Optimal timing decisions in financial markets

Martin Vannestål
Abstract


This thesis consists of an introduction and five articles. A common theme in all the articles is optimal timing when acting on a financial market. The main topics are optimal selling of an asset, optimal exercising of an American option, optimal stopping games and optimal strategies in trend following trading. In all the articles, we consider a financial market different from the standard Black-Scholes market. In two of the articles this difference consists in allowing for jumps of the underlying process. In the other three, the difference is that we have incomplete information about the drift of the underlying process. This is a natural assumption in many situations, including the case of a true buyer of an American option, trading in a market which exhibits trends, and optimal liquidation of an asset in the presence of a bubble. These examples are all addressed in this thesis.

*Keywords*: optimal stopping, American options, optimal stopping games, incomplete information

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List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.


IV H. Dyrssen and M. Vannestål. "Optimal stopping games for a process with jumps." Submitted for publication.


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1. Introduction

This thesis consists of an introduction and five articles. In the introduction, we briefly present the areas, models and techniques which are treated and used in the articles.

All five articles are devoted to the study of stochastic optimization in continuous time finance. Common to all the articles is that we try to determine the optimal timing of an action in a financial market. An underlying asset which is traded at a financial market moves in a random way specified by a model. Our objective is to find a strategy that, based on these movements, optimizes some gain or minimizes some cost. Typical examples, which are also treated in this thesis in various guises, are i) optimal selling of a stock, ii) optimal exercising of an American option and, iii) trend following trading. The first two of these examples are so-called optimal stopping problems, and Articles I, II, IV and V in this thesis are devoted to analyzing problems from this class. Here the strategy to be found is a point in time, called an optimal stopping time, at which some gain (cost) is maximized (minimized) if we choose to stop. Of interest is to find both this optimal stopping time (in case it exists!), and the optimal value attained when using this strategy, i.e. stopping at this time. The third example is a so-called optimal switching problem, and in Article III in this thesis we analyze a problem from this class from an alternative angle. Here the strategy to be found is not a single point in time, but a sequence of time points at which we choose to go from one state to another. In the example of trend following trading, one state could correspond to "hold the stock" and the other to "not hold the stock", and the sequence of time points would constitute times at which we buy and sell the stock, respectively. Again, of interest is also the value attained when using the optimal strategy.

The rest of this introduction is organized as follows. In Section 1.1 we present various examples of problems from stochastic optimization relevant for this thesis. For the sake of clarity, we define the problems in the standard Black-Scholes model. In Section 1.2 we discuss different extensions of the Black-Scholes model in which the problems in this thesis are defined. In particular, we will extend the model in order to allow for jumps in the underlying process, as well as imposing incomplete information about the parameters in the model. We will also demonstrate various techniques that can be used to solve the problems addressed in this thesis. These include a dynamic programming operator to analyze a jump-diffusion, and filtering techniques to handle uncertainty about the model parameters.
1.1 Timing Optimality

We present here some different types of problems from stochastic optimization in general, and optimal stopping in particular. We choose to focus on the type of problems relevant for this thesis, so by necessity the list given here is highly incomplete when it comes to covering the area of mathematical finance and stochastic optimization. For example, we do not discuss European options, including the celebrated Black-Scholes formula for the European call option, see instead [3]. Neither do we discuss the theory for stochastic optimal control, including portfolio optimization, see instead [34].

One of the key concepts in stochastic optimization is the notion of a stopping time. If \( \tau \) is a random variable with values in \([0, \infty]\), such that \( \{ \tau \leq t \} \in \mathcal{F}_t \) for all \( t \), then we say that \( \tau \) is a stopping time with respect to the filtration \( \mathcal{F}_t \) for all \( t \geq 0 \). In words, this means that the decision whether or not to choose to stop at time \( t \) may depend on the past behavior of the process only, and not on the future. An example of a stopping time is the first exit time from a bounded region for a Brownian motion. A time which is not a stopping time is the instant at which a Brownian motion attains its maximum on the unit interval.

Throughout this section, unless specified otherwise, we will assume that the problems are defined in a Black-Scholes market, with two traded assets with dynamics

\[
\begin{align*}
    dX_t &= \alpha X_t dt + \sigma X_t dW_t \\
    dB_t &= r B_t dt.
\end{align*}
\]

(1.1)

Here \( B \) is a risk-free bank account with interest rate \( r \geq 0 \), and \( X \) is an asset, e.g. a stock or a house, modeled by a Geometric Brownian Motion (henceforth GBM) starting at \( x > 0 \), with drift parameter \( \alpha \) and diffusion parameter (volatility) \( \sigma > 0 \). A solution \( X \) in (1.1) is given by

\[
    X_t = x e^{(\alpha - \sigma^2/2)t + \sigma W_t},
\]

(1.2)

and for the expected value, we have

\[
    \mathbb{E}_x [X_t] = x e^{\alpha t}.
\]

(1.3)

Here the subscript on the left hand side is there to explicitly emphasize the starting point of the process.

1.1.1 Optimal Stopping

We consider a (discounted) Markovian optimal stopping problem of the form

\[
    V(t, x) = \sup_{t \leq \tau \leq T} \mathbb{E}_x [e^{-\beta(\tau - t)} G(X_\tau)],
\]

(1.4)

where \( T \leq \infty \). Of interest is to find \( V \) as well as an optimal stopping time \( \tau^* \) for which this value is attained. In this thesis we only study discounted problems. Since most of the potential applications we consider come from finance,
this is natural since the value of a discounted payoff has the interpretation as the present value of some future gain (in which case $\beta$ might be the sum of interest, inflation, taxes etc.). We also omit formulating (1.4) with an integral (running reward) term. This is not a major restriction, since such a term can be incorporated in $G$ under quite mild assumptions, see [30].

There are several ways to tackle the problem in (1.4) when $X$ is a diffusion, not necessarily as explicitly defined as in (1.1). One way is by means of excessive and superharmonic functions, see e.g. [13] or [30]. Another way is by means of (generalized) concave functions, see e.g. [8]. A third way is by formulating a free-boundary problem for $V$ and the optimal stopping boundary $b(t)$, see [32] and the references therein. Under some regularity and integrability conditions on $G$, it follows that

$$\tau^* = \inf\{s \geq t : V(s,X_s) = g(X_s)\}. \quad (1.5)$$

Alternatively, $\tau^*$ is the first exit time from the so-called continuation region, defined by

$$\mathcal{C} = \{(t,x) : V(t,x) > g(x)\}, \quad (1.6)$$

wherein $V$ solves a partial differential equation. In order to find the value $V$ and the optimal stopping boundary, one solves the equation and imposes certain boundary conditions.

### 1.1.2 Optimal Selling of an Asset

One of the most natural optimal stopping problems is the following. If an agent has an asset that he wants to sell, then he asks himself at what point in time this should be done in order to maximize the gain from the sell, i.e. how to make as much money as possible. Assume that the asset follows the GBM defined in (1.1), where $\alpha$ and $\sigma$ are known constants, and assume a constant cost $a \geq 0$ has to be payed when selling. The agent then wants to find

$$V(x) = \sup_{0 \leq \tau \leq T} \mathbb{E}_x \left[ e^{-\beta \tau} (X_\tau - a) \right] \quad (1.7)$$

as well as finding the optimal stopping time $\tau^*$ for which this value is attained. Note that $V$ here is on the form in (1.4), with $G(x) = x - a$ and $t = 0$. In the case $T = \infty$ and $a = 0$, the problem is trivial. Indeed, if $\alpha > \beta$ then it follows from (1.3) that the supremum would be obtained along the sequence $\tau_n = n$ of deterministic stopping times, in which case $V = \infty$ and no optimal stopping time exists ($\tau^* = \infty$). If, on the other hand, $\beta \leq \alpha$ then $V = x$ by optional sampling and $\tau^* = 0$.

The case where $T = \infty$ and $a > 0$ is slightly more involved. In [34, Example 5.2.5] the following explicit solution is derived (see also [30, Examples 10.2.2 and 10.4.2]):
$V(x) = \begin{cases} 
\infty & \text{if } r < \alpha \\
x & \text{if } r = \alpha 
\end{cases} \tag{1.8}$

and

$V(x) = \begin{cases} 
x^* \left( \frac{x}{x^*} \right)^n, & x < x^* \text{ if } r > \alpha, \\
x - a, & x \geq x^* 
\end{cases} \tag{1.9}$

where $n > 1$ is one solution of a quadratic equation and $x^* = an/(n - 1)$ is the optimal stopping boundary, which in this example is constant. The corresponding optimal stopping time does not exist ($\tau^* = \infty$), is given by $\tau^* = 0$ in (1.8) respectively, and $\tau^* = \inf\{t \geq 0 : X_t \geq x^*\}$ in (1.9).

In [12], the case $T < \infty$ is studied, where optimality is measured with respect to $M_t := \max_{0 \leq s \leq t} X_s$ for $t \in [0, T]$ (see also [11] for a related problem). For an alternative formulation of the stock selling problem, see [17].

In these examples, the drift coefficient $\alpha$ and the volatility $\sigma$ are assumed to be known constants. For the drift coefficient, this is a rather strong assumption. In Section 3 below, we will explain why this is, and we will suggest an alternative model where the assumption of complete information about the drift is relaxed.

1.1.3 The American Put Option

An American put option is a financial contract that gives the holder of the option the right, but not the obligation, to sell one unit of the underlying asset for a prespecified price $K$, called the strike price, at any time before or equal to a prespecified time $T$, called the exercise date. Occasionally one studies the case where $T = \infty$, in which case the problem is called perpetual. In this thesis we consider both perpetual problems and problems with finite exercise dates. The put option is probably the most studied option of American type, see e.g. [19], [29] and [31]. It can be shown, see for example Chapter 2.7 in [22], that the arbitrage free price of an American put option is given by

$$V(t,x) = \sup_{t \leq \tau \leq T} \mathbb{E}_x^Q[e^{-r(\tau-t)}(K - X_\tau)^+],$$

where the expected value is taken with respect to a measure under which the drift of the asset equals $r$. Furthermore, $V$ and $b(t)$ solve the free-boundary problem

$$\begin{cases} 
V_t + \mathbb{L}_X V - rV = 0 & \text{if } x > b(t), \\
V(t,x) = (K - x)^+ & \text{if } x = b(t), \\
V_x(t,x) = -1 & \text{if } x = b(t), \\
V(t,x) > (K - x)^+ & \text{if } x > b(t), \\
V(t,x) = (K - x)^+ & \text{if } 0 < x < b(t). 
\end{cases} \tag{1.10}$$
Here $\mathbb{L}_X$ is the infinitesimal operator of $X$. By an application of Itô’s formula (this is allowed since $V$ is regular enough, see [32, chapter 3.5]), we arrive at the early exercise premium representation of the value function:

$$V(t, x) = p(t, x) + \int_t^T rKe^{-r(u-t)}\mathbb{P}_{t,x}(X_u \leq b(u))\,du. \quad (1.11)$$

Here $p(t, x)$ is the price of the corresponding European put option, and the integral represents the added value of the option when one is allowed to exercise at any time before $T$. Setting $x = b(t)$, we get from (1.10) that

$$K - b(t) = p(t, x) + \int_t^T rKe^{-r(u-t)}\mathbb{P}_{t,b(t)}(X_u \leq b(u))\,du. \quad (1.12)$$

It is known that the solution to (1.12) is the unique solution to $b(t)$ in (1.10) in the class of continuous increasing functions $c: [0, T] \to \mathbb{R}$ satisfying $0 < c(t) < K$ for all $0 < t < T$. In Article V we study this integral equation when $X$ is a jump-diffusion.

1.1.4 Optimal Stopping Games

An optimal stopping game (OSG), also known as Dynkin game, is a more general optimal stopping problem in that it involves two agents, here called players. We refer to Player 1 as the maximizer and to Player 2 as the minimizer. We consider here only zero-sum games, i.e. games where the sum of the two players’ gains equals zero. Player $i$ chooses a stopping $\tau_i$ with respect to the filtration generated by a Markov process $X = (X_t)_{t \geq 0}$ starting at $X_0 = x$. The rule of the game is that at $\tau_1 \wedge \tau_2$, the minimizer pays the amount $G_1(X_{\tau_1})\mathbb{1}_{\tau_1 \leq \tau_2} + G_2(X_{\tau_2})\mathbb{1}_{\tau_1 > \tau_2}$ to the maximizer, for $G_1 \leq G_2$. Note that one is punished if one is the first one to stop in the sense that your payoff is lower in that case. Hence the objective of Player 1 (Player 2) is to maximize (minimize)

$$R_x(\tau_1, \tau_2) := \mathbb{E}_x[G_1(X_{\tau_1})\mathbb{1}_{\tau_1 \leq \tau_2} + G_2(X_{\tau_2})\mathbb{1}_{\tau_1 > \tau_2}].$$

If we define

$$\underline{V}(x) := \sup_{\tau_1} \inf_{\tau_2} R_x(\tau_1, \tau_2)$$

and

$$\overline{V}(x) := \inf_{\tau_2} \sup_{\tau_1} R_x(\tau_1, \tau_2),$$

then we clearly have $\underline{V} \leq \overline{V}$. If we also have $\underline{V} \geq \overline{V}$, it makes sense to talk about the value of the game, and we define $V := \underline{V} = \overline{V}$. It is shown in [16] that the value of an OSG exists under rather weak assumptions.

From a financial point of view, there is a natural interpretation of an OSG as an option of American type, where the issuer has the right to cancel the
contract before the buyer chooses to exercise. This interpretation was first introduced in [23]. In Article IV we show, by means of an iterative scheme, how to find the value of an OSG with a jump-diffusion as the underlying asset.

1.2 Model Extensions and Solution Methods

In this section we show some ways that one can extend the model in which the problems presented in Section 1 are defined.

1.2.1 Jump-diffusions

In the seminal paper [28], the standard Black-Scholes model was extended to allow for jumps in the asset price process. This was a first step to model the behavior one sometimes observes at the stock market, where the prices can move a lot in a very short time period. This behavior is not consistent with the stock being modeled by a diffusion, with continuous sample paths, unless the volatility is assumed to be very large. Allowing for jumps of the stock price also makes the stock returns more consistent with empirical data. For an introduction to financial modeling with jump-processes, see [5] and the numerous references therein.

Let \( \mathbb{N} \) be a Poisson random measure on \( \mathbb{R}_+ \times A \), for \( A \subseteq \mathbb{R} \). Let \( \lambda dt \nu(dz) \) be the mean measure of \( \mathbb{N} \), for a probability measure \( \nu \) on \( A \). We model \( X \) by the jump-diffusion process satisfying

\[
dX_t = \mu(X_t) dt + \sigma(X_t) dW_t + \int_A \phi(X_t-; z) \mathbb{N}(dt, dz).
\]

Let \( \{T_n\}_{n=1}^\infty \) be the jump times of \( \mathbb{N} \). At time \( T_n \) the process \( X \) jumps from \( X_{T_n-} \) to \( X_{T_n-} + \phi(X_{T_n-}; Z_n) \), where \( Z_n \) are i.i.d. random variables with distribution \( \nu \) (compare [28], in which \( \phi(X_{T_n-}; Z_n) = X_{T_n-}(e^{Z_n} - 1), A = \mathbb{R} \) and \( \nu \) is the Normal distribution).

With \( X \) as defined in (1.13) as the underlying asset in (1.4), one could write down a free-boundary problem for \( V \) and the optimal boundary \( b \). However, due to the integral term in the equation, one would have to solve an integro-differential equation, and hence there would be little hope of finding an explicit solution. To circumvent this, a method inspired by dynamic programming can be used, see e.g. [18], [1], [9], [36], as well as Articles I and IV in this thesis. To simplify the notation, we assume that \( \int_A \phi(X_t-; z) \mathbb{N}(dt, dz) = \gamma X_t dN_t \), i.e. \( X \) jumps with a constant relative jump size \( \gamma \) when \( \mathbb{N} \) jumps. The idea is the following. Let \( \tau \) be an arbitrary but fixed \( \mathcal{F}_t \)-stopping time, and let \( T_1 \) denote the first time the process \( \mathbb{N} \) jumps. Assume that we use the following strategy: if \( \tau < T_1 \), stop at \( \tau \). If \( \tau \geq T_1 \), update \( X \) at \( T_1 \) to \( X_{T_1} = (1 + \gamma)X_{T_1-} \) and perform...
optimally thereafter. The value of this strategy is then given by
\[
E_x \left[ e^{-r \tau} G(X_\tau) \mathbb{1}_{\{\tau < T_1\}} + e^{-r T_1} V(X_{T_1}) \mathbb{1}_{\{\tau \geq T_1\}} \right] 
\]
\[
= E_x \left[ e^{-r \tau} G(X_\tau) \mathbb{1}_{\{\tau < T_1\}} + e^{-r T_1} V((1 + \gamma)X_{T_1-}) \mathbb{1}_{\{\tau \geq T_1\}} \right] 
\]
\[
= E_x \left[ e^{-(r+\lambda) \tau} G(X_\tau) + \lambda \int_0^\tau e^{-(r+\lambda)t} V((1 + \gamma)X_{T_1-}) \, dt \right],
\]
Hence, if we for functions \( h \) in some suitable class, e.g. bounded and continuous, define the operator \( J \) on \( h \) by
\[
Jh(y) = \sup_{\tau} E_x \left[ e^{-(r+\lambda) \tau} G(X_\tau) + \lambda \int_0^{\tau_1 \wedge \tau_2} e^{-(r+\lambda)t} V((1 + \gamma)X_{T_1-}) \, dt \right],
\]
then we expect the value function \( V \) to be a fixed point of the operator \( J \). This is a useful observation, both from a theoretical and a numerical point of view. It leads to a constructive way of finding the value function, without having to solve an equation with a non-local term. Under suitable conditions on the space of functions being considered, it is in many cases possible to show that \( J \) is a contraction on that space, and hence has a unique fixed point by the Banach fixed-point theorem. One can then start with any function \( h_0 \) in the space, and then iteratively define \( h_n = Jh_{n-1}, n \geq 0 \). The value function \( V \) is then the point-wise limit of this sequence. This method is utilized in Articles I and IV in this thesis.

1.2.2 Incomplete Information

In this section we extend the standard Black-Scholes model in three different cases to incorporate incomplete information about the drift of the underlying asset. One of the key tools used is the Wonham filter, see [39] and [25, Theorem 9.1].

In Section 1.1.2, the drift coefficient \( \alpha \) and the diffusion coefficient (volatility) \( \sigma \) were assumed to be known constants. For \( \sigma \), this is a reasonable assumption, since by observing the trajectory of the asset price for a short time period, one can estimate the volatility of the asset with arbitrary precision. However, the drift parameter is much harder to estimate. Thanks to the log-normality of \( X \), one easily derives a confidence interval for \( \alpha \), and one finds that even for a small volatility one needs to observe the price fluctuations for hundreds of years in order to get a decent estimate. The following example is borrowed from [35, p. 144]: If the true parameters are \( \mu \) and \( \sigma \), respectively, then
\[
\hat{\mu} := \frac{\ln \left( \frac{X_t}{x} \right)}{t} + \frac{\sigma^2}{2} \sim \mathcal{N} \left( \mu, \frac{\sigma^2}{t} \right),
\]
so a 95% confidence interval for \( \mu \) is given by \( (\hat{\mu} - 1.96\sigma/\sqrt{t}, \hat{\mu} + 1.96\sigma/\sqrt{t}) \). Even for a rather modest value for \( \sigma, t \) has to be huge. For example, if \( \sigma = 0.2 \)
we have to observe the process for approximately 1580 years to get a confidence interval of length 0.02!

Since the drift of the asset is infeasible to estimate in practice, it is natural to incorporate incomplete information about the drift in the model. Below, we extend the standard Black-Scholes model to a model with incomplete information about the drift. As it turns out, even for a simple model incorporating this, we see examples of phenomena very different from the standard case.

**Optimal liquidation/exercising**

Suppose we want to find

\[
V = \sup_{\tau} \mathbb{E}_x [e^{-r\tau}G(X_\tau)],
\]

where the asset price follows a GBM

\[
dX_t = \mu X_t dt + \sigma X_t dW_t, \quad t \geq 0,
\]

starting at $x > 0$. Let the volatility $\sigma$ be a known positive constant, but the drift $\mu$ be a random variable independent of $W$ that can take two values, $\mu_l$ and $\mu_h$, where $\mu_l < \mu_h$. Although $\mu$ is not known a priori, the modeler has an initial guess for the probabilities of the events $\{\mu = \mu_l\}$ and $\{\mu = \mu_h\}$. We denote the initial guess of the probability of the event $\{\mu = \mu_h\}$ by $\pi$, and hence the estimated probability of the event $\{\mu = \mu_l\}$ equals $1 - \pi$, where $\pi \in (0, 1)$. The modeler then updates this initial belief as he observes the evolution of the price, and can incorporate this in his optimal strategy. Indeed, if $\Pi_t := \mathbb{P}(\mu = \mu_h|\mathcal{F}_t^X)$, it follows from [25, Theorem 9.1] that

\[
\left(\frac{dX_t}{\Pi_t}\right) = \left(\mu_l + \Pi_t(\mu_h - \mu_l)\right)dt + \left(\frac{\sigma}{\omega \Pi_t(1 - \Pi_t)}\right)d\tilde{W}_t,
\]

where $\omega = (\mu_h - \mu_l)/\sigma$ and $\tilde{W}$ is a standard $\mathbb{P}$-Brownian motion defined by

\[
d\tilde{W}_t = d\tilde{W}_t + \frac{\mu - (1 - \Pi_t)\mu_l - \Pi_t\mu_h}{\sigma}dt.
\]

Note that $X$ now depends on $\Pi$, so solving (1.14) amounts to solving a two-dimensional optimal stopping problem. It turns out that after a certain change of measure, this can be reduced to a one-dimensional problem. In [10] and [24] these techniques are used when determining the optimal investment timing in a project, and in [14] for optimal selling of an asset with finite time horizon. Article II in this thesis is devoted to optimal exercising of American options in this model. Among other things, we show that for the holder of an American put option, it is sometimes optimal to exercise the option the first time the asset price crosses a boundary *from below*, a situation that does not occur in the standard Black-Scholes model.
Momentum
Another situation where incomplete information about the drift comes into play is when modeling the momentum of an asset. This is the notion that an asset that has performed well in the recent past tends to continue to do so for some period. Utilizing this effect when trading is sometimes referred to as the strategy of "buying past winners, selling past losers". The momentum effect has been statistically verified, see e.g. [20], [21], and to model this phenomenon, several mathematical models have been suggested, see e.g. [2], [15], [37] and Article I in this thesis.

We will now reconsider problem (1.7), with \( a = 0 \) and \( T = \infty \). An agent wants to sell an asset optimally by making use of the momentum effect. A simple model that captures this is the following, which is used in [2] and [15]: suppose \( \theta \) is a random variable with distribution \( P(\theta = 0) = \pi \) and \( P(\theta \geq t|\theta > 0) = e^{-\lambda t} \) for some \( \lambda > 0 \) and \( \pi \in [0,1) \). Let \( X \) be a GBM with dynamics

\[
dX_t = \mu(t)X_t dt + \sigma X_t dW_t,
\]

where \( W \) is independent of \( \theta \) and

\[
\mu(t) = \mu_2 - (\mu_2 - \mu_1)I_{\{t \geq \theta\}},
\]

hence when \( \theta \) occurs, the drift of the asset changes from \( \mu_2 \) to \( \mu_1 \). To avoid trivial cases, assume \( \mu_1 < r < \mu_2 \). The objective is to find

\[
V(x) = \sup_{\tau \geq 0} E_x [e^{-r\tau}X_\tau]
\]

as well as an optimal \( \mathcal{F}^X \)-stopping time.

The agent does not observe when \( \theta \) occurs (in that case the problem would be trivial). However, based on the evolution of the asset price, he might come up with an educated guess about when the drift has switched from the higher one to the lower, i.e. when it is time to sell. Hence, this is another example of incomplete information about the drift, and in order to solve the problem it is again possible to use filtering theory and a change of measure. In Article I in this thesis we consider a more general model, which contains the model described above as a special case.

Trend Following Trading
A third situation where incomplete information of the asset plays a role is when the market of the underlying asset exhibits trends.

In Section 1.2.2, the problem was when to sell an asset optimally in the presence of momentum. With the interpretation of the asset \( X \) being a stock, it is natural to allow for sequential trading in \( X \), i.e. to buy and sell the stock repeatedly at a sequence of time points. This is an example of an optimal switching problem with two states (hold the stock and not hold the stock, respectively). Since the general theory of optimal switching is not the main focus
of the thesis, and since we in Article III use an alternative approach rather than more standard techniques to analyze the problem, we refrain from presenting these techniques, and we refer to the literature for a thorough treatment.

A commonly used trading strategy is trend following trading. An investor following this strategy tries to buy the stock at the beginning of an uptrend, and sell it at the beginning of a downtrend. Other examples of trading strategies analyzed in the literature are contra-trend trading, see e.g. [27] and [26], and the buy and hold strategy, see e.g. [38] and [37]. We now consider the following model for trend following trading (see also [6], [7] as well as Article III in this thesis): Assume the asset price satisfies

\[ dX_t = \mu(t)X_t dt + \sigma X_t dW_t, \]

where \( \sigma > 0 \) is a constant, \( W \) is a Brownian motion and the drift \( \mu(t) \) is a continuous time Markov chain, independent from \( W \), taking values in \( \{ \mu_1, \mu_2 \} \). Here \( \mu_1 < \mu_2 \), and with intensity \( \lambda_i > 0 \) the Markov chain jumps from state \( \mu_i \) to \( \mu_{3-i}, i = 1, 2 \). These parameters are all assumed to be known to the investor.

Suppose for simplicity that we enter the regime switching market defined above with no stock. Define a sequence of \( \mathcal{F}^X \)—stopping times

\[ \mathcal{J} = (\nu_1, \tau_1, \nu_2, \tau_2, \ldots), \]

where

\[ 0 \leq \tau_1 \leq \nu_1 \leq \tau_2 \leq \nu_2 \leq \ldots \leq T, \]

and where a buying decision is made at \( \tau_i \) and a selling decision at \( \nu_i \). Let \( \epsilon \) denote the percentage cost per transaction, and let \( T \leq \infty \) be the time horizon under which we trade. The objective is to maximize

\[ J(x, \mu, \mathcal{J}) := \mathbb{E}_x \left[ \sum_{n=1}^{\infty} \left[ e^{-\beta \nu_n} X_{\nu_n} (1 - \epsilon) - e^{-\beta \tau_n} X_{\tau_n} (1 + \epsilon) \right] 1_{\{ \tau_n < T \}} \right], \]

where \( \mu_1 < \beta < \mu_2 \) to avoid degeneracy. Again, the filtering techniques due to Wonham [39] allow us to transform this problem with incomplete information to a two-dimensional problem with complete information. Indeed, by applying [25, Theorem 9.1] we want to maximize \( J(x, \pi, \mathcal{J}) = J(x, \mu, \mathcal{J}) \), where the conditional probability \( \Pi_t := \mathbb{P} [ \mu(t) = \mu_2 | \mathcal{F}^X_t ] \) and the asset price satisfy

\[ d\Pi(t) = (\lambda_1 (1 - \Pi_t) - \lambda_2 \Pi_t) dt + \omega \Pi_t (1 - \Pi_t) d\hat{W}_t \]

and

\[ dX_t = \hat{\mu}(t)X_t dt + \sigma X_t d\hat{W}_t. \]

Here \( \omega := (\mu_2 - \mu_1)/\sigma, \hat{\mu}(t) := \mu_1 + (\mu_2 - \mu_1) \Pi_t \) and the innovation process

\[ \hat{W}_t := \frac{1}{\sigma} \int_0^t (\mu(s) - \hat{\mu}(s)) ds + W_t \]
is a standard Brownian motion, see e.g. [30]. This problem can be solved numerically, compare [6] and [7], by solving a corresponding system of variational inequalities. In Article III in this thesis a variant of this problem is treated, and optimal (constant) boundaries for the belief process $\Pi$ are found, where optimality is defined in terms of the long-term mean return.
2. Summary of papers

2.1 Paper I

We study the optimal liquidation strategy for a momentum trade in a setting where the drift of the asset drops from a high value to a smaller one at some random point in time $\theta$. This problem has been studied when the underlying asset is an arithmetic Brownian motion in [2], and a geometric Brownian motion in [15], respectively. In the case where $\theta$ is observable, i.e., where one has complete information about when $\theta$ occurs, the problem becomes trivial. Indeed, to maximize the gain from the sell one should simply sell the asset at the point in time at which the drift changes from the higher one to the lower.

In Paper I we consider a situation which is strictly between the case with complete information and the case with incomplete information studied in [15] and [2]. We study a momentum trade modeled as a geometric Brownian motion with a drift that drops from one constant value to another at an unobservable time. However, the time of change of the drift is assumed to coincide with one of the jump times of a Poisson process, and these jump times are observable. The Poisson process represents external shocks to the system (an unfavorable political decision, a financial crisis, the release of bad news, etc.) that possibly cause the momentum effect to disappear. The probability that $\theta$ occurs is assumed to be known. More precisely, we let $N$ be a Poisson process with intensity $\lambda/p$, and we let $\mathbb{P}(\theta = 0) = \pi$ and $\mathbb{P}(\theta = T_i | \theta > 0) = p(1 - p)^{i-1}$, where $T_i$ are the jump times of $N$. We refer to this situation, intermediate between the complete information case and the incomplete information setting, as a model with partial information. It was shown in [36] how to reduce such a problem to a two-dimensional problem. In Paper I, we show how to find a change of measure that reduces this two-dimension problem into a one-dimensional problem, with a jump-diffusion as the underlying asset, and we provide a thorough analysis of this new problem. In particular, we show that the value is a fixed point of a certain operator, defined on a suitable function space. We show that iterating this operator, by starting with the smallest function in the class, produces an increasing sequence of functions, with the value of the original problem as its point-wise limit. That this is indeed the case follows from a verification argument. We conclude by establishing rate of convergence of the value function, as well as parameter dependencies. We also compare our findings with [15], which is contained in the present model as the special case $p \searrow 0$. We show that knowing that the disappearance of the momentum effect is triggered by observable external shocks significantly improves the optimal strategy.
2.2 Paper II

We make a few observations of unexpected phenomena that may occur when exercising options under incomplete information. In the Black-Scholes model, optimal exercise of vanilla options is very well documented. For example, it is well known that the optimal stopping boundary for an American put option is an increasing curve, \( b(t) \), and that the optimal strategy is to exercise the option the first time the asset value falls below \( b(t) \), see e.g. [19], [4] and [32]. In Paper II we extend the standard Black-Scholes model to a model with incomplete information about the drift of the asset. We consider the model as described in Section 1.2.2, where the drift can take either of two possible values \( \mu_l \), \( \mu_h \), such that \( \mu_l < r < \mu_h \) where \( r \) is the interest rate. We use filtering techniques to transform the original problem with incomplete information to a two-dimensional problem with complete information. As it turns out, this new formulation of the problem is not fully two-dimensional, since it is the same Brownian motion driving the asset process \( X \) and the belief process \( \Pi \). This insight, first made in [24], allows us to transform the two-dimensional problem to an auxiliary optimal stopping problem with one dimension. We show that certain parameter values give rise to stopping regions very different from the standard case. For example, we show that for the American put (call) option it is sometimes optimal to exercise the option once the asset price process reaches an upper (lower) boundary. This has to do with the uncertainty of the drift. For e.g a put option, a low value is profitable for the holder of the option. However, if the value increases a lot, this might indicate that the drift equals \( \mu_h \), and hence one should exercise (at the upper boundary) in order to minimize the loss. This is in stark contrast to the standard case, where one should always wait for a low value of the asset.

2.3 Paper III

We study perpetual two-state switching problems in connection with trend following trading strategies. More precisely, we study the long-term performance of time-homogeneous strategies, when applied to different models of a financial market. The corresponding value functions are infinite, so standard techniques using dynamic programming are of no use, since these require the value functions to be finite. With infinite value functions, optimality cannot be defined in terms of the maximal value obtained when acting optimally. Instead, we focus on optimality of the strategies. In the class of time-homogeneous strategies, we measure optimality in terms of the maximal growth rate of the expected value and expected growth rate of the value function. We study the strategies using techniques from renewal theory. This turns out to be particularly useful for perpetual problems in a mean-reverting or regime switching model, since the expected time of an excursion is finite, as opposed to in a model with an arithmetic or geometric Brownian motion as the underlying as-
set. We demonstrate two examples where our findings are applied. Firstly, optimal currency trading in a mean-reverting model, such that $e^{X_t}$ models the interest rate where $dX_t = -\beta X_t dt + \sigma dW_t$ is an Ornstein-Uhlenbeck process. Secondly, trend following trading in a regime switching model similar to the one used in [6] and [7]. Here, the underlying process is a GBM such that $dX_t = \mu(t) S_t dt + \sigma S_t dW_t$. However, the drift function is an unobservable Markov chain with two states, on opposite sides of the interest rate. We hence have to use filtering techniques in order to derive the dynamics of the belief for the higher drift. The time-homogeneous optimal switching boundaries are then calculated for this belief process.

2.4 Paper IV
We study a general two-person zero-sum optimal stopping game, where the underlying asset is a jump-diffusion. In order to avoid having to solve variational inequalities with non-local terms, we instead find the value by means of an iterative procedure, by applying a certain operator sequentially. By the dynamic programming principle, we expect the value function to be a fixed point of this operator. That this is indeed the case is proved by a direct approach. Furthermore, we show that applying the operator $n$ times gives the value of the optimal stopping game, when restricted to $[0, T_n]$, where $T_n$ is the $n$:th jump of the underlying process. We then prove that the operator is a contraction on a certain function space, and by an application of Banach fixed-point theorem, we find that the convergence to the original value function is exponential. Finally, we illustrate our findings with an example. We consider the game version of the American put option, where the underlying process is a geometric Brownian motion between positive jumps, proportional to the asset price. We provide conditions on the parameters determining whether the optimal stopping time for the issuer of the option is infinite (i.e when the value coincides with the value of a perpetual American put option) or finite, in which case the issuer should stop the first time the process hits the strike price.

2.5 Paper V
The American put option is the most studied vanilla option of American type. The arbitrage free price of the option is known to satisfy a free-boundary problem, where the boundary to be found, the optimal exercise boundary, satisfies an integral (Volterra) equation. The equation for the boundary $b(t)$ is obtained by setting $x = b(t)$ in the early exercise premium representation of the option value. In a general jump-diffusion model, a similar equation can be obtained, see e.g. [33]. However, this equation depends on the value itself, so one has to know the value in order to obtain an equation for the boundary. We consider
different ways to circumvent this, firstly by only allowing for negative jumps of the process. In such a model, one can only leave the stopping region by diffusing up through the boundary, and in this way we get an equation for the boundary similar to the standard case. More precisely, we show that an early exercise premium representation similar to the standard case holds, and we obtain an equation for the boundary by setting \( x = b(t) \) in this representation. Furthermore, we prove that the optimal exercise boundary is the unique solution to this equation, in the class of continuous functions \( \gamma : [0, T) \to (0, K) \) bounded by the strike price \( K \). We conclude by deriving the boundary equation in the case where the underlying asset has negative, constant jumps proportional to the asset price. In the case of positive jumps, we study the corresponding integral equation where the term involving the value function has been replaced with an upper and a lower bound of the value function. In particular, we show that a certain approximation of the integral equation gives an upper bound for the true boundary.
3. Summary in Swedish


En vanlig lösningsmetod för den här typen av problem är att formulera och lösa ett fritt randproblem för det optimala värdet, och sedan säkerställa, med hjälp av ett verifikationsargument, att lösningen till detta problem ger lösningen till ursprungproblemet.


I artikel I studerar vi när man optimalt ska sälja en aktie då man vet att märkaren kommer att svänga och gå ner vid någon slumpmässig tidpunkt i framtiden. Man observerar inte tidpunkten då marknaden svänger, om så vore fallet vore problemet trivialt. Däremot antar vi att denna tidpunkt sammanfaller med en extern chock, vilken vi observerar.

I artikel II studerar vi när man optimalt ska lösa in en amerikansk option, i en modell med ofullständig information om driften på den underliggande tillgången. Från början vet vi inte om den underliggande tillgången tenderar
att utvecklas bättre eller sämre än ett bankkonto, men av att observera hur tillgången utvecklas kan vi bilda oss en bättre och bättre uppfattning om hur detta förhåller sig, och på så sätt hur vi ska agera optimalt.

I artikel III studerar vi optimalbytesproblem med hjälp av tekniker från förnyelseteori. Av intresse är den asymptotiska tillväxthastigheten av värdet vi erhåller då vi köper och säljer en tillgång sekvensiellt.

I artikel IV studerar vi optimala stopptidsspel. Detta är en mer generell klass av problem än optimala stopptidsproblem, då två personer spelar mot varandra, och båda har möjlighet att stoppa.

Slutligen, i artikel V, studerar vi den randekvation till en amerikansk sålpoption som uppkommer när den underliggande tillgången modelleras som en hopp-diffusion. Det visar sig att en ekvation liknande den som uppkommer i det vanliga fallet med en Black-Scholes-modell gäller även i det här fallet.
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References

Numerical results indicate that we will have
\begin{equation}
\tilde{c}(t) \leq b(t) \leq \hat{c}(t),
\end{equation}
and moreover, as might be expected, for small values of $\lambda \gamma$ the boundary $\tilde{c}$ seems to be a good approximation of the real exercise boundary. Establishing the veracity of the conjecture constituted by (19) remains an interesting open problem.

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