-port matrix for traf+line
nollmat = [1 Ztrapu1 ; 0 1]*[ALz BLz ; CLz DLz] ;
Az= nollmat(1,1) ;
Bz= nollmat(1,2) ;
Cz= nollmat(2,1) ;
Dz= nollmat(2,2) ;
ZThCz=Bz/Az ;
Zs=[ZThC1 0 0 ; 0 ZThC2 0 ; 0 0 ZThCz] ;

%--- Calculate the per-unit val. for ind. load imp. under normal conditions
Zindpu=uind^2/(pind/cosind)*(cosind+j*sqrt(1-cosind^2))/Zb10 ;
Zlds=inv(tmat)*[fasafaktor*Zindpu 0 0 ; 0 Zindpu 0 ; 0 0 Zindpu]*tmat ;

%--- Calculate symm. comp. for currents at the industry
Is=inv(Zs+Zlds)*[UThc ;0 ;0] ;

%--- Calculate symm. comp. for voltages at the industry
Uc=Zlds*Is ;

%--- Calculate the pos. seq. voltage and current from traf to line
UBIB1 = [AL1 BL1 ; CL1 DL1]*[Uc(1) ; Is(1)] ;
UB1 = UBIB1(1) ;
IB1 = UBIB1(2) ;
SB1=UB1*conj(IB1)*Sb ;

%--- Calculate the neg. seq. voltage and current from traf to line
UBIB2 = [AL1 BL1 ; CL1 DL1]*[Uc(2) ; Is(2)] ;
UB2 = UBIB2(1) ;
IB2 = UBIB2(2) ;
SB2=UB2*conj(IB2)*Sb ;

%--- Calculate the zero seq. voltage and current from traf to line
UBIBz = [ALz BLz ; CLz DLz]*[Uc(3) ; Is(3)] ;
UBz = UBIBz(1) ;
IBz = UBIBz(2) ;
SBz=UBz*conj(IBz)*Sb ;

%--- Calculate the phase components for the voltage at the industry
Ucf=tmat*Uc ;

%--- Calculate total delivered power from transformer to line
Stot=SB1+SB2+SBz ;

%--- Example 13.3
alfa = exp(j*120*pi/180) ;
tmat = [1 1 1 ; alfa*alfa alfa 1 ; alfa alfa*alfa 1 ] ;

l1=2 ;
l2=1 ;
zled=0.17+j*0.3 ;
yled=j*3.2e-6 ;
zledz=3*zled ;
yledz=0.5*yled ;
%
pld1=0.500 ;
cosl1d=0.8 ;
uld1=10 ;
pld2=0.200 ;
cosl2d=0.95 ;
uld2=0.4 ;
%fasafaktor=2 ;
%
st1=0.8 ;
xt1proc=7 ;
st2=0.3 ;
xt2proc=8 ;
%
%--- Select base values
Sb=0.5 ;
Ub10=10 ;
Ub04=0.4 ;
Ib10=Sb/sqrt(3)/Ub10 ;
Ib04=Sb/sqrt(3)/Ub04 ;
Zb10=Ub10^2/Sb ;
Zb04=Ub04^2/Sb ;
%
%--- Calculate pur-unit-values for the strong grid
u1=1 ;
%
%--- Calculate pur-unit-values for transformer T1
zT1_1=j*xt1proc/100*Sb/st1 ;
zT1_z=zT1_1 ;
%
%--- Calculate pur-unit-values for transformer T2
zT2_1=j*xt2proc/100*Sb/st2 ;
zT2_z=zT2_1 ;
%
%--- Calculate pur-unit-values for line L1
zL1_1 = zled*11/Zb10 ;
zL1_2 = zL1_1 ;
yL1_1 = yled*11*Zb10 ;
zL1_z = zledz*11/Zb10 ;
yL1_z = yledz*11*Zb10 ;
%
%--- Calculate pur-unit-values for line L2
zL2_1 = zled*12/Zb10 ;
zL2_2 = zL2_1 ;
yL2_1 = yled*12*Zb10 ;
zL2_z = zL_2 * 12/Zb10;
yL2_z = yL_2 * 12 * Zb10;
%
%--- Calculate pur-unit-values for the LD1 impedance
zLD1_1=uld1^2/(pld1/cosld1)*(cosld1+j*sqrt(1-cosld1^2))/Zb10;
%
%--- Formulate the Y-bus matrix for the positive sequence grid
y22_1=1/zT1_1+1/zL1_1+(yL1_1/2)+1/zL2_1+(yL2_1/2);
y33_1=1/zL1_1+(yL1_1/2)+1/zLD1_1;
y44_1=1/zL2_1+(yL2_1/2)+1/zT2_1;
ybus_1=[1/zT1_1 -1/zT1_1 0 0 0;
        -1/zT1_1 y22_1 -1/zL1_1 -1/zL2_1 0;
        0 -1/zL1_1 y33_1 0 0;
        0 -1/zL2_1 0 y44_1 -1/zT2_1;
        0 0 0 -1/zT2_1 1/zT2_1];
%
%--- 2a) Calculate the Thevenin voltage in node 5
Zbus_1=inv(Ybus_1);
U_Th5=Zbus_1(5,1)*u1/Zbus_1(1,1);
%
%--- 2b) Calculate the positive sequence Thevenin impedance
Ybus_D1=Ybus_1(2:5,2:5);
Zbus_D1=inv(Ybus_D1);
Z_Th5_1=Zbus_D1(4,4);
%
%--- 2c) Calculate the negative sequence Thevenin impedance
Zbus_D2=Zbus_D1;
Z_Th5_2=Zbus_D2(4,4);
%
%--- 2d) Calculate the zero sequence Thevenin impedance
y22_z=1/zT1_z+1/zL1_z+(yL1_z/2)+1/zL2_z+(yL2_z/2);
y33_z=1/zL1_z+(yL1_z/2);
y44_z=1/zL2_z+(yL2_z/2)+1/zT2_z;
ybus_Dz=[y22_z -1/zL1_z -1/zL2_z 0;
        -1/zL1_z y33_z 0 0;
        -1/zL2_z 0 y44_z -1/zT2_z;
        0 0 -1/zT2_z 1/zT2_z];
Zbus_Dz=inv(Ybus_Dz);
Z_Th5_z=Zbus_Dz(4,4);
%
%--- 3) Calculate the per-unit values for the LD2 impedance
zLD2pu=uld2^2/(pld2/cosld2)*(cosld2+j*sqrt(1-cosld2^2))/Zb04;
ZLD2s=inv(tmat)*[fasafaktor*zLD2pu 0 0;
                        0 zLD2pu 0;
                        0 0 zLD2pu]*tmat;
Zs=[Z_Th5_1 0 0;
        0 Z_Th5_2 0;
        0 0 Z_Th5_z];
I5s=-inv(Zs+ZLD2s)*[U_Th5;
                     0;
                     0];
%
%--- 4) Calculate the symmetrical components for all voltages
UTh2_1=Zbus_1(1,2)*u1/Zbus_1(1,1) + Zbus_D1(1,4)*I5s(1);
UTh2_2=Zbus_D2(1,4)*I5s(2);
UTh2_z=Zbus_Dz(1,4)*I5s(3) ;  
UTh3_1=Zbus_1(3,1)*u1/Zbus_1(1,1) + Zbus_D1(2,4)*I5s(1) ;  
UTh3_2=Zbus_D2(2,4)*I5s(2) ;  
UTh3_z=Zbus_Dz(2,4)*I5s(3) ;  
UTh4_1=Zbus_1(4,1)*u1/Zbus_1(1,1) + Zbus_D1(3,4)*I5s(1) ;  
UTh4_2=Zbus_D2(3,4)*I5s(2) ;  
UTh4_z=Zbus_Dz(3,4)*I5s(3) ;  
UTh5_1=Zbus_1(5,1)*u1/Zbus_1(1,1) + Zbus_D1(4,4)*I5s(1) ;  
UTh5_2=Zbus_D2(4,4)*I5s(2) ;  
UTh5_z=Zbus_Dz(4,4)*I5s(3) ;  
%
%--- 5) Calculate pos., neg., and zero sequence currents through lines
IL1_1=(UTh2_1-UTh3_1)/zL1_1 ;  
IL1_2=(UTh2_2-UTh3_2)/zL1_2 ;  
IL1_z=(UTh2_z-UTh3_z)/zL1_z ;  
IL2_1=(UTh2_1-UTh4_1)/zL2_1 ;  
IL2_2=(UTh2_2-UTh4_2)/zL2_2 ;  
IL2_z=(UTh2_z-UTh4_z)/zL2_z ;  
ILD1_1=UTh3_1/zLD1_1 ;  
ILD1_2=UTh3_2/zLD1_2 ;  
ILD1_z=UTh3_z/zLD1_1 ;  
%
%--- Calculation of efficiency
SL1_n2_1=UTh2_1*conj(IL1_1) ;  
SL1_n2_2=UTh2_2*conj(IL1_2) ;  
SL1_n2_z=UTh2_z*conj(IL1_z) ;  
SL1_n2=SB*(SL1_n2_1+SL1_n2_2+SL1_n2_z) ;  
SL1_n3_1=UTh3_1*conj(IL1_1) ;  
SL1_n3_2=UTh3_2*conj(IL1_2) ;  
SL1_n3_z=UTh3_z*conj(IL1_z) ;  
SL1_n3=SB*(SL1_n3_1+SL1_n3_2+SL1_n3_z) ;  
SL2_n2_1=UTh2_1*conj(IL2_1) ;  
SL2_n2_2=UTh2_2*conj(IL2_2) ;  
SL2_n2_z=UTh2_z*conj(IL2_z) ;  
SL2_n2=SB*(SL2_n2_1+SL2_n2_2+SL2_n2_z) ;  
SL2_n4_1=UTh4_1*conj(IL2_1) ;  
SL2_n4_2=UTh4_2*conj(IL2_2) ;  
SL2_n4_z=UTh4_z*conj(IL2_z) ;  
SL2_n4=SB*(SL2_n4_1+SL2_n4_2+SL2_n4_z) ;  
verkn=100*( real(SL1_n3)+real(SL2_n4 ))/(real(SL1_n2)+real(SL2_n2))
Appendix B
Matlab-codes for Example 8.10

% Start of file
clear,
clear global

deg=180/pi;
maxiter=10;
EPS=1e-4;

k1=-0.2; k2=1.2; k3=-0.07; k4=0.4;

% Step 1
converged=0; iter=0; x=3/deg;

while ~converged & iter < maxiter,

% Step 2
delta_gx=k4-(k1*x+k2*cos(x-k3));

% Step 3
if all(abs(delta_gx)< EPS),
    converged=1;
    iter=iter;
    xdeg=x*deg
else
    % Step 4
    Jac=k1-k2*sin(x-k3); %Jac=dfx/dx;
    % Step 5
delta_x=inv(Jac)*delta_gx;
    % Step 6
    x=x+delta_x;
    iter=iter+1;
end, % if all

if iter=maxiter,
    iter=iter,
    disp('The equation has no solutions')
    disp('or')
    disp('bad initial value, try with another initial value')
end, % iter
end, % while
% End of file
B. Matlab-codes for Example 8.10
Appendix C
Matlab-codes for Example 8.12

% Start of file
clear,
clear global
deg=180/pi;
rad=1/deg;

nbus=3; %Number of buses

%Step 1
% 1a)
U1=1; theta1=0; PD1=0; QD1=0;
U2=1; PG2=2; PD2=1; QD2=0.2;
PG3=0; QG3=0; PD3=2; QD3=-0.4;

Z12=j*0.2; y12=1/Z12;
Z13=j*0.4; y13=1/Z13;
Z23=j*0.5; y23=1/Z23;

y11=y12+y13; y22=y12+y23; y33=y13+y23;

%1b)
Y=[y11 -y12 -y13 ; -y12 y22 -y23 ; -y13 -y23 y33];
G=real(Y);
B=imag(Y);
PGD2=PG2-PD2;
PGD3=PG3-PD3;
QGD3=QG3-QD3;

%1c)
%Bus 1 is slack bus
%Bus 2 is PU-bus
%Bus 3 is PQ-bus

bus1=1; bus2=2; bus3=3;

n_pu_pq=2; %number of PU-buses and PQ-buses

PGD=[PGD2;PGD3];
QGD=[QGD3];
C. Matlab-codes for Example 8.12

theta2=0;
U3=1;
theta3=0;
iter=0;
VOLT=[U1;U2;U3];
ANG=[theta1;theta2;theta3];

while iter < 4
  % Step 2
  %2a)
  for m=1:nbus
    for n=1:nbus
      PP(m,n)=VOLT(m)*VOLT(n)*(G(m,n)*cos(ANG(m)-ANG(n))+B(m,n)*sin(ANG(m)-ANG(n)));
      QQ(m,n)=VOLT(m)*VOLT(n)*(G(m,n)*sin(ANG(m)-ANG(n))-B(m,n)*cos(ANG(m)-ANG(n)));
    end, %for n
  end, % for m
  P=sum(PP'); Q=sum(QQ');
  %2b)
deltaP=PGD-P([bus2 bus3])';
deltaQ=QGD-Q([bus3])';

  % Step 4
  %4a)
  for m=1:nbus
    for n=1:nbus
      if m==n,
        H(m,m)=-Q(m)-B(m,m)*VOLT(m)*VOLT(m);
        N(m,m)= P(m)+G(m,m)*VOLT(m)*VOLT(m);
        J(m,m)= P(m)-G(m,m)*VOLT(m)*VOLT(m);
        L(m,m)= Q(m)-B(m,m)*VOLT(m)*VOLT(m);
      else
        H(m,n)= VOLT(m)*VOLT(n)*(G(m,n)*sin(ANG(m)-ANG(n))-B(m,n)*cos(ANG(m)-ANG(n)));
        N(m,n)= VOLT(m)*VOLT(n)*(G(m,n)*cos(ANG(m)-ANG(n))+B(m,n)*sin(ANG(m)-ANG(n)));
        J(m,n)= -VOLT(m)*VOLT(n)*(G(m,n)*cos(ANG(m)-ANG(n))+B(m,n)*sin(ANG(m)-ANG(n)));
        L(m,n)= VOLT(m)*VOLT(n)*(G(m,n)*sin(ANG(m)-ANG(n))-B(m,n)*cos(ANG(m)-ANG(n)));
      end, %if
    end, %for n
  end, % for m
  H(bus1,:)=[]; H(:,bus1)=[]; N(bus1,:)=[]; N(:,[bus1 bus2])=[];
  J([bus1 bus2],:)=[]; J(:,bus1)=[]; L([bus1 bus2],:)=[]; L(:,[bus1 bus2])=[];
  JAC=[H N; J L];
DX=inv(JAC)*[deltaP;deltaQ];

delta_theta=DX(1:n_pu_pq);
delta_U=DX(n_pu_pq+1:length(DX));

iter=iter+1;
ANG([bus2 bus3])=ANG([bus2 bus3])+delta_theta;
VOLT([bus3])=VOLT([bus3]).*(1+delta_U);

for m=1:nbus
    for n=1:nbus
        PP(m,n)=VOLT(m)*VOLT(n)*(G(m,n)*cos(ANG(m)-ANG(n))+B(m,n)*sin(ANG(m)-ANG(n)));
        QQ(m,n)=VOLT(m)*VOLT(n)*(G(m,n)*sin(ANG(m)-ANG(n))-B(m,n)*cos(ANG(m)-ANG(n)));
    end, %for n
end, %for m
P=sum(PP');
Q=sum(QQ');

PG1=P(bus1)+PD1,
QG1=Q(bus1)+QD1,
QG2=Q(bus2)+QD2,

% End of file