Equilibrium Strategies for Time-Inconsistent Stochastic Optimal Control of Asset Allocation

JOHAN DIMITRY EL BAGHDADY
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Abstract

We have examined the problem of constructing efficient strategies for continuous-time dynamic asset allocation. In order to obtain efficient investment strategies, a stochastic optimal control approach was applied to find optimal transaction control. Two mathematical problems are formulated and studied: Model I; a dynamic programming approach that maximizes an isoelastic functional with respect to given underlying portfolio dynamics and Model II; a more sophisticated approach where a time-inconsistent state dependent mean-variance functional is considered. In contrast to the optimal controls for Model I, which are obtained by solving the Hamilton-Jacobi-Bellman (HJB) partial differential equation; the efficient strategies for Model II are constructed by attaining subgame perfect Nash equilibrium controls that satisfy the extended HJB equation, introduced by Björk et al. in [1]. Furthermore; comprehensive execution algorithms where designed with help from the generated results and several simulations are performed. The results reveal that optimality is obtained for Model I by holding a fix portfolio balance throughout the whole investment period and Model II suggests a continuous liquidation of the risky holdings as time evolves. A clear advantage of using Model II is concluded as it is far more efficient and actually takes time-inconsistency into consideration.

Keywords: Stochastic optimal control, dynamic programming, asset allocation, non-cooperative games, subgame perfect Nash equilibrium, time-inconsistency, dynamic portfolio optimization, mean-variance, state dependent risk aversion, extended Hamilton-Jacobi-Bellman, execution algorithms.
Jämviktsstrategier för tidsinkonsistent stokastisk optimal styrning av tillgångsallokering
**Sammanfattning**


**Keywords:** Stokastisk optimal styrning, dynamisk programmering, tillgånsallokering, icke-kooperativa spel, Nashjämvikt, tidsinkonsistens, dynamisk portföljoptimering, avvägning mellan förväntad avkastning och varians, tillståndsberoende riskhantering, utökad Hamilton-Jacobi-Bellman, exekveringsalgoritmer.
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Dedicated to the strongest woman I know; my mother
Tereza Saad Faltuous.
A fundamental problem encountered by investors is to find an efficient investment strategy, an action policy that is optimal with respect to some risk-return preference. That is; we seek to construct an optimal portfolio of assets and instruments subject to that an amount of pre-determined constraints and objectives are satisfied. Usually, an investment strategy must consider several market related parameters and constraints including expected returns, asset correlations, volatilities, transaction costs and asset price spreads.

In this thesis we investigate a small investor’s continuous time dynamic investment strategy for portfolios of underlying assets. By small investor we mean that the actions conducted by the investor do not affect the market prices. As opposed to static portfolio management, where the wealth allocation is determined from the first day of investment, dynamic investment strategies involve continuous rebalancing of the portfolios and thus provide a dynamic allocation amongst the assets at several time instances during the portfolio’s existence. The asset re-allocation is required to be conducted in an optimal fashion. Namely; follow a policy that ensures the satisfaction of constraints and exceed minimum performance requirements at all times. It is natural to see that this type of challenge could be viewed as an optimal control problem and therefore also be handled in an adequate fashion.
Chapter 2

Background

For as long as people have been able to steer processes, it has been desirable to steer them in an optimal fashion. Regardless of the type of process it might be; a queuing system, a payroll administration, a plantation, a satellite orbit change, missile trajectories or even financial transactions as well as asset management. We will try to give an intuitive proposal of interconnection between the mathematical theory of optimal control and its applications, particularly dynamic portfolio optimization.

In this chapter an appropriate introduction to the fundamental concepts and previously conducted research is introduced in order to facilitate the reader to follow what is treated in the upcoming chapters.

2.1 Optimal control theory

The branch of mathematics known as optimal control theory is an extension of the calculus of variations and is a mathematical method for deriving control strategies. The theory surfaced during the cold war in the 1950s, when the Union of Soviet Socialist Republics and the United States of America competed for dominance in space flight proficiency [2]. The ramifications of optimal control makes it adaptable to a large variety of fields including:

- Aerospace and aeronautical engineering
- Robotics and automation
- Biological engineering
- Process control
- Management sciences
- Finance
- Economics

Many problems can be formulated as control problems and most control problems are difficult enough to be considered nearly unsolvable when approached with ad-hoc techniques that work for several other types of engineering problems. Furthermore; there usually exists more than one solution for a control problem and not all of them are feasible and/or easily obtained with ordinary problem solving techniques. This is where optimal control becomes convenient, due to available systematic approaches for finding solutions and the ability to reduce redundancy and finding optimal controllers with help of existing theory [3].
The standard optimal control problem is concerned with minimizing or maximizing a performance functional, dependent of the state of a system, given the dynamics of the state. The state is controlled by some admissible control function, which is to be determined in order to fulfill the necessary objectives. The minimizing problem with finite time horizon can in a deterministic setting be expressed as

\[
\inf_{\nu \in \mathcal{V}} \left\{ \int_0^T \varphi(s, x_s^\nu, \nu_s) \, ds + \Phi(T, x_T^\nu) \right\}
\]

s.t. \[\dot{x}_s^\nu = \psi(s, x_s^\nu, \nu_s),\]
\[x_t^\nu \in \mathcal{S}_t, \quad x_T^\nu \in \mathcal{S}_T,\]
\[\nu(s, x) \in \mathcal{V} \subseteq \mathbb{R}^m, \quad \forall (s, x) \in \mathcal{X} \subseteq \mathbb{R}^+ \times \mathbb{R}^n,\]

where the notation \(x_s^\nu = x(s, \nu_s)\) is used. The vector field \(\psi\) describes how the relationship between the derivative of the state with respect to time depends on the time, the state and the control. Both the initial and terminal state of the system belong to the smooth manifolds \(\mathcal{S}_t\) and \(\mathcal{S}_T\) respectively, representing the boundary conditions of the system’s dynamics. \(\mathcal{V}\) is the set of admissible controls that can be used to take the system from state \(x_t\) to \(x_T\). Within the braces there is a so called terminal cost, \(\Phi\), penalizing deviation from a predetermined terminal state. There is also a trajectory cost, \(\varphi\), describing the cost of the currently chosen trajectory in the state space, thus penalizing the state’s deviation from an optimal trajectory from \(\mathcal{S}_t\) to \(\mathcal{S}_T\) through the time-state space \(\mathcal{X}\). Example of solution trajectories between manifolds in three spatial dimensions is illustrated in Figure 2.1.

![Figure 2.1: Three state trajectories when controlling from \(\mathcal{S}_t\), defined by the line in three spatial dimensions that corresponds to the intersection \(\mathcal{S}_t := \{ x \in \mathbb{R}^3 \mid g_1(x) = g_2(x) = 0 \}\), to some surface \(\mathcal{S}_T = \{ x \in \mathbb{R}^3 \mid g_3(x) = 0 \}\). Here \(\nu^\dagger\) and \(\nu^*\) are arbitrary admissible controls while \(\hat{\nu}\) is the optimal control corresponding to the optimal trajectory \(\hat{x}\).](image-url)

A real life example could be the problem of missile guidance with optimal trajectories. Where the essence of the optimality could be that it is desired to eliminate a target with specified coordinates; with as little cost as possible and within a limited period of time. Where cost may for example be defined as fuel consumption. The dynamics of the system, which tell us how the current coordinates of the missile change with respect to time, can be expressed as some function regarding information about the system and its environment. The terminal state...
Johan Dimitry

is then the coordinates of a target which is to be eliminated. The control variables, deciding the trajectory, could then be thrust and rotation.

2.2 Problem formulation and interpretations

The main problem of this thesis is to investigate and determine an efficient way of performing dynamic wealth allocations amongst risk-free and risky assets with consideration for risk management. The risk management part requires that the investor is able to choose a risk-profile that stipulates and quantifies the amount of risk in relation to the eventual reward. This is interpreted as a dynamic portfolio optimization problem, or equivalently a dynamic asset allocation problem, with the desire of determining efficient trading strategies. In Figure 2.2 a flow schedule of the transactions shows some of the parameters that will play a major role in the modelling and analysis of this problem.

![Figure 2.2: An illustration of some of the important parameters that are involved when investigating dynamic portfolio strategies.](image)

The problem as described will be modelled and analysed in two steps, generating two different models that are separated by fundamental changes in each models’ respective structure. Thereafter in section 7.2.1 a third model is suggested for further research and development as a natural extension of the studies conducted in this thesis.

- **Model I**: The objective is to maximize the expected utility of the terminal portfolio wealth. An isoelastic utility function, also known as constant relative risk aversion (CRRA) is considered. In this model, wealth is instantaneously transferred between the accounts without any costs. This is essentially Merton’s portfolio problem [4] without consumption. Here a dynamic programming approach is used in a stochastic setting to solve the optimal control problem.

- **Model II**: The setup is the same as in Model I with exception for the definition of the objective function. This is a dynamic state dependent mean-variance portfolio optimization problem where the objective function has risk aversion as a function of the initial wealth. This is introduced due to the straightforward ability to, in a practical way, quantify and thus choose exactly how much the reluctance to risk affects the expected return. Interpretation of the efficient frontiers connection to the practical risk choice is easier and the consequences of the efficient strategies thus become more comprehensible for an investor. For this problem, it is realised that Bellman’s principle of optimality is violated.
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due to time-inconsistency that is caused by the objective function’s non-linearity and its
dependence of the initial state of the system. In other words, regular dynamic program-
ming is not available and we turn to a game theoretic framework to extend the previously
mentioned solution approach for the stochastic optimal control problem.

When it is assumed that there are no transaction costs and spread, in accordance with the
framework of Model I, the problem is to determine the optimal portfolio balance. In other
words, each control function at each point in time will be the fraction of the wealth placed in
its corresponding asset. The change in the wealth allocation of portfolio is thus thought of as
displayed in Figure 6.1 where the fractions, representing our "steering wheel", are adjusted as
time moves along in order to let the wealth follow an optimal trajectory in the state space.

![Wealth allocation as time evolves (example)](image)

Figure 2.3: An illustration of a possible strategy for how the allocation of wealth between a
stock and a bank account could be performed in order to be achieve efficiency with respect to
given performance parameters.

In Model II, we will use the absolute amount of wealth in each asset as the control functions.

2.3 Previous work

problem, more precisely he introduced the classic mean-variance portfolio selection problem
which is one of the fundamental building blocks in modern finance. The proposed problem
is concerned with an investor’s goal to maximize the expected return while simultaneously
minimizing the risk, measured by the variance of the expected portfolio return. The solutions to
this type of problems are often called efficient strategies or allocations and vary as the individual
risk profile of each investor varies. Due to the fact that difference in risk preference leads to
different opinions on what optimality for the portfolio actually is. The efficient strategies give
rise to a so-called efficient frontier, a curve composed of all pairs of expected return and variance
that satisfy the mean-variance problem. An analytic expression of the efficient frontier, for the
case when the covariance matrix is non-negative definite and short-selling is allowed, is derived
in 1984. The way that one measures risk in these type of problems was criticized by many and
measures as the lower partial moment, the semi-variance and the downside risk are discussed
in [10], [11], [12] and [13].
When the model is extended to consider multi-period optimization, or equivalently continuous-time optimization as each time period’s length becomes infinitesimal, time-inconsistency is often introduced as the models take realistic interpretations into account. Since these type of problems almost always are considered as stochastic optimal control problems, the time-inconsistency leads to complications when trying to apply known and reliable methods for solving optimal control problems. As discussed later on, in sections 3.2.2 and 3.3 the time-inconsistency conflicts Bellman’s principle of optimality and in this context the notion of optimality is vague. In general, two approaches are available for handling the type of problems that are considered to be time-inconsistent.

- **Pre-commitment strategy**: Optimize the objective function at the initial time and disregard whether the strategy’s optimality is valid for future points in time or not. In other words, in view of the initial time, the policy is optimal.

- **Equilibrium strategy**: Take the time-inconsistency seriously and investigate the problem from a game theoretic perspective. The asset allocation problem is viewed as game where each point in time corresponds to a player, or a reincarnations of the same player, and the optimal strategy is provided by a subgame perfect Nash equilibrium.

The first authors to come up with a continuous-time mean-variance model with pre-commitment strategies were Richardson [14] in 1989 and Bajeux-Besnainou and Portait [15] in 1998. And further work was conducted by Li and Ng [16] in 2000, where they used several stochastic linear-quadratic control problems to study the original time-inconsistent mean-variance optimization. In a comparable fashion a continuous-time solution is provided by [17], [18], [19], [20] and [21].

The approach of looking for Nash equilibrium points was initiated in 1955 by Strotz in [22] and further studies where conducted by Goldman [23], Krusell and Smith [24], Pollak [25] and Vieille and Weibull [26]. In the work [27] by Peleg and Yaari in 1973, the time-inconsistent problem was treated as a non-cooperative game where the optimal strategies are described by Nash equilibrium. In 2006 Ekeland and Lazrak’s work [28] use subgame perfect Nash equilibrium to take non-commitment seriously, as the name of their article suggests. Thereafter, in 2008 Ekeland and Pirvu [29] considered Merton’s problem [4] with variable hyperbolic discounting, in a time-inconsistent context, with a rigorous definition of the continuous-time Nash equilibrium concept. In 2010, a mean-variance portfolio in an incomplete market was studied in [30] by Basak and Chabakauri where a time-consistent strategy was obtained as a closed-form solution derived with a dynamic programming approach.

In [1] and [31] Björk et al. provide discrete-time and continuous-time approaches for a general class of time-inconsistent objective functions for stochastic optimal control problems, this is conducted by deriving an extended version of the Hamilton-Jacobi-Bellman equation. In [32] the same authors argued that the model that was regarded and solved by Basak and Chabakauri is not reasonable from an economic point of view. Their arguments is that the risk aversion need to depend on the initial endowment of the investor, since they are looking at absolute parts of the portfolio and not fractions, a constant amount in risky assets regardless of initial investment does not make sense. An investor with a million units of currency will not have the same risk profile as an investor with a hundred units of the same currency. Further, it is easy to see that a dimensionless coefficient of risk aversion, as proposed by Basak and Chabakauri, is irrational in accordance with the following reasoning. If we denote wealth by \( W \) then the mean-variance objective function is

\[
J = E[W] - \gamma \text{Var}[W],
\]

where \( \gamma \) is the coefficient of risk aversion. Obviously, the dimension of the expectation must be money and naturally the dimension of the variance must be money\(^2\), thus elementary dimensional analysis reveals that \( \gamma \) must be of dimension money\(^{-1}\) for the function to be accurately
stated. Björk et al. derive a solution for the time-inconsistent mean-variance problem for a general function of risk aversion $\gamma(w)$, where $w$ denotes the investor’s initial endowment, in [32].

Recently, in [33], Höfers and Wunderlich study dynamic risk-constraints, where the objective is to obtain strategies that reduce dynamic shortfall risk measures in both discrete and continuous time. Further, dynamic portfolio selection without risk-free assets is investigated by Lam et al. in [34].
Chapter 3

Mathematical background

The purpose of this chapter is to gather important results from previous studies conducted in relevant areas in order to construct a mathematical framework that justifies the prior and subsequent analysis concerning the modelling in chapter 4. The methods in this chapter are taken and interpreted from [35], [36], [1] and [31]. Throughout, we will work with a filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}, \mathbb{P})\), where \(\Omega\) is the outcome space with elements \(\tilde{\omega}\), \(\mathcal{F}\) is the trivial \(\sigma\)-algebra and \(\mathcal{F}\) is the \(\mathbb{P}\)-augmentation of the filtration generated by a Brownian motion under the probability measure \(\mathbb{P}\). Regarding related mathematical preliminaries; the reader is encouraged to look through Appendix A.

3.1 Stochastic optimal control

At each point in time \(s \in [0, T]\), the state \(X_s \in \mathbb{R}^n\) of an \(n\)-dimensional dynamic system form a pair \((s, X_s)\) that represents trajectories through the time-state space \(\mathcal{X}\), defined by

\[
\mathcal{X} := [t, T) \times \mathbb{R}^n,
\]

with closure

\[
\bar{\mathcal{X}} = \mathcal{X} \cup \partial \mathcal{X},
\]

for any \(T \in [0, \infty)\) and \(t \geq 0\). Let the stochastic process \(\nu_s = \nu(s, \tilde{\omega})\) be our control process, chosen at each point in time to control the state \(X_s\), satisfying the system’s law of evolution. Let \(\mathcal{V}\) be the class of all adapted and \(\mathcal{F}_s\)-measurable controls \(\nu_s\) mapping to the Borel set \(\mathcal{V} \subset \mathbb{R}^k\), for each \(s \in [t, T]\). This means that the controls only depend on the current state and time. The dynamics describing the system’s law of evolution are given by the controlled stochastic differential equation

\[
dX_{s}^{\nu} = \xi_{s}^{\nu}(X_{s}^{\nu}) \, ds + \eta_{s}^{\nu}(X_{s}^{\nu}) \, dB_{s}, \quad X_{t}^{\nu} = x. \tag{3.1}
\]

Where \(B_s\) is an \(m\)-dimensional Brownian Motion, denoted \(B_s \in \mathcal{BM}(\mathbb{R}^m)\) and the notations \(f^q_{\nu}(p) := f(s, p, q)\) and \(X_s^{\nu} = X(s, \nu_s)\) are used. We assume that the coefficients

\[
\xi : \mathcal{X} \times \mathcal{V} \to \mathbb{R}^n,
\]

and

\[
\eta : \mathcal{X} \times \mathcal{V} \to \mathbb{R}^{n \times m},
\]

satisfy the linear growth and Lipschitz conditions (\((A.0.4)\) and \((A.0.5)\) in Theorem \((A.4)\) and hence the SDE \((3.1)\) has a unique strong solution

\[
X_{s}^{\nu} \in \mathcal{D}^{2}(\Omega, \mathcal{F}, \mathbb{P}),
\]

for each initial point \((t, x) \in \mathcal{X}\). Let \(\mathcal{V}_0\) be the set of every control process on the form \(\nu_s = \nu(s, X_s) \in \mathcal{V}\) that satisfy problem-specific control constraints and ensures a unique square integrable solution for \((3.1)\). I.e. \(\mathcal{V}_0\) is the set of admissible state feedback control laws.
3.1.1 Defining the problem

In plain English, the objective is to find the optimal control law $\nu(s, X_s)$ that maximizes the expected performance of the system during its total running time.

Consider a performance rate function, $\varphi: X \times V \rightarrow \mathbb{R}$, which in some sense describes the systems instantaneous performance per unit time as a function of the control law $\nu$. Consider as well a bequest function $\Phi: X \rightarrow \mathbb{R}$ describing the systems terminal yield at the terminal state. Then the stochastic optimal control problem is formulated as follows,

$$\sup_{\nu \in V_0} \mathbb{E}_{t,x} \left[ \int_t^T \varphi(s, X_s^\nu, \nu(s, X_s^\nu)) \, ds + \Phi(T, X_T^\nu) \right]$$

s.t. $\begin{align*}
    dX^\nu_s &= \xi_s^\nu(X_s^\nu) \, ds + \eta_s^\nu(X_s^\nu) \, dB_s, \quad \forall s \in [t, T], \\
    X_t &= x, \\
    \nu(s, X_s) &\in V_0, \quad \forall (s, X_s^\nu) \in X.
\end{align*}$

Note that the expectation in (3.2) is conditioned on the information and state at $s = t$, hence the notation $\mathbb{E}_{t,x} [\cdot] = \mathbb{E} [\cdot | \mathcal{F}_t, X_t^\nu = x, s = t]$, was used and in the forthcoming calculations will be the standard notation.

3.1.2 Dynamic programming

Given that there exists an optimal control law for the problem in (3.2), our weapon of choice for the task of obtaining it is known as dynamic programming.

Let us introduce the performance functional $J(t, x, \nu): X \times V_0 \rightarrow \mathbb{R}$, defined as the conditional expectation in equation (3.2) with the given dynamics. In other words, $J(t, x, \nu)$ is the expected performance during the period $[t, T]$, due to the usage of the control law $\nu$, given that we start at state $x$ and time $t \geq 0$. Analogously with the performance functional, the optimal value function $\psi: X \rightarrow \mathbb{R}$ given by

$$\psi(t, x) := \sup_{\nu \in V_0} \{ J(t, x, \nu) \},$$

describes the optimal expected performance. In the imminent analysis it will be assumed that the optimal value function is smooth enough; particularly, twice continuously differentiable on $X$ and continuously differentiable on $\overline{X}$,

$$\psi \in C^2(X) \cap C(\overline{X}).$$

By fixing the pair $(t, x) \in X$, introducing the real constant $\varepsilon > 0$ and letting the set $\mathcal{W} \subset X$ be of the form

$$\mathcal{W} = \{(s, p) \in X | t \leq s \leq t + \varepsilon < T\},$$

and therefore

$$X \setminus \mathcal{W} = \{(s, p) \in X | t + \varepsilon < s \leq T\}.$$

Then the following control law is constructed, a switching policy such that

$$\nu(s, p) = \begin{cases} 
\nu^*(s, p), & \forall (s, p) \in \mathcal{W}, \\
\hat{\nu}(s, p), & \forall (s, p) \in X \setminus \mathcal{W},
\end{cases}$$

where $\hat{\nu}$ denotes the optimal control law and $\nu^*$ is any arbitrary admissible control law. Now consider the following two strategies:
1. Implementing the optimal control law for the entire time interval, which by definition leads to the following performance functional

\[ \mathcal{J}(t, x, \nu) = \psi(t, x). \] (3.4)

2. Switch control policy from an arbitrarily chosen control law to the optimal at some time \( t + \varepsilon \), in other words use the law \( \nu \) described in equation (3.3). The logic here is that during the first time period, corresponding to the set \( W \), the expected performance must be

\[ \mathbb{E}_{t,x} \left[ \int_t^{t+\varepsilon} \varphi(s, X_s^{\nu^*}, \nu^*_s) \, ds \right], \quad \forall (t, x) \in W. \]

During the remaining time the system is in the stochastic state \( X_{t+\varepsilon}^{\nu^*} \) and we said that the optimal control law will be used during \( (t + \varepsilon, T] \), which implies that the expected performance at \( t + \varepsilon \) is given by \( \psi(t + \varepsilon, X_{t+\varepsilon}^{\nu^*}) \). Thus, when conditioning that \( X_{t+\varepsilon}^{\nu^*} = x \), this leads to the conditional expected performance

\[ \mathbb{E}_{t,x} \left[ \psi(t + \varepsilon, X_{t+\varepsilon}^{\nu^*}) \right], \quad \forall (t, x) \in \mathcal{X} \setminus W. \]

This in turn gives us the functional

\[ \mathcal{J}(t, x, \nu) = \mathbb{E}_{t,x} \left[ \int_t^{t+\varepsilon} \varphi(s, X_s^{\nu^*}, \nu^*_s) \, ds + \psi(t + \varepsilon, X_{t+\varepsilon}^{\nu^*}) \right], \] (3.5)

for each \( (t, x) \in \mathcal{X} \).

By construction of the strategies (3.4) and (3.5) one realises that it always must hold that

\[ \mathcal{J}(t, x, \nu) \leq \mathcal{J}(t, x, \hat{\nu}), \]

explicitly suggesting that

\[ \psi(t, x) \geq \mathbb{E}_{t,x} \left[ \int_t^{t+\varepsilon} \varphi(s, X_s^{\nu^*}, \nu^*_s) \, ds + \psi(t + \varepsilon, X_{t+\varepsilon}^{\nu^*}) \right], \] (3.6)

where equality is obtained if and only if \( \nu \equiv \hat{\nu} \), which does not need to be a unique control strategy. Expansion of the \( \psi \) term in equation (3.6) with help from Itô’s formula yields

\[
\psi(t + \varepsilon, X_{t+\varepsilon}^{\nu^*}) - \psi(t, X_t^{\nu^*}) = \int_t^{t+\varepsilon} \left\{ \frac{\partial \psi}{\partial s}(s, X_s^{\nu^*}) + \left( \mathcal{L}^{\nu^*} \psi \right)(s, X_s^{\nu^*}) \right\} \, ds \\
+ \int_t^{t+\varepsilon} \nabla_p \psi(s, X_s^{\nu^*}) \eta_s^{\nu^*} \, dB_s, \] (3.7)

where the infinitesimal operator

\[ \mathcal{L}^{\nu} := \sum_{i=1}^n \xi_i^{\nu} \frac{\partial}{\partial p_i} + \frac{1}{2} \sum_{i,j=1}^n (\eta_{ij}^{\nu}) \frac{\partial^2}{\partial p_i \partial p_j}, \]

is utilized for notational purposes. Insertion of equation (3.7) into (3.6), dividing by \( \varepsilon \) and realising that the Itô integral vanishes due to the appliance of the expectation operator gives us the inequality

\[ 0 \geq \mathbb{E}_{t,x} \left[ \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \left\{ \varphi(s, X_s^{\nu^*}, \nu^*_s) + \frac{\partial \psi}{\partial s}(s, X_s^{\nu^*}) + \left( \mathcal{L}^{\nu^*} \psi \right)(s, X_s^{\nu^*}) \right\} \, ds \right]. \]

\(^{1}\eta'\) denotes the matrix transpose of \( \eta \).
After letting $\varepsilon \to 0$, the fundamental theorem of integral calculus yields

$$0 \geq \frac{\partial \psi}{\partial s}(t, x) + \varphi(t, x, \nu) + (L^\nu \psi)(t, x),$$

where $\nu := \nu^*(t, x)$ and equality is obtained if and only if

$$\nu^*(t, x) \equiv \hat{\nu}(t, x).$$

This now leads us to an important PDE serving as a necessary condition, known as the Hamilton-Jacobi-Bellman Equation (HJBE)

$$\frac{\partial \psi}{\partial s}(t, x) + \sup_{\nu \in V_0} \{ \varphi(t, x, \nu) + (L^\nu \psi)(t, x) \} = 0, \quad \forall (t, x) \in \mathcal{X} \quad (3.8)$$

with terminal value given by

$$\psi(t, x) = \Phi(t, x), \quad \forall (t, x) \in \partial \mathcal{X}. \quad (3.9)$$

This result states that if $\nu$ is the optimal control for problem (3.2), then the optimal value function $\psi$ satisfies the Hamilton-Jacobi-Bellman Equation. The following theorem shows that the HJBE serves as a sufficient condition as well as a verification theorem.

**Theorem 3.1.** (Verification theorem for dynamic programming)

Assume that there exists a function $\Gamma(s, p, q)$ that satisfies the HJBE (3.8) and is such that

$$\nabla_p \Gamma(s, X_s^\nu, X_s^\hat{\nu}) \in L^2.$$

Further let

$$f(t, x) := \arg \sup_{\nu \in V_0} \{ \varphi(t, x, \nu) + (L^\nu \Gamma)(t, x, \nu) \}.$$

Then the optimal value function is $\Gamma \equiv \psi$ and there exists an optimal control law given by $f \equiv \hat{\nu}$.

**Proof.** The formal proof is omitted but can be viewed in both [35] and [36]. \qed

### 3.2 Game theory and time-consistent optimization strategies

This section briefly discusses important notions that are used in the forthcoming concepts.

#### 3.2.1 Game theory

Generally speaking, a game consists of involved players with the ability to choose between a set of strategies in order to maximize their expected utility, respectively. All definitions that are introduced in this section are brought from the notes of Jörgen Weibull’s game theory seminars at The Royal Institute of Technology [37].

**Definition 3.1.** (Game)

A game is defined as

$$\mathcal{G} := \left\langle \mathcal{P}, \hat{\mathcal{V}}, \mathcal{J} \right\rangle,$$

where $\mathcal{P}$ is the set of $r$ players, the set

$$\hat{\mathcal{V}} := \bigotimes_{p \in \mathcal{P}} \hat{V}_p,$$

is the Cartesian product of all players’ feasible strategy sets and $\mathcal{J} : \hat{\mathcal{V}} \to \mathbb{R}^r$ is the augmented utility function. Let $\nu_p \in \hat{V}_p$ be the strategy of player $p$, for $p \in \{1, \ldots, r\}$, then $\nu = (\nu_1, \ldots, \nu_r)$ is said to be a strategy profile.
Definition 3.2. (Non-cooperative game)
A game where the involved players make independent decisions because they are not able to (or not allowed to) form irrevocable teams.

Definition 3.3. (Subgame)
If one utilizes decision trees where nodes correspond to stages, illustrating extensive-form sequential games; then if \( G \) is a game and \( z \) is any node in the tree, except for an end node, then the subgame \( G(z) \) is a game containing its initial node \( z \) and every node that is reachable from \( z \).

Definition 3.4. (Nash equilibrium)
The strategy profile \( \nu^* \in \tilde{V} \) is said to be a Nash equilibrium of \( G = \langle P, \tilde{V}, J \rangle \) if for all \((p, \nu_p) \in P \times \tilde{V}_p\) it holds that
\[
J_p(\nu_p^*, \nu_{-p}^*) \geq J_p(\nu_p, \nu_{-p}^*),
\]
where the strategy profile of everyone except player \( p \) is denoted \( \nu_{-p}^* \).

Definition 3.5. (Subgame perfect Nash equilibrium)
A strategy profile is said to be a subgame perfect Nash equilibrium if it corresponds to a Nash equilibrium of every subgame within the original game.

3.2.2 Time-consistency in an optimization setting

Typically, time-consistency refers to preservation of the legitimacy of a policy or claim, throughout the evolution of time \([1][32]\). That is, an optimal control strategy conducted at time \( t_n \geq t_0 \) is time-consistent if it is said to be optimal when considered at times \( t_{n-1}, t_{n-2}, \ldots, t_1, t_0 \). Thus, an inconsistent strategy inflicts the problem of not being able to guarantee that the decision made at \( t_n \) is still optimal when going back to \( t_{n-1} \), which is exactly what the dynamic programming approach for optimal control problems essentially relies on, as the fundamental assumption is that Bellman’s principle of optimality holds.

"An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision." - Richard E. Bellman [38]

We have provided an example that intuitively illustrates time-inconsistency in real life.

Example 3.1. (Exam studies)
Consider a student that is participating in a university course with a two month long curriculum ending with a written exam. Say that the student commits to a strategy that allows him/her to study and memorise an adequate amount of the course literature per day until the exam date, in order to be able to pass the exam. Now, almost always the student overestimates his/her ability to cover literature and as they approach the exam date they end up increasing the amount of hours that is spent per day on studying. One can argue that this behaviour is due to the student’s recognition of the situation’s need of resources, therefore this yields an increased willingness to commit and hence invest more of their resources.

If the student at the the first day of the curriculum is offered to extend the amount of days until the exam by one day, for a fee of e.g. $30, he/she will most likely not take the offer. But if the same offer is presented to the same student just one day before the exam, he/she is very likely to take the offer and is probably willing to pay much more. Thus there is an inconsistency in the student’s willingness to commit and invest as time moves along.
3.3 Handling time-inconsistency - A game theoretic structure

Some stochastic control problems are formulated such that their objective function has a dependence of the initial state and even a non-linear function of the expectation operation. These mentioned changes of the performance functional may in some sense generalize the previous framework and thus providing a greater ability to solve more complex stochastic optimal control problems. The changes however introduce a complication and the new problem is said to be time-inconsistent in the sense that Bellman’s principle of optimality can not be used [1] [32]. Particularly the time-inconsistency prohibits us from using the regular dynamic programming approach since this leads to confusion when defining what optimality is. An optimal policy relative to an initial point in $\mathcal{X}$ is not guaranteed to be optimal when time evolves and viewed from the future. Here, in this section, we will build upon the previous section’s theory to construct a way to find a subgame perfect Nash equilibrium strategy, instead of standard optimal control strategies as before. Thus, we are looking for a so called equilibrium control law.

Consider the same controlled SDE (3.1) as before, but now our objective is to maximize a performance functional

$$J (t, x, \nu) := \mathbb{E}_{t,x} [V (x, X_T^\nu)] + W (x, \mathbb{E}_{t,x} [X_T^\nu]),$$

(3.10)

for each initial pair $(t, x)$. In other words, the problem we are facing is expressed as

$$\sup_{\nu \in \mathcal{V}_0} \mathbb{E}_{t,x} [V (x, X_T^\nu)] + W (x, \mathbb{E}_{t,x} [X_T^\nu])$$

$$\text{s.t. } dX_s^\nu = \xi_s^\nu (X_s^\nu) ds + \eta_s^\nu (X_s^\nu) dB_s, \; \forall s \in [t, T],$$

$$X_t = x,$$

$$\nu (s, X_s) \in \mathcal{V}_0, \; \forall (s, X_s) \in \mathcal{X}.$$

(3.11)

The state $x$ is present in both functions $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $W : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, the last-mentioned is a non-linear function operating on the expectation. For the mentioned functions it is assumed that the following holds for each $(s, p) \in \mathcal{X}$,

$$V (p, X_T^\nu) \in \mathcal{L} (\Omega, \mathcal{F}, \mathbb{P}),$$

implying that

$$W (p, \mathbb{E}_{s,p} [X_T^\nu]) < \infty,$$

since $X_T^\nu \in \mathcal{L} (\Omega, \mathcal{F}, \mathbb{P})$ as well.

Consider a non-cooperative game where each fixed point in time, $t$, corresponds to a player with the ability of controlling only the state $X_t$ by choosing an appropriate control $\nu (t, X_t)$. The augmentation of all players’ control functions yields a feedback control law

$$\nu : \mathcal{X} \to \mathbb{R}^m,$$

giving player $t$ the performance functional described in equation (3.10). If we define the control law $\nu$ as previously in equation (3.3), then $\hat{\nu}$ is said to be a subgame perfect Nash equilibrium strategy for player $t$, if for every player $s > t$ that chooses the strategy $\hat{\nu} (s, X_s)$, player $t$ chooses the same strategy $\hat{\nu} (t, X_t)$. We define that for each arbitrary control law $\nu \in \mathcal{V}_0$ and equilibrium control $\hat{\nu} \in \mathcal{V}_0$, it holds that

$$\liminf_{\varepsilon \to 0} \frac{J (t, x, \hat{\nu}) - J (t, x, \nu)}{\varepsilon} \geq 0,$$

and thus defining the equilibrium value function $\psi \in \mathcal{C}^2 (\mathcal{X}) \cap \mathcal{C} (\bar{\mathcal{X}})$ as

$$\psi (t, x) := J (t, x, \hat{\nu}).$$
3.3.1 Extended Hamilton-Jacobi-Bellman equation

In order illustrate the extended HJBE, an informal derivation is conducted and we refer to Björk et al. in [31] and [1] for a more formal and rigorous derivation. The proper way to go would be to consider the discrete time case first and then use some of the results in the continuous time derivation, as described by Björk et al. in [1]. The main idea is in some sense the same as in the dynamic programming case, two strategies are considered and then the optimal one is chosen and then the PDE arises when dividing by $\varepsilon$ and going to the limit as $\varepsilon \to 0$. Consider the definitions and introductions below, that are adopted from [1].

i) Introduce $\vartheta^q : \mathcal{X} \to \mathbb{R}$ given by

$$
\vartheta^q(s,p) := \mathbb{E}_{s,p}\left[V(q, X_T^\hat{\nu})\right], \quad \forall q \in \mathbb{R}^n,
$$

then $\vartheta : \mathcal{X} \times \mathbb{R}^n \to \mathbb{R}$ is defined as

$$
\vartheta(s,p,q) := \vartheta^q(s,p).
$$

ii) The infinitesimal operator $\hat{L}^\nu$ is almost defined as $L^\nu$ with exception for the inclusion of the time-derivative,

$$
\hat{L}^\nu := \frac{\partial}{\partial s} + \sum_{i=1}^n \xi_i^\nu \frac{\partial}{\partial p_i} + \frac{1}{2} \sum_{i,j=1}^n (\eta^\nu)^{ij} \frac{\partial^2}{\partial p_i \partial p_j},
$$

and for any function $\kappa(s,p)$ we set that

$$
(\hat{L}_\varepsilon^\nu \kappa)(s,p) := \mathbb{E}_{s,p}\left[\kappa(s+\varepsilon, X_{s+\epsilon}^\nu)\right] - \kappa(s,p).
$$

iii) $\omega : \mathcal{X} \to \mathbb{R}^n$ is defined as

$$
\omega(s,p) := \mathbb{E}_{s,p}\left[X_T^\hat{\nu}\right].
$$

iv) Introduce the operator $\mathcal{M}$ such that

$$
(\mathcal{M}^\nu \omega)(s,p) := \frac{\partial W}{\partial q}(p, \omega(s,p)) \left(\hat{L}^\nu \omega\right)(s,p),
$$

and

$$
(\mathcal{M}^\nu \omega)(s,p) := W(p, \mathbb{E}_{t,x}\left[\omega(s+\varepsilon, X_{s+\epsilon}^\nu)\right]) - W(p, \omega(s,p)).
$$

v) Finally, we introduce

$$
(W \circ \omega)(s,p) := W(p, \omega(s,p)).
$$

Now, Theorem 3.13 in [31] states that the following inequality holds,

$$
(\hat{L}_\varepsilon^\nu \psi)(t,x) - (\hat{L}_\varepsilon^\nu \theta)(t,x,x) + (\hat{L}_\varepsilon^\nu \vartheta^x)(t,x) - \hat{L}_\varepsilon^\nu (W \circ \omega)(t,x) + (\mathcal{M}_\varepsilon^\nu \omega)(t,x) \leq 0. \quad (3.13)
$$

Analogously with before, we want to divide the expression in (3.13) with $\varepsilon$ and let $\varepsilon \to 0$. First we realise that

$$
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (\hat{L}_\varepsilon^\nu \kappa)(t,x) = (\hat{L}^\nu \kappa)(t,x),
$$

but before considering the limit for $\mathcal{M}_\varepsilon^\nu$, some investigation is required. The following approximation can be made

$$
\mathbb{E}_{t,x}\left[\omega(t+\varepsilon, X_{t+\epsilon}^\nu)\right] = \omega(t,x) + (\hat{L}^\nu \omega)(t,x) + o(\varepsilon),
$$
and a Taylor series expansion of $W$ yields

$$W(x, E_t, x \[\omega(t + \varepsilon, X_{t+\varepsilon}^\nu)]) = W(x, \omega(t, x)) + \frac{\partial W}{\partial q}(x, \omega(t, x)) \left(\tilde{L}^{\nu}\omega\right)(t, x) + o(\varepsilon).$$

It is now evident that the limit indeed becomes

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (M^{\nu}\omega)(t, x) = \frac{\partial W}{\partial q}(x, \omega(t, x)) \left(\tilde{L}^{\nu}\omega\right)(t, x),$$

hence the limit as $\varepsilon \to 0$ for the expression in equation (3.13), after dividing by $\varepsilon$, leads to the extended Hamilton-Jacobi-Bellman equation for the subgame perfect Nash equilibrium strategy problem.

**Definition 3.6. (Extended HJBE system)**

For each $(t, x) \in \mathcal{X}$, it holds that

$$0 = \sup_{\nu \in \mathcal{V}_0} \left\{ \left(\tilde{L}^{\nu}\psi\right)(t, x) - \left(\tilde{L}^{\nu}\vartheta\right)(t, x, x) + \left(\tilde{L}^{\nu}\varrho^q\right)(t, x) - \tilde{L}^{\nu}(W \circ \omega)(t, x) + \left(M^{\nu}\omega\right)(t, x) \right\},$$

$$\left(\tilde{L}^{\nu}\varrho^q\right)(t, x) = 0, \quad \forall q \in \mathbb{R}^n,$$

$$\left(\tilde{L}^{\nu}\omega\right)(t, x) = 0,$$

$$\psi(T, x) = V(x, x) + W(x, x),$$

$$\varrho^q(T, x) = V(q, x), \quad \forall q \in \mathbb{R}^n,$$

$$\omega(T, x) = x.$$

Before presenting a verification theorem for the results introduced in Definition 3.6, we define the function space $\mathfrak{L}^2(X^\nu)$. We say that the function $\Gamma(s, p) : \mathcal{X} \to \mathbb{R}$ belongs to $\mathfrak{L}^2(X^\nu)$ if the condition

$$E_t, x \left[ \int_t^T \|\frac{\partial \Gamma}{\partial p}(s, X^\nu_s) \eta^\nu(s, X^\nu_s)\|^2 ds \right] < \infty, \quad \forall (t, x) \in \mathcal{X},$$

holds.

**Theorem 3.2. (Theorem 7.1 in [1], a verification theorem for the extended HJBE system)** Assume that for each $(t, x, q) \in \mathcal{X} \times \mathbb{R}^n$ it holds that

i) The triple $(\psi, \varrho^q, \omega)$ is a solution to the extended HJBE in Definition 3.6.

ii) We have enough smoothness; $\psi \in C^{1,2}$ and $\vartheta \in C^{1,2,2}$.

iii) The function $\tilde{\nu}$ is an admissible control law and

$$\tilde{\nu} = \arg \sup_{\nu \in \mathcal{V}_0} \left\{ \left(\tilde{L}^{\nu}\psi\right)(t, x) - \left(\tilde{L}^{\nu}\vartheta\right)(t, x, x) + \left(\tilde{L}^{\nu}\varrho^q\right)(t, x) - \tilde{L}^{\nu}(W \circ \omega)(t, x) + \left(M^{\nu}\omega\right)(t, x) \right\},$$

where as before, $\mathcal{V}_0$ is the set of admissible control laws.

iv) All the functions $\psi, \vartheta, q, \omega$ and $W \circ \omega$ belong to $\mathfrak{L}^2(X^\nu)$.
Then, \( \hat{\nu} \) is an equilibrium control law, \( \psi \) is its equilibrium value function and the functions \( \vartheta \) and \( \omega \) have the probabilistic interpretations for each \((s,p) \in X\):

\[
\vartheta(q) = \mathbb{E}_{s,p}[V(q, X^\phi_T)], \quad \forall q \in \mathbb{R}^n, 
\]

and

\[
\omega(s, p) := \mathbb{E}_{s,p}[X^\phi_T].
\]

Proof. See [1].
Chapter 4

Modelling and analysis

In this chapter we start off by introducing a more general setting for the fundamental dynamics of the system. The setting includes transaction costs and asset price spread to illustrate how the structure should be viewed in order to understand and thus justify how and why it is reduced to the setting that is utilised both Model I and II.

4.1 Generalised portfolio dynamics

Consider the overview in Figure 4.1 illustrating the fundamental structure used to formulate the models. It shows that liquidity can flow out of the bank account, with proportional transaction cost \( c_O \), as the risky asset is purchased at price \((1 + a)P_s\) at time \( s \in [t, T] \). The theoretical price process of the risky asset is denoted \( P_s \) and the real constant \( a \geq 0 \) is the relative increase in price of the risky asset, which yields the price that the market maker is prepared to sell at, the ask price. It further reveals that liquidity can flow in to the bank account, with proportional transaction cost \( c_I \), as the risky asset is sold at price \((1 - b)P_s\) at time \( s \in [t, T] \). The real constant \( b \in [0, 1) \) denotes the relative decrease in price, corresponding to the bid price, which is what the market maker is prepared to sell the asset for. The processes \( O_s \) and \( I_s \) are both increasing càdlàg and denote the accumulated number of transactions that have been performed up to and at time \( s \), in each direction respectively.

Figure 4.1: A schematic overview of the flow of liquidity to and from both the bank account and the risky asset, revealing the presence of model-imperative parameters.

For each one of the forthcoming models, the following introductions and descriptions define the groundwork which is built upon and used throughout the whole chapter.

\footnote{Continue à droite, limite à gauche, which in English means right continuous with left limits.}
The bank account’s value process is under influence of a risk-free\footnote{The term risk-free indicates here the absence of noise, which consequently suggests predictable future values.} rate of return \( r \in \mathbb{R} \), and thus satisfy the scalar deterministic differential equation
\[
d\theta_s = r \theta_s ds, \quad \theta_t = x, \quad \forall s \in [t, T]. \tag{4.1}
\]

The price process dynamics for the risky asset follows a geometric Brownian motion described by the scalar stochastic differential equation
\[
dP_s = \mu P_s ds + \sigma P_s dB_s, \quad P_t \in \mathbb{R} \setminus (-\infty, 0], \quad \forall s \in [t, T], \tag{4.2}
\]
where \( B_s \in \mathcal{BM}(\mathbb{R}) \) and \((\mu, \sigma) \in \mathbb{R} \times \mathbb{R} \). The parameters \( \mu \) and \( \sigma \) represents mean rate of return and volatility, of the price \( P_s \), respectively. Now, we can investigate the dynamics of the amount of currency that is held in the bank account and in the risky asset; specifically, the dynamics of \( X \) and \( Y \). The infinitesimal change of holdings in the bank account must satisfy the following relationship which is obtained when demanding conservation of cash flow in each node of the network.
\[
dx_s = \frac{d\theta_s}{\theta_s} X_s - (1 + cO + a) P_s dO_s + (1 - cI + b) P_s dI_s, \quad X_t = x, \tag{4.3}
\]
where \( dO_s \) and \( dI_s \) are the ongoing number of transactions in each direction at time \( s \). Analogously as for the bank account, the infinitesimal change of holdings in the risky asset is given by the relationship
\[
dY_s = \frac{dP_s}{P_s} Y_s + P_s dO_s - P_s dI_s, \quad Y_t = y. \tag{4.4}
\]

The same idea is behind this as for \textit{the law of conservation of mass} in fluid dynamics;

“\textit{The net rate of fluid flow into a control volume must be equal to the rate of change of fluid mass within the control volume.”}

Comparably; an account’s current holdings must be equal to the sum of the net rate of change within the account and the net flow into that account.

Deciding that short-selling will not be permitted, the current wealth of the portfolio is defined as
\[
\Pi_s := X_s + Y_s, \quad \Pi_t = \pi, \quad Y_s \geq 0, \quad X_s > 0, \quad \forall s \in [t, T], \tag{4.5}
\]
where \( \pi := x + y \) and the liquidated terminal wealth of the portfolio is
\[
\ell(X_T, Y_T) := X_T + (1 - cI - b) Y_T. \tag{4.6}
\]

Let us, for simplicity, introduce
\[
k_O := (1 + cO + a),
\]
and
\[
k_I := (1 - cI - b).
\]

Furthermore; consider the auxiliary \( C^2 \) mapping \( \phi : [t, T] \times \mathbb{R}^2 \to \mathbb{R} \) given by
\[
\phi(s, p, q) := p + q.
\]
then it holds that equation (4.3) is equivalent to

$$\Pi_s = \phi(s, X_s, Y_s), \forall s \in [t, T].$$

The differential for the liquidated portfolio wealth at time $s$ is thus in accordance with results from stochastic calculus expressed as

$$d\Pi_s = \partial_\phi(s, X_s, Y_s) ds + \frac{1}{2} \partial^2_\phi(s, X_s, Y_s) d\langle X \rangle_s + \frac{1}{2} \partial^2_\phi(s, X_s, Y_s) d\langle Y \rangle_s,$$

which neatly reduces to

$$d\Pi_s = dX_s + dY_s. \quad (4.7)$$

After inserting (4.1) and (4.2) into (4.3) and (4.4), followed by substituting the two lastly mentioned equations into (4.7), we arrive at the portfolio process dynamics,

$$d\Pi_s = (rX_s + \mu Y_s) ds + (1 - \kappa_O) P_s dO_s + (\kappa_I - 1) P_s dI_s + \sigma Y_s dB_s. \quad (4.8)$$

### 4.2 Model I

Consider the scenario with no transaction costs and the spread is zero, simultaneously. In other words, let the bid-ask spread as well as the transaction costs in each direction go to 0, leading to

$$(\kappa_O, \kappa_I) = (1, 1) \quad \text{as} \quad (c_O, c_I, a, b) = (0, 0, 0, 0),$$

which in turn leads to the dynamics

$$d\Pi_s = (rX_s + \mu Y_s) ds + \sigma Y_s dB_s. \quad (4.9)$$

By realising that we are bounded by a logical barrier; the fractions of the portfolio wealth must add up to a whole, the following variables are introduced. The fraction of the wealth held in the risky asset, $\nu_s^P$, at time $s$ and the remaining proportion that is allocated in the bank account, $\nu_s^\theta$, at the same time are defined by

$$\nu_s^P := \frac{Y_s}{\Pi_s}, \quad \nu_s^\theta := \frac{X_s}{\Pi_s}, \quad \nu_s^P + \nu_s^\theta = 1, \quad \forall s \in [t, T].$$

Set $\nu_s := \nu_s^P$, then the controlled portfolio process dynamics is expressed by the Itô diffusion process

$$d\Pi'_s = (r + (\mu - r) \nu_s) \Pi'_s ds + \sigma \nu_s \Pi'_s dB_s, \quad \Pi'_s = \pi, \quad \forall s \in [t, T]. \quad (4.10)$$

### 4.2.1 Risk management with isoelastic utility

Consider the isoelastic utility function, also known as a constant relative risk aversion (CRRA) function,

$$\Phi(p) = \frac{p^\gamma}{\gamma}, \quad \gamma \in (0, 1),$$

where the real constant $\gamma$ is the coefficient of risk aversion, which is proportional to the willingness to take risk. In other words, an increase in $\gamma$ reflects a larger risk appetite, and vice versa. In Figure 4.2 we can see how the function looks like for three different values of $\gamma$, thus illustrating curves for three risk preferences.
4.2.2 Control problem

The purpose is to decide how to perform the portfolio balancing in order to obtain maximum utility of the liquidated wealth \((4.10)\) at the terminal time \(T\), given the dynamics of the portfolio wealth for each \(s \in [t, T]\). Let the current wealth denote the state of our system and the fraction of the wealth that is allocated in the risky asset be our control function. Then we are interested in maximizing the functional

\[
J(t, \pi, \nu) = \mathbb{E}_{t, \pi}[\Phi(\Pi_T^\nu)].
\]

Since transaction costs and spreads are neglected and are all therefore equal to zero, it holds that the liquidated wealth is equivalent to the wealth itself and the control problem for this model is given by

\[
\begin{align*}
\sup_{\nu \in [0,1]} \mathbb{E}_{t, \pi}[\Phi(\Pi_T^\nu)] \\
\text{s.t.} \quad d\Pi_s^\nu = (r + (\mu - r)\nu_s)\Pi_s^\nu ds + \sigma\nu_s\Pi_s^\nu dB_s, \\
\Pi_t^\nu = \pi, \quad \forall (s, \Pi_s^\nu) \in [t, T] \times \mathbb{R} \setminus (-\infty, 0].
\end{align*}
\]

(4.11)

In accordance with previous theory regarding dynamic programming, the problem described by (4.11) gives rise to the following HJB equation, previously stated in (3.8), for a value function \(\psi(t, \pi)\).

\[
0 = \frac{\partial \psi}{\partial s}(t, \pi) + \sup_{\nu \in [0,1]} \left\{ (r + (\mu - r)\nu)\pi \frac{\partial \psi}{\partial \pi}(t, \pi) + \frac{1}{2}\sigma^2\nu^2\pi^2 \frac{\partial^2 \psi}{\partial \pi^2}(t, \pi) \right\},
\]

(4.12)

for each \((s, p) \in [t, T] \times \mathbb{R} \setminus (-\infty, 0]\), with terminal condition \(\psi(T, p) = \Phi(p)\). The static non-linear optimization problem of finding the supremum of the expression within braces in (4.12), is easily solved by finding a stationary point that is obtained by differentiating with respect to \(\nu\). Let us denote the partial derivatives as subscripts

\[
\frac{\partial \psi}{\partial \pi}(t, \pi) \equiv \psi_p,
\]

to avoid confusion. Then it must hold that

\[
0 = \frac{\partial}{\partial \nu} \left\{ (r + (\mu - r)\nu)\pi \psi_p + \frac{1}{2}\sigma^2\nu^2\pi^2 \psi_{pp} \right\} = (\mu - r)\pi \psi_p + \nu\sigma^2\pi^2 \psi_{pp},
\]

\[
\begin{aligned}
\text{Johan Dimitry} \\
\text{Figure 4.2: Visualisation of the utility function } \Phi(p) \text{ for three different risk profiles.}
\end{aligned}
\]
suggesting that the optimum \( \hat{\nu} \) is obtained at
\[
\hat{\nu} = -\frac{\mu - r}{\sigma^2} \frac{\psi_p}{\psi_{pp}}.
\] (4.13)

Substituting (4.13) into (4.12) now yields the non-linear partial differential equation
\[
0 = \psi_s + r \pi \psi_p - \frac{(\mu - r)^2 \psi_p^2}{2\sigma^2} \frac{\psi_{pp}}{\psi_{pp}}.
\] (4.14)

To solve the problem in (4.14), the ansatz
\[
\psi (t, \pi) := \delta (t) \frac{\pi^\gamma}{\gamma},
\]
is introduced, hence transforming the problem to
\[
0 = \pi^\gamma \left( \frac{1}{\gamma} \delta' + \delta \left( r - \frac{(\mu - r)^2}{2\sigma^2 (\gamma - 1)} \right) \right).
\]

Since \( x > 0 \) implies that \( x^\gamma > 0 \) for each \( \gamma \in (0, 1) \), we arrive at the first-order ODE problem
\[
\dot{\delta} + \delta \gamma \left( r - \frac{(\mu - r)^2}{2\sigma^2 (\gamma - 1)} \right) , \quad \delta (T) = 1,
\] (4.15)
where the terminal condition is derived from the fact that \( \psi (T, \pi) = \delta (T) \gamma^{-1} \pi^\gamma = \gamma^{-1} \pi^\gamma \).

Elementary calculations involving an integrating factor gives us the solution to (4.15) as
\[
\delta (t) = e^{-\gamma(t-T) \left( r - \frac{2(\mu-r)^2}{\gamma^2(\gamma-1)} \right)},
\]

Thus, the PDE in (4.14) is satisfied by
\[
\psi (t, \pi) = \frac{\pi^\gamma}{\gamma} e^{-\gamma(t-T) \left( r - \frac{2(\mu-r)^2}{\gamma^2(\gamma-1)} \right)},
\] (4.16)
which is the optimal value function for the stochastic optimal control problem (4.11) with optimal control given by substitution into equation (4.13),
\[
\hat{\nu} = \frac{r - \mu}{\sigma^2 (\gamma - 1)},
\]
suggesting that the optimal relative wealth allocations at each point in time are
\[
\hat{\nu}_s^P = \frac{r - \mu}{\sigma^2 (\gamma - 1)},
\] (4.17)
and
\[
\hat{\nu}_s^\theta = 1 - \frac{r - \mu}{\sigma^2 (\gamma - 1)}.
\] (4.18)

---

3After solving the HJB PDE, we will explicitly see that \( \hat{\nu} \) actually is a global maximizer of the non-linear optimization problem.
Remark 4.1. Note that if we take the second derivative, with respect to \( \nu \), of the expression subject to the supremum in equation (4.12), we get that
\[
\frac{\partial^2}{\partial \nu^2} \left\{ (r + (\mu - r)\nu) \pi \psi_p + \frac{1}{2} \sigma^2 \nu^2 \pi \psi_{pp} \right\} = \sigma^2 \pi \psi_{pp},
\]
which indeed suggests that \( \hat{\nu} \) is a global maximizer for the static non-linear optimization problem.

Remark 4.2. As seen in equation (4.17), the optimal control function is time-invariant and therefore suggests that the optimal strategy is to hold the same fraction of the total wealth in the risky asset all the time. Consequently; the same is true for the bank account holdings. Note that; it is not suggested that a constant amount of money is held in the risky asset or bank account, it is the ratios
\[
\frac{Y_s}{\Pi_s} \quad \text{and} \quad \frac{X_s}{\Pi_s},
\]
that are fixed for each \( s \in [t, T] \). There is however a direct dependence on the model imperative parameters, including the coefficient of risk aversion.

Given the optimal control (4.17) we can insert it into the dynamics in equation (4.10) and explicitly express the optimal portfolio wealth dynamics on integral form as
\[
\Pi_{s}^{\hat{\nu}} = \pi + \left( r - \frac{(\mu - r)^2}{\sigma^2 (\gamma - 1)} \right) \int_{t}^{s} \Pi_{u}^{\hat{\nu}} du - \frac{\mu - r}{\sigma (\gamma - 1)} \int_{t}^{s} \Pi_{u}^{\hat{\nu}} dB_u.
\]
Consider the function
\[
m(s) := E_{t, \pi} \left[ \Pi_{s}^{\hat{\nu}} \right],
\]
where \( m(t) = \pi \). Hence; it holds that
\[
m(s) = m(t) + E_{t, \pi} \left[ \left( r - \frac{(\mu - r)^2}{\sigma^2 (\gamma - 1)} \right) \int_{t}^{s} \Pi_{u}^{\hat{\nu}} du \right] - E_{t, \pi} \left[ \frac{\mu - r}{\sigma (\gamma - 1)} \int_{t}^{s} \Pi_{u}^{\hat{\nu}} dB_u \right],
\]
or equivalently
\[
m(s) = m(t) + \left( r - \frac{(\mu - r)^2}{\sigma^2 (\gamma - 1)} \right) \int_{t}^{s} m(u) du - \frac{\mu - r}{\sigma (\gamma - 1)} E_{t, \pi} \left[ \int_{t}^{s} \Pi_{u}^{\hat{\nu}} dB_u \right].
\]
Now consider the following definition.

Definition 4.1. Let \( \mathcal{B}([0, \infty)) \) be the Borel \( \sigma \)-algebra on the non-negative real line and define the set \( \mathcal{T} := [t, T] \subseteq [0, \infty) \). Then \( \mathcal{W} = \mathcal{W}(\mathcal{T}) \) is defined to be the class of functions
\[
\zeta(s, \tilde{\omega}) : \mathcal{T} \times \Omega \to \mathbb{R},
\]
satisfying
i) \( (s, \tilde{\omega}) \to \zeta(s, \tilde{\omega}) \in \mathcal{B}([0, \infty)) \times \mathcal{F}, \)
ii) \( \zeta(s, \tilde{\omega}) \) is \( \mathcal{F}_s \)-adapted,
iii) \( \zeta(s, \tilde{\omega}) \in \mathcal{L}^2(ds \otimes d\mathbb{P}) \).
If we assume that $\Pi^\pi_u \in W(\mathcal{T})$, then it holds for the stochastic integral that

$$E_{t,\pi} \left[ \int_t^s \Pi^\pi_u dB_u \right] = 0,$$

and is therefore leading to

$$m(s) = m(t) + \left( r - \frac{(\mu - r)^2}{\sigma^2 (\gamma - 1)} \right) \int_t^s E_{t,\pi} \left[ \Pi^\pi_u \right] du.$$

Differentiating $m$ with respect to $s$ gives us the following initial value problem, a first-order ordinary differential equation

$$\frac{dm}{ds} = \left( r - \frac{(\mu - r)^2}{\sigma^2 (\gamma - 1)} \right) m(s), \quad m(t) = \pi,$$

which is satisfied by

$$m(s) = \pi e^{(s-t) \left( r - \frac{(\mu - r)^2}{\sigma^2 (\gamma - 1)} \right)}.$$  \hspace{1cm} (4.20)\hspace{1cm}

The expression in equation (4.20) is by definition the expected value of the optimal portfolio wealth, i.e.

$$E \left[ \Pi^\pi_s \right] = \pi e^{(s-t) \left( r - \frac{(\mu - r)^2}{\sigma^2 (\gamma - 1)} \right)}.$$  \hspace{1cm} (4.21)\hspace{1cm}

In order to obtain the optimal portfolio variance, consider the function

$$n(s) := E_{t,\pi} \left[ \left( \Pi^\pi_u \right)^2 \right], \quad n(t) = \pi^2,$$

then the variance is given by

$$\text{Var}_{t,\pi} \left[ \Pi^\pi_s \right] = n(s) - m^2(s).$$  \hspace{1cm} (4.22)\hspace{1cm}

The squared optimal portfolio dynamics is with help from stochastic integration by parts expressed as

$$\left( \Pi^\pi_s \right)^2 = \pi^2 + 2 \int_t^s \Pi^\pi_u \Pi^\pi_u dB_u + \left( \Pi^\pi_s, \Pi^\pi_s \right)_s$$

$$= \pi^2 + 2 \left( r - \frac{(\mu - r)^2}{\sigma^2 (\gamma - 1)} \right) \int_t^s \left( \Pi^\pi_u \right)^2 du - \frac{\mu - r}{\sigma (\gamma - 1)} \int_t^s \left( \Pi^\pi_u \right)^2 dB_u$$

$$+ \left\{ \frac{\mu - r}{\sigma (\gamma - 1)} \int_t^s \Pi^\pi_u dB_u, \frac{\mu - r}{\sigma (\gamma - 1)} \int_t^s \Pi^\pi_u dB_u \right\}_s$$

$$= \pi^2 + 2 \left( r - \frac{(\mu - r)^2}{\sigma^2 (\gamma - 1)} \right) \int_t^s \left( \Pi^\pi_u \right)^2 du$$

$$- \frac{\mu - r}{\sigma (\gamma - 1)} \int_t^s \left( \Pi^\pi_u \right)^2 dB_u + \left( \frac{\mu - r}{\sigma (\gamma - 1)} \right)^2 \int_t^s \left( \Pi^\pi_u \right)^2 d\langle B \rangle_u$$

$$= \pi^2 + 2 \left( r - \frac{\mu - r}{\gamma - 1} \frac{2\gamma - 3}{\sigma^2} \right) \int_t^s \left( \Pi^\pi_u \right)^2 du - \frac{\mu - r}{\sigma (\gamma - 1)} \int_t^s \left( \Pi^\pi_u \right)^2 dB_u,$$

where $\langle \cdot \rangle$ denotes the quadratic variation. The function $n(s)$ now satisfies the relationship

$$n(s) = n(t) + 2 \left( r - \frac{(\mu - r)^2}{(\gamma - 1)} \frac{2\gamma - 3}{\sigma^2} \right) \int_t^s n(u) du,$$
due to Itô isometry in accordance with the assumption that \((\Pi^\nu_t)^2 \in \mathcal{W}(T)\). Differentiation with respect to \(s\) leads to the initial value problem
\[
\frac{dn}{ds} = 2 \left( r - \left( \frac{\mu - r}{\gamma - 1} \right)^2 \frac{2\gamma - 3}{\sigma^2} \right) n(s), \quad n(t) = \pi^2,
\]
which is satisfied by
\[
n(s) = \pi^2 e^{2(s-t) \left( r - \left( \frac{\mu - r}{\sigma^2(\gamma - 1)} \right)^2 \right)}.
\]
By substituting equations (4.20) and (4.23) into the relationship (4.22) the variance of the optimal portfolio wealth becomes
\[
\text{Var}_{t,\pi} \left[ \Pi^\nu_s \right] = \left( \pi e^{(s-t) \left( r - \left( \frac{\mu - r}{\sigma^2(\gamma - 1)} \right)^2 \right)} \right)^2 \left( e^{(s-t) \left( \frac{\mu - r}{\sigma^2(\gamma - 1)} \right)^2} - 1 \right). \tag{4.24}
\]

4.3 Model II

The dynamics of the portfolio is obtained as before, but here we let our control be the absolute amount of wealth placed in the risky assets, not the fraction of the total wealth as considered in Model I. Accordingly,
\[
\nu_s := Y_s,
\]
suggesting that
\[
X_s = \Pi_s - Y_s = \Pi_s - \nu_s,
\]
and equation (4.9) thus translates to
\[
d\Pi^\nu_s = (r\Pi^\nu_s + (\mu - r) \nu_s) \, ds + \sigma \nu_s dB_s, \quad \Pi^\nu_t = \pi. \tag{4.25}
\]

4.3.1 Risk management with state-dependent mean-variance functional

This model is considering the problem of maximizing the functional
\[
\mathcal{J}(t,\pi,\nu) = \mathbb{E}_{t,\pi} \left[ \Pi^\nu_T \right] - \frac{\gamma}{2\pi} \text{Var}_{t,\pi} \left[ \Pi^\nu_T \right], \tag{4.26}
\]
where \(\Pi^\nu_s\) satisfy the SDE in (4.25). The idea here is to follow the rationale presented by Björk et al. in [32], where it is interesting to investigate risk aversion per initial endowment of the investor. Further, a dimensional analysis shows that it makes perfect sense to have \(\text{money}^{-1}\) as dimension of the risk aversion, since this suggests a dimension-wise balance in the performance functional. The functional (4.26) leads to the fact that the equilibrium control problem we are faced with fits into special case 4.3.3 in [1].

4.3.2 Equilibrium control problem

Again, the absence of transaction costs and spreads neutralizes the difference between wealth and liquidated wealth. Having this in mind, we start off by re-writing the performance functional in (4.26) in accordance with
\[
\mathbb{E}_{t,\pi} \left[ \Pi^\nu_T \right] - \frac{\gamma}{2\pi} \text{Var}_{t,\pi} \left[ \Pi^\nu_T \right] = \mathbb{E}_{t,\pi} \left[ \Pi^\nu_T - \frac{\gamma}{2\pi} (\Pi^\nu_T)^2 \right] + \frac{\gamma}{2\pi} (\mathbb{E}_{t,\pi} \left[ \Pi^\nu_T \right])^2,
\]
where a simple rearrangement has been performed after taking advantage of the relationship
\[
\text{Var} \left[ (\cdot) \right] = \mathbb{E} \left[ (\cdot)^2 \right] - (\mathbb{E} \left[ (\cdot) \right])^2.
\]
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The control problem can now be formulated as the following time-inconsistent stochastic control problem,

\[
\sup_{\nu \in (0, \Pi)} E_{t, \pi} \left[ \Pi_T^\nu - \frac{\gamma}{2\pi} (\Pi_T^\nu)^2 \right] + \frac{\gamma}{2\pi} (E_{t, \pi} [\Pi_T^\nu])^2
\]

s.t. \[d\Pi_s = (r\Pi_s + (\mu - r) \nu_s) \, ds + \sigma \nu_s \, dB_s, \quad \Pi_0 = \pi, \quad \forall (s, \Pi_s^\nu) \in [t, T] \times \mathbb{R} \setminus (-\infty, 0]. \tag{4.27}\]

By comparing with \[\text{(3.10)}, \text{we realize that}\]

\[
V(p, q) = q - \frac{q^2}{2p}, \tag{4.28}\]

\[
W(p, q) = \gamma \frac{q^2}{2p}, \tag{4.29}\]

\[
\vartheta(s, p, q) = E_{s, p} \left[ V(q, \Pi_T^p) \right], \tag{4.30}\]

and

\[
\omega(s, p) = E_{s, p} \left[ \Pi_T^p \right]. \tag{4.31}\]

With the given information in \[\text{(4.28), (4.31),}\] a couple of tedious calculations enables us to explicitly express the operations present in \[\text{(3.15)}\] as follows, where each subscript still corresponds to a partial derivative with respect to the subscripted variable.

\[
\left( \hat{\mathcal{L}}^\nu \vartheta \right)(t, \pi) = \vartheta_s(t, \pi, \pi) + (r\pi + (\mu - r) \nu) \vartheta_p(t, \pi, \pi) + \frac{1}{2} \sigma^2 \nu^2 \vartheta_{pp}(t, \pi, \pi),
\]

\[
\left( \hat{\mathcal{L}}^\nu \vartheta \right)(t, \pi, \pi) = \vartheta_s(t, \pi, \pi) + (r\pi + (\mu - r) \nu) \left\{ \vartheta_p(t, \pi, \pi) + \vartheta_q(t, \pi, \pi) \right\}
\]

\[
+ \frac{1}{2} \sigma^2 \nu^2 \left\{ \vartheta_{pp}(t, \pi, \pi) + 2\vartheta_{pq}(t, \pi, \pi) + \vartheta_{qq}(t, \pi, \pi) \right\},
\]

\[
\hat{\mathcal{L}}^\nu (W \odot \omega)(t, \pi) = W_q(\pi, \omega(t, \pi)) \omega_s(t, \pi)
\]

\[
+ (r\pi + (\mu - r) \nu) \left\{ W_p(\pi, \omega(t, \pi)) + W_q(\pi, \omega(t, \pi)) \omega_p(t, \omega) \right\}
\]

\[
+ \frac{1}{2} \sigma^2 \nu^2 \left\{ W_{pp}(\pi, \omega(t, \pi)) + W_q(\pi, \omega(t, \pi)) \omega_{pp}(t, \pi)
\]

\[
+ 2W_{pq}(\pi, \omega(t, \pi)) \omega_p(t, \pi) + W_{qq}(\pi, \omega(t, \pi)) \omega_p(t, \pi)^2 \right\},
\]

and

\[
(M^\nu \omega)(t, \pi) = W_q(\pi, \omega(t, \pi)) \left\{ (r\pi + (\mu - r) \nu) \omega_p(t, \pi) + \frac{1}{2} \sigma^2 \nu^2 \omega_{pp}(t, \pi) \right\}.
\]

By embedding these expressions into the equations of Definition \[\text{3.6}, \text{and attaining that}\]

\[
\psi(t, \pi) = \vartheta(t, \pi, \pi) + \frac{\gamma}{2\pi} \omega(t, \pi)^2, \tag{4.32}\]

the HJB type equation for this particular control problem reduces to

\[
0 = \vartheta_s + \frac{\omega_s}{\pi} + \sup_{\nu \in (0, \Pi)} \left\{ (r\pi + (\mu - r) \nu) \left[ \vartheta_p + \frac{\omega_p}{\pi} \right] + \frac{1}{2} \sigma^2 \nu^2 \left[ \vartheta_{pp} + \frac{\omega_{pp}}{\pi} \right] \right\}, \tag{4.33}\]

\[
0 = \vartheta_s + (r\pi + (\mu - r) \nu) \vartheta_p + \frac{1}{2} \sigma^2 \nu^2 \vartheta_{pp}, \quad q \equiv \pi, \tag{4.34}\]
and
\[ 0 = \omega_s + (r\pi + (\mu - r) \hat{\nu}) \omega_p + \frac{1}{2} \sigma^2 \hat{\nu}^2 \omega_{pp}. \]  
(4.35)

Where the points of evaluation of each function have been suppressed and \( \hat{\nu} \) is the global maximizer of the static optimization problem.

Now, the plan is to determine the optimal equilibrium control candidate by solving the static non-linear optimization problem given by the supremum in equation (4.33). Before we solve the optimization problem explicitly, we need to understand what the expression actually means in terms of derivatives of the probabilistically interpreted functions \( \vartheta \) and \( \omega \). For this reason and for the purpose of actually being able to solve the non-linear PDE we can introduce the following ansatz, which in some sense suggests that the portfolio dynamics is following a Geometric Brownian Motion. Let
\[ \hat{\nu}(s, p) := \delta(s) p, \]

suggesting that
\[ E_{s, p} \left[ \Pi_T^\hat{\nu} \right] := \alpha(s) p, \]  
(4.36)
and
\[ E_{s, p} \left[ (\Pi_T^\hat{\nu})^2 \right] := \beta(s) p^2. \]  
(4.37)

The ansatz will transform our system of partial differential equations into a system of ordinary differential equations subsequently to the recognitions
\[ \vartheta(s, p, q) = \alpha(s) p - \frac{\gamma}{2\pi} \beta(s) p^2, \]  
(4.38)
and
\[ \omega(s, p) = \alpha(s) p. \]  
(4.39)

An implicit expression for the equilibrium control is obtained by finding the stationary point that satisfies
\[ 0 = \frac{\partial}{\partial \nu} \left\{ (r\pi + (\mu - r) \nu) \left[ \vartheta_p + \frac{\gamma \omega_p}{\pi} \right] + \frac{1}{2} \sigma^2 \nu^2 \left[ \vartheta_{pp} + \frac{\gamma \omega_{pp}}{\pi} \right] \right\}, \]

suggesting that
\[ \hat{\nu} = -\frac{(\mu - r)}{\sigma^2} \frac{\vartheta_p \pi + \gamma \omega_p}{\vartheta_{pp} \pi + \gamma \omega_{pp}}. \]  
(4.40)

After taking (4.38) and (4.39) into consideration with the result (4.40), we arrive at
\[ \hat{\nu}(t, \pi) = \frac{\mu - r}{\gamma \sigma^2} \cdot \frac{\alpha(t) + \gamma \left[ \alpha^2(t) - \beta(t) \right]}{\beta(t)} \pi, \]  
(4.41)
giving us the following relationship between \( \alpha \), \( \beta \) and \( \delta \),
\[ \delta(s) = \frac{\mu - r}{\gamma \sigma^2} \cdot \frac{\alpha(s) + \gamma \left[ \alpha^2(s) - \beta(s) \right]}{\beta(s)}. \]  
(4.42)

Inserting the equilibrium control into (4.33) yields the PDE
\[ 0 = \vartheta_s + \frac{\gamma \omega_p}{\pi} + (r\pi + (\mu - r) \hat{\nu}) \left[ \vartheta_p + \frac{\gamma \omega_p}{\pi} \right] + \frac{1}{2} \sigma^2 \hat{\nu}^2 \left[ \vartheta_{pp} + \frac{\gamma \omega_{pp}}{\pi} \right], \]

which is a linear combination of the remaining two equations of the extended HJB system, specifically it is one of the second equation, (4.34), plus \( \gamma \omega/\pi \) of the third equation, (4.35).
problem is thus reduced to solving the non-linear PDEs corresponding to (4.34) and (4.35), due to the ansatz the equations are transformed as follows. Equation (4.34) becomes

\[ 0 = \left\{ \dot{\alpha} - \gamma \beta + \left[ r + \frac{(\mu - r)^2}{\gamma \sigma^2} \cdot \frac{\alpha + \gamma (\alpha^2 - \beta)}{\beta} \right] (\alpha - \gamma \beta) - \frac{(\mu - r)^2}{2 \gamma \sigma^2} \cdot \frac{(\alpha + \gamma (\alpha^2 - \beta))^2}{\beta} \right\} \pi, \]  

(4.43)

and equation (4.35) becomes

\[ 0 = \left\{ \dot{\alpha} + \alpha \left[ r + \frac{(\mu - r)^2}{\gamma \sigma^2} \cdot \frac{\alpha + \gamma (\alpha^2 - \beta)}{\beta} \right] \right\} \pi. \]  

(4.44)

Equation (4.43) can be re-written as a system of ordinary differential equations

\[
\begin{align*}
0 &= \dot{\alpha} + \left( r + \frac{(\mu - r)^2}{\gamma \sigma^2} \cdot \frac{\alpha + \gamma (\alpha^2 - \beta)}{\beta} \right) \alpha, \\
0 &= \dot{\beta} + \beta \left[ 2r + 2 \frac{(\mu - r)^2}{\gamma \sigma^2} \cdot \frac{\alpha + \gamma (\alpha^2 - \beta)}{\beta} + \frac{(\mu - r)^2}{\gamma^2 \sigma^2} \cdot \frac{(\alpha + \gamma (\alpha^2 - \beta))^2}{\beta^2} \right],
\end{align*}
\]

(4.45a, 4.45b)

suggesting that equation (4.44) is redundant since it is actually equivalent to (4.45a). The boundary conditions become

\[ \alpha (T, \pi) = 1, \]

and

\[ \beta (T, \pi) = 1, \]

and all we have to do is solve (4.45a)-(4.45b) and insert into (4.41) and (4.32) to obtain the equilibrium control and the optimal value function respectively. Due to the non-linearity of the system (4.45a)-(4.45b), the Lipschitz and linear growth conditions are not satisfied, in other words uniqueness and existence can not be guaranteed for an eventual solution.

Fortunately there is a way around the recently mentioned problem, we will express an equation for \( \delta (t) \) and then we can take advantage of the relationship (4.42) to develop a numerical algorithm for determining the expressions containing \( \alpha (t) \) and \( \beta (t) \). Consider the Geometric Brownian Motion that describes the portfolio’s wealth process

\[
d\Pi^\nu_s = (r + (\mu - r) \delta (s)) \Pi^\nu_s ds + \sigma \delta (s) \Pi^\nu_s dB_s, \quad \Pi^\nu_t = \pi, 
\]

which on integral form is

\[
\Pi^\nu_s = \pi + \int_t^s (r + (\mu - r) \delta (u)) \Pi^\nu_u du + \sigma \int_t^s \delta (u) \Pi^\nu_u dB_u. 
\]

Assuming that \( \delta (u) \Pi^\nu_u \in \mathcal{W} (T) \) and \( \delta (u) (\Pi^\nu_u)^2 \in \mathcal{W} (T) \), we would like to express the relationships (4.36) and (4.37) in terms of \( \delta (s) \) in order to reveal how \( \alpha, \beta \) and \( \delta \) relate explicitly. Let us define

\[ m (s) := E_{t, \pi} \left[ \Pi^\nu_s \right], \quad m (t) = \pi, \]

which is equivalent to

\[
m (s) = m (t) + \int_t^s (r + (\mu - r) \delta (u)) E_{t, \pi} \left[ \Pi^\nu_u \right] du + \sigma E_{t, \pi} \left[ \int_t^s \delta (u) \Pi^\nu_u dB_u \right]. 
\]
Since it is assumed that \( \delta (u) \Pi_u^\nu \in \mathcal{W} (T) \), the stochastic integral vanishes and we arrive at
\[
m (s) = m (t) + \int_t^s \left( r + (\mu - r) \delta (u) \right) m_u du.
\]
Differentiation with respect to \( s \) yields the first order ODE
\[
\frac{dm}{ds} = (r + (\mu - r) \delta (s)) \ m (s), \ \ m (t) = \pi.
\]
The ODE is satisfied by
\[
m (s) = \pi e^{\int_t^s (r + (\mu - r) \delta (u)) du},
\]
leading to
\[
E_{t, \pi} \left[ \Pi_s^\nu \right] = \pi e^{\int_t^s (r + (\mu - r) \delta (u)) du}, \tag{4.46}
\]
giving us, in accordance with the relationships \((4.36)\), the expectation \((4.46)\) evaluated at time \( s = T \),
\[
E_{t, \pi} \left[ \Pi_T^\nu \right] = \pi e^{\int_t^T (r + (\mu - r) \delta (s)) ds}.
\]
Moving on to consider the expectation of the squared wealth process, define
\[
n (s) := E_{t, \pi} \left[ \left( \Pi_s^\nu \right)^2 \right], \ \ n (t) = \pi^2.
\]
The squared portfolio wealth process is, with help from stochastic integration by parts and The Itô Isometry, expressed as
\[
E_{t, \pi} \left[ \left( \Pi_s^\nu \right)^2 \right] = E_{t, \pi} \left[ \pi^2 + 2 \int_t^s \Pi_u^\nu d\Pi_u^\nu + \left\langle \Pi^\nu, \Pi^\nu \right\rangle_s \right] = \pi^2 + 2 \int_t^s \Pi_u^\nu d\Pi_u^\nu + \left\langle \sigma \int_t^s \delta (u) \Pi_u^\nu dB_u, \sigma \int_t^s \delta (u) \Pi_u^\nu dB_u \right\rangle_s
\]
\[
= \pi^2 + 2 \int_t^s \left( r + (\mu - r) \delta (u) + \frac{1}{2} \sigma^2 \delta^2 (u) \right) E_{t, \pi} \left[ \left( \Pi_u^\nu \right)^2 \right] du + 2\sigma E_{t, \pi} \left[ \int_t^s \delta (u) \left( \Pi_u^\nu \right)^2 dB_u \right] _\|_{\|0}
\]
due to the assumption that \( \delta (u) \left( \Pi_u^\nu \right)^2 \in \mathcal{W} (T) \). Thus we arrive at
\[
n (s) = n (t) + 2 \int_t^s \left( r + (\mu - r) \delta (u) + \frac{1}{2} \sigma^2 \delta^2 (u) \right) n_u du,
\]
and in turn, after differentiated with respect to \( s \), we get the ODE
\[
\frac{dn}{ds} = 2 \left( r + (\mu - r) \delta (s) + \frac{1}{2} \sigma^2 \delta^2 (s) \right) n (s), \ \ n (t) = \pi^2. \tag{4.47}
\]
The ODE in \((4.47)\) is satisfied by
\[
n (s) = \pi^2 e^{2 \int_t^s \left( r + (\mu - r) \delta (u) + \frac{1}{2} \sigma^2 \delta^2 (u) \right) du},
\]
which directly translates to
\[
E_{t, \pi} \left[ \left( \Pi_s^\nu \right)^2 \right] = \pi^2 e^{2 \int_t^s \left( r + (\mu - r) \delta (u) + \frac{1}{2} \sigma^2 \delta^2 (u) \right) du}, \tag{4.48}
\]

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and by the relationship (4.37), we know that evaluating (4.48) at \( s = T \), we arrive at
\[
E_{t, \pi} \left[ \left( \Pi_T^\nu \right)^2 \right] = \pi^2 \frac{2 \int_t^T \left( r + (\mu - r) \delta(s) + \frac{1}{2} \sigma^2 \delta^2(s) \right) ds}{\beta(t)},
\]
and now \( \alpha \) and \( \beta \) are expressed in terms of \( \delta \) as requested.

**Remark 4.3.** Note that the variance of the portfolio’s wealth process now is easily obtained from the relationship

\[
\text{Var}_{t, \pi} \left[ \Pi_T^\nu \right] = E_{t, \pi} \left[ \left( \Pi_T^\nu \right)^2 \right] - \left( E_{t, \pi} \left[ \Pi_T^\nu \right] \right)^2 = \left( \pi e^{\int_t^T (r + (\mu - r) \delta(s)) ds} \right)^2 \left( e^{\int_t^T \sigma^2 \delta^2(s) ds} - 1 \right),
\]

which is going to be useful for further analysis.

In order for \( \alpha \) and \( \beta \) to satisfy (4.42) it must hold that \( \delta \) satisfies the integral equation

\[
\delta(t) = \frac{\mu - r}{\gamma \sigma^2} \left( e^{-\int_t^T (r + (\mu - r) \delta(s)) ds} + \gamma e^{-\int_t^T \sigma^2 \delta^2(s) ds} - \gamma \right),
\]

making it clear that once knowing what \( \delta(t) \) is, the equilibrium control is obtained by multiplying with \( \pi \) and the equilibrium value function is thus obtained from the following expression, after substituting \( \alpha \) and \( \beta \) into equation (4.32).

\[
\psi(t, \pi) = \pi \left\{ e^{\int_t^T (r + (\mu - r) \delta(s)) ds} + \frac{\gamma}{2} e^{\int_t^T \sigma^2 \delta^2(s) ds} \left( 1 - e^{\int_t^T \sigma^2 \delta^2(s) ds} \right) \right\}.
\]

In the upcoming chapter we will show how equation (4.50) can be solved numerically with a convergent recursive approach.
Chapter 5

Algorithm design

In order to be able to conduct any actions with help from the suggestions by our performed analysis, we need to understand where every piece of the puzzle fits, where it comes from and in which order it should be implemented. Therefore, an algorithm design is necessary for each model, before an implementation can be made in a trading system. The first sections reveal neat ways of calibrating the model-imperative parameters as well as a numerical solution of equation (4.50), followed by potential designs for each model’s execution algorithm.

5.1 Parameter calibration

Since we assume that the risky asset follows a Geometric Brownian Motion with constant coefficients, as described in by the stochastic differential equation (4.2), we can obtain an explicit expression for the asset’s price at each point in time by solving the SDE. From (4.2) we deduce that

\[
\int_1^x \frac{dP_t}{P_t} = \mu \int_t^x ds + \sigma \int_t^x dB_s = \mu (x - t) + \sigma (B_x - B_t) - \frac{1}{2} \sigma^2 (x - t),
\]

(5.1)

where \(B_x - B_t \sim \mathcal{N}(0, x - t)\). Note that the second integral in (5.1) is a stochastic Itô integral and has been evaluated accordingly, further reading is available in Appendix A. By introducing the standard normally distributed random variable \(Z \sim \mathcal{N}(0, 1)\), the expression in equation (5.1) becomes

\[
\ln \frac{P_x}{P_t} = \left( \mu - \frac{1}{2} \sigma^2 \right) (x - t) + Z \sqrt{x - t},
\]

or equivalently

\[
P_x = P_t e^{\left( \mu - \frac{1}{2} \sigma^2 \right) (x - t) + Z \sqrt{x - t}}.
\]

(5.2)

If historical data, observations of prices, have been collected for a specific asset and if we denote its price on day \(j\) by \(P_j\), then in accordance with (5.2) it holds that

\[
P_{j+1} = P_j e^{(\mu - \frac{1}{2} \sigma^2) \Delta s + \sigma Z \sqrt{\Delta s}}, \quad Z \sim \mathcal{N}(0, 1),
\]

(5.3)

where we have partitioned the time line

\[
t = s_1 < s_2 < \cdots < s_j < s_{j+1} < \cdots < s_{N-1} < s_N = T,
\]

and defined \(\Delta s := s_{j+1} - s_j\) is the time-difference between the collected data points, \(P_{j+1}\) and \(P_j\). Now, we want to determine expressions for \(\mu\) and \(\sigma\), from equation (5.3) it is obtained that

\[
\ln \frac{P_{j+1}}{P_j} = \left( \mu - \frac{1}{2} \sigma^2 \right) \Delta t + \sigma Z \sqrt{\Delta s},
\]
which in turn leads to the system of equations
\[ \mathbb{E} \left[ \ln \frac{P_{j+1}}{P_j} \right] = \left( \mu - \frac{1}{2} \sigma^2 \right) \Delta s, \]  
and
\[ \text{Var} \left[ \ln \frac{P_{j+1}}{P_j} \right] = \sigma^2 \Delta s. \]  
Equation (5.5) yields
\[ \tilde{\sigma} = \frac{1}{\sqrt{\Delta s}} \sqrt{\text{Var} \left[ \ln \frac{P_{j+1}}{P_j} \right]}, \]  
and equation (5.4) thus transforms into
\[ \tilde{\mu} = \frac{1}{\Delta s} \left( \mathbb{E} \left[ \ln \frac{P_{j+1}}{P_j} \right] + \frac{1}{2} \text{Var} \left[ \ln \frac{P_{j+1}}{P_j} \right] \right). \]  
Let us, for discrete data points \( \{ \varpi_j \}_{j=1}^N \) that are observations of a stochastic variable \( \varpi \), we have that
\[ \mathbb{E} [\varpi] = \frac{1}{N} \sum_{j=1}^{N-1} \varpi_j, \]
and
\[ \text{Var} [\varpi] = \frac{1}{N-1} \sum_{j=1}^{N-1} (\varpi_j - \mathbb{E} [\varpi])^2. \]
Using these expressions, we can finally estimate \( \mu \) and \( \sigma \) from the given observations in accordance with the following relationships
\[ \tilde{\mu} = \frac{1}{\Delta s} \left\{ \frac{1}{N} \sum_{j=1}^{N-1} \ln \frac{P_{j+1}}{P_j} + \frac{1}{2} \frac{1}{N-1} \sum_{j=1}^{N-1} \left( \ln \frac{P_{j+1}}{P_j} - \frac{1}{N} \sum_{j=1}^{N-1} \ln \frac{P_{j+1}}{P_j} \right)^2 \right\}, \]  
and
\[ \tilde{\sigma} = \frac{1}{\sqrt{\Delta s}} \sqrt{\frac{1}{N-1} \sum_{j=1}^{N-1} \left( \ln \frac{P_{j+1}}{P_j} - \frac{1}{N} \sum_{j=1}^{N-1} \ln \frac{P_{j+1}}{P_j} \right)^2}. \]  
Note that,
\[ \sum_{j=1}^{N-1} \ln \frac{P_{j+1}}{P_j} = \ln \frac{P_N}{P_1} + \ln \frac{P_{N-1}}{P_{N-2}} + \cdots + \ln \frac{P_3}{P_2} + \ln \frac{P_2}{P_1} = \ln P_N - \ln P_{N-1} + \ln P_{N-1} - \ln P_{N-2} + \cdots + \ln P_3 - \ln P_2 + \ln P_2 - \ln P_1 = \ln P_N - \ln P_1 = \ln \frac{P_N}{P_1}, \]
that is, equations (5.8) and (5.9) can be simplified to
\[ \tilde{\mu} = \frac{1}{\Delta s} \left\{ \frac{1}{N} \ln \frac{P_N}{P_1} + \frac{1}{2} \frac{1}{N-1} \sum_{j=1}^{N-1} \left( \ln \frac{P_{j+1}}{P_j} - \frac{1}{N} \ln \frac{P_N}{P_1} \right)^2 \right\}, \]  
and
\[ \tilde{\sigma} = \frac{1}{\sqrt{\Delta s}} \sqrt{\frac{1}{N-1} \sum_{j=1}^{N-1} \left( \ln \frac{P_{j+1}}{P_j} - \frac{1}{N} \ln \frac{P_N}{P_1} \right)^2}. \]
5.2 Numerical approach to solving the integral equation

Assuming that $\mu > r$ we can construct a sequence $\{\delta_n(t)\}_{n \geq 0} \in C[0, T]$, where $\delta_0 = 1$ and
\[
\delta_n(t) = \frac{\mu - r}{\gamma \sigma^2} \left( e^{-\int_t^T (r + (\mu - r)\delta_n-1(s) + \sigma^2 \delta_n-1(s) \, ds)} + \gamma e^{-\int_t^T \sigma^2 \delta_n-1(s) \, ds} - \gamma \right), \tag{5.12}
\]

Björk et al. in [32] uses the recursive relationship in (5.12) to show that $\delta(t)$ has a unique solution in $C[0, T]$ by first proving that $\{\delta_n(t)\}_{n \geq 0}$ as well as $\{\dot{\delta}_n(t)\}_{n \geq 0}$ are uniformly bounded in $C[0, T]$, further they use the Arzela-Ascoli Theorems and the Grönwall Inequality to prove existence and uniqueness of a solution $\delta(t)$ as $n \to \infty$ for each $t \in [0, T]$. In other words we know that $\delta_n(t) \to \delta(t)$ as $n \to \infty$ for all $t \in [0, T]$, which in turn suggests that we can construct an algorithm that takes advantage of the recursive relation (5.12) and computes arbitrarily accurate approximations of $\delta$ as we choose $n$ to be sufficiently large.

5.2.1 Approximating $\delta(t)$

Using the recursive relationship provided by equation (5.12), we are able to obtain a sufficiently good approximation of the function $\delta(t)$, more precisely we find that 10 iterations yields an error of approximately $10^{-6}$. In Figure 5.1 it is possible to see the behaviour of the process of convergence for 8 iterations. This was performed with parameter values chosen such that the curves are generated in a way that emphasizes their interesting behaviour. Particularly, the parameter values for this case are set to: $r = 4\%$, $\mu = 6.2\%$, $\sigma = 41\%$, $\gamma = 1$ and $T = 20$.

![Convergence of $\delta_n(t)$](image)

Figure 5.1: An illustration of how fast each iteration closes in on the solution. Note how every iteration is alternating sides (below/above) the higher step’s curve, $\delta_7$ is below $\delta_6$ which is above $\delta_5$ which in turn is below $\delta_4$ and so on.
5.3 Algorithm - Model I

The main idea is that the algorithm should enforce the strategy that is imposed by the analysis performed for Model I. That is, create a mechanism that keeps everything on track as time evolves by continuous rebalancing, so that the suggested optimal portfolio balance, given by \( \hat{\nu} \), is maintained at all times. It is the algorithm’s job to make sure that every change in the value of the holdings, both in the bank and in the risky asset, is accounted for and leads to an immediate reallocation of wealth.

In the schematic overview of the algorithm in Figure 5.2, it is seen that the first step is to calibrate the markets’ parameters, for which the calibration of the drift \( \mu \) and volatility \( \sigma \) is performed in accordance with equations (5.10) and (5.11). The risk-free rate of return on the bank holdings can be calibrated by considering a non-arbitrage market consisting of government bonds and equate the risk free rate with the discount rate of the bonds’ generated cash-flows, it is the same idea as for the zero-rate bootstrapping method suggested by Hult et al. in [39]. Although; for the purpose of this thesis we will just assign any real valued constant, smaller than \( \mu \), for the risk-free rate when producing the results presented in chapter 6. Thereafter the optimal balance, determined with the help of the optimal fraction of wealth to be invested in the risky asset (equation (4.17)), is computed. Assuming that the trading is initiated with all the wealth allocated in the bank account, the natural next move would be to acquire the prescribed amount of the risky asset that would correspond to the optimal balance. Thus, the holdings in the risky asset at the initial time must be

\[
Y_t = \hat{\nu} \pi.
\]

After a pre-determined time interval \( \Delta t \), an update-rate chosen to be e.g. 15 minutes or even 15 days, the bank holdings \( X_t \) will have changed value as well as the risky asset’s price \( P_t \), implying a change in the holdings \( Y_t \). Thus, if \( t \neq T \), the algorithm will have to investigate if and how the portfolio balance has deviated from the optimal value. In other words, the interesting quantity to measure is

\[
\left| \frac{\hat{\nu} - Y_t}{P_t} \right| = \left| \hat{\nu} - \frac{Y_t}{X_t + Y_t} \right| \Leftrightarrow \left| Y_t - \frac{\hat{\nu}}{1 - \hat{\nu}} X_t \right|,
\]

to understand how much we need to acquire or liquidate at each point in time to commit to the optimal strategy, until time \( T \) where we will liquidate everything. Further detail, although still on a relatively high level, regarding the decision processes and execution procedure is available in the schematic overview in Figure 5.2.

5.4 Algorithm - Model II

As for the previous algorithm, for Model I, the initiation procedure for this algorithm contains a parameter calibration. However; before computing the optimal amount to be acquired of the risky asset, the algorithm must use the calibrated parameters to compute an approximation of \( \delta(t) \), in the way that is suggested by equation (5.12). In contrast to the previous algorithm, the main objective is not to maintain an optimal balance; the optimal strategy to enforce is a time dependent optimal holding amount of the risky asset, \( \hat{\nu}_t \), see equations (4.41) and (4.42) where we can see how \( \delta \) comes into play. The practical implications of this is that for each point in time such that \( t \neq T \), the algorithm needs to compute the holding amount of the risky asset that is strategically optimal, thereafter measure the magnitude of the deviation due to movement in the underlying assets. The deviation is simply expressed as the difference between the optimal value and the current holdings, that is

\[
|\hat{\nu}_t - Y_t|,
\]

which explicitly defines how much to liquidate or purchase at each point in time. In Figure 5.3, a schematic overview of the complete algorithm is available.
Equilibrium Strategies for Time-Inconsistent Stochastic Optimal Control of Asset Allocation

Parameter calibration → Compute $\hat{\nu}$ → Acquire $Y_t = \hat{\nu}\pi$ worth of the risky asset → Wait until $t = t + \Delta t$

$t = T$? Yes → Liquidate everything → STOP

No → $Y_t < \hat{\nu}\pi$? No → $Y_t = \hat{\nu}\pi$? No → $Y_t = \hat{\nu}\pi$? Yes → Acquire $\frac{\hat{\nu}}{1-\hat{\nu}}X_t - Y_t$ worth of the risky asset → Wait until $t = t + \Delta t$

No → $Y_t < \hat{\nu}\pi$? Yes → Acquire $\frac{\hat{\nu}}{1-\hat{\nu}}X_t - Y_t$ worth of the risky asset → Liquidate $Y_t - \frac{\hat{\nu}}{1-\hat{\nu}}X_t$ worth of the risky asset

Figure 5.2: A schematic overview of an algorithm that performs a portfolio balancing in accordance with the analysis for Model I.
Parameter calibration → Compute $\delta(t)$ → Compute $\nu_t$ → Acquire $Y_t = \nu_t$ worth of the risky asset

Wait until $t = t + \Delta t$

$t = T$?

Yes → Liquidate everything → STOP

No → Compute $\nu_t$

$Y_t < \nu_t$?

Yes → Acquire $\nu_t - Y_t$ worth of the risky asset

No → $Y_t = \nu_t$?

Yes → Wait until $t = t + \Delta t$

No → Liquidate $Y_t - \nu_t$ worth of the risky asset

Yes → $Y_t = \nu_t$?
Chapter 6

Results

In this chapter we investigate the results generated by the previously conducted modelling and
analysis. Regardless of the model that is investigated; we have utilized the methods suggested
in section 5.1 to generate numerical values for the drift $\mu$ and volatility $\sigma$. However; the risk-free
rate $r$ and the coefficients of risk aversion were chosen arbitrarily for each case except for the
smart investor.

For the calibrated parameters we have used historical data of a stock. Specifically; 240 daily
closing prices per year for Nordea Bank (publ) on the Frankfurt Stock Exchange, dating 16
years back, see Appendix B for more details. For Model I; both conservative and aggressive
investor profiles where considered to generate the results and simulations in this chapter. For
Model II; a so-called smart investor profile is considered as well, in Table 6.1 the parameter
values utilized for each case and model are presented. Note that the values in Table 6.1 are
kept constant regardless of the choice of investment period, $T$, with exception for the smart
investor profile where the value of the coefficient of risk-aversion depends on the length of the
investment period. That is; $\gamma$ is presented for investment periods $T = 1/5/10/16$ years. The
several values of $\gamma$ in the smart investor’s case are a consequence of the existence of a maximum
expected return point that is discussed in 6.2.3 it is in fact the reason for introducing the smart
investor profile in the first place. The initial endowment $\pi$ is given in units of currency, i.e. it
has dimension money and the initial time $t = 0$. The coefficient of risk aversion is dimensionless.

Table 6.1: This table displays the numerical values of the involved parameters.

<table>
<thead>
<tr>
<th>Model</th>
<th>Conservative</th>
<th>Aggressive</th>
<th>Smart</th>
<th>Aggressive</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi$</td>
<td>6.2%</td>
<td>6.2%</td>
<td>6.2%</td>
<td>6.2%</td>
</tr>
<tr>
<td>$\mu$</td>
<td>41.0%</td>
<td>41.0%</td>
<td>41.0%</td>
<td>41.0%</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>5.0%</td>
<td>5.0%</td>
<td>5.0%</td>
<td>5.0%</td>
</tr>
<tr>
<td>$r$</td>
<td></td>
<td></td>
<td></td>
<td>1.345/1.345/1.34/1.33</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>0.1</td>
<td>0.8</td>
<td>1.345</td>
<td>1</td>
</tr>
</tbody>
</table>

6.1 Model I

According to equation (4.17), the optimal control functions is as mentioned in Remark 4.2
constant with respect to time. That is; the optimal strategy is to hold a constant fraction,
not amount, of the current portfolio wealth at each point in time $s$. Using the values that are
presented in Table 6.1 we obtain the optimal strategy for both a conservative investor and an
aggressive investor, corresponding to $\gamma = 0.1$ and $\gamma = 0.8$ respectively. We can clearly see
that the investor with the bigger risk appetite invests a larger part of the wealth in the risky
Figure 6.1: These two pie charts illustrate the optimal portfolio balance that is obtained for a conservative investor ($\gamma = 0.1$) and aggressive investor ($\gamma = 0.8$), respectively.

asset, more precisely 35.55%, which is significantly more than 7.9%, which is the conservative investor’s optimal strategy.

### 6.1.1 Expected performance and efficient frontier

For a deeper understanding of what the fruits of our conducted analysis are; we need to consider what we actually are interested in looking at in order to define and measure the performance of the model. Equation (4.16) gives us $\psi(t, \pi)$, the optimal value function that satisfies the HJB partial differential equation and is thus the optimal performance. However; for the purpose of this investigation, we are actually not as interested in the optimal value function as we are interested in the expected value of the terminal portfolio wealth. This choice of interest is due to the fact that the objective function only is a a way for us to constrain the dynamic programming such that an investor’s preference is reflected. Particularly; the most important metric is the expected value of the portfolio wealth at the end of the investment period, since this value literally tells the investor what the mean return of his/her strategy will become. For the purpose of choosing strategy, the expected portfolio wealth by itself would yield incomplete information. In order to make an adequate assessment of the different investment strategies, the investor needs to know how much risk he/she is being exposed to for every increase/decrease in expected portfolio return. If the variance is considered simultaneously as the expected value of the terminal portfolio wealth, we would get a sense for how much the return is with respect to the taken risk. With this information at hand, it is easy to construct an efficient frontier, a curve that plots the pairs $(\sigma_{\Pi}, \mu_{\Pi})$ as $\gamma$ is varied in the interval $(0, 1)$, where

$$\sigma_{\Pi} := \sqrt{\text{Var}_{t, \pi} \left[ \Pi_T^\pi \right]} = \pi e^{r_T - \frac{(\mu - r)^2}{2 \sigma^2 (\gamma - 1)} T} \sqrt{e^{\frac{(\mu - r)^2}{2 \sigma^2 (\gamma - 1)} T} - 1},$$

and

$$\mu_{\Pi} := E_{t, \pi} \left[ \Pi_T^\pi \right] = \pi e^{r_T - \frac{(\mu - r)^2}{2 \sigma^2 (\gamma - 1)} T},$$

where we have inserted equations (4.21) and (4.24) into the definitions, with $t = 0$ and $s = T$.

We can define the efficient frontier as the mapping $\Theta : \mathbb{R} \to \mathbb{R}^2$. Particularly; it is the parametrised curve

$$\Theta : \gamma \to (\sigma_{\Pi}, \mu_{\Pi}),$$
and is plotted for different values of $T$ in Figure 6.2.

![Graphs showing efficient frontiers for different investment periods.](image)

(a) The one-year investment period’s efficient frontier. Almost linear behaviour.

(b) The five-year investment period’s efficient frontier.

(c) The ten-year investment period’s efficient frontier.

(d) The six-year investment period’s efficient frontier.

Figure 6.2: These plots illustrate how the efficient frontier for Model I looks like for the four different investment periods. The curve in (a) suggests an almost linear relationship between the expected return and the standard deviation. This changes as $T$ increases and thus leading to the more characteristic behaviour of efficient frontiers in general.

### 6.1.2 Simulations

The conducted parameter calibration, executed with the collected historical data[^1] enables us to perform simulations for several different cases. The same setting is used in each model where we have simulated four times (changing the random number generator’s seed) for four time periods and two different risk preferences. Namely; there is a separation between the conservative investor versus the growth investor, as seen in Table 6.1.

A simulation of the evolution of an initial endowment of one unit of currency when invested in the underlying stock, bank account or the portfolio determined by the optimal control for Model I is seen in Figures 6.3, 6.4, 6.5 and 6.6. The portfolio with preference $\gamma = 0.1$ corresponds

[^1]: The data source is available in Appendix B.
to a conservative investor and the investor with preference $\gamma = 0.8$ is willing to take more risk.

Figure 6.3: Four different scenarios, each corresponding to a specific series of samples, are simulated and plotted for a one-year investment period using Model I.
Equilibrium Strategies for Time-Inconsistent Stochastic Optimal Control of Asset Allocation

(a) Five years simulation with seed 2 in MATLAB’s random number generator.

(b) Five years simulation with seed 4 in MATLAB’s random number generator.

(c) Five years simulation with seed 6 in MATLAB’s random number generator.

(d) Five years simulation with seed 8 in MATLAB’s random number generator.

Figure 6.4: Four different scenarios, each corresponding to a specific series of samples, are simulated and plotted for a five-year investment period using Model I.
(a) Ten years simulation with seed 2 in MATLAB’s random number generator.

(b) Ten years simulation with seed 4 in MATLAB’s random number generator.

(c) Ten years simulation with seed 6 in MATLAB’s random number generator.

(d) Ten years simulation with seed 8 in MATLAB’s random number generator.

Figure 6.5: Four different scenarios, each corresponding to a specific series of samples, are simulated and plotted for a ten-year investment period using Model I.
Figure 6.6: Four different scenarios, each corresponding to a specific series of samples, are simulated and plotted for a sixteen-year investment period using Model I.
6.2 Model II

As previously conducted for Model I; we have used the values in Table 6.1 in order to investigate the results of different scenarios, corresponding to an investor profile each. As seen in one of the upcoming sections, Section 6.2.3, the efficient frontiers suggest that there exists a maximum expected value risk appetite level. The investor profile that corresponds to the maximum expected value of the portfolio will be referred to as the *smart* investor. Hence; the following results are generated for three investor profiles, in contrast to the two profiles that where considered for Model I.

6.2.1 The generated $\delta(t)$

With help from the calibrated and set values in Table 6.1, as well as the recursive relationship in equation (5.12), $\delta(t)$ is computed with sufficiently high accuracy (ten iterations) for each different case and several investment periods. The generated curves can be seen in Figure 6.7.

![Curves](image)

Figure 6.7: These plots illustrate how the $\delta(t)$ curve, that is approximated with 10 iterations, looks like for the four different time-intervals that are investigated in this thesis.
Equilibrium Strategies for Time-Inconsistent Stochastic Optimal Control of Asset Allocation

6.2.2 Portfolio balance over time

In order to obtain the type of information that is presented for Model I in Figure 6.1, some things are considered differently. Since for this model, the optimal control is the amount of money, not the fraction, that is supposed to be held in the risky asset at each point in time. For this reason, we chose to consider the ratio

$$Y_s = \frac{\hat{\nu}_s}{E_{t,\pi}[\Pi']_s} = \frac{\hat{\nu}_s}{\pi e^{\int_t^s (r+(\mu-r)\delta(u)) du}} = \delta(s) e^{-\int_t^s (r+(\mu-r)\delta(u)) du},$$

to describe the fraction of the expected portfolio wealth that at each point in time should be held in stock. This is fraction, which is a function of time, is displayed in Figure 6.8.

(a) One-year curves for the fraction of wealth in the risky asset.
(b) Five-year curves for the fraction of wealth in the risky asset.
(c) Ten-year curves for the fraction of wealth in the risky asset.
(d) Sixteen-year curves for the fraction of wealth in the risky asset.

Figure 6.8: These plots illustrate how the fraction of wealth that is invested in the risky asset changes during a whole investment period. We have set $t = 0$ and all the other parameters are as given in Table 6.1.

For further intuition; snapshots are taken at times $s = 0$ years (today), $s = 4$ years, $s = 8$ years and $s = 16$ years, to create pie-charts that illustrates the change of the portfolio’s balance over
the sixteen-year investment period, following the smart investment strategy. This evolution of wealth allocation is seen in Figure 6.9.

Wealth allocation with a *smart* strategy over 16 years.

![Wealth allocation chart](chart.png)

**Today**
- Stock: 5.72%
- Bank: 94.28%

**In four years**
- Stock: 4.74%
- Bank: 95.26%

**In eight years**
- Stock: 3.93%
- Bank: 96.07%

**In sixteen years**
- Stock: 2.70%
- Bank: 97.30%

Figure 6.9: An illustration of how the balance changes over time for a portfolio that is controlled by a *smart* investor during a sixteen-year investment period. The stock holding percentage corresponds to the value of the curve in Figure 6.8d at each point in time where the snapshots are taken.

In contrast to what is suggested by the results for Model I, these results reveal that a continuous change of the balance is the optimal strategy regardless of risk profile and length of investment period. Specifically; the optimal strategy suggests that liquidation of the stock is to be performed as time evolves. Keep in mind that it is the fraction of the expected portfolio wealth that is computed here and not the fraction of the actual portfolio wealth.

### 6.2.3 Expected performance and efficient frontier

Analogously with the reasoning behind the measurement and understanding of the performance of Model I; we are interested in how much return on initial wealth we can expect per extra unit of risk that is added to the portfolio. Just as before, the risk is measured as the overall standard deviation of the portfolios’ performance. Consequently; the efficient frontier for Model II is constructed as the mapping $\Theta : \mathbb{R} \rightarrow \mathbb{R}^2$ as well, explicitly described by

$$\Theta : \gamma \rightarrow (\sigma_{II}, \mu_{II}).$$

Where with help from equations (4.46) and (4.49) we define the expected return as

$$\mu_{II} := \mathbb{E}_{t,\pi} \left[ \Pi_T^{\delta} \right] = \pi e^{\int_0^T (r + (\mu - r) \delta(u)) du},$$

and the standard deviation as

$$\sigma_{II} := \sqrt{\text{Var}_{t,\pi} \left[ \Pi_T^{\delta} \right]} = \pi e^{\int_0^T (r + (\mu - r) \delta(s)) ds} \sqrt{e^{\int_0^T \sigma^2 \delta^2(s) ds} - 1},$$

where we have set $t = 0$ and $s = T$. The efficient frontier for several investment periods is seen in Figure 6.10. An intriguing observation that is made when investigating the plots in Figure 6.10 is that there is a clear limit to how much risk taking that is actually leading to a higher expected return. Particularly; after exceeding the mentioned limit, the expected return decreases remarkably.
Equilibrium Strategies for Time-Inconsistent Stochastic Optimal Control of Asset Allocation

(a) The one-year investment period’s efficient frontier with a maximum expected return reached with $\gamma = 1.345$.

(b) The five-year investment period’s efficient frontier with a maximum expected return reached with $\gamma = 1.345$.

(c) The ten-year investment period’s efficient frontier with a maximum expected return reached with $\gamma = 1.340$.

(d) The sixteen-year investment period’s efficient frontier with a maximum expected return reached with $\gamma = 1.330$.

Figure 6.10: These plots illustrate how the efficient frontier for Model II looks like for the four different investment periods. The curves in (a)-(d) all suggest that once the investor passes a certain level of risk-taking, he/she is not awarded for it. In fact; the investor gets punished.

6.2.4 Simulations

The simulated evolution of wealth, starting with one unit of currency, when invested in either the underlying stock, bank account or the portfolio determined by the equilibrium control for Model II is seen in Figures 6.11, 6.12, 6.13 and 6.14. The portfolio with preference $\gamma = 6$ corresponds to a conservative investor and the investor with preference $\gamma = 1$ is taking the most risk. The values of $\gamma \in \{1.33, 1.34, 1.345, 1.345\}$ correspond to the so-called smart investor for each of the four investment periods.
(a) One year simulation with seed 2 in MATLAB’s random number generator.

(b) One year simulation with seed 4 in MATLAB’s random number generator.

(c) One year simulation with seed 6 in MATLAB’s random number generator.

(d) One year simulation with seed 8 in MATLAB’s random number generator.

Figure 6.11: Four different scenarios, each corresponding to a specific series of samples, are simulated and plotted for a one-year investment period using Model II.
Equilibrium Strategies for Time-Inconsistent Stochastic Optimal Control of Asset Allocation

Figure 6.12: Four different scenarios, each corresponding to a specific series of samples, are simulated and plotted for a five-year investment period using Model II.
(a) Ten years simulation with seed 2 in MATLAB’s random number generator.

(b) Ten years simulation with seed 4 in MATLAB’s random number generator.

(c) Ten years simulation with seed 6 in MATLAB’s random number generator.

(d) Ten years simulation with seed 8 in MATLAB’s random number generator.

Figure 6.13: Four different scenarios, each corresponding to a specific series of samples, are simulated and plotted for a ten-year investment period using Model II.
Equilibrium Strategies for Time-Inconsistent Stochastic Optimal Control of Asset Allocation

Figure 6.14: Four different scenarios, each corresponding to a specific series of samples, are simulated and plotted for a sixteen-year investment period using Model II.

(a) Sixteen years simulation with seed 2 in MATLAB’s random number generator.

(b) Sixteen years simulation with seed 4 in MATLAB’s random number generator.

(c) Sixteen years simulation with seed 6 in MATLAB’s random number generator.

(d) Sixteen years simulation with seed 8 in MATLAB’s random number generator.
Chapter 7

Discussion

The notion of optimality is deeply dependent on what you would consider within a specific setting. When speaking of optimality it is imperative to emphasize in what sense an action is considered to be optimal and how the optimality is defined in each case. It is not unusual to vaguely, and thus incorrectly, state optimization problems as follows:

“What is the optimal driving route to the store?”.

This question could be either interpreted as;

“Which route to the store minimizes the fuel consumption?”,

or

“Which route to the store minimizes the travelling time?”,

or even both formulations could be implied simultaneously by the initial ambiguous one. Drawing parallels to the topics of this thesis; analogously, the optimization that is conducted here does not e.g. take the cost of asset movement into consideration. Neither is the transaction time of asset movement considered. It is desirable to perform a dynamic portfolio optimization, but when are we allowed to call a strategy optimal or efficient?

By generally considering the framework of the modelling and analysis, our utilized settings lead to some directly observable defects regarding the validity of the taken approaches’ optimal strategies. Particularly one thing that stands out, as previously mentioned, is the absence of transaction costs. This defect is giving the algorithms the freedom to purchase and liquidate arbitrarily large volumes, as long as it is considered to be a necessary action. I.e. an action that maintains the optimal trajectory for the portfolio wealth. Most likely the absence of friction in our models lead to a significant over-estimation of how well the models perform.

7.1 Model performance

In order to get a sense of how well each model actually performs, we need to define and measure the performance itself. One way to do it is to consider a ratio that explicitly tells us how much expected return per unit risk there is for each scenario. That is, the performance is a dimensionless efficiency that is defined by the relationship

\[
\text{Efficiency} := \frac{E_{t,\pi} \left[ \Pi_T^\nu \right]}{\sqrt{\text{Var}_{t,\pi} \left[ \Pi_T^\nu \right]}}
\]

and the explicit value for each scenario is available in Table 7.1.
Table 7.1: The models’ efficiency computed per model for each risk profile and four different investment periods.

<table>
<thead>
<tr>
<th>T</th>
<th>Model I Conservative</th>
<th>Model I Aggressive</th>
<th>Model II Conservative</th>
<th>Model II Smart</th>
<th>Model II Aggressive</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>21.82</td>
<td>4.80</td>
<td>215.97</td>
<td>48.45</td>
<td>36.03</td>
</tr>
<tr>
<td>5</td>
<td>9.74</td>
<td>2.06</td>
<td>43.28</td>
<td>9.71</td>
<td>7.22</td>
</tr>
<tr>
<td>10</td>
<td>6.87</td>
<td>1.37</td>
<td>21.65</td>
<td>4.82</td>
<td>3.58</td>
</tr>
<tr>
<td>16</td>
<td>5.41</td>
<td>1.01</td>
<td>13.54</td>
<td>2.96</td>
<td>2.19</td>
</tr>
</tbody>
</table>

As seen in Table 7.1, there exists further evidence for the fact that the investment periods length has a major impact on the performance. By comparing the performances model-wise it is seen that regardless of the investor’s risk preference; Model II outperforms Model I. E.g. by looking at the worst-case scenario, corresponding to an Aggressive investment strategy over sixteen years; it is obvious that the unit return per unit risk is almost non-existent for Model I and for Model II the investor receives more than double the unit return per unit risk. Taking the best-case scenario into consideration, a one-year Conservative investment strategy; Model II out-performs Model I by more than ten times. Furthermore; as previously predicted, the Smart investment strategy for Model II is always better then the Aggressive.

Another approach for measuring the performance of the models in a comprehensible way would be to take the inverse ratio into consideration. Namely; compute the unit risk taken per unit expected return, which should always be less than 1 and larger than 0. Let us call this the inefficiency ratio and define it as

\[
\text{Inefficiency} := \frac{\sqrt{\text{Var}_{t,\pi}[\Pi^\nu_T]}}{\text{E}_{t,\pi}[\Pi^\nu_T]} = \frac{1}{\text{Efficiency}},
\]

where the computed values are presented in Table 7.2. Values close to 0 indicate low inefficiency, implying a lower level of risk-taking per unit expected return and the investor is thus expected to be compensated well for the risk. Moreover; values near 1 indicate that the level of risk-taking is poorly compensated, in terms of expected return.

Table 7.2: The models’ inefficiency computed per model for each risk profile and four different investment periods.

<table>
<thead>
<tr>
<th>T</th>
<th>Model I Conservative</th>
<th>Model I Aggressive</th>
<th>Model II Conservative</th>
<th>Model II Smart</th>
<th>Model II Aggressive</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.05</td>
<td>0.21</td>
<td>0.005</td>
<td>0.02</td>
<td>0.03</td>
</tr>
<tr>
<td>5</td>
<td>0.10</td>
<td>0.49</td>
<td>0.02</td>
<td>0.10</td>
<td>0.14</td>
</tr>
<tr>
<td>10</td>
<td>0.15</td>
<td>0.73</td>
<td>0.05</td>
<td>0.21</td>
<td>0.28</td>
</tr>
<tr>
<td>16</td>
<td>0.18</td>
<td>0.99</td>
<td>0.07</td>
<td>0.34</td>
<td>0.46</td>
</tr>
</tbody>
</table>

7.2 Future R&D and alternative areas of use

The scope of the current set up can be expanded significantly and lead researchers and modellers in several directions; in-depth theoretical research as well as practical modification of the set up itself. To get a glimpse of future potential enhancements of the investigation conducted in this thesis we have presented concise examples below, starting with a specific suggestion of what we consider to be a natural extension to the work in this thesis.
7.2.1 Proportional transaction costs and bid-ask spread

A natural extension of Model II is taking proportional transaction costs and bid-ask spread into consideration. This setup takes the model a step closer to a realistic scenario where the strategy needs to be concerned with how the rebalancing is performed and not only what portfolio balance is optimal. In fact, the optimal balance in each point in time is affected by the trading behaviour and vice versa.

Consider the portfolio dynamics given by the SDE in equation (4.8) and let us reason as follows. The accumulated amount of currency that has been transferred out of the bank account during a time interval $[t, \tau]$ must satisfy

$$
\int_t^\tau P_s dO_s = \int_t^\tau P_s \frac{dO_s}{ds} \, ds = \int_t^\tau \nu_s^O \, ds,
$$

where the introduced function $\nu_s^O \geq 0$ denotes the currency per unit time, transaction rate, that flows out of the bank account at any time $s \in [t, T]$. Analogously for the money flowing into the bank account, we define

$$
\nu_s^I := P_s \frac{dI_s}{ds} \geq 0.
$$

The idea is to use the introduced transaction rates as control functions and thus steer the state of the system by deciding how much money should flow in each direction at each point in time. Equations (4.3) and (4.4) can now be re-written as

$$
dX_s^\nu = \left( rX_s^\nu - \kappa_O \nu_s^O + \kappa_I \nu_s^I \right) \, ds, \quad X_t^\nu = x, \tag{7.1}
$$

and

$$
dY_s^\nu = \left( \mu Y_s^\nu + \nu_s^O + \nu_s^I \right) \, ds + \sigma Y_s^\nu dB_s, \quad Y_t^\nu = y, \tag{7.2}
$$

with $x > 0$ and $y \geq 0$ indicating that the bank account never goes bankrupt and short selling of the risky asset is not permitted and the notation $\nu = (\nu^O, \nu^I)$ is used in the superscripts.

The performance functional we are interested in maximizing, given the dynamics (7.1) and (7.2), is expressed as

$$
\mathcal{J} (t, x, y, \nu^O, \nu^I) = \mathbb{E}_{t,x,y} \left[ \ell (X_T^\nu, Y_T^\nu) \right] - \frac{\gamma}{2 (x + y)} \text{Var}_{t,x,y} \left[ \ell (X_T^\nu, Y_T^\nu) \right]
$$

$$
= \mathbb{E}_{t,x,y} \left[ \ell (X_T^\nu, Y_T^\nu) - \gamma \left( \ell (X_T^\nu, Y_T^\nu) \right)^2 \right] + \gamma \left( \mathbb{E}_{t,x,y} \left[ \ell (X_T^\nu, Y_T^\nu) \right]^2 \right) \frac{2 (x + y)}{2 (x + y)}
$$

$$
= \mathbb{E}_{t,x,y} \left[ X_T^\nu + \kappa I Y_T^\nu - \gamma \left( X_T^\nu + \kappa I Y_T^\nu \right)^2 \right] + \gamma \left( \mathbb{E}_{t,x,y} \left[ X_T^\nu + \kappa I Y_T^\nu \right]^2 \right) \frac{2 (x + y)}{2 (x + y)},
$$

contemplating that we want to maximize the expected value of the liquidated wealth at time $T$ at the same time as we, with some preference $\gamma$, want to minimize the variance of the liquidated wealth at the same time. The equilibrium control problem for this setup is expressed as

$$
\sup \mathbb{E}_{t,x,y} \left[ X_T^\nu + \kappa I Y_T^\nu - \gamma \left( X_T^\nu + \kappa I Y_T^\nu \right)^2 \right] + \gamma \left( \mathbb{E}_{t,x,y} \left[ X_T^\nu + \kappa I Y_T^\nu \right]^2 \right) \frac{2 (x + y)}{2 (x + y)}
$$

s.t. $dX_s^\nu = \left( rX_s^\nu - \kappa_O \nu_s^O + \kappa_I \nu_s^I \right) \, ds, \quad X_t^\nu = x,

$$
dY_s^\nu = \left( \mu Y_s^\nu + \nu_s^O + \nu_s^I \right) \, ds + \sigma Y_s^\nu dB_s, \quad Y_t^\nu = y,
$$

$\nu \in \mathcal{V}_0 (s, X_s^\nu, Y_s^\nu), \quad \forall (s, X_s^\nu, Y_s^\nu) \in \mathcal{X},

$$
\mathcal{X} \subseteq [t, T] \times \mathbb{R}^2$ denotes a time-state space and and $\mathcal{V}_0$ a suitable set of admissible controls that can be dependent on time and the state of the system.
7.2.2 Further suggestions of model enhancements and extension of the scope

Below are a few candidates for future research subjects, modifications to the set up and alternative areas of use. Keep in mind that most of these can be combined as well, creating more research gaps that needs to be filled.

Prove existence and uniqueness of the extended HJBE

As suggested in [1] by Björk et al. the existence and uniqueness of the extended Hamilton-Jacobi-Bellman equation is yet to be proven. Thus further research within this area of this part of mathematics is required in order to draw general conclusions from this beautiful theory.

Time-dependent parameters

Instead of assuming that the drift and volatility are constant coefficients in the Itô process that describes the dynamics of the underlying asset; it is more realistic to assume that they are functions of time, \( \mu_s = \mu(s) \) and \( \sigma_s = \sigma(s) \),

\[
dP_s = \mu_s P_s ds + \sigma_s P_s dB_s, \quad P_t = P_0.
\]

This construction could e.g. be beneficial if the modeller is using some type of autoregressive time-series to describe the volatility, GARCH volatility models, i.a. The reason for the benefit is that the generated time-series can be seen as a function of time despite the discontinuity that is a consequence of the stochastic nature yielding the values.

Stochastic volatility

In contrast to having time-dependent volatility, it is also possible to model the volatility as a stochastic process itself and thus end up with a system of stochastic differential equations that describe the underlying asset’s dynamics. This could e.g. be Heston’s volatility model

\[
dP_s = \mu_s P_s ds + \sqrt{\nu_s} P_s dB_s^P, \quad P_t = P_0, \\
d\nu_s = \alpha (\beta - \nu_s) ds + \gamma \sqrt{\nu_s} dB_s^\nu, \quad \nu_t = 0,
\]

which is a so-called mean-reverting stochastic process, where \( \nu_t \) is the instantaneous variance, \( \alpha \) is describes how fast the reversion occurs towards the long-term mean value, \( \beta \), \( \gamma \) is the volatility of the volatility and the Brownian motions \( B_s^P \) and \( B_s^\nu \) drive the noise of the price process and volatility process respectively.

Stochastic interest rate models

Analogously with the reasoning for stochastic volatility models, it is not unusual to assume that the interest rate is following a mean-reverting stochastic process, e.g. an Ornstein-Uhlenbeck process,

\[
dr_s = \alpha (\beta - r_s) ds + \sigma dB_s,
\]

where \( \alpha \) is describes how fast the reversion occurs, \( \beta \) is the long-term mean value and \( \sigma \) is the instantaneous volatility.
Multiple risky assets both with and without a risk-free asset

The set up is easily expanded to handle $n$ assets by considering $n$-dimensional vector valued Itô diffusion processes driven by multidimensional Brownian motions to describe the price processes’ dynamics.

$$dP_t = \mu ds + \sigma dB_s, \quad P_t = P_0 \in \mathbb{R}^n,$$

where $\mu \in \mathbb{R}^n$, $\sigma \in \mathbb{R}^{n \times d}$ and $B_s \in \mathcal{B} \mathcal{M}(\mathbb{R}^d)$. No that risk-free assets are easily introduced to the set by letting its corresponding matrix-element be equal to 0, thus annihilating the effect of the present noise induced by the Brownian motion.

Including derivatives in the portfolio

The scope of the analysis could be expanded to include derivatives as well as stock in the portfolio. This problem becomes particularly interesting as the derivatives can be viewed as non-linear functions of the price process of the available stocks on the market.

Asset and liability management

Stochastic optimal control can be used to create a dynamic hedging strategy to handle future liabilities; where the logic is built upon the idea that you can create a derivative contract, or buy a basket of options that more or less annihilates the liability, cash flow-wise. The common set up would to regard the minimization of some pre-defined distance between the assets’ values and the liabilities’ at time of maturity. Namely;

$$\inf_{A \in \mathcal{A}} \mathbb{E}[\Phi (A_T - L_T) \mid \mathcal{F}_t],$$

where $\mathcal{A}$ i some admissible set and $\Phi$ is the function that measures the distance in question. Typically; the squared distance is used as a basic example, i.e. $\Phi (\cdot) = (\cdot)^2$, which is a well known method that generates the so-called quadratic hedge.

Optimal consumption and investment

Let us say that the investor wants to withdraw money from the portfolio; then the optimal amount to withdraw at each point in time can be modelled as another control variable. In other words, the optimal consumption rate $c_s \geq 0$ as well as the portfolio weights $(\nu_s^j)_{j \geq 0}$ are the control variables and the objective is to minimize a functional of the form

$$J(t, \pi, \nu, c) = \mathbb{E}_{t, \pi} \left[ \int_t^T \varphi (s, c_s) ds + \Phi (\Pi_T^{c_s}) \right].$$

Infinite horizon optimal control

Instead of fixing a specific length $T$ of the investment period, it could be useful to investigate what optimal equilibrium investment strategies that emerges when one lets $T$ tend to infinity. The problem one is faced with when trying to find these types of strategies is known as an infinite horizon optimal control problem. Due to natural reasons the objective function does not have terminal cost and the performance functional can e.g. be written as follows

$$J(t, \pi, \nu) = \mathbb{E}_{t, \pi} \left[ \int_t^\infty \varphi (s, \Pi_s^{\nu}, \nu (s, \Pi_s^{\nu})) ds \right].$$
Appendices
Appendix A

Mathematical preliminaries

In this appendix we aim to briefly introduce some mathematical concepts that are considered to be relevant prerequisites for the mathematics that is used in this thesis.

**Definition A.1.** (Probability measure and probability space)
Let the space $\Omega$ be an outcome space of possible outcomes $\omega$ and $\mathcal{F}$ be the $\sigma$-algebra of subsets of $\Omega$. If a finite measure $P : \mathcal{F} \to [0, 1]$ is introduced on the measurable space $(\Omega, \mathcal{F})$, such that

i) for every set $S \subset \Omega$ being a member of $\mathcal{F}$,

$$P(S) = \int_S dP(\omega) \geq 0,$$

ii) $P(\emptyset) = 0$,

iii) $P(\Omega) = 1$,

iv) $P$ is countably additive, i.e. if $(S_j)_{j=1}^{\infty}$ is a collection of pairwise disjoint members of $\mathcal{F}$, then

$$P\left(\bigcup_{j=1}^{\infty} S_j\right) = \sum_{j=1}^{\infty} P(S_j).$$

then $P$ is a probability measure and the measure space $(\Omega, \mathcal{F}, P)$ is thus a probability space.

**Definition A.2.** (Measurability w.r.t. a $\sigma$-algebra)
Consider the measure space $(X, \mathcal{F}, \mu)$, where $\mu$ is some finite measure. The mapping $\phi : X \to Y$ is said to be $\mathcal{F}$-measurable if for each Borel set $S \subset Y$, it holds that

$$\phi^{-1}(S) = \{x \in X \mid \phi(x) \in S\} \in \mathcal{F}.$$

A shorthand notation for this is $\phi \in \mathcal{F}$.

**Definition A.3.** (Random variable)
A random variable is a mapping $X : \Omega \to \mathbb{R}$ such that $X$ is $\mathcal{F}$-measurable for each. In other words, such that

$$X^{-1}(S) = \{\omega \in \Omega \mid X(\omega) \in S\} \in \mathcal{F},$$

for each Borel set $S$. 

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**Definition A.4.** (Stochastic process)
A stochastic process \(X(t, \omega), \Omega, \mathcal{F}, \mathbb{P}\) is a mapping
\[
X : [0, \infty) \times \Omega \to \mathbb{R}
\]
such that
\[
X(t, \cdot) : \Omega \to \mathbb{R}
\]
is \(\mathcal{F}\)-measurable \(\forall t \in [0, \infty)\).

**Definition A.5.** (Integrability)
The function \(\phi \in \mathcal{F}\) is said to be \(p\)-integrable, \(\phi \in L^p(X, \mathcal{F}, \mu)\), if
\[
\left( \int_X |\phi(x)|^p \, d\mu(x) \right)^{1/p} < \infty.
\]

**Definition A.6.** (Expected value)
Let \(X \in L^1(\Omega, \mathcal{F}, \mathbb{P})\), then its expected value is
\[
E[X] := \int_\Omega X(\omega) \, d\mathbb{P}(\omega).
\]

**Proposition A.1.** If \(\psi : \mathbb{R} \to \mathbb{R}\) is an integrable Borel function, then
\[
E[\psi(X)] = \int_\Omega \psi(X(\omega)) \, d\mathbb{P}(\omega).
\]
This is also known as the Law of the Unconscious Statistician.

**Remark A.1.** In accordance with Definitions A.5, A.6 and Proposition A.1 it becomes clear that
\[
X \in L^1(\Omega, \mathcal{F}, \mathbb{P}) \quad \Leftrightarrow \quad E[|X|] < \infty.
\]

**Definition A.7.** (Filtration)
Let the smallest \(\sigma\)-algebra generated by the stochastic process \(X_t\) on \((\Omega, \mathcal{F}, \mathbb{P})\) over \([0, t]\) be
\[
\mathcal{F}_t = \sigma\{X_r \mid r \leq t\}.
\]
Then a filtration \(\mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0}\) on \((\Omega, \mathcal{F}, \mathbb{P})\) is a family of \(\sigma\)-algebras on \(\Omega\) such that
\begin{itemize}
  \item[i)] \(\mathcal{F}_t \subseteq \mathcal{F}, \forall t \geq 0,\)
  \item[ii)] \(r \leq t \Rightarrow \mathcal{F}_r \subseteq \mathcal{F}_t.\)
\end{itemize}

**Definition A.8.** (Adapted stochastic process)
Given a filtration \(\mathcal{F}\), a stochastic process \(X_t\) on \((\Omega, \mathcal{F}, \mathbb{P})\) is said to be adapted to \(\mathcal{F}\) if \(X_t \in \mathcal{F}_t\) for each \(t \geq 0\).

**Definition A.9.** (Brownian motion)
The stochastic process \(B_t\), defined on the filtered probability space \((\Omega, \mathcal{F}, \mathbb{P})\), is a real-valued Brownian motion, \(B_t \in \mathcal{B} \mathcal{M}(\mathbb{R})\), if it holds that
\begin{itemize}
  \item[i)] \(B_0 = 0,\)
  \item[ii)] \(B_t \sim \mathcal{N}(0, 1), \forall t \geq 1,\)
  \item[iii)] \(B_t\) is adapted to \(\mathcal{F}_.\)
\end{itemize}
iv) the mapping \( t \to B_t \) is continuous \( \mathbb{P} \)-almost surely,

v) its time increments are \( (B_t - B_r) \sim \text{i.i.d. } \mathcal{N}(0, t-r) \) for each \( r \in [0, t] \).

**Definition A.10.** (Multidimensional Brownian motion)
An \( \mathbb{R}^p \)-valued process \( B_t = (B^1_t, \ldots, B^p_t) \), on \( (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}) \), is a \( p \)-dimensional Brownian motion if its components are independent and \( B^j_t \in \mathcal{B} \mathcal{M}(\mathbb{R}) \) for each \( j \in [1, p] \cap \mathbb{Z} \). Analogously, with the notation in Definition A.9, \( B_t \in \mathcal{B} \mathcal{M}(\mathbb{R}^p) \).

**Definition A.11.** (Conditional expectation)
Consider the \( \mathbb{R}^n \)-valued random variable \( X \in L(\Omega, \mathcal{F}, \mathbb{P}) \). If there exists a \( \sigma \)-algebra \( \mathcal{G} \subset \mathcal{F} \), then the conditional expectation of \( X \) given \( \mathcal{G} \), denoted \( \mathbb{E}[X | \mathcal{G}] \), is a \( \mathbb{P} \)-a.s. unique function such that

i) \( \mathbb{E}[X | \mathcal{G}] : \Omega \to \mathbb{R}^n \),

ii) \( \mathbb{E}[X | \mathcal{G}] \in \mathcal{G} \),

iii) and

\[
\int_A \mathbb{E}[X | \mathcal{G}] (\omega) \, d\mathbb{P}(\omega) = \int_A X(\omega) \, d\mathbb{P}(\omega), \quad \forall A \in \mathcal{G}.
\]

**Definition A.12.** (First variation)
Let \( \Gamma = \{t_0, t_1, \ldots, t_{n-1}, t_n\} \) be a partition of the interval \( [0, T] \) such that \( 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T \) and let the partitions mesh be

\[
\|\Gamma\| = \max_{0 \leq i \leq n-1} (t_{i+1} - t_i).
\]

Then the first variation of a function \( \phi \) is defined as

\[
\mathfrak{F}_{[0,T]}(\phi) := \lim_{\|\Gamma\| \to 0} \sum_{j=0}^{n-1} |\phi(t_{j+1}) - \phi(t_j)|.
\]

Furthermore; \( \phi \) is said to be

i) of finite variation if \( \mathfrak{F}_{[0,T]}(\phi) < \infty \),

ii) of bounded variation if \( \sup_t \mathfrak{F}_{[0,T]}(\phi) < \infty \).

**Lemma A.2.** If \( \phi \) is differentiable and \( \phi'(t) \in L[0,T] \), then

\[
\mathfrak{F}_{[0,T]}(\phi) = \int_0^T |\phi'(t)| \, dt.
\]

**Definition A.13.** (Quadratic variation)
Let \( \Gamma \) be the same partition as defined previously. The quadratic variation of a function \( \phi \) over \( [0, T] \) is

\[
\langle \phi \rangle_T = \langle \phi, \phi \rangle_T := \lim_{\|\Gamma\| \to 0} \sum_{j=0}^{n-1} |\phi(t_{j+1}) - \phi(t_j)|^2,
\]

and the covariation of \( \phi \) and \( \psi \) is defined as

\[
\langle \phi, \psi \rangle_T := \lim_{\|\Gamma\| \to 0} \sum_{j=0}^{n-1} (\phi(t_{j+1}) - \phi(t_j)) (\psi(t_{j+1}) - \psi(t_j)).
\]

Moreover; analogously as for the first variation, \( \phi \) is said to be
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i) of finite quadratic variation if \( \langle \phi \rangle_t < \infty, \ \forall t \in \mathbb{R} \setminus (-\infty, 0) \),

ii) of bounded quadratic variation if \( \sup_t \langle \phi \rangle_t < \infty \).

**Definition A.14.** (Itô process)

Let \( B_t \in \mathcal{BM}(\mathbb{R}) \) on \((\Omega, \mathcal{F}, P)\), then an Itô process \( X_t \) on \((\Omega, \mathcal{F}, P)\) is a stochastic process described by

\[
X_t = X_0 + \int_0^t \xi_s(\omega) \, ds + \int_0^t \eta_s(\omega) \, dB_s,
\]

where the adapted processes \( \eta_s \) and \( \xi_s \) are called the diffusion coefficient and the drift coefficient respectively. The coefficients are \( P \)-a.s. square-integrable, i.e. they satisfy

\[
P \left( \int_0^t \xi_s^2(\omega) \, ds < \infty, \ \forall t \geq 0 \right) = 1,
\]

and

\[
P \left( \int_0^t \eta_s^2(\omega) \, ds < \infty, \ \forall t \geq 0 \right) = 1.
\]

**Lemma A.3.** (Itô formula)

Let \( X_t \) be an Itô process that on differential form is described by

\[
dX_t = \xi_t dt + \eta_t dB_t,
\]

and introduce the function \( \phi(t, x) \in C^{1,2}([0, \infty) \times \mathbb{R}) \). Then if there exists a stochastic process \( Y_t = \phi(t, X_t) \), the process is an Itô process with differential

\[
dY_t = \frac{\partial \phi}{\partial t}(t, X_t) \, dt + \frac{\partial \phi}{\partial x}(t, X_t) \, dX_t + \frac{1}{2} \frac{\partial^2 \phi}{\partial x^2}(t, X_t) \, d\langle X, X \rangle_t.
\]

**Theorem A.4.** (Existence and uniqueness of solutions of stochastic differential equations)

Consider a random variable \( R \in L^2(\Omega, \mathcal{F}, P) \) independent of the \( \sigma \)-algebra \( \mathcal{F}_B^B \), where \( B_t \in \mathcal{BM}(\mathbb{R}^m) \). Let \( \xi : [0, T] \times \mathbb{R}^n \times \mathcal{Y} \to \mathbb{R}^n \) and \( \eta : [0, T] \times \mathbb{R}^n \times \mathcal{Y} \to \mathbb{R}^{n \times m} \) be \( \mathcal{F} \)-measurable functions satisfying the following for \( T \in [0, \infty) \).

i) Linear growth in \( x \) uniformly in \((t, y)\), i.e.

\[
|\xi(t, x, y)| + |\eta(t, x, y)| \leq \alpha (1 + |x|), \quad \forall (t, x, y) \in [0, T] \times \mathbb{R}^n \times \mathcal{Y},
\]

for some real constant \( \alpha > 0 \),

ii) and the Lipschitz condition

\[
|\xi(t, x + \Delta x) - \xi(t, x)| + |\eta(t, x + \Delta x, y) - \eta(t, x, y)| \leq \beta |\Delta x|,
\]

for each \((t, x, y) \in [0, T] \times \mathbb{R}^n \times \mathcal{Y} \) and some real constant \( \beta \), where \( \Delta x \geq 0 \).

Then, for the stochastic differential equation

\[
\begin{cases}
    dX_t = \xi(t, X_t, y_t) dt + \eta(t, X_t, y_t) dB_t, \\
    X_0 = R.
\end{cases}
\]

there exists an \( \mathcal{F}^Z_t \)-adapted unique solution \( X_t(\omega) \in L^2(\Omega, \mathcal{F}, P) \).

**Proof.** Please see [35].
Definition A.15. (The class $W$ of functions)
Let $\mathcal{B}([0, \infty))$ be the Borel $\sigma$-algebra on the non-negative real line and define the set $\mathcal{T} := [t_1, t_2] \subseteq [0, \infty)$. Then $W = W(\mathcal{T})$ is defined to be the class of functions
\[ \zeta(t, \omega) : \mathcal{T} \times \Omega \rightarrow \mathbb{R}, \]
satisfying

i) $(t, \omega) \rightarrow \zeta(t, \omega) \in \mathcal{B}([0, \infty)) \times \mathcal{F}$,

ii) $\zeta(t, \omega)$ is $\mathcal{F}_t$-adapted,

iii) $\zeta(t, \omega) \in L^2(dt \otimes d\mathbb{P})$.

Lemma A.5. (Itô isometry)
\[
\mathbb{E} \left[ \left( \int_{\mathcal{T}} \phi(t, \omega) \, dB_t \right)^2 \right] = \mathbb{E} \left[ \int_{\mathcal{T}} \phi^2(t, \omega) \, dt \right], \quad \forall \phi \in W(\mathcal{T}).
\]
Appendix B

Data

The data used in this thesis is easily obtained as a comma separated file from the Yahoo finance API. The following address can be directly inserted into a web-browser’s address field to download the historical daily prices for Nordea Bank (publ), from 2000-08-31 to 2016-08-31, on the Frankfurt Stock Exchange (Börse Frankfurt):


Note that the stock symbol for this specific stock is NDB.F.
Bibliography


Equilibrium Strategies for Time-Inconsistent Stochastic Optimal Control of Asset Allocation


