Application of biorthogonal eigenfunction system for extraction of Tollmien-Schlichting waves in acoustic receptivity simulations

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Internal report

Acoustic receptivity of a two-dimensional boundary layer on a flat plate with elliptic leading edge is studied through direct numerical simulation (DNS). Sound waves are modelled by a uniform oscillation of freestream boundaries in time which results to an infinite-wavelength acoustic wave. Acoustic disturbances interact with strong streamwise gradients at the leading edge or surface non-homogeneities and create Tollmien-Schlichting (TS) waves inside the boundary layer. Measuring amplitude of TS waves created by sound waves is challenging due to presence of Stokes wave (acoustic boundary layer) with the same temporal frequency of TS waves. In this study biorthogonal eigenfunction system of local linear stability equations has been utilised to extract TS wave amplitudes. This method is based on the concept of using adjoint mode as a projector where the TS amplitude is obtained by projecting the DNS solution onto adjoint TS modes. However, the computed TS wave amplitude employing this method found to be modulated. It is shown that the modulation is due to existence of a small amplitude wave in the DNS data that is not expandable onto the basis of local linear stability equations.

Key words: Boundary layers, instability, acoustic receptivity, biorthogonal eigenfunction system

1. Introduction

Understanding the laminar-turbulent transition process is crucial for design optimisation in numerous aerodynamic applications. The practical and fundamental importance of transition process has attracted extensive studies in the past decades, however, it is not yet fully understood. Boundary layer flows are open systems where disturbances in the freestream, such as sound waves or freestream turbulence enter the boundary layer as steady and/or unsteady fluctuations of the basic state. The response of boundary layer to these fluctuations creates the initial amplitude of the instability waves. This process is the first stage
of transition process and is called the receptivity stage. The term receptivity was first used by Morkovin (1969) to describe the mechanism by which energy from the freestream enters and excites instability waves inside the boundary layer. Goldstein & Hultgren (1989) categorised the experimental studies aimed at understanding the receptivity of boundary layer flows into two groups: first, those in which disturbances are introduced locally within the boundary layer (forced disturbances) and second, those in which disturbances are introduced at some distance from the boundary layer (natural disturbances). The latter case generates intermediate freestream disturbances of usually acoustic or vortical type (Nishioka & Morkovin 1986). The natural disturbances generally do not contain a wavelength that coincides with the instability that is generated within the boundary layer. Murdock (1980) investigated the generation of Tollmien-Schlichting (TS) wave by acoustic disturbances on a flat-plate boundary layer with sharp leading edge through numerical simulations. He shows that interaction between the sound wave and the TS waves does not occur over the entire boundary layer and plane sound waves feed energy into TS waves only in the leading-edge region. Goldstein (1983) shows through asymptotic analysis that long-wavelength acoustic freestream disturbances are coupled to short-wavelength TS waves via a length scale reduction mechanism near the leading edge. Wave-length adjustment or wave-length conversion can be attributed to rapid streamwise variations of the mean flow, e.g. at the leading edge, and sudden changes in surface boundary condition.

This study is concerned with boundary layer response to acoustic disturbance on a finite-thickness flat plate with elliptic leading edge. Acoustic waves are introduced as time-periodic oscillations in the freestream. Acoustic disturbances that enter the boundary layer excite different unsteady modes. These include a Stokes wave and instability waves of both discrete and continuous spectrum type. Measuring the amplitude of TS waves excited by acoustic disturbances is a challenging task due to the presence of Stokes wave with the same temporal frequency of TS waves. Hence, Fourier amplitude of perturbations in time contain both TS and Stokes waves amplitudes. This problem has been tackled by different approaches in both experimental and numerical studies. Lin et al. (1992) investigated receptivity to planar sound waves of the boundary layer over a finite-thickness flat plate with elliptic leading edge through direct numerical simulations. They introduced the modified super ellipse (MSE) leading-edge geometry with zero curvature at the junction to the flat plate in order to minimise undesired localised receptivity due to geometric discontinuities (Goldstein & Hultgren 1989). The mean flow is at Reynolds number $Re_b = 2400$ with $b$ being the plate half-thickness. To calculate acoustic receptivity, they subtract theoretical Stokes-wave solution from the total perturbation field obtained in the numerical simulation. Furthermore, wavenumber information of the TS wave present in unsteady solutions is extracted by differentiating the Fourier phase of the wall-normal perturbation velocity in the streamwise direction. Fuciarelli et al. (2000) obtained branch I acoustic receptivity coefficient for the same base flow as Lin et al. (1992). They use a decomposition method based
on phase speed difference between TS and Stokes waves, proposed by Wlezien (1994), to separate the two. Because the phase and amplitude of the Stokes wave are relatively constant over several TS wavelengths, this decomposition is possible. However, near the leading-edge region due to curvature and local pressure gradients the amplitude and phase of Stokes wave modulate and is not distinguishable from the TS wave. Therefore this decomposition technique fails in this region.

Wanderley & Corke (2001) consider two MSE leading edge geometries at the Reynolds number $Re_b = 2400$ and calculate the leading-edge and branch I receptivity coefficient over a wide range of acoustic frequencies. In order to obtain TS wave amplitude, a separately calculated Stokes wave on the same geometry is subtracted from the total perturbation velocity. After subtraction of Stokes wave they still observe a long wavelength oscillation present in the perturbation velocity. A one-dimensional spatial filter is designed to reject fluctuations with wavelength larger than the expected TS mode. After filtering they observe a single-frequency oscillation, without amplitude modulation. In a recent numerical work, Turner (2012) employed DNS and parabolised stability equations (PSE) to study TS wave amplitude on a flat plate and a parabolic body. He follows the numerical method of Haddad & Corke (1998) which solves linearised Navier-Stokes equations and can be used for parabolic bodies as well as flat-plate geometries. The filtering process is the same as Wanderley & Corke (2001) who has based their work on Haddad & Corke (1998) as well. Turner (2012) shows that DNS results contain non-modal modes or remnants of the Stokes layer not removed by the filtering process. This phenomenon is most significant in the lower branch region and appears as an oscillation in the curve of TS amplitude when plotted versus streamwise coordinate. This behaviour has not been reported in previous works.

Casalis et al. (1997) investigated leading edge and localised acoustic receptivity of two-dimensional boundary layer through DNS. In order to remove the Stokes layer from the DNS solutions, they solved linearised unsteady boundary layer equations (LUBLE) with a forcing term which corresponds to the freestream oscillations. Far from the leading edge, the solution of LUBLE is equivalent to the analytical solution of Stokes wave. However, in the leading edge region the base flow influence on the perturbations is incorporated in LUBLE. Hence, the Stokes wave calculated by LUBLE is different from the analytical one. Its worth to mention, the Stokes wave calculated by Wanderley & Corke (2001) does not incorporate the influence of base flow on the perturbations.

In experimental investigations of acoustic receptivity, several techniques have been developed to measure TS wave amplitude. Wlezien (1994) studied the receptivity of leading edge and suction slits experimentally. He used the complex plane spiral analysis to obtain TS wave amplitude. White et al. (2000) use a pulsed-sound technique to separate Stokes and TS waves. The pulsed-sound technique utilises the difference in group velocity of the two waves. The Stokes waves travel at the speed of sound compared to TS waves which travel
at a fraction of freestream velocity. Sound is emitted from speakers in a short burst of two to four cycles. The Stokes wave rapidly passes the measurement location, and after a short delay, the TS wave passes the measurement location. The measurements are timed such that data are acquired when the Stokes wave has passed. In this technique large numbers of ensemble averages are used to account for low-frequency oscillations on the flow. The reader is referred to Saric et al. (2002) for more comprehensive review on boundary-layer receptivity to freestream disturbances.

Recently, Monschke et al. (2016) used biorthogonal decomposition method to separate Stokes and TS waves in an experimental study to measure acoustic receptivity of a boundary layer over a flat plate with MSE leading edge. This method is based on biorthogonal eigenfunction system (BES) introduced by Tumin (2003). The biorthogonality relation between direct and adjoint modes of linear stability equations allows the use of adjoint mode as a projector. TS amplitude is obtained by projecting the DNS solution onto adjoint of TS waves. Application of BES for analysis of both experimental and numerical data in various receptivity problems have shown this method is a powerful tool for gaining insight into the flow dynamics (Piot et al. 2007; Tumin 2007, 2011).

In a recent numerical investigation, Shahriari et al. (2016) computed the acoustic receptivity of flow around a flat plate geometry with MSE leading edge. They considered both compressible and incompressible frameworks and used different high-order flow solvers. Moreover, they obtain TS waves amplitude using independent methods suggested by Murdock (1980) and Wlezien (1994). The results of their work show the level of acoustic receptivity is one order of magnitude lower than what was previously reported in both numerical and experimental works.

In this study, we consider a finite-thickness flat plate with MSE leading edge at Reynolds number of $Re_b = 2400$. Following the work by Shahriari et al. (2016) who established the correct acoustic receptivity coefficient for this flow configuration, we employ biorthogonal eigenfunction system to obtain TS wave amplitudes and compute the acoustic receptivity coefficient. The aim is to offer a more robust technique to obtain TS waves amplitude in acoustic receptivity problems and gain more insight about the dynamics of this flow.

2. Flow configuration and numerical approach

2.1. Geometry and flow parameters

Flow around a semi-infinite flat plate with a modified super ellipse (MSE) leading edge is considered for investigation. This geometry follows the suggestion by Lin et al. (1992) and is defined as

$$\left(\frac{y}{b}\right)^2 + \left(\frac{a-x}{a}\right)^m = 1, \text{ with } m = 2 + \left(\frac{x}{a}\right)^2,$$  \hspace{1cm} (1)

where, $x$ and $y$ are the streamwise and wall-normal coordinates respectively. Parameters $a$ and $b$ are the semi-major and semi-minor axis of the ellipse. The
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Figure 1: (a) Leading edge shape with aspect ratio of $AR = 20$ for MSE (---) and regular ellipse (----). (b) Surface curvature of the shapes in (a). Vertical dotted line shows juncture between leading edge and flat plate.

aspect ratio of an ellipse is defined as $AR \equiv a/b$ measuring curvature of the leading edge. Large value of $AR$ results to sharp leading-edge nose and small $AR$ values correspond to more blunt leading-edge shape. MSE leading edge shape eliminates the curvature discontinuity at the junction of the ellipse and the flat plate which in return, reduces undesirable localised receptivity. The exponent $m$ rises smoothly from value of 2 at the tip of leading edge ($x = 0$) to 3 at the joint to flat plate ($x = a$), providing geometrical smoothness at the juncture. Figure 1(a) demonstrates the MSE leading edge shape compared to a leading edge with regular elliptic curvature (constant exponent $m = 2$). The curvature coefficient of a plane curve defined as

$$\kappa = \frac{|y''|}{(1 + y'^2)^{3/2}},$$

is shown in figure 1(b) for MSE and regular ellipse. It is apparent that MSE has continuous curvature at the junction to the flat plate while there is a jump in the curvature for the regular ellipse at the joint location.

The semi-minor axis of the ellipse, $b$, is chosen as reference length scale in this study. The reference velocity is chosen to be the freestream velocity $U_\infty$ and the kinematic viscosity $\nu$ is chosen in a way that the Reynolds number becomes $Re = U_\infty b/\nu = 2400$. This flow configuration follows various numerical and experimental investigations, (e.g. Lin et al. 1992; Saric & White 1998; Fuciarelli et al. 2000; Wanderley & Corke 2001; Monschke et al. 2016; Shahriari et al. 2016). The length of computational domain in the downstream direction is $L = 450$ units (in terms of the plate half-thickness), which corresponds to a Reynolds number of $Re_L = U_\infty L/\nu = 1.08 \times 10^6$ at the end of domain. The extent of the the domain in the wall-normal direction and upstream of the leading edge is 400 units, which is 200 times larger than the maximum boundary layer thickness in the considered domain.
2.2. Direct numerical simulations

Incompressible Navier-Stokes equations together with the continuity equation subject to constant fluid properties,

\begin{align}
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} &= -\nabla p + \frac{1}{Re} \nabla^2 \mathbf{u} + \mathbf{f}, \quad (3a) \\
\nabla \cdot \mathbf{u} &= 0, \quad (3b)
\end{align}

are integrated in time using \texttt{NEK5000} code developed by Fischer \textit{et al.} (2008). Here $\mathbf{u} = \{u, v\}^T$ is the vector of velocity components in the streamwise and wall-normal directions, $p$ represents the pressure field, $Re$ denotes the Reynolds number and $\mathbf{f}$ is the body force term. \texttt{NEK5000} is based on spectral element method (SEM) proposed by Patera (1984) which combines geometrical flexibility of finite element method with spectral accuracy. The physical domain in SEM is decomposed into spectral elements where the local approximation of flow field is obtained as a sum of Lagrange interpolants defined by an orthogonal basis of Legendre polynomials up to degree $N$. Gauss-Lobatto-Legendre (GLL) nodes are used to build velocity Lagrange polynomial interpolants and Gauss-Legendre (GL) nodes for pressure Lagrange polynomial interpolants (two orders less than the velocity field). Polynomial order $N$ is the same in all spatial directions. Here we have used $N = 8$ for velocity grid and and $N = 6$ for pressure grid. The use of staggered pressure grid makes it unnecessary to specify explicit boundary condition for pressure. Pressure is evaluated through the natural boundary conditions at the outflow. This spatial discretisation is referred to $P_N-P_{N-2}$ formulation introduced by Maday & Patera (1989).

Equations are advanced in time using a third-order conditionally stable backward differentiation and extrapolation scheme (BDFk/EXTk), employing an implicit treatment of the diffusion term and explicit treatment of the advection term. \texttt{NEK5000} is highly parallelised and scalable on thousands of threads (Tufo & Fischer 2001). Current results are obtained using up to 512 processors. Two types of simulation have been carried out in this work. First, steady-state base flow solution is computed. Further, nonlinear simulations with acoustic disturbances are performed for a range of frequencies.

For the base flow computation, no-slip and no-penetration boundary conditions are imposed on the solid surface. The far-field boundary conditions (inflow and upper boundary) are of Dirichlet type, obtained from a potential-flow solution around a corresponding body thickened by the displacement thickness of the evolving boundary layer. Along the stagnation streamline, symmetric boundary condition has been employed to reduce the computational cost. The outflow boundary condition is the natural condition derived from the weak form of Navier-Stokes equations. They read as

\begin{align}
\frac{1}{Re} \frac{\partial u}{\partial x} - p &= 0, \quad \frac{\partial v}{\partial x} = 0. \quad (4)
\end{align}

It is worth mentioning that conditions (4) are not enforced explicitly but satisfied intrinsically by the numerical solver used in this work.
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Figure 2: Computational grid for leading edge with $AR = 20$. (a) Distribution of the spectral elements in the upstream part of the body. (b) Close-up view of the nose region. Gray dots: Gauss-Lobatto-Legendre (GLL) points.

Figure 2 shows the computational mesh around the leading edge geometry with $AR = 20$. Spectral elements in the upstream part of the computational domain are depicted in figure 2(a). Elements in the wall-normal direction are clustered towards the surface to resolve the boundary layer more efficiently. A close up view of the leading-edge nose region is shown in figure 2(b) along with the distribution of GLL points within each element. The mesh is generated using the open source code gridgen–c by Sakov (2011). This grid generator’s main feature is to preserve orthogonality of the grid over curved surfaces as seen in figure 2. The spacing of the elements in the streamwise direction is equidistant on the flat plate part and the whole computational domain consists of 70000 two-dimensional spectral elements.

2.3. Perturbed flow

Acoustic wave disturbances within an incompressible framework are modelled by superposing a periodic fluctuation in the form of $u' = \varepsilon \cos(\omega t)$, with angular frequency $\omega/2\pi$ and amplitude $\varepsilon$, on the streamwise velocity component of the far-field boundary. Therefore, the far-field boundary condition in a nonlinear simulation would be to enforce

$$u = U_\infty (1 + \varepsilon \cos(\omega t)),$$

where $U_\infty$ is the freestream velocity. This model have been broadly used in numerical studies using incompressible Navier-Stokes formulation (e.g. Murdock 1980; Fuciarelli et al. 2000; Wanderley & Corke 2001; Shahriari et al. 2016). The ‘acoustic’ waves of such form have zero spatial wavenumber, resulting in an infinite-wavelength acoustic wave model. The acoustic perturbation amplitude is typically equal to $\varepsilon = 10^{-2}$ or $10^{-3}$. This range of perturbations amplitude
generates a uniform oscillation with maximum amplitude of $1\% U_\infty$, which is sufficiently small to assume linear behaviour for perturbations.

DNS solution fields of simulations with acoustic disturbances contain Stokes wave and instability waves of both discrete and continuous spectrum type. Fourier transform in time can not separate these waves since they exhibit the same frequency as the excitation (acoustic) frequency. Further details on the method to extract TS waves is described in section §3.

Uniform pulsation of flow over a semi-infinite flat plate generates Stokes wave. Assuming parallel base flow, it admits to analytical solution of form

$$u(y, t) = U_\infty \varepsilon \left[ \cos(\omega t) - e^{-ky} \cos(\omega t - ky) \right], \quad \text{with} \quad k = \sqrt{\frac{\omega Re}{2}}, \quad (6)$$

which does not depend on the streamwise coordinate. Figure 3(a) demonstrates the Stokes wave solution (6) at different samples of non-dimensional time, $\omega t = [0 : \pi/4 : 7\pi/4]$, in one period. The Stokes mode shape is obtained by Fourier transformation of the solutions in time and is shown in figure 3(b). The Stokes layer thickness is thinner than and embedded in the Blasius boundary layer and is constant in the homogeneous streamwise direction.

3. Methodology

In order to decompose arbitrary small perturbations in a boundary layer into normal modes of linearised Navier-Stokes equations, an orthogonality relation between the normal modes has to be derived so the amplitude of each mode can be evaluated (Tumin 2003). Orr-Sommerfeld (OS) equations (linearised Navier-Stokes equations with parallel base flow assumption) are nonself adjoint. This means the eigenmodes of OS operator are not orthogonal to each other, hence, they do not form a set of orthogonal basis. To formulate an orthogonality
relation, adjoint of OS equations should be taken into account. The eigenfunctions of both direct and adjoint OS operators provide a biorthogonal basis which allows decomposition of a perturbation into normal modes. Such a system is called biorthogonal eigenfunction system (BES). Tumin (2011) demonstrates the applications of BES of linear stability equations for various receptivity problems in boundary layer flows. Monschke et al. (2016) utilised the same technique to extract amplitude of TS waves in their acoustic receptivity experiment. In this work, we use the OS equations in primitive variables form. BES consists of the set of direct and adjoint eigenfunctions with a biorthogonality relation between them. In the following, first the direct problem is defined and then the adjoint of the direct equations is derived. Further, the biorthogonality relation between the set of direct and adjoint eigenfunctions is obtained. The method to extract amplitude of TS waves using BES is described afterwards.

3.1. Formulation of direct and adjoint problems

As mentioned above, we consider OS equations as the direct problem. OS equations are based on linear local stability theory and are common for studying modal behaviour of perturbations. Local stability theory simplifies the linear stability equations by parallel base flow approximation. Parallel base flow assumption describes a flow in the homogeneous direction that only depends on the inhomogeneous (wall-normal) $y$-coordinate. Furthermore, the variation of base flow in the homogeneous direction is neglected and the wall-normal velocity component of base flow is assumed to be zero, i.e. $U = [U(y), 0]$. For two-dimensional perturbations, the solution can be expressed as Fourier modes of form

$$q(x, y, t) = \hat{q}(y) e^{i \alpha x - i \omega t} + c.c.$$

where $x$ is the streamwise coordinate, $y$ is the wall-normal coordinate, $\alpha$ stands for streamwise wavenumber and the angular frequency is expressed by $\omega$. Introducing perturbation ansatz (7) into linearised Navier-Stokes equations, we arrive at the following system of equations for perturbations

$$-i \omega \hat{u} + i \alpha U \hat{u} + U_y \hat{v} + i \alpha \hat{p} - \frac{1}{Re} \left( -\alpha^2 \hat{u} + \hat{u}_{yy} \right) = 0,$$

$$-i \omega \hat{v} + i \alpha U \hat{v} + \hat{p}_y - \frac{1}{Re} \left( -\alpha^2 \hat{v} + \hat{v}_{yy} \right) = 0,$$

$$i \alpha \hat{u} + \hat{v}_y = 0,$$

where a subscript denotes partial derivative with respect to that variable. System of equations (8) with appropriate boundary conditions are recast to an eigenvalue problem which can be categorised into temporal or spatial stability problem. In spatial stability analysis, $\omega$ is prescribed as real-valued number and complex eigenvalue $\alpha$ is obtained. The real part of eigenvalue $\alpha$ denotes the streamwise perturbation wavenumber and its imaginary part corresponds to spatial growth rate in the streamwise direction.
In this study angular frequency is prescribed by the frequency of acoustic perturbations and we aim to find complex eigenvalue $\alpha$ which corresponds to TS eigenvector. System of equations (8) is recast in the form of a generalised linear eigenvalue problem for $\alpha$ in the form of

$$Nq = i\alpha Mq,$$

(9)

where $q = [\hat{u}, \hat{v}, \hat{p}, i\alpha \hat{v}]^T$ and the individual operators are

$$M = \begin{pmatrix}
U & \frac{1}{Re} D & 1 & 0 \\
0 & U & 0 & -\frac{1}{Re} \\
1 & 0 & 0 & 0 \\
0 & -\frac{1}{Re} & 0 & 0
\end{pmatrix},$$

(10a)

$$N = \begin{pmatrix}
i\omega + \frac{1}{Re} D^2 & -U_y & 0 & 0 \\
0 & i\omega + \frac{1}{Re} D^2 & -D & 0 \\
0 & -D & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{Re}
\end{pmatrix}.$$  

(10b)

Here, operator $D$ denotes partial derivative with respect to wall-normal $y$-coordinate. The general eigenvalue problem (9) is solved using MATLAB eigenvalue solver $\text{eig}$, provided a base flow and boundary conditions at the wall and at the far field, i.e.

$$\hat{u} = \hat{v} = 0 \quad \text{at} \quad y = 0,$$

(11a)

$$\hat{u}, \hat{v} \quad \text{bounded as} \quad y \to \infty.$$  

(11b)

The operators $M$ and $N$ are discretised using Chebyshev polynomials in the wall normal direction for each station $x$. MATLAB differentiation matrix suit by Weideman & Reddy (2000) is used for building operator $D$ and base flow derivative in the wall-normal direction.

The adjoint equation of spatial eigenvalue problem (9) can be written in the form of (for detailed derivation see Appendix A)

$$N^+ \psi = -i \alpha^+ M^+ \psi$$

(12)
where a $^+$ sign denotes adjoint property with adjoint eigenvector $\psi = [\psi_1, \psi_2, \psi_3, \psi_4]^T$ and individual adjoint operators

\[
M^+ = \begin{pmatrix}
U & 0 & 1 & 0 \\
-\frac{1}{Re}D & U & 0 & -\frac{1}{Re} \\
1 & 0 & 0 & 0 \\
0 & -\frac{1}{Re} & 0 & 0
\end{pmatrix},
\]

\[
N^+ = \begin{pmatrix}
-i\omega + \frac{1}{Re}D^2 & 0 & 0 & 0 \\
-U_y & -i\omega + \frac{1}{Re}D^2 & D & 0 \\
0 & D & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{Re}
\end{pmatrix}.
\]

3.1.1. Boundary conditions of direct and adjoint systems

The boundary terms remaining from integration by parts procedure in derivation of adjoint equations are (see Appendix B)

\[
b.t. = \left[ \psi_2 \hat{p} + \psi_3 \hat{v} + \frac{1}{Re} \left( \psi_1 \hat{v} - \psi_1 \frac{\partial \hat{u}}{\partial y} - \psi_2 \frac{\partial \hat{v}}{\partial y} + \hat{u} \frac{\partial \psi_1}{\partial y} + \hat{v} \frac{\partial \psi_2}{\partial y} \right) \right]_0^\infty
\]  
(14)

The following choice of boundary conditions, for both direct and adjoint equations, ensures (14) to vanish at the boundaries.

\[
\hat{u} = \hat{v} = 0, \quad \psi_1 = \psi_2 = 0 \quad \text{at the wall,}
\]
(15)

\[
\frac{\partial \hat{u}}{\partial y} = \hat{v} = 0, \quad \frac{\partial \psi_1}{\partial y} = \psi_2 = 0 \quad \text{at freestream.}
\]
(16)

3.2. Biorthogonality condition

To derive the biorthogonality relation between eigenvectors of direct and adjoint problem one should evaluate the expression

\[
\langle L \mathbf{q}, \psi \rangle - \langle \mathbf{q}, L^+ \psi \rangle = 0,
\]  
(17)

for two arbitrary direct and adjoint eigenfunctions. Here, $\langle , \rangle$ is defined as unweighted inner product,

\[
\langle \mathbf{a} , \mathbf{b} \rangle = \int_0^\infty \mathbf{b}^H \mathbf{a} \, dy,
\]  
(18)

where the superscript $^H$ denotes Hermitian of the vector. Operators $L$ and $L^+$ are the compact form of direct and adjoint operators, respectively (see Appendix A).

\[
L = i\alpha M_1 + i\alpha M_2 \frac{\partial}{\partial y} - N_1 - N_2 \frac{\partial}{\partial y} - N_3 \frac{\partial^2}{\partial y^2},
\]  
(19)

\[
L^+ = -i\alpha^+ M_1^H + i\alpha^+ M_2^H \frac{\partial}{\partial y} - N_1^H + N_2^H \frac{\partial}{\partial y} - N_3^H \frac{\partial^2}{\partial y^2}.
\]  
(20)
The right hand side of (17) is set to zero because the boundary conditions (15) and (16), makes the boundary terms to vanish at the boundaries. By substituting operators (19) and (20) into the relation (17) for an arbitrary direct eigenvector with index $k$ and an adjoint eigenvector with index $j$, we have

$$\int_{0}^{\infty} \psi_{j}^{H} \left( i \alpha_{k} M_{1} + i \alpha_{k} M_{2} \frac{\partial}{\partial y} - N_{1} - N_{2} \frac{\partial}{\partial y} - N_{3} \frac{\partial^{2}}{\partial y^{2}} \right) q_{k} \, dy = 0 \quad (21a)$$

$$\int_{0}^{\infty} \left( -i \alpha_{j}^{+} M_{1}^{H} + i \alpha_{j}^{+} M_{2}^{H} \frac{\partial}{\partial y} - N_{1}^{H} + N_{2}^{H} \frac{\partial}{\partial y} - N_{3}^{H} \frac{\partial^{2}}{\partial y^{2}} \right) \psi_{j} \right)^{H} q_{k} \, dy = 0 \quad (21b)$$

The second integral (21b) can be rewritten using integration by parts and considering the fact that the boundary terms vanishes by our choice of boundary conditions. Therefore, the integral becomes

$$\int_{0}^{\infty} \psi_{j}^{H} \left( i \alpha_{k} M_{1} + i \alpha_{k} M_{2} \frac{\partial}{\partial y} - N_{1} - N_{2} \frac{\partial}{\partial y} - N_{3} \frac{\partial^{2}}{\partial y^{2}} \right) q_{k} \, dy = 0 \quad (22a)$$

$$\int_{0}^{\infty} \left( i \alpha_{j} M_{1} + i \alpha_{j} M_{2} \frac{\partial}{\partial y} - N_{1} - N_{2} \frac{\partial}{\partial y} - N_{3} \frac{\partial^{2}}{\partial y^{2}} \right) \psi_{j} \right) q_{k} \, dy = 0 \quad (22b)$$

Collecting the terms, we arrive at the integral of form

$$(\alpha_{k} - \alpha_{j}) \int_{0}^{\infty} \psi_{j}^{H} M q_{k} \, dy = 0, \quad (23)$$

which holds for any direct and adjoint eigenvector and results to the biorthogonality condition

$$\int_{0}^{\infty} \psi_{j}^{H} M q_{k} \, dy = C \delta_{kj}. \quad (24)$$

Here, $\delta_{kj}$ denotes the Kronecker delta symbol if either of the direct or adjoint eigenvectors is a discrete mode and $C$ is the normalisation constant. The biorthogonality relation (24) can be interpreted as a weighted inner product with operator $M$ as the weight operator.

$$\langle a, b \rangle_{M} = \langle Ma, b \rangle = \int_{0}^{\infty} b^{H} M a \, dy. \quad (25)$$

Expression (24) states that the eigenvectors of direct and adjoint equations are mutually orthogonal under this inner product. In other words, each direct eigenvector of the system is orthogonal to all but one adjoint eigenfunction.

### 3.3. Decomposition method

The direct and adjoint spatial eigenvalue problems (9) and (12), with the biorthogonality relation (24) form a bi-orthogonal eigenfunction system. The set of eigenfunctions of both direct and adjoint problems provide a biorthogonal basis allowing for decomposition of a perturbation into normal mode (Tumin 2003). In this study we are interested to extract amplitude of TS waves excited by acoustic disturbances. The DNS solutions contain TS waves, Stokes waves...
and freestream modes at the same time. Therefore, we use adjoint mode as a projector where the TS amplitude is obtained by projecting the DNS solution onto adjoint of TS waves. The amplitude of TS wave, \( C_{TS} \), at each \( x \) location can be evaluated through

\[
C_{TS} = \frac{\langle \hat{\phi}, \psi_{TS} \rangle_{M}}{\langle q_{TS}, \psi_{TS} \rangle_{M}}.
\]

Here, \( \psi_{TS} \) is the adjoint TS eigenfunction, \( q_{TS} \) is the TS wave solution to the direct problem and \( \hat{\phi} = [\hat{u}_{DNS}, \hat{v}_{DNS}, \hat{p}_{DNS}, \frac{d}{d\tau} \hat{v}_{DNS}]^T \) is the vector of DNS data after Fourier transform in time. The denominator in (26) can be considered as a normalisation constant for adjoint eigenfunction at each station. It is also important to note that the computed TS amplitude depends on the normalisation of the direct TS eigenmode. Here, we have normalised TS eigenmode, \( q_{TS} \), with maximum streamwise mode velocity.

4. Results

4.1. Base flow

A steady-state solution which is sufficiently resolved in space and converged in time is calculated prior to performing the simulations with the incident perturbations. The governing equations are integrated in time until a steady state base flow \( (U, P) \) is reached. The computation of steady state solutions via simple time-stepping technique requires a criteria for stopping the simulation. We consider the base flow converged in time when the difference of two subsequent flow fields does not change significantly, e.g. \( O(10^{-11}) \). The converged steady base flow is visualised in figure 4 for the leading edge with aspect ratio of \( AR = 20 \). Presence of surface curvature induces pressure gradient especially in the region close to the leading edge as depicted in figure 4(c).

Figures 5(a) and (b) show the distribution of the surface pressure coefficient \( C_p \) and its chordwise derivative as functions of the streamwise coordinate. As is seen in these figures, there is a local region of favourable pressure gradient near the leading edge where the flow accelerates followed by a region of adverse pressure gradient where the flow decelerates moving downstream. Sufficiently far downstream of the leading edge, the pressure gradient is small and the flow can be considered to be of Blasius type. A comparison between the Blasius similarity solution and the velocity profiles obtained by DNS is shown in figure 5(c,d) for streamwise positions of \( x = 50 \) and 200. At the former location, a deviation from the Blasius profile is apparent due to the presence of the pressure gradient. At the latter location both velocity components closely match the Blasius profile.

4.2. Perturbed flow

A number of simulations with small-amplitude acoustic perturbations and different non-dimensional frequencies, \( F = 2\pi f \nu / U_{\infty}^2 \times 10^{-6} \), were performed. As a result, the perturbed flow is harmonic in time with the prescribed acoustic
frequency. Fourier transform in time is employed on series of flow-field snapshots to obtain Fourier amplitudes of the perturbations $\hat{\phi}$

$$
\hat{\phi}(x, y, t) = \hat{\phi}(x, y) e^{-i\omega t} + c.c.
$$

Here, $\hat{\phi} = [\hat{u}_{\text{dns}}, \hat{v}_{\text{dns}}, \hat{p}_{\text{dns}}]^T$ is the vector of perturbations Fourier amplitude and $c.c.$ refers to complex conjugate term. Figure 6 displays the real part of Fourier amplitude of perturbations velocity, $\text{Re}\{\hat{u}_{\text{dns}}\}$ and $100 \times \text{Re}\{\hat{v}_{\text{dns}}\}$, for acoustic wave frequency $F = 100 \times 10^{-6}$ and $AR = 20$. The thin structure embedded in the boundary layer in figure 6(a) is the Stokes layer which has nearly constant thickness (see figure 3 as well). The structure of waves in figure 6(b) clearly indicates the existence of TS waves which go under exponential growth between the lower and upper branches of neutral stability curve.

Amplitude distribution in the streamwise direction at a constant height of 0.2 from the wall surface is shown in figure 7(a). In this near wall distance, the Fourier amplitude of disturbances oscillate spatially with a wavelength equal to the TS wavelength. Murdock (1980) suggests that the Fourier amplitude of the

Figure 4: Snapshots of converged steady base flow (a) streamwise velocity, (b) wall-normal velocity and (c) pressure distribution. The computational domain is 450 units long in the downstream direction and 400 units long in the wall-normal direction and upstream of the leading edge.
Figure 5: (a) Pressure coefficient distribution $C_p$ and (b) its chordwise derivative as a function of the streamwise coordinate. The vertical dashed lines depict the junction of the leading edge and the flat plate. Comparison of the streamwise and wall-normal velocity profiles with the Blasius similarity solution, (c) at the streamwise location $x = 50, Re_x = 1.2 \times 10^5$ and (d) at the streamwise location $x = 200, Re_x = 4.8 \times 10^5$; 20:1 MSE leading edge.

total disturbance oscillates spatially around a mean value with the amplitude and wavelength of the TS wave. Magnitude of the spatial mean of disturbances is associated with the sound wave and the magnitude of the envelope about the mean is associated with TS wave. This is shown in more detail in Shahriari et al. (2016). The amplitude distribution in the freestream at a distance of 20 units away from the wall, is shown in figure 7(b). In the freestream, the TS wave amplitude is close to zero while the amplitude of Stokes wave is around unity. The vertical dashed lines in figure 7 depicts the location of leading edge and flat plate junction and the first and second branches of the neutral stability curve, respectively. The leading edge curvature and associated pressure gradient in the nose region generate a large amplitude perturbation which decays rapidly after the junction. However, the leading edge effect on the perturbations amplitude remains visible in the branch I region. Leading edge effect is most strong close to the wall surface, however, it is present even in the far away regions.
The structures indicate coexistence of TS and Stokes waves inside the boundary layer. Non-dimensional frequency of acoustic wave is $F = 100 \times 10^{-6}$. The dashed line shows the boundary layer thickness $\delta_{99}$ of underlying base flow.

4.3. Projection of TS wave

In this section we focus on determining the amplitude of TS waves present in the DNS of perturbed flow employing the biorthogonal eigenfunction system (9) and (12), with the biorthogonality relation (24). As stated in section §3.3, the amplitude of TS wave at each $x$ location can be obtained by evaluating the weighted inner product of adjoint TS mode and DNS field

$$C_{TS} = \frac{\langle \hat{\phi}, \psi_{TS} \rangle_{M}}{\langle q_{TS}, \psi_{TS} \rangle_{M}},$$

where $\psi_{TS}$ is the adjoint TS eigenfunction, $q_{TS}$ is the TS wave solution to the direct problem and $\hat{\phi} = [\hat{u}_{dns}, \hat{v}_{dns}, \hat{p}_{dns}, \partial_{x} \hat{v}_{dns}]^{T}$ is the vector of Fourier amplitudes of DNS field in time.

Figure 8 shows the extracted amplitude of TS wave (black line) versus the $x$ coordinate for acoustic disturbances with non-dimensional frequency $F = 100 \times 10^{-6}$ and MSE leading edge geometries with aspect ratio of $AR = 6$ and 20. The decomposition has been performed from location $x = 20$ to the end of computational domain in both cases. TS wave amplitude obtained by the complex spiral method proposed by Wlezien (1994) is also shown in figure 8 (red line). The computed amplitude by both methods is not accurate up to the first branch region. Downstream of the branch I region, it is apparent that the extracted TS amplitude oscillates with a wavelength equal to wavelength of TS wave.
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Figure 7: Fourier amplitude distribution in the streamwise direction at different heights from the wall surface. (a) at a constant height of 0.2 units, close to the wall surface, the total disturbance amplitude spatially oscillates with TS wavelength. (b) freestream wave far away from the wall with a phase velocity close to unity. Non-dimensional frequency is $F = 100 \times 10^{-6}$. Vertical dashed lines, from left to right, indicate the junction of leading edge and flat plate and the location of first and second branches of the neutral stability curve.

Figure 8: Distribution of TS wave amplitude (black line) along the streamwise coordinate obtained using BES of linear stability equations for (a) 6:1 MSE and (b) 20:1 MSE leading edge geometry. Red line depicts the TS wave amplitude computed using the method proposed by Wlezien (1994).

This oscillatory behaviour of TS amplitude is seen by Turner (2012) in the lower branch region when plotted versus streamwise coordinate. He suggests that the DNS results contain remnants of the Stokes layer which are not removed by the filtering process. As a result the amplitude curve is oscillatory and contaminated in the lower branch region. The filtering process used by Turner (2012) follows the work of Haddad & Corke (1998). They obtain the Stokes wave solution on the same geometry and subtract it from the DNS solution fields. The
governing equation of Stokes flow is obtained by removing the convective-inertia terms of Navier-Stokes equation. Consequently, any influence of base flow on the perturbation field is eliminated. On the other hand, perturbations in the DNS are evolved on the underlying base flow with non-zero pressure gradients in the lower branch region. Therefore, the computed Stokes wave by this method is different than the actual Stokes wave present in the DNS solution.

In this study, the projected TS amplitude exhibit oscillatory behaviour within the whole domain, as depicted in figure 8. In order to understand the source of oscillations in the projected TS amplitude with BES decomposition method, several test cases are considered in next section.

4.4. Validation of decomposition method

In this section we try to understand the oscillatory behaviour of the TS wave amplitude and gain more insight about the physics of this flow. To this end, we verify the biorthogonal decomposition method by applying it to synthetic DNS fields which are constructed by combining discrete and continuous eigenvectors of OS equations as well as the analytical solution of Stokes wave. Further, we perform DNS with oscillatory localised forcing applied at different regions in the flow. The amplitude of excited TS waves by the localised forcing is then extracted through biorthogonal decomposition method.

Spatial spectrum of both direct and adjoint problems is shown in figure 9 for $\sqrt{Re_x} = 700$ and $F = 100$. It is evident that complex conjugate adjoint eigenvalues (black circles) match perfectly with the direct ones (red dots). Moreover, upstream and downstream travelling Stokes modes are found with
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wavenumbers close to zero. Mode shapes of direct and adjoint TS eigenfunctions as well as downstream travelling Stokes mode are plotted in figure 10.

4.4.1. Synthetic DNS fields

In this part synthetic DNS fields are constructed using a combination of discrete and continuous OS modes along with the analytical solution of Stokes wave in the form of

$$\phi = \text{Re} \left\{ \varepsilon \frac{q_{TS}}{\max |\hat{u}_{TS}|} e^{i \alpha_{TS} x - i \omega t} + \sum \frac{q_{cont}}{\max |\hat{u}_{cont}|} e^{i \alpha_{cont} x - i \omega t} + \text{Stokes wave} \right\}.$$  

Here, \(q_{TS}\) denotes the TS mode with its corresponding wavenumber \(\alpha_{TS}\). The second term includes a sum of continuous spectrum modes, \(q_{cont}\), with corresponding wavenumber \(\alpha_{cont}\). These modes are the solution of direct spatial eigenvalue problem (9) at each streamwise coordinate \(x\), given a real valued \(\omega\) and using the computed steady base flow. Figures 10 and 11 display the mode shapes of TS wave and selected continuous spectrum modes at \(\sqrt{Re_x} = 700\), for \(\omega = 0.24\) which corresponds to non-dimensional frequency \(F = 100 \times 10^{-6}\). Each mode is normalised by the maximum streamwise velocity to be comparable with each other. The amplitude of TS mode is chosen to be \(\varepsilon = 10^{-2}\) in the following test cases.

The most basic test that the decomposition method must pass is to correctly project the TS wave amplitudes out of a synthetic DNS field constructed only by TS eigenvectors, i.e.

$$\phi_1 = \text{Re} \left\{ \varepsilon \frac{q_{TS}}{\max |\hat{u}_{TS}|} e^{i \alpha_{TS} x - i \omega t} \right\}.$$  

(29)

Figure 12(a) shows the streamwise velocity amplitude distribution of synthetic DNS field \(\phi_1\), inside the boundary layer. The amplitude of constructed DNS field in the inviscid outer layer decreases dramatically toward zero. Since \(\phi_1\) contains only TS waves, the maximum amplitude at each streamwise position, \(\max |\phi_1|\), is equal to the maximum amplitude of TS waves. This quantity can be evaluate easily and used to verify the decomposition method in the following test problems. The decomposition method is applied on field \(\phi_1\) (Fourier amplitudes in time) and the maximum TS wave amplitude in the wall-normal direction at each station is obtained by evaluating \(A = |C_{TS}|\) (since direct TS eigenvector is normalised with maximum streamwise velocity component). Figure 12(d) compares the extracted amplitude \(A\) through decomposition method and the maximum TS wave amplitude in DNS field \(\phi_1\). The two amplitude curves are on top of each other which demonstrates that the decomposition method has successfully projected the TS waves.

In the next test case, the synthetic DNS field is constructed by combining TS waves and continuous modes of OS equations in the form of

$$\phi_2 = \text{Re} \left\{ \varepsilon \frac{q_{TS}}{\max |\hat{u}_{TS}|} e^{i \alpha_{TS} x - i \omega t} + \sum \frac{q_{cont}}{\max |\hat{u}_{cont}|} e^{i \alpha_{cont} x - i \omega t} \right\}.$$  

(30)
Figure 10: (a) components of direct TS eigenvector $q_{TS}$ with wavenumber $\alpha_{TS} = 0.665 - 0.003i$, (b) components of adjoint TS eigenvector $\psi_{TS}$ with wavenumber $\alpha_{TS}^+ = 0.665 + 0.003i$ and (c) components of direct Stokes eigenvector. These modes are obtained by solving the direct and adjoint spatial eigenvalue problems at $\sqrt{Re_x} = 700$, with $\omega = 0.24$ which corresponds to $F = 100$. Direct eigenvectors are normalised by the maximum value of streamwise velocity.
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Figure 11: Components of continuous mode $q_{\text{cont}}$ (a) from pressure branch with wavenumber $\alpha_{\text{cont}} = -0.005 + 0.157i$ and (b) from vorticity branch with wavenumber $\alpha_{\text{cont}} = 0.239 + 0.0011i$. These modes are obtained by solving the direct spatial eigenvalue problem at station $\sqrt{Re}_x = 700$, with $\omega = 0.24$ which corresponds to $F = 100$. Each eigenvector is normalised by the maximum value of streamwise velocity.

Here we consider one mode from the pressure branch and one mode from the vorticity branch of continuous spectrum, as shown in figure 11 (see Tumin (2011) for branches of continuous spectrum). Both chosen modes are downstream travelling modes with negative growth rate ($\text{Im}ag(\alpha) > 0$). This means the initial amplitude of these waves decreases when travelling downstream. On the other hand, TS waves amplitude decays up to the lower branch of neutral stability curve and then grows exponentially until branch II where the amplitude starts to decay again.

Figure 12(b) shows the Fourier amplitude of streamwise velocity for synthetic field $\phi_2$ at a constant distance from wall inside the boundary layer. The initial
amplitude is dominated by the continuous pressure branch mode and is in the order of $O(1)$. Further downstream, the pressure branch mode decays rapidly and dies out owing to its large negative spatial growth rate. The chosen vorticity branch mode has a very small negative spatial growth rate, therefore, it takes long distances for this wave to decay and die out. Thus, the amplitude distribution of synthetic field in downstream positions is dominated by TS waves together with the vorticity branch mode which appears as an small oscillation on top of TS wave. By applying the decomposition method on $\phi_2$, amplitude of TS waves are extracted. The maximum amplitude of extracted TS waves are identical to the maximum amplitude of TS waves used to build $\phi_2$. Figure 12(d) shows that the two amplitude curves are, once again, on top of each other which demonstrates that the decomposition method successfully filters continuous spectrum modes.
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Figure 13: (a) Fourier amplitude of streamwise perturbation velocity at a constant distance from the wall, inside the boundary layer, for synthetic DNS field $\phi_4$ constructed by two TS waves with a constant phase shift $\varphi = \pi/3$. (b) the projected amplitude of TS waves.

For the next test problem, the synthetic DNS field, $\phi_3$, is constructed using TS waves and analytical solution of Stokes wave, i.e.

$$\phi_3 = \text{Re} \left\{ \frac{q_{\text{TS}}}{\max |\dot{u}_{\text{TS}}|} e^{i\alpha_{\text{TS}} x - i\omega t} + q_{\text{st}} e^{i\omega t} \left[ 1 - e^{-(i+1)ky} \right] \right\},$$

(31)

where $k = \sqrt{\omega Re/2}$ and $q_{\text{st}} = [1, 0, 0, 0]^T$ is the Stokes eigenvector with only streamwise velocity non-zero component. Figure 12(c) shows the Fourier amplitude distribution of $\phi_3$ at a constant distance from wall inside the boundary layer. This amplitude distribution is very similar to the one in the actual DNS solutions, shown in figure 7(a). Disturbances oscillate spatially with TS wavelength around a constant mean owing to the constant thickness of Stokes layer in the streamwise direction. The extracted amplitude of TS waves through decomposition method for synthetic field $\hat{\phi}_3$, results to a smooth curve like the previous test cases, shown in figure 12(d). Therefore, it is shown that the decomposition method filters the analytical Stokes wave as well. Note that the analytical Stokes wave is obtained with parallel base flow assumption, i.e. $U(y)$. Therefore, it is a solution of OS equations (with zero wavenumber $\alpha_{\text{st}} = 0$) and can be expanded onto basis of OS modes.

The presented test cases with synthetic DNS fields demonstrate that the projection method is able to filter out the analytical Stokes wave as well as continuous spectrum modes and project the amplitude of the TS waves correctly. However, when the decomposition method is applied on the actual DNS solution fields, the projected amplitude of TS waves is oscillatory with a wave length equal to TS wavelength, shown in figure 8. Wanderley & Corke (2001) observe an oscillatory behaviour on the curve of receptivity amplitude plotted versus frequency. They state the origin of the oscillation can be traced to an interaction between instability waves which originate at two sites of receptivity in this flow; leading edge and juncture of flat plate and leading edge. The TS waves
excited at the juncture are superimposed on those triggered at the leading edge. At a given frequency, the wavelength of the waves originating at the two sites is the same. It is also expected that these waves have the same initial phase difference with respect to freestream disturbances. If there is an integer number of wavelengths between the leading edge and the junction (which is equal to semi-major axis of MSE, $a$), the waves will be in phase and their amplitude will add travelling downstream of the junction. However, if the distance between leading edge and junction does not accommodate integer number of wavelengths, there will be a phase shift between the two waves and the amplitude of the superimposed wave downstream of the junction will oscillate spatially. This idea sounds promising, however, one can see that the phase shift does not cause an oscillatory behaviour in the results. Lets consider two TS waves that are superposed with a constant phase shift $\varphi$

$$\phi_4 = \text{Re} \left\{ \frac{q_{TS}}{\max|\hat{u}_{TS}|} e^{i\varphi_{TS}} x - i \omega t \left[ 1 + e^{i\varphi} \right] \right\}. \tag{32}$$

The phase shift term, $e^{i\varphi}$, is essentially a constant value added to the amplitude of TS wave at each station (unless $\varphi$ is not constant and varies with streamwise coordinate). Figure 13(a) shows the Fourier amplitude of $\phi_4$ inside the boundary layer where a constant phase shift $\varphi = \pi/3$ is used to construct the field. As expected, the amplitude curve is smooth as well as the projected amplitude through decomposition method plotted in figure 13(b).

In order to take the non-parallel effects of the base flow into account, non-local stability analysis is performed using linear parabolised stability equations (PSE). Introducing an initial disturbance mode $\hat{q}$ at a streamwise position $x_0$, PSE can be solved by marching downstream. The initial disturbance mode is obtained using the solution of local stability problem (OS equation). PSE and
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Figure 15: Amplitude shape of the localised force applied at (a) upstream of the leading edge at \( x = -3 \), (b) streamwise coordinate \( x = 3 \) and (c) \( x = 5 \) in the leading edge region and (d) at \( x = 50 \) close to the lower branch of neutral stability curve. The amplitude of force pulsates in time with a frequency \( F = 100 \times 10^{-6} \). The dashed line in (d) depicts boundary layer thickness.

OS equation are solved using the same underlying base flow to obtain TS mode shapes and growth rates at each station. The N-factor based on the maximum velocity at each station

\[ N_u = \ln \left( \frac{\max(\hat{u})}{\max(\hat{u}_0)} \right), \]

for OS equation and PSE is plotted in figure 14(a). Here, \( \hat{u}_0 \) is the velocity at first station. Figure 14(a) shows that the non parallel effects are not significant for TS waves. A synthetic field is then constructed using the PSE mode shapes in the form of

\[ \phi_5 = \text{Re} \left\{ \varepsilon \bar{q}_{TS} e^{i\alpha_{TS} x - i\omega t} \right\}, \]

where \( \bar{q}_{TS} \) denotes the TS mode shape from PSE calculation with respective wavenumber \( \alpha_{TS} \). Figure 14(b) shows the projected amplitude of TS waves in \( \phi_5 \) which matches very well with the expected TS waves amplitude.

4.4.2. Localised forcing of the flow

The aim of this section is to investigate the effect of leading edge on the evolution of perturbations through localised forcing of the flow. To this end, the boundary
Figure 16: Contours of Fourier amplitude of streamwise velocity, Re $\{\hat{u}_{dns}\}$, for localised force applied at (a) upstream of the leading edge at $x = 3$, (b) leading edge region at $x = 3$, (c) leading edge region at $x = 5$ and (d) at $x = 50$ on the flat plate part. The localised force amplitude pulsates in time with frequency $F = 100 \times 10^{-6}$. Boundary layer thickness is depicted by dashed line. Contour levels of the first three plots are the same.

conditions for perturbations in the freestream are set to zero and the flow is perturbed locally by an oscillatory body force. We consider four cases with different locations in the streamwise direction where the localised force is applied. One case with the force applied in an upstream location of the leading edge, two cases on the leading edge region and one case on the flat plate region. Figure 15 shows the location and the amplitude shape of the localised force. The amplitude of force pulsates in time with a frequency $F = 100 \times 10^{-6}$. The leading edge geometry is 20:1 MSE. The radius of maximum force amplitude is larger than boundary layer thickness to make sure that perturbations are excited outside the boundary layer as well.
Figure 16 shows the response of boundary layer to the localised excitation of the flow at different locations. Contours of streamwise velocity are visualised to demonstrate the structures present in the flow. For the cases where force is applied in the leading edge region, similar contour levels are used, so, the structures can be compared to each other. Figure 16(d) displays large amplitude TS waves excited directly by the force applied on the flat plate. In this case, the force location is close to the first branch of neutral stability curve, hence, the excited perturbations go under exponential growth and have large amplitudes. Moving the force location to upstream positions, the TS waves structures become less apparent compared to the other waves excited at the leading edge. When the force is applied in front of the leading edge, TS waves are excited but too small in amplitude to be visible with the chosen contour levels in figure 16(a).

Biorthogonal decomposition method is applied on the DNS solution fields with localised force as presented above. The result of the projection procedure is visualised in figure 17. The projected amplitude of TS waves when the force is applied in front of the leading edge shows an oscillatory behaviour in the branch I region, depicted in figure 17(a). The amplitude of this oscillation decreases moving downstream, with high amplitudes in the first branch region and extremely small amplitudes in the second branch region. Figure 17(b) and (c) show the projected TS waves amplitude when the force is applied in the leading edge region. There is an oscillation present in the amplitude curve when the force is applied at $x = 3$ and localised in the branch I region. When the force is moved to $x = 5$ location, the oscillation is almost not apparent anymore. Perturbing the flow locally on the flat plate region generates clean TS waves. The projected amplitude is shown in figure 17(d) where the decomposition method correctly gives zero amplitude upstream of the forcing location and a smooth amplitude curve downstream of the force location. In order to compare the growth of TS waves in all cases, the amplitude curves are normalised to have unit amplitude at the branch II location and are plotted together in figure 17(e). The normalised amplitude curves match perfectly in the second branch region.

The amplitude distribution curves in figure 17 suggest that there is a wave present in DNS solution fields, excited in the freestream or the leading edge region which is not a solution of OS equations. Therefore, it is not expandable onto the basis of OS eigenfunctions and consequently, it is not orthogonal to the adjoint TS eigenvector. This can be shown by assuming that the particular solution of Navier-Stokes equations contains a small amplitude wave, $\varepsilon \cos(\omega t)$ in addition to the Stokes wave. We construct a synthetic DNS field, $\phi_6$, using TS wave, Stokes wave and $\varepsilon \cos(\omega t)$, i.e.

$$
\phi_6 = \text{Re} \left\{ \varepsilon \frac{q_{TS}}{\max|\hat{u}_{TS}|} e^{i\omega_{TS} x - i\omega t} + q_{st} e^{i\omega t} \left[ \varepsilon + 1 - e^{-(i+1)ky} \right] \right\},
$$

where $k = \sqrt{\omega Re/2}$ and $q_{st} = [1, 0, 0, 0]^T$ is the Stokes eigenvector with only streamwise velocity non-zero component. Figure 18(a) shows the Fourier amplitude distribution of $\phi_6$ at a constant distance of 0.2 from the wall, inside
Figure 17: Projected amplitude of TS waves by BES for localised forcing applied at (a) \(x = -3\) upstream of the leading edge, (b) \(x = 3\) in the leading edge region, (c) \(x = 5\) in leading edge region and (d) \(x = 50\) on the flat plate. (e) Normalised amplitude curves with maximum amplitude of one at branch II location. The black solid line, red star, blue square and green circle symbols represent cases with the localised force applied at \(x = -3, 3, 5,\) and 50, respectively.

the boundary layer. This amplitude distribution is very similar to the one in the actual DNS solution, shown in figure 7(a). The small amplitude wave, \(\varepsilon \cos(\omega t)\), does not have a visible effect on the total amplitude evolution. The extracted amplitude of TS waves through decomposition method for synthetic field \(\phi_6\) is shown in figure 18(b) with black line. The amplitude of TS wave used
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Figure 18: (a) Fourier amplitude of streamwise perturbation velocity at a constant distance of 0.2 from the wall, inside the boundary layer, for synthetic DNS field $\phi_6$ constructed by TS waves, analytical Stokes wave and wave-like particular solution $\varepsilon \cos(\omega t)$. (b) the projected amplitude of TS waves in black. The red line depicts amplitude of TS wave used in construction of $\phi_6$.

in construction of $\phi_6$ is shown in red. The projected TS amplitude behaves similarly to the one obtained using the actual DNS solution field. This confirms the existence of such a wave in the DNS solution fields, however, the exact origin of such a wave is unknown to us. This wave could be the so-called freestream mode with a phase velocity close to unity and $\alpha = \omega$, which is excited by the time harmonic boundary condition or localised pulsating force upstream of the leading edge. Another possibility is that such a wave is part of the Stokes wave in the DNS, which is the particular solution of Navier-Stokes equations satisfying the oscillating boundary condition. The particular solution deviates from the analytical Stokes wave in DNS due to non-zero pressure gradients and non-parallelism of base flow.

5. Summary and outlook

Acoustic receptivity of a two-dimensional boundary-layer flow about a flat plate geometry with elliptic leading edge is studied through DNS. Incompressible formulation of Navier-Stokes equations is considered in this work and sound waves are modelled by a uniform oscillation of freestream boundaries in time which results to an infinite-wavelength ‘acoustic’ wave. Acoustic disturbances enter the boundary layer and through receptivity process create TS waves inside the boundary layer. Obtaining the amplitude of TS waves is a challenging task due to presence of Stokes wave with the same temporal frequency of TS waves. Therefore, Fourier amplitude of perturbations in time consists of both TS and Stokes waves as well as continuous spectrum modes. In this study biorthogonal eigenfunction system of local linear stability equations has been utilised to extract TS wave amplitudes. This method is based on the concept of using
adjoint mode as a projector where the TS amplitude is obtained by projecting the DNS solution onto adjoint TS modes.

The computed TS wave amplitude employing this method found to be modulated. Two set of test cases are designed in order to understand this phenomenon. In the first set of test cases, TS wave amplitude is successfully computed using BES from synthetic DNS fields. The synthetic DNS fields are constructed using discrete and continuous modes of OS equations including Stokes wave. This test demonstrates that if the DNS field is expandable on the basis of OS modes, then the TS wave amplitude can be successfully projected out. In the second set of tests, DNS computations are performed with localised pulsating force. Applying such a force on the flat plate region, with the extent of the local force being both inside and outside of the boundary layer, excites TS waves and its amplitude does not exhibit oscillation using BES. However, if the localised force is applied upstream of the leading edge or in the leading edge region, the resulting amplitude of the TS wave is modulated. This results show that the DNS solution field contains a small amplitude wave, excited in the freestream or in the leading edge region, that is not expandable on the basis of OS modes and therefore, it is not orthogonal to the adjoint TS eigenfunction. Using synthetic DNS field which contains a wave of the form $\varepsilon \cos(\omega t)$, shows that the projected TS amplitude exhibit similar oscillation with the one obtained from the actual DNS data. Further investigations on this method using compressible DNS fields (Shahriari et al. 2016) and compressible formulation of BES shows similar behaviour to the incompressible case.

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Appendix A. Adjoint derivation

Consider the direct spatial problem (9) in compact form of

\[ \mathcal{L}q = 0 \]  

(36)

with the compact operator \( \mathcal{L} \)

\[ \mathcal{L} = i\alpha M_1 + i\alpha M_2 \frac{\partial}{\partial y} - N_1 - N_2 \frac{\partial}{\partial y} - N_3 \frac{\partial^2}{\partial y^2} \]  

(37)

and the individual operators

\[ M_1 = \begin{pmatrix} U & 0 & 1 & 0 \\ 0 & U & 0 & -\frac{1}{Re} \\ 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{Re} & 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & \frac{1}{Re} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \]

\[ N_1 = \begin{pmatrix} i\omega & -U_y & 0 & 0 \\ 0 & i\omega & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{Re} \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \]

\[ N_3 = \begin{pmatrix} \frac{1}{Re} & 0 & 0 & 0 \\ 0 & \frac{1}{Re} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \]

The aim is to derive the adjoint operator \( \mathcal{L}^+ \) which satisfies the following relation

\[ \langle \mathcal{L}q, \psi \rangle = \langle q, \mathcal{L}^+\psi \rangle + \text{boundary terms.} \]  

(38)

The form of adjoint problem depends on the inner product that is used. We choose an standard unweighted inner product defined as

\[ \langle a, b \rangle = \int_0^\infty b^H a \ dy. \]  

(39)

Here, the superscript \(^H\) denotes Hermitian of the vector. Expanding the leftmost inner product of (38) and taking into account that the inner product is a linear
operator results to

\[
\langle \mathcal{L} \mathbf{q} , \psi \rangle = \langle (i \alpha \mathbf{M}_1 + i \alpha \mathbf{M}_2 \frac{\partial}{\partial y} - \mathbf{N}_1 - \mathbf{N}_2 \frac{\partial}{\partial y} - \mathbf{N}_3 \frac{\partial^2}{\partial y^2}) \mathbf{q} , \psi \rangle \\
= \langle (i \alpha \mathbf{M}_1 - \mathbf{N}_1) \mathbf{q} , \psi \rangle \\
+ \langle (i \alpha \mathbf{M}_2 - \mathbf{N}_2) \frac{\partial \mathbf{q}}{\partial y} , \psi \rangle \\
- \langle \mathbf{N}_3 \frac{\partial^2 \mathbf{q}}{\partial y^2} , \psi \rangle
\]

Each inner product can be expanded by performing integration by parts

\[
\langle (i \alpha \mathbf{M}_1 - \mathbf{N}_1) \mathbf{q} , \psi \rangle = \int_0^\infty \psi^H (i \alpha \mathbf{M}_1 - \mathbf{N}_1) \mathbf{q} \, dy \\
= \int_0^\infty (-i \alpha ^+ \mathbf{M}_1^H - \mathbf{N}_1^H)^H \psi^H \mathbf{q} \, dy \\
= \langle \mathbf{q} , (-i \alpha ^+ \mathbf{M}_1^H - \mathbf{N}_1^H) \psi \rangle
\]

\[
\langle i \alpha \mathbf{M}_2 \frac{\partial \mathbf{q}}{\partial y} , \psi \rangle = \int_0^\infty \psi^H i \alpha \mathbf{M}_2 \frac{\partial \mathbf{q}}{\partial y} \, dy \\
= \left[ i \alpha \psi^H \mathbf{M}_2 \mathbf{q} \right]_0^\infty - \int_0^\infty \frac{\partial}{\partial y} (\psi^H i \alpha \mathbf{M}_2) \mathbf{q} \, dy \\
= \left[ i \alpha \psi^H \mathbf{M}_2 \mathbf{q} \right]_0^\infty - \int_0^\infty \frac{\partial \psi^H}{\partial y} i \alpha \mathbf{M}_2 \mathbf{q} \, dy + \\
- \int_0^\infty i \alpha \psi^H \frac{\partial \mathbf{M}_2}{\partial y} \mathbf{q} \, dy \\
= \left[ i \alpha \psi^H \mathbf{M}_2 \mathbf{q} \right]_0^\infty - \int_0^\infty \left( -i \alpha ^+ \mathbf{M}_2^H \frac{\partial \psi}{\partial y} \right)^H \mathbf{q} \, dy + \\
- \int_0^\infty \left( -i \alpha ^+ \frac{\partial \mathbf{M}_2^H}{\partial y} \psi \right)^H \mathbf{q} \, dy \\
= \left[ i \alpha \psi^H \mathbf{M}_2 \mathbf{q} \right]_0^\infty + \langle \mathbf{q} , i \alpha ^+ \mathbf{M}_2^H \frac{\partial \psi}{\partial y} \rangle + \langle \mathbf{q} , i \alpha ^+ \frac{\partial \mathbf{M}_2^H}{\partial y} \psi \rangle
\]
\[ \langle N_2 \frac{\partial q}{\partial y}, \psi \rangle = \int_0^\infty \psi^H N_2 \frac{\partial q}{\partial y} \, dy \]

\[ = [\psi^H N_2 q]_0^\infty - \int_0^\infty \frac{\partial}{\partial y} (\psi^H N_2) \, q \, dy \]

\[ = [\psi^H N_2 q]_0^\infty - \int_0^\infty \frac{\partial}{\partial y} (\psi^H N_2) \, q \, dy - \int_0^\infty \psi^H \frac{\partial N_2}{\partial y} \, q \, dy \]

\[ = [\psi^H N_2 q]_0^\infty - \int_0^\infty \left( N_2^H \frac{\partial N_2}{\partial y} \right) \, q \, dy \]

\[ = [\psi^H N_2 q]_0^\infty + \langle q, -N_2^H \frac{\partial \psi}{\partial y} \rangle + \langle q, -\frac{\partial N_2^H}{\partial y} \psi \rangle \]

\[ \langle N_3 \frac{\partial^2 q}{\partial y^2}, \psi \rangle = \int_0^\infty \psi^H N_2 \frac{\partial^2 q}{\partial y^2} \, dy \]

\[ = [\psi^H N_3 \frac{\partial q}{\partial y}]_0^\infty - \int_0^\infty \frac{\partial}{\partial y} (\psi^H N_3) \, \frac{\partial q}{\partial y} \, dy \]

\[ = [\psi^H N_3 \frac{\partial q}{\partial y}]_0^\infty - \left[ \frac{\partial}{\partial y} (\psi^H N_3) \right]_0^\infty + \int_0^\infty \frac{\partial}{\partial y} \left( \frac{\partial}{\partial y} (\psi^H N_3) \right) \, q \, dy \]

\[ = [\psi^H N_3 \frac{\partial q}{\partial y} - \frac{\partial}{\partial y} (\psi^H N_3) \, q]_0^\infty + \int_0^\infty \left( \frac{\partial^2 \psi^H}{\partial y^2} N_3 + 2 \frac{\partial \psi^H}{\partial y} \frac{\partial N_3}{\partial y} + \psi^H \frac{\partial^2 N_3}{\partial y^2} \right) \, q \, dy \]

\[ = [\psi^H N_3 \frac{\partial q}{\partial y} - \frac{\partial}{\partial y} (\psi^H N_3) \, q]_0^\infty + \langle q, N_3^H \frac{\partial^2 \psi}{\partial y^2} \rangle + \langle q, 2 \frac{\partial N_3^H}{\partial y} \frac{\partial \psi}{\partial y} \rangle + \langle q, \frac{\partial^2 N_3^H}{\partial y^2} \psi \rangle \]

Since the operators \( M_2, N_2 \) and \( N_3 \) have constant coefficient, their partial derivative with respect to \( y \) coordinate vanishes and the adjoint operator \( \mathcal{L}^+ \)
can be written in the form of
\[
\langle L \mathbf{q}, \psi \rangle = \langle \mathbf{q}, (-i \alpha^+ M_1^H - N_1^H + i \alpha^+ M_2^H \frac{\partial}{\partial y} + N_2^H \frac{\partial}{\partial y} - N_3^H \frac{\partial^2}{\partial y^2}) \psi \rangle + \text{boundary terms}
\]
\[
= \langle \mathbf{q}, L^+ \psi \rangle + \text{boundary terms}
\]
Therefore, the compact adjoint operator \( L^+ \) can be written in the form of
\[
L^+ = -i \alpha^+ \left( M_1^H - M_2^H \frac{\partial}{\partial y} \right) - N_1^H + N_2^H \frac{\partial}{\partial y} - N_3^H \frac{\partial^2}{\partial y^2}
\]
\[
(41)
\]
**Appendix B. Boundary terms of adjoint problem**

To derive adjoint equations integration by parts is employed and the following boundary terms remain (for detailed derivation see Appendix A)
\[
\text{b.t.} = [i \alpha \psi^H M_2 \mathbf{q}]_0^{\infty} - [\psi^H N_2 \mathbf{q}]_0^{\infty} - \left[ \psi^H N_3 \frac{\partial \mathbf{q}}{\partial y} - \frac{\partial}{\partial y} (\psi^H N_3) \mathbf{q} \right]_0^{\infty}
\]
\[
(42)
\]
where \( \mathbf{q} = [\hat{u}, \hat{v}, \hat{\rho}, i \alpha \hat{v}]^T \) is the direct eigenvector and \( \psi = [\psi_1, \psi_2, \psi_3, \psi_4]^T \) represents the adjoint eigenvector. The operators \( M_2, N_2 \) and \( N_3 \) are
\[
M_2 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad N_2 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad N_3 = \begin{pmatrix}
\frac{1}{\nu e} & 0 & 0 & 0 \\
0 & \frac{1}{\nu e} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]
Since \( N_3 \) has constant coefficients the boundary terms can be written as
\[
\text{b.t.} = \left[ \psi^H M_2 \mathbf{q} - \psi^H N_2 \mathbf{q} - \psi^H N_3 \frac{\partial \mathbf{q}}{\partial y} + \frac{\partial \psi^H N_3}{\partial y} \mathbf{q} \right]_0^{\infty}
\]
\[
(43)
\]
By performing the matrix-vector multiplication, the boundary terms are written in the final form of
\[
\text{b.t.} = \left[ \psi_2 \hat{p} + \psi_3 \hat{v} + \frac{1}{\nu e} \left( \psi_1 \hat{u} - \psi_1 \frac{\partial \hat{u}}{\partial y} - \psi_2 \frac{\partial \hat{v}}{\partial y} + \hat{u} \frac{\partial \psi_1}{\partial y} + \hat{v} \frac{\partial \psi_2}{\partial y} \right) \right]_0^{\infty}
\]
\[
(44)
\]
Extraction of TS waves using BES in acoustic receptivity simulations

REFERENCES


