VIBRATION OF SANDWICH BEAMS

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Doctoral Thesis

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Abstract

Some aspects and properties of the lateral vibration of sandwich beams are investigated, including the concept of apparent bending stiffness and shear modulus, allowing the sandwich beam dynamics to be approximately described by classical beam theory. A sixth order beam model is derived including boundary conditions, and the free and forced response of some beam configurations analyzed. The possibility of computing material parameters from measured eigenfrequencies, i.e. inverse analysis, is considered. The higher order model is also utilized for investigation of the energy propagation through sandwich composite beams and the transmission over different junctions.

Conference presentations

The results published in this thesis have been partly presented at the following conferences and workshops:

- Annual national PhD student meeting, Lund, Sweden, 2002 (briefly)
- Machine Acoustics, Aalborg, Denmark, 2002 (briefly)
- International Conference of Sound and Vibration, ICSV 10, Stockholm, Sweden, 2003
- Nordic Vibration Research in Stockholm, Sweden, 2004
- International Conference of Sound and Vibration, ICSV 11, St Petersburg, Russia, 2004
- Annual national PhD student meeting, Stockholm, Sweden, 2004 (briefly)
- International Conference of Sound and Vibration, ICSV 12, Lisbon, Portugal, 2005
- Seventh international conference on sandwich structures, Aalborg, Denmark, 2005

In addition, the modified conference paper "Modelling Flexural Vibration of a Sandwich Beam Using Modified 4th Order Theory" (not included in the thesis) has been accepted for publication in Journal of Sandwich Structures & Materials, and "Modelling the vibration of sandwich beams using frequency dependent parameters" has been preliminarily accepted for publication in Journal of Sound and Vibration.

Partition of work between the authors

Daniel Backström has written all papers and performed the measurements. Anders Nilsson has provided general direction and guidance through the work.
“Here I learn everything about vibrating beams and then no one wants to talk about it.”

The author’s life described by Sweden’s best cartoonist. Copyright Jan Berglin, printed with the permission of the artist.
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1 Introduction

During World War II, the British made De Havilland Mosquito served in Europe, the Middle and Far East and on the Russian front. Designed as a bomber, it excelled not only in this field but also as a fighter aircraft, as a mine layer, pathfinder, in military transport and in photo reconnaissance. It had been constructed during the Battle of Britain and the first prototype made its maiden flight in November 1940, less than a year after the design project had started. From an engineering viewpoint, it had one spectacular feature – the fuselage was made of a molded plywood-balsa sandwich material, which was strong yet lightweight, and, equally important in times of war, its components were readily available in contrast to aluminium. The importance of the Mosquito in the war effort proved the value of the new sandwich materials.

Today, the amount of lightweight composite materials implemented in the aircraft and vehicle industries is increasing, as is the need for effective and accurate modelling of complex structures comprising such elements. In this thesis, some of the vibro-acoustical properties governing lateral vibration of sandwich beams are investigated. As ordinary theory is not applicable for describing the flexural dynamics of sandwich beams and plates, other higher order theories have been derived. These models take into account the effects of shear in the core, and often also include rotational inertia terms neglected in the Bernoulli-Euler beam theory and the corresponding Kirchhoff plate equation. The first step in this direction is given by the Timoshenko beam theory, derived in for example [1], in which the lateral shear deformation of a beam is described by means of an effective shear stress uniformly distributed over the beam cross-section. The word “effective” is used here to emphasize the fact that the real stress distribution cannot be constant over the cross-section; the boundary conditions requires it to vanish at the top and bottom of the beam. For a homogenous beam, the shear stress distribution is parabolic rather than constant over the cross-section. As is known from the mechanics of solids, shear deformation becomes important when the distance between opposing transverse forces becomes smaller. Since this distance is a half of a wavelength when considering wave propagation, it can be concluded that shear deformation becomes more important for short wavelengths, i.e. high frequencies. Timoshenko theory is however not capable of correctly describing sandwich beam vibration as the deformation of the laminates of the sandwich is coupled to the shear deformation of the core. This coupling becomes important in the high frequency range and effectively stiffens the beam as compared to a homogenous beam with identical shear modulus and

![Figure 1: A sandwich plate. Picture from www.ie-sps.com.](image-url)
bending stiffness; in principle, the laminates limit the shear deformation of the beam core. Thus, it can be concluded that there are three major mechanisms of deformation governing the vibration of a typical sandwich beam; namely that of pure bending of the entire cross-section, which is a low frequency phenomenon taken into account by classical beam theory; pure shear of the sandwich beam core which dominates the mid-frequency range; and finally that of the bending of the laminates due to shear deformation of the core. Naturally, at yet higher frequencies additional mechanisms become important; however, for typical sandwich beam structures subject to vibration in the audible frequency range these may often be considered of negligible importance.

Figure 2: Sandwich beam deformation mechanisms. $\gamma$ is the angle of shear deformation and $\beta$ is the angle of bending deformation.
2 Modified homogenous beam theories - paper I

2.1 Introduction

The wave propagation in a three-layer sandwich structure can be computed in two ways; either by using the general theory of elasticity, which results in very complicated formulations, see for example [2], [3] or [4]; or by making some general assumptions regarding the stress and deformation fields, and thereby greatly decreasing the complexity of the problem. The latter approach is inherently subject to error due to the assumptions made; however, in the frequency region generally of interest in engineering problems, i.e. the audible range, these simplified methods can yield satisfactory results if derived and applied in a correct way. The well-known Bernoulli-Euler model deals with thin homogenous beams subject to pure bending deformation, see for example [5] or [6]. Rayleigh added the effects of rotational inertia to the Bernoulli-Euler beam in the classical Theory of Sound, [7]. The effect of thick beams, i.e. shear deformation, was first considered by Stepan Timoshenko in his textbooks on mechanics, see for example [1]. Since then numerous research papers and textbooks have applied similar approaches in the analysis of various beam and plate structures.

In this thesis, the case of the three-layered sandwich beam is considered. The beam consists of two thin and relatively stiff sheets, denoted laminates, bonded to a thick and lightweight core. The result is a lightweight beam structure with a high static bending stiffness, properties which are of great interest to the industry. Some of the previous work on this type of structures include [8], where the effects of constrained viscoelastic layers are investigated, and [9] were the equation of motion of a three-layer sandwich beam is derived using force and moment equilibrium techniques. In [10], some different sandwich beam models are analyzed and compared, and a model including shear deformation of the laminates is introduced. The applicability of ordinary 4th order Timoshenko theory to the problem of sandwich beam vibration is investigated in for example [11], [12] and [13].

The 6th order theory utilized in this thesis is a refined version of the model presented by E. Nilsson and A. C. Nilsson in [14], and is capable of handling asymmetric beam structures.

2.2 Modified Bernoulli-Euler beam theory

When measuring for example the eigenfrequencies of a sandwich composite beam subject to some given boundary conditions, the eigenfrequencies will be distributed in such a way that the total bending stiffness of the structure, using classical Bernoulli-Euler beam theory, seemingly decreases with frequency in a reversed S-shape. This is due to the increasing importance of other mechanisms of deformation, such as shear deformation in the core; these effects are not included in elementary beam theory, resulting in the apparent non-constant behaviour of the bending stiffness or elasticity modulus. The obvious conclusion would be to discard this simple model and derive instead a higher-order model taking into account the core shear effects, i.e. the 6th order model derived previously. However, there is also the possibility of utilizing the classical theory with a frequency dependent “apparent” bending stiffness, thereby describing at least approximately these effects with the advantage of greatly reduced complexity. It is shown that in the case of forced response, the regions around the resonance frequencies are reasonably accurately described by this approach whereas in high-impedance regions the error is in general large. However, the resonance peaks contain the dominating
part of the kinetic energy of a vibrating beam or plate, indicating that the method of apparent bending stiffness often could be utilized in order to estimate for example the sound transmission loss of a sandwich panel. The procedure would simply be to utilize a frequency dependent bending stiffness in an existing model developed for thin homogenous plates. In Figure 3, the measured apparent bending stiffness of a sandwich beam specimen is shown. This frequency dependent bending stiffness curve was used to estimate the sound transmission loss of a panel made of the same composite material, see Figure 4. The predicted curve was obtained from a model designed for thin homogenous and finite plates subject to given boundary conditions, see [15]. As can be seen, the calculated transmission loss agrees well with measured data.
2.3 Modified Timoshenko theory

As an alternative to Bernoulli-Euler theory, the more advanced Timoshenko beam theory can be utilized. The advantage of the modified Timoshenko beam model (or the corresponding Midlin plate theory) is that the apparent parameters, the bending stiffness \( D_{\text{app}}^T(f) \) and the core shear modulus \( G_{\text{app}}^T(f) \), do not depend on boundary conditions as is the case for the \( D_{\text{app}}(f) \) parameter used in the modified Bernoulli-Euler theory. Instead, the parameters are obtained indirectly from the dispersion curves of the sixth order theory. Both parameters have similar frequency dependencies, being approximately constant below a certain frequency limit, and approaching a linear asymptote above. The intersection of the asymptotes indicates the frequency limit above which ordinary Timoshenko theory fails to describe the propagation of waves in sandwich beams. In this region, either a higher order model or a modified homogenous beam theory needs to be utilized. The low frequency asymptotes of \( D_{\text{app}}^T(f) \) and \( G_{\text{app}}^T(f) \) are \( D_{\text{tot}} \) and \( G_c \), respectively, while in the high frequency limit both parameters approach a linear behaviour described by

\[
\lim_{f \to \infty} \frac{G_{\text{app}}^T(f)}{G_c} = \lim_{f \to \infty} \frac{D_{\text{app}}^T(f)}{D_{\text{tot}}} = \frac{\sqrt{\mu(D_1 + D_2)}}{G_c h_c} 2\pi f. \tag{1}
\]

In order to obtain simple explicit expressions for the apparent Timoshenko parameters in the entire frequency range, the asymptotes could be blended using some well-chosen interpolating function.

Figure 5: The frequency dependence of the apparent Timoshenko bending stiffness and shear modulus for a typical sandwich beam. Both entities are normalized with respect to their static values.

2.4 Measurement results and conclusions

In order to verify the apparent bending stiffness approach, measurements on a freely suspended asymmetric beam were performed. The obtained transfer accelerance functions display excellent agreement with the results of the sixth order model.
Figure 6: Measured and predicted transfer accelerance. —, measured; - - - , 6th order model; · · · , apparent bending stiffness approach.

case of the modified Bernoulli-Euler beam approach, the resonance peaks are generally described well while in the high-impedance regions there are larger deviations as previously mentioned. A typical transfer accelerance plot is shown in Figure 6, indicating the high degree of agreement between the measurements and the 6th order model in the entire considered frequency range.
3 Material parameter estimation - paper II

3.1 Introduction

One interesting application for the 6th order theory is material parameter characterization. In principle, the response of the plate depends on two material parameters, the core shear modulus $G_c$ and the elasticity modulus of the laminates $E_{\text{lam}}$. Here, only symmetric beams are considered, i.e. $D_1 = D_2 = D_{\text{lam}}$. The task of finding the material parameters is then defined by the optimization of a two-dimensional error function $\epsilon(f_n, D_{\text{lam}}, G_c)$, where $f_n$ is an array of measured eigenfrequencies. The calculated eigenfrequencies $f_n$ of a sandwich beam subject to known boundary conditions is obtained from the roots of the determinant of the boundary condition matrix,
\[ \det M_{BC}(f_n) = 0, \] (2)
which, unless the boundary conditions are of a simple nature such as simply supported ends, needs to be solved numerically using for example the Newton-Raphson method. Thus, we can define the error function as
\[ \epsilon = \| \det M_{BC}(\tilde{f}_n) \|, \] (3)
where $\| \cdot \|$ is some well-chosen norm, or, which is maybe more intuitive, as
\[ \epsilon = \frac{1}{N} \sum_n \left| \frac{f_n - \tilde{f}_n}{f_n} \right|, \] (4)
where $\left| \cdot \right|$ is the absolute value operator. This is simply the mean of the absolute relative error of the calculated eigenfrequencies. A possible advantage of Eq. (3) is that it does not involve as an intermediate step the procedure of numerically obtaining the eigenfrequencies. However, error functions based on the difference between measured and predicted eigenfrequencies represents as mentioned a more intuitive formulation and is also more commonly found in the litterature, see for example [16] or [17].

3.2 Sensitivity to measurement error

As all measurements involve some degree of error, it is interesting to analyze the effect of a random error on the input data. This was done by defining two different sandwich beams, calculating numerically their eigenfrequencies, contaminating these with random error and then apply the inverse algorithm in order to see the effect on the output.

A normally distributed absolute error with a standard deviation of 5 Hz was added to the calculated eigenfrequencies, resulting in the “shot groups” shown in Figs 7 and 8. In these plots, the dashed line indicate the “bull’s eyes”, i.e. the original material parameters of the beams. It is evident from these plots that for the considered beams, the elasticity modulus $E_{\text{lam}}$ of the laminates is subject to higher sensitivity to measurement error, which is likely due to the fact that for these beams (and most other common sandwich beams) the main part of the measured eigenfrequencies lie in the frequency range dominated by shear deformation.

Further, the negative slope of the shot groups – i.e. if $E_{\text{lam}}$ is underestimated, there is an opposite error in $G_c$ – can be explained as a result of “stiffness compensation”. In order to maintain a constant set of eigenfrequencies, the algorithm increases one of the stiffness parameters if the other is too low.
Figure 7: “Shot group”, beam 1.

Figure 8: “Shot group”, beam 2.
Figure 9: A typical error function, obtained from measured eigenfrequencies. Note the well-defined error minimum.

3.3 Results and conclusions

The proposed method for the estimation of the material parameters of sandwich beams yields excellent agreement with measured eigenfrequencies. The error function displays a well-defined minimum, see Figure 9, indicating that any simple numerical minimization algorithm should be capable of resolving the material parameters. The method thus provides an effective mean of non-destructive material identification for three-layered symmetric sandwich beams.
4 Energy propagation over junctions - Paper III

4.1 Introduction

The transmission across joints connecting structures play an important role in structural dynamics, as vibration in one system transmit into neighboring systems by means of coupling of forces and deformations. Classical problems include different types of beam junctions, such as the L-, T- and X-junctions were two or more beams are connected, attached damping masses and spring loaded joints. For ordinary homogenous beams, the properties of these types of junctions have been explored in for example [5], [18] and [19]. Analyzing the transmission loss, defined as the ratio of incident to transmitted energy flow in dB, requires that the energy flow for a sandwich beam is defined, as well as the coupling conditions of the considered junction.

![Mass-spring loaded junction](image)

Figure 10: A mass-spring loaded junction. The solid arrows represent propagating waves, and the dashed arrows nearfields.

4.2 Energy flow in sandwich beams

In order to calculate the transmission of energy across junctions, an expression for the energy flow in a sandwich beam needs to be derived. The time average of the energy flow \( \Pi \) becomes

\[
\bar{\Pi} = \Re \left( -F \frac{\partial w^*}{\partial t} + M \frac{\partial \beta^*}{\partial t} + M_s \frac{\partial \gamma^*}{\partial t} \right),
\]

(5)

where \( F \) is the cross-sectional transverse force, \( w \) is the total transverse deflection of the beam, \( M \) the total cross-sectional bending moment, \( M_s \) is the bending moment in the laminates due to shear deformation of the core, \( \beta \) is the angle of pure bending and \( \gamma \) is the angle due to pure shear deformation. \( \Re(\cdot) \) is the real part and \( ^* \) denotes complex conjugate. Considering a displacement wave with amplitude \( A \), Eq. (5) yields

\[
\bar{\Pi} = C |A|^2,
\]

(6)

where \( C \) is a constant depending on the properties of the beam. This implies that if the transmission loss \( R \) over a junction connecting identical beams is considered, \( C \) will
cancel out and the $R$ becomes

$$R = 10 \log_{10} \left| \frac{A_{\text{incident}}}{A_{\text{transmitted}}} \right|^2. \quad (7)$$

### 4.3 Energy transmission over junctions

The transmission of energy across various junctions connecting identical beams is considered. The symmetry implies Eq. (7) can be used directly. Assuming a unit incident displacement wave, i.e. $A_{\text{incident}} = 1$, and applying the correct coupling conditions yields $A_{\text{transmitted}}$ as a function of frequency. For a sandwich beam described by $6^{th}$ order theory, there will in general be a total of 6 unknown amplitudes – the amplitudes of the reflected and transmitted propagating waves and corresponding nearfields – requiring 6 coupling conditions. The coupling conditions are expressed in terms of the displacement variables $w$, $\beta$ and $\gamma$ and the cross-sectional forces and moments $F$ and $M$. For the junction shown in Figure 10, continuity in the total deflection $w$ and slope $w'$ are the first and most obvious conditions. Additionally, continuity in the bending angle $\beta$ and the shear angle derivative $\gamma'$ are imposed. If the moment of inertia of the mass is neglected, the total bending moment around the mass should vanish. Finally, the transverse cross-sectional force resultant $F$ must satisfy the following condition regarding vertical equilibrium,

$$\Delta F = K w = m \frac{\partial^2 w}{\partial t^2}, \quad (8)$$

where $\Delta F$ is the force difference over the junction, and $K$ and $m$ are the spring stiffness and mass terms as indicated in Figure 10. The resulting transmission loss, for a typical sandwich beam and some chosen values of $K$ and $m$ is shown in Figure 11.

![Transmission loss over a mass–spring loaded junction](image)

**Figure 11:** The transmission loss over a spring-mass junction versus frequency. The dashed line indicates the fundamental frequency of the mass-spring system, $\omega_0 = \sqrt{K/m}$.

The blocking phenomenon, where $R \to \infty$, is located at a frequency which can be estimated by using the apparent bending stiffness approach.
4.4 Measurement results and conclusions

Measurements on a sample sandwich beam with an blocking mass in the shape of a steel cylinder inserted in the beam core were performed in order to validate the theory. The junction corresponds to that described in the previous section, with $K = 0$. This implies the blocking mechanism disappears and the resulting transmission loss is a smooth curve varying from 0 dB as $f \to 0$ to approximately 3 dB in the high frequency range, with a bell-shaped maximum in between. The measurement procedure involves embedding the beam ends in sand in order to suppress reflection, and measuring transfer accelerance functions between an force input and four positions located symmetrically around the junction, see Figures 12.

From the four accelerance functions, the propagating wave amplitudes in the beam can be estimated. From these amplitudes, the transmission coefficient and thus the transmission loss over the junction can be obtained. The results are shown in Figure 13. In order to evaluate the quality of the measurements, different indicators such as the coherence functions and the condition numbers of the system matrices were considered. Although the measured transmission loss is highly contaminated by noise, it is obvious that the 6th order model agrees better than ordinary thin beam theory.

In order to improve the measurement results, a longer beam specimen could be used. This would decrease the influence of nearfields assumed negligible in the analytical model for the estimation of the transmission loss. Further, reflections could be more effectively suppressed at even lower frequencies if larger portions of the beam ends would be embedded.
Figure 13: Measured and predicted transmission loss over a mass-loaded junction. The solid line represents the measured data in third octave bands, the dashed line represents the 6th order model and the dotted line was obtained from classical Bernoulli-Euler theory.

Figure 14: Ongoing transmission loss measurements.
References


Modelling the vibration of sandwich beams using frequency dependent parameters

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Abstract

Various types of sandwich beams with foam or honeycomb cores are currently used in the industry, indicating the need for simple methods describing the dynamics of these complex structures. By implementing frequency dependent parameters, the vibration of sandwich composite beams can be approximated using simple fourth order beam theory. A higher order sandwich beam model is utilized in order to obtain estimates of the frequency dependent bending stiffness and shear modulus of the equivalent Bernoulli-Euler and Timoshenko models. The resulting predicted eigenfrequencies and transfer accelerance functions are compared to the data obtained from the higher order model and from measurements.

Key words: sandwich, beam, bending, vibration, Bernoulli-Euler, Timoshenko

1 Introduction

The type of sandwich structures considered in this article consist of two thin but relatively stiff sheets bonded to each side of a thick and light-weight core, see Fig. 1. A typical setup could be two aluminium sheets glued to a foam core, or – present in for example ship-building – steel plates bonded to a plastic core. Examples of widely used laminate materials are glass-reinforced plastic, abbreviated GRP, and carbon fibre. The purpose of the core is to maintain the distance between the laminates and to resist shear deformation, thus ideally maintaining pure bending of the beam around the neutral axis. This achieves

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the sought-for high static bending stiffness. Typical core configurations include plastic or metal foams and honeycomb materials. A different category of “sandwich” materials comprise plates consisting of two laminates separated by a visco-elastic layer designed to increase damping. The visco-elastic core layer deforms in shear parallel to the beam, i.e. in-plane deformation, whereas for the type of thick-core sandwich beams considered in this paper the principal shear deformation is in the lateral direction. This type of shear deformation was considered by S. Timoshenko in his derivation of thick-beam theory, see [1], [2] and [3]. Constrained layer sandwich beams have been analyzed extensively in for example [4] and [5].

The dynamic response of a sandwich beam differs from that of an ordinary 4th order beam in the high frequency region. This is due to the more complex constitution of the structure, being a coupled system of two thin beams and a thick core. The ordinary Bernoulli-Euler beam model neglects shear and rotational inertia and can be shown to rapidly deviate from the real dynamics of a sandwich. More refined theories taking these effects into account were presented by Lord Rayleigh and Timoshenko in [6] and [1], respectively. However, neither of these models is suitable for describing transverse vibration of composite beams. Bernoulli-Euler theory fails rapidly due to the assumption of an infinite shear modulus, and also Timoshenko theory fails since the vibration of a sandwich beam will be governed by the bending dynamics of the laminates at high frequencies; in terms of propagating waves, Timoshenko theory predicts that for high frequencies the deformation of the beam will be governed by the effects of shear and rotation, while for a three-layer sandwich beam with thin laminates and a soft core the high frequency region is governed by pure bending of the laminates. Modelling the dynamics of layered beams and plates can be achieved in three different ways:

- Using exact formulation for the core deformation and solving the governing equations by means of numerical analysis.
- Imposing assumptions on the internal stress and deformation fields and thus obtaining finite-order equations which can be solved analytically.
- By utilizing models developed for homogenous beams and plates, in combination with effective or apparent parameters.

The first approach has been analysed in for example [7] and [8] and compared to “elementary” methods in [9]. It is concluded that for most applications in the industry, finite-order models yield satisfactory solutions which make them attractive considering the increase of computational cost associated with “exact” models utilizing the general wave equation. The last item will be explored in this article and represents the simplest way of obtaining estimates for the various dynamic properties of layered beams and plates, such as vibration response, sound transmission loss and radiation.
Historically, research focusing on the flexural dynamics of composite beams and plates gained momentum after the war, when sandwich structures became an important part of aircraft construction:

- Mead and Markus [5] presented a 6th order (bi-cubic) theory neglecting rotational inertia. In their paper, they investigated the response of a sandwich beam subject to “damped normal loadings” and presented orthogonality relations for the displacement of the beam.
- Mead [10] made a short review of different theories and also presented a model taking into account the effects of inertia and shear deformation in the laminates. 6th and 8th order (bi-cubic and bi-quartic) equations of motion for symmetric and asymmetric sandwich beams were presented. However, no boundary conditions or exciting force distributions were imposed.
- A. C. Nilsson presented in [7] an “exact” field equations model of a sandwich with thin laminates. The properties of this model were extensively investigated in [8], using laborious numerical methods to solve ill-conditioned sets of equations.
- E. Nilsson and A. C. Nilsson [11] instead suggested a 6th order (bi-cubic) model derived using Hamilton’s principle which describes the dynamics of a symmetric sandwich beam for frequencies below the mass-spring-mass frequency where the laminates start to move independently. This model was shown to yield accurate results when compared to measured data.

In the following section, a 6th order sandwich beam theory for asymmetric beams is derived. This model will provide the basis for the development of modified lower-order models. The purpose of these simpler models is to provide means of analyzing the vibration of sandwich composite structures using existing tools developed for ordinary beams, for example formulae for the estimation of the sound transmission loss or vibration level of a sandwich panel. Please see [12] for a more detailed presentation, including the derivation of the 10th order model mentioned briefly in section 2.4.
Fig. 2. Deformation mechanisms; $\gamma$ due to pure shear of the core and $\beta$ due to pure bending of the entire beam section.

2 High order theory

The model presented in [13] and [11] is generalized to cover asymmetric sandwich structures, i.e. when the laminates are not equal. The governing set of equations are derived by applying Hamilton’s principle, expressed as

$$\delta \int_{t_0}^{t_1} \int_0^L (U - T + A)dxdt = 0,$$

where $\delta$ here denotes the variation operator, $U$ is the potential energy per unit length of the beam, $T$ the kinetic energy per unit length and $A$ the potential energy due to external forces (boundary conditions and applied forces and bending moments) per unit length. The spatial integration is over the length $L$ of the beam, and the time integration limits $t_0$ and $t_1$ are arbitrary for an assumed stationary harmonic time dependence.

The total deformation $w$ depends on $x$ only since the core is assumed to be laterally incompressible, thus forcing the laminates to move in phase. This is a valid assumption well below the cut-on frequency for antiphase motion of the laminates, see section 2.4. The geometric relation between $w$, the angle of pure bending $\beta$ and the angle of pure shear $\gamma$ is now given by

$$\frac{\partial w}{\partial x} = \beta + \gamma,$$

see Figure 2.

The potential energy density $U$ consists of three parts; energy due to pure bending $\beta$ of the entire beam, shear in the core $\gamma$ and pure bending of the thin
laminates due to the shear deformation of the core. Thus, we have

\[ U_1 = \frac{1}{2} D_{\text{tot}} \left( \frac{\partial \beta}{\partial x} \right)^2, \]

\[ U_2 = \frac{1}{2} (D_1 + D_2) \left( \frac{\partial \gamma}{\partial x} \right)^2, \]

\[ U_3 = \frac{1}{2} G_c h_c \gamma^2, \]

where \( D_{\text{tot}} \) is the total bending stiffness per unit width of the sandwich beam, \( D_1 \) and \( D_2 \) are the bending stiﬀnesses of the laminates, and \( G_c \) is the effective core shear modulus. The total bending stiffness \( D_{\text{tot}} \) can be written as a weighted sum of the elasticity moduli of the laminates and the core,

\[ D_{\text{tot}} = c_0 E_1 + c_1 E_2 + c_2 E_c, \]

where the \( c \) coefficients are given by

\[ c_0 = -y_0 h_1^2 + y_0^2 h_1 + \frac{1}{3} h_1^3, \]

\[ c_1 = \frac{1}{3} \left( (h_1 + h_c + h_2)^3 - (h_1 + h_c)^3 \right) + y_0^2 h_2 - y_0 \left( (h_1 + h_c + h_2)^2 - (h_1 + h_c)^2 \right), \]

\[ c_2 = -y_0 \left( (h_1 + h_c)^2 - h_1^2 \right) + y_0^2 h_c + \frac{1}{3} \left( (h_1 + h_c)^3 - h_1^3 \right), \]

and the neutral layer coordinate \( y_0 \) is

\[ y_0 = \frac{h_1^2 E_1 + (2h_1 + h_c)E_c h_c + (2h_1 + 2h_c + h_2)E_2 h_2}{2 (E_1 h_1 + E_c h_c + E_2 h_2)} \]

The bending stiﬀnesses per unit width of the laminates are

\[ D_i = \frac{E_i h_i^3}{12}, \quad \text{for } \ i = 1, 2. \]

The total kinetic energy density \( T \) is due to the velocity distribution \( \dot{w} \) of the beam, and the angular velocity \( \dot{\beta} \) of the sandwich cross-section:

\[ T_1 = \frac{1}{2} \mu_{\text{tot}} \left( \frac{\partial \dot{w}}{\partial t} \right)^2, \]

\[ T_2 = \frac{1}{2} I_{\text{tot}} \left( \frac{\partial \dot{\beta}}{\partial t} \right)^2, \]

where \( \mu_{\text{tot}} \) is the total mass per unit area of the beam, and \( I_{\text{tot}} \) is the mass moment of inertia per unit width of the cross-section.
Finally, the boundary conditions and applied forces are accounted for in $A$ as

$$\int_0^L A \, dx = - \int_0^L p(x)w(x) \, dx - [Fw - M\beta - M_s \gamma]_0^L,$$  

(7)

where $p(x)$ is an external pressure exciting the beam, $F$ is the shear force, $M$ is the total bending moment and $M_s$ is the bending moment acting on the laminates due to shear deformation in the core, all per unit width.

2.1 **Governing equations**

By performing the variation in Eq. (1), the governing equations are obtained:

$$\begin{align*}
(D_1 + D_2) \left( \frac{\partial^4 w}{\partial x^4} - \frac{\partial^3 \beta}{\partial x^3} \right) & - G_c h_c \left( \frac{\partial^2 w}{\partial x^2} - \frac{\partial \beta}{\partial x} \right) + \mu_{\text{tot}} \frac{\partial^2 w}{\partial t^2} \\
- p & = 0,
\end{align*}$$  

(8)

$$- D_{\text{tot}} \frac{\partial^2 \beta}{\partial x^2} + (D_1 + D_2) \left( \frac{\partial^3 w}{\partial x^3} - \frac{\partial^2 \beta}{\partial x^2} \right) - G_c h_c \left( \frac{\partial w}{\partial x} - \beta \right)$$

$$+ I_{\text{tot}} \frac{\partial^2 \beta}{\partial t^2} = 0,$$

(9)

with boundary conditions

$$\begin{align*}
F & = - D_{\text{tot}} \frac{\partial^2 \beta}{\partial x^2} + I_{\text{tot}} \frac{\partial^2 \beta}{\partial t^2} \quad \text{or} \quad w = 0, \\
M & = - D_{\text{tot}} \frac{\partial \beta}{\partial x} \quad \text{or} \quad \beta = 0, \\
M_s & = -(D_1 + D_2) \left( \frac{\partial w}{\partial x} - \frac{\partial \beta}{\partial x} \right) \quad \text{or} \quad \frac{\partial w}{\partial x} = 0.
\end{align*}$$

(10)

(11)

(12)

It can be verified that the above shear force and total bending moment satisfy basic equilibrium equations. Now, by assuming a time and space dependence of $e^{i(\omega t - kx)}$, where $\omega$ is the angular frequency and $k$ the wavenumber for flexural waves, and combining Eqs. (8) and (9) with $p = 0$, the characteristic equation – or *dispersion relation* – gives the relationship between the wavenumber $k$ and the frequency $\omega$ as

$$\begin{align*}
(D_1 + D_2)D_{\text{tot}} k^6 + \left( G_c h_c D_{\text{tot}} - \omega^2 I_{\text{tot}} (D_1 + D_2) \right) k^4 \\
- \omega^2 (\mu_{\text{tot}} (D_1 + D_2 + D_{\text{tot}}) + G_c h_c I_{\text{tot}}) k^2 + \mu_{\text{tot}} \omega^2 \\
\times \left( \omega^2 I_{\text{tot}} - G_c h_c \right) & = 0.
\end{align*}$$

(13)

This is a 6th order (bi-cubic) polynomial equation of even powers of $k$, and can be transformed into a 3rd order equation using a simple substitution. The
Fig. 3. Wavenumber magnitudes. —, \(\kappa_1\), propagating; ——, \(\kappa_2\), evanescent; --, \(\kappa_3\), propagating; \(\cdots\), \(\kappa_3\), evanescent.

Magnitudes of the wavenumbers of a typical sandwich beam are shown in Fig. 3. The solid curve represents the main propagating mode, whereas the dashed curve represents a nearfield below the cut-on frequency for rotational waves. The dotted curve represents a pure nearfield. The solutions to the dispersion relation are denoted \(k = \pm \kappa_1\pm i\kappa_2\) and \(\pm i\kappa_3\). The \(\kappa\) quantities are real below the cut-on frequency for rotational waves, for systems without losses. This frequency, denoted by \(\omega_{\text{rot}}\), is obtained by setting \(k = 0\) in the dispersion relation. This yields

\[
\omega_{\text{rot}} = \sqrt{\frac{G_c h_c}{I_{\text{tot}}}}. \tag{14}
\]

The displacement and shear solutions are now given as

\[
w(x, t) = e^{i\omega t} \sum_{n=1}^{6} \hat{A}_n e^{-i\kappa_n x}, \quad \beta(x, t) = e^{i\omega t} \sum_{n=1}^{6} \hat{B}_n e^{-i\kappa_n x},
\]

or, using a different notation, as

\[
w(x, t) = \{A_1 \sin \kappa_1 x + A_2 \cos \kappa_1 x + A_3 e^{-\kappa_2 x} + A_4 e^{\kappa_2(x-L)} \\
+ A_5 e^{-\kappa_3 x} + A_6 e^{\kappa_3(x-L)}\} e^{i\omega t}, \tag{15}
\]

\[
\beta(x, t) = \{B_1 \sin \kappa_1 x + B_2 \cos \kappa_1 x + B_3 e^{-\kappa_2 x} + B_4 e^{\kappa_2(x-L)} \\
+ B_5 e^{-\kappa_3 x} + B_6 e^{\kappa_3(x-L)}\} e^{i\omega t}. \tag{16}
\]

The spatial dependences of Eqs. (15) and (16) are real for loss-free systems for frequencies below \(\omega_{\text{rot}}\). Inserting the above solutions into Eq. (9) and equating all coefficients of \(x\)-depending functions to zero yields a relationship between the \(A\) and \(B\) coefficients (or the corresponding \(\hat{A}\) and \(\hat{B}\) coefficients), reducing
the problem to 6 unknowns, for example $A_n$. A good reason for preferring Eqs. (15) and (16) is that they are numerically more favourable.

2.2 Boundary conditions and mode shapes

The boundary conditions are given by Eqs. (10) to (12). Written in vector notation, where $\mathbf{A}$ and $\mathbf{B}$ contain the unknown coefficients, we have

$$Z_1 \mathbf{A} + Z_2 \mathbf{B} = \mathbf{0},$$

(17)

where the $Z_1$ and $Z_2$ matrices depend on the boundary conditions. The first three rows in $Z_1$ and $Z_2$ can be chosen to contain the boundary conditions for $x = 0$, and the last three rows the conditions for $x = L$. Further,

$$\mathbf{B} = \mathbf{X} \mathbf{A},$$

(18)

where the matrix $\mathbf{X}$ is obtained by substituting Eqs. (15) and (16) into Eq. (9) and equating all coefficients of $x$-dependent functions to zero. Thus, for free vibration, the non-trivial solutions are given by the frequency condition

$$\det(Z_1 + Z_2 \mathbf{X}) = 0,$$

(19)

and the corresponding null space vector. As an example, consider the case of clamped boundary conditions. From Eq. (10) we know that $w = 0$ at both ends, so the first row in $Z_1$ will be given by the values of each subfunction in Eq. (15) for the displacement $-i.e. \sin(\kappa_1 x), \cos(\kappa_1 x), e^{-\kappa_2 x} \ldots$ evaluated at $x = 0$. Similarly, the fourth row would contain the corresponding values evaluated at $x = L$. The second and fifth rows of $Z_2$ are connected to Eq. (11). In this case, we have $\beta = 0$ at $x = 0$ and $L$. The final condition is given by Eq. (12) as $\frac{\partial w}{\partial x} = 0$, which occupies the third and sixth rows of $Z_1$. Hence, with the rest of the elements being equal to zero, we have

$$Z_1 = \begin{pmatrix}
0 & 1 & 1 & e^{-\kappa_2 L} & 1 & e^{-\kappa_3 L} \\
0 & 0 & 0 & 0 & 0 & 0 \\
\kappa_1 & 0 & -\kappa_2 & \kappa_2 e^{-\kappa_2 L} & -\kappa_3 & \kappa_3 e^{-\kappa_3 L} \\
\sin(\kappa_1 L) & \cos(\kappa_1 L) & e^{-\kappa_2 L} & 1 & e^{-\kappa_3 L} & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\kappa_1 \cos(\kappa_1 L) & -\kappa_1 \sin(\kappa_1 L) & -\kappa_2 e^{-\kappa_2 L} & \kappa_2 & -\kappa_3 e^{-\kappa_3 L} & \kappa_3
\end{pmatrix},$$

(20)
and

\[
Z_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & e^{-\kappa_2 L} & 1 & e^{-\kappa_3 L} \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
sin(\kappa_1 L) \cos(\kappa_1 L) & e^{-\kappa_2 L} & 1 & e^{-\kappa_3 L} & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

The sought-for relationship between the \(A\) and \(B\) coefficients, yielding the \(X\) matrix, is obtained as

\[
B_1 = X_2 A_2, \quad B_2 = X_1 A_1, \quad B_3 = X_3 A_3,
\]

\[
B_4 = X_4 A_4, \quad B_5 = X_5 A_5, \quad B_6 = X_6 A_6,
\]

(20)

where

\[
X_1 = -X_2 = -\frac{\kappa_1 \{(D_1 + D_2)\kappa_1^2 + G_c h_c\}}{(D_{tot} + D_1 + D_2)\kappa_1^2 + G_c h_c - \omega^2 I_{tot}},
\]

\[
X_3 = -X_4 = -\frac{\kappa_2 \{(D_1 + D_2)\kappa_2^2 + G_c h_c\}}{(D_{tot} + D_1 + D_2)\kappa_2^2 + G_c h_c - \omega^2 I_{tot}},
\]

\[
X_5 = -X_6 = -\frac{\kappa_3 \{(D_1 + D_2)\kappa_3^2 + G_c h_c\}}{(D_{tot} + D_1 + D_2)\kappa_3^2 + G_c h_c - \omega^2 I_{tot}}.
\]

Now, by computing the roots of Eq. (19) — using some numerical method like the secant method or Newton’s method — the eigenfrequencies \(\omega_m\) of the sandwich configuration are obtained.

For simply supported boundary conditions, it can be seen directly from the corresponding system matrix \(M = Z_1 + Z_2 X\) that the eigenfrequencies will be given by inserting

\[
kL = m\pi, \quad m \in \mathbb{N}
\]

into the dispersion relation Eq. (13), yielding

\[
\omega_m = m^2\pi^2 \frac{D_{tot}\{(D_1 + D_2)\pi^2m^2 + G_c h_c L^2\}}{E^2 \mu_{tot}\{(D_{tot} + D_1 + D_2)\pi^2m^2 + G_c h_c L^2\}}.
\]

(21)

By comparing Eq. (21) with the well-known expression for the eigenfrequencies of simply supported Bernoulli-Euler beams,

\[
\omega_{BE} = \frac{m^2\pi^2}{L^2} \sqrt{\frac{D}{\mu}},
\]

(22)
Fig. 4. Mode shapes, clamped conditions: —, mode 1; - - , mode 2; - - - , mode 3.

where $D$ is the bending stiffness and $\mu$ is mass per unit area of the homogenous beam, one can readily identify a discrete estimate of the apparent bending stiffness $D_{\text{app}}$ of the simply supported sandwich beam as

$$D_{\text{app}}^{(m)} = \frac{D_{\text{tot}} \left\{ (D_1 + D_2)\pi^2 m^2 + G_c h_c L^2 \right\}}{(D_{\text{tot}} + D_1 + D_2)\pi^2 m^2 + G_c h_c L^2}. \quad (23)$$

$D_{\text{app}}^{(m)}$ is the apparent value of the bending stiffness of an ordinary Bernoulli-Euler beam, with the same dimensions and mass, at the $m^{th}$ resonance. These modified lower order beam models will be more carefully analyzed later in this article.

Calculating the mode shapes of a sandwich configuration involves solving the nullspace problem $M\bar{A} = \bar{0}$, where $M = Z_1 + Z_2 X$ is a known matrix. A simple way of doing this is to assume a unit first element of the $\bar{A}$ vector, so that the system can be written

$$\begin{pmatrix} \alpha & \bar{W}^T \\ \bar{V} & M_{\text{sub}} \end{pmatrix} \begin{pmatrix} 1 \\ \bar{A}_{\text{sub}} \end{pmatrix} = \bar{0},$$

where $\alpha$ is a scalar, $\bar{V}$ and $\bar{W}$ are vectors, $M_{\text{sub}}$ a matrix and $\bar{A}_{\text{sub}}$ a vector. Now, by expanding, we obtain

$$\bar{V} + M_{\text{sub}}\bar{A}_{\text{sub}} = \bar{0}. \quad (24)$$

This system can be solved in order to obtain the unknown elements given by the $\bar{A}_{\text{sub}}$ elements, defining the $\bar{A}$ and $\bar{B}$ coefficients. Predicted normalized modes for a clamped sandwich beam are shown in Fig. 4.
2.3 Forced response

The inhomogenous problem – when a force or bending moment distribution is exciting the beam – can be solved in terms of Green’s functions. Consider a beam excited by a lateral point force per unit width \( F(x) e^{i\omega t} \) at \( x = x_0 \), see Figure 5. Denote the deformation solutions for \( 0 < x < x_0 \) by \( w_- \) and \( \beta_- \), and the corresponding solutions for \( x_0 < x < L \) by \( w_+ \) and \( \beta_+ \). For numerical reasons, it is preferable to use the following sets of base functions for the solutions, as compared to that used in Eqs. (15) and (16):

\[
\begin{align*}
\bar{\mathbf{b}}_- &= \left( \sin \kappa_1 x, \cos \kappa_1 x, e^{-\kappa_2 x}, e^{\kappa_2 (x-x_0)}, e^{-\kappa_3 x}, e^{\kappa_3 (x-x_0)} \right)^T, \\
\bar{\mathbf{b}}_+ &= \left( \sin \kappa_1 (L-x), \cos \kappa_1 (L-x), e^{-\kappa_2 (L-x)}, e^{\kappa_2 (x_0-x)}, e^{-\kappa_3 (L-x)}, e^{\kappa_3 (x_0-x)} \right)^T,
\end{align*}
\] (25)

We have to consider the fact that their respective \( \mathbf{X} \) matrices are different; it is easy to see that in this particular case \( \mathbf{X}_+ = -\mathbf{X}_- = -\mathbf{X} \). Now, the boundary conditions at \( x = 0 \) and \( x = L \) together with the coupling conditions at \( x = x_0 \) yields a system of 12 equations, sufficient to solve for the unknown coefficient vectors \( \bar{\mathbf{A}}_- \) and \( \bar{\mathbf{A}}_+ \) as functions of the exciting force amplitude \( F_0 \). In matrix notation, where \( \mathbf{M}_- \) and \( \mathbf{M}_+ \) are the \( 3 \times 6 \) boundary condition matrices at \( x = 0 \) and \( x = L \), respectively,

\[
\begin{pmatrix}
\mathbf{M}_- & 0 \\
\mathbf{M}_c & \mathbf{M}_+ \\
0 & \mathbf{M}_-
\end{pmatrix}
\begin{pmatrix}
\bar{\mathbf{A}}_- \\
\bar{\mathbf{A}}_+
\end{pmatrix}
= \bar{\mathbf{F}},
\] (26)

where \( \mathbf{M}_c \) is a \( 6 \times 12 \) matrix describing the coupling conditions. \( \bar{\mathbf{F}} \) contains the force amplitude \( F_0 \) as its only non-zero element. The coupling conditions are
obtained from Eqs. (10) to (12), together with conditions regarding continuity in total deflection $w$, the total deflection angle $\frac{\partial w}{\partial x}$ and the bending angle $\beta$.

This leads to the six coupling conditions at $x = x_0$, defining $M_c$ and $\bar{F}$:

$$w_- = w_+, \quad (27)$$

$$\frac{\partial w_-}{\partial x} = \frac{\partial w_+}{\partial x}, \quad (28)$$

$$\beta_- = \beta_+, \quad (29)$$

$$\frac{\partial \beta_-}{\partial x} = \frac{\partial \beta_+}{\partial x}, \quad (30)$$

$$\frac{\partial^2 w_-}{\partial x^2} = \frac{\partial^2 w_+}{\partial x^2}, \quad (31)$$

$$(D_1 + D_2) \left( \frac{\partial^3 w_-}{\partial x^3} - \frac{\partial^3 \beta_-}{\partial x^3} \right) - G_c h_c \left( \frac{\partial w_-}{\partial x} - \beta_- \right) = F_0. \quad (32)$$

$$-(D_1 + D_2) \left( \frac{\partial^3 w_+}{\partial x^3} - \frac{\partial^3 \beta_+}{\partial x^3} \right) + G_c h_c \left( \frac{\partial w_+}{\partial x} - \beta_+ \right) = F_0.$$  

Here, Eqs. (30) and (31) are obtained indirectly from combining Eqs. (11) and (12). Implementing the coupling conditions in this order implies that the $9^{th}$ element of $\overline{F}$ contains $F_0$, the other elements being equal to zero. The result is a governing $12 \times 12$ matrix system, which can be solved using some numerical software tool like Matlab, obtaining the displacement fields $w$, $\beta$ and indirectly $\gamma$ for any given point force exciting the beam:

$$w(x, t) = \begin{cases} e^{i\omega t} \tilde{A}_- \tilde{b}_-, & x \leq x_0 \\ e^{i\omega t} \tilde{A}_+ \tilde{b}_+, & x \geq x_0 \end{cases}$$

$$\beta(x, t) = \begin{cases} e^{i\omega t} \tilde{B}_- \tilde{b}_-, & x \leq x_0 \\ e^{i\omega t} \tilde{B}_+ \tilde{b}_+, & x \geq x_0 \end{cases} \quad (33)$$

For more complex force distributions, the solution is given as an integral over the length of the beam:

$$w(x, t) = e^{i\omega t} \int_0^L G_w(x, \chi) F'(\chi) d\chi, \quad (34)$$

where $G_w(x, \chi)$ is the displacement at $x$ corresponding to a unit point force applied at $\chi$, obtained from Eq. (33), and $F'(\chi)$ is the force distribution (force per unit length and width) as a function of the position variable $\chi$. Analogously, the bending angle field $\beta$ can be obtained from the corresponding Green’s function $G_\beta(x, \chi)$.

The case of simply supported end conditions can be treated separately by expressing the solutions for $w$ and $\beta$ in terms of sin and cos series expansions, respectively. This is due to the fact that these sets of functions satisfy the
boundary conditions as well as provide a full base for the solutions. Thus, considering point excitation, we let

\[ w = e^{\omega t} \sum_{n=1}^{\infty} A_n \phi_n(x), \quad \beta = e^{\omega t} \sum_{n=1}^{\infty} B_n \psi_n(x), \]

where the orthogonal \( \phi \) and \( \psi \) functions are given by

\[ \phi_n(x) = \sin \frac{n \pi x}{L}, \quad \psi_n(x) = \cos \frac{n \pi x}{L}. \]

\( A_n \) and \( B_n \) are coefficient sets to be determined, and should not be confused with the \( A_n \) and \( B_n \) sets introduced earlier. It can be noted that since \( n > 0 \), the cosine series expansion is incomplete. However, the \( B_0 \) term corresponds to rigid body rotation and is of no interest.

Inserting these ansatze into the governing Eqs. (8) and (9) yields, with time dependence excluded,

\[ \sum_{n=1}^{\infty} \left\{ \frac{n \pi}{L} \left( \frac{n^2 \pi^2}{L^2} (D_1 + D_2) + G_c h_c \right) \left( \frac{n \pi}{L} A_n - B_n \right) \right. \]

\[ - \omega^2 \mu_{\text{rot}} A_n \right\} \phi_n(x) = p(x), \]

\[ \sum_{n=1}^{\infty} \left\{ \frac{n^2 \pi^2}{L^2} \left( D_{\text{tot}} B_n - (D_1 + D_2) \left( \frac{n \pi}{L} A_n - B_n \right) \right) - G_c h_c \left( \frac{n \pi}{L} A_n - B_n \right) \right\} \psi_n(x) = 0. \]

By assuming a point excitation on the form \( p(x) = F_0 \delta(x - x_0) \), where \( F_0 \) is per unit width of the beam, and taking the inner products over the length of the beam of Eqs. (37) and (38) with \( \phi_q(x) \) and \( \psi_q(x) \), respectively, we obtain

\[ \left\{ \frac{n \pi}{L} \left( \frac{n^2 \pi^2}{L^2} (D_1 + D_2) + G_c h_c \right) \left( \frac{n \pi}{L} A_n - B_n \right) - \omega^2 \mu_{\text{rot}} A_n \right\} \frac{L}{2} \]

\[ = F_0 \phi_n(x_0), \]

\[ \frac{n^2 \pi^2}{L^2} \left( D_{\text{tot}} B_n - (D_1 + D_2) \left( \frac{n \pi}{L} A_n - B_n \right) \right) - G_c h_c \left( \frac{n \pi}{L} A_n - B_n \right) \]

\[ - \omega^2 I_{\text{rot}} B_n = 0. \]

The result is a \( 2 \times 2 \) algebraic system which yields the \( A_n \) and \( B_n \) coefficients as functions of the exciting force amplitude per unit width, \( F_0 \). Approximate mobility functions can be obtained numerically in terms of truncated mode sums. Note also that by calculating the determinant of the matrix system – i.e. the denominator of each mode in the mode expansion – we obtain the ordinary frequency relationship of \( \omega_n^2 - \omega^2 \), where \( \omega_n \) is given by Eq. (21). The expression for the total displacement \( w \) can be shown to be identical to that
of the ordinary Bernoulli-Euler theory, the only difference being the values of the eigenfrequencies $\omega_n$:

$$w(x, t) = \frac{2F_0 e^{i\omega t}}{L\mu_n} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi r}{R}\right)}{\omega_n^2 - \omega^2}. \quad (41)$$

2.4 Notes on a 10th order model including the effects of core compressibility

The effects of core compressibility was included in a more advanced model, where the core deflection and shear deformation were described by linearly interpolating the deformations of the laminates. However, this 10th order model turned out to be numerically cumbersome and therefore not of great interest. It could potentially become valuable for predicting the vibration of sandwich beams with heavy laminates and soft cores, where the mass-spring-mass frequency of the system, i.e. the cut-on frequency for independent vibration of the laminates, is forced down into the considered frequency range. Another application could be for fluid-loaded plates.

2.5 Implementing losses

Losses can be easily implemented into the model by means of complex elasticity moduli, in accordance with common practice. Thus,

$$E_n = \Re(E_n)(1 + i\eta_n), \quad n = 1, 2,$$

$$E_c = \Re(E_c)(1 + i\eta_c),$$

$$G_c = \Re(G_c)(1 + i\eta_c), \quad (42)$$

where $E_n$ and $\eta_n$ denote the elasticity moduli and loss factors of the laminates, respectively, and $E_c$, $G_c$ and $\eta_c$ the elasticity modulus, shear modulus and loss factor of the core.

3 Apparent bending stiffness

A different approach to modelling the flexural response of sandwich composites is provided by the concept of apparent bending stiffness. The idea is to implement a frequency dependent bending stiffness parameter, denoted $D_{\text{app}}(\omega)$, in the ordinary Bernoulli-Euler beam theory or the corresponding Kirchhoff plate theory. The frequency dependence of $D_{\text{app}}$ is chosen such that the thin-beam theory as accurately as possible mimics the true response of the composite beam.
However, it should be stressed that apparent bending stiffness is an approximate tool. Naturally, it is impossible to correctly describe the combined effects of the propagating mode and the nearfields using only one wavenumber. The most intuitive way to deal with this problem is to assume that $D_{\text{app}}$ depends on boundary conditions, while another option is to discard the $\kappa_3$ wavenumber and implement Timoshenko theory instead of Bernoulli-Euler theory. The latter possibility will be analyzed later.

It can also be mentioned that instead of implementing a single, frequency dependent bending stiffness parameter $D_{\text{app}}(\omega)$ to be shared by all modes, it is possible to assign a unique bending stiffness $D_{\text{app}}^{(m)}$ to each mode, using for example Eq. (23) in the case of a simply supported sandwich beam. The advantage of this approach is that calculated mobilities will not suffer from “narrowing” of its peaks, which is otherwise the case when using a frequency dependent smooth bending stiffness curve. See the discussions in sections 3.3 and 6.

In summary, the apparent bending stiffness concept is an approximate engineering approach, “translating” a sandwich beam into an ordinary homogeneous beam. Various ways of obtaining estimates of $D_{\text{app}}(\omega)$ will be given in the following section.

3.1 Different estimates of $D_{\text{app}}$

3.1.1 Method of displacement error minimization

There are several possible estimates of the $D_{\text{app}}$ parameter. One definition could require the minimization of the mean displacement error with respect to $D_{\text{app}}$,

$$E(D_{\text{app}}, \omega) = \frac{1}{L} \int_0^L |w - w_{\text{app}}| \, dx,$$

(43)

where $w$ is the “true” deflection as calculated by Eq. (33) and $w_{\text{app}}$ is the deflection yielded by applying Bernoulli-Euler theory with a frequency dependent bending stiffness. However, this is a rather cumbersome approach as it involves the full solution to the 6th order problem. Further, once the full solution is obtained, a numerical approach to solving the minimization problem is probably required if the absolute error definition is to be used. As an alternative, the least-squares method could be considered.

1 In a wide frequency region, the magnitude of $\kappa_3$ is much larger than those of the other wavenumbers. This implies that the nearfield associated with $\kappa_3$ is highly localized to edges and discontinuities, and will not influence the displacement some distance away from these.
Fig. 6. Different estimates of $D_{\text{app}}$ for the sample sandwich beam specified in Table 3, with both ends clamped. —, numerical error minimization; - - - , method of equating eigenfrequencies; ..., $\kappa_1$-method; o, eigenfrequencies.

Besides from the computational disadvantages of this definition, another problem lies in the fact that the obtained estimate of $D_{\text{app}}$ must depend on the chosen type of excitation force distribution. In Fig. 6, point excitation has been utilized. Also note that the chosen cost function could be replaced by for example kinetic energy error, yielding a different estimate.

3.1.2 $\kappa_1$-method

In contrast to the previous definition, the simplest estimate of $D_{\text{app}}$ is obtained by inserting the main propagating wavenumber $\kappa_1$ into the Bernoulli-Euler dispersion relation, solving for the bending stiffness. This yields

$$D_{\text{app}} = \frac{\mu \omega^2}{\kappa_1^4}. \quad (44)$$

The problem with this approach is the discarding of the two other wave numbers, implying that the effects of nearfields are not described as accurately as they could be. As mentioned earlier, the apparent bending stiffness – as applicable to Bernoulli-Euler theory – should depend on boundary conditions, in order to achieve an accurate description of the problem. For example, using this approach one will obtain the same eigenfrequencies for both clamped and free end conditions. This is true for Bernoulli-Euler beams, but not for sandwich composites. Also, the low-frequency asymptote will in general be overestimated. This problem can be illustrated as follows: consider the maximum static deflection for a sandwich beam subject to a central point force $F$
<table>
<thead>
<tr>
<th>Simply supported (central load)</th>
<th>$q_b^{-1}$</th>
<th>$q_s^{-1}$</th>
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<tbody>
<tr>
<td>Clamped-clamped (central load)</td>
<td>48</td>
<td>4</td>
</tr>
<tr>
<td>Clamped-free (end load)</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 1
Bending and shear coefficients for different boundary conditions, static deflection, per unit width (obtained from thick beam statics, see for example [14]):

$$
\delta_{\text{max}} = F \left( \frac{q_b L^3}{D_{\text{tot}}} + \frac{q_s L}{C_c h_c} \right).
$$

(45)

Here, $q_b$ and $q_s$ are the bending deflection coefficient and shear deflection coefficient, respectively, and depend on boundary conditions (see Table 1). For an equivalent Bernoulli-Euler beam the corresponding expression is given by

$$
\delta_{\text{max}}^{\text{BE}} = F \frac{q_b L^3}{D_{\text{app}}}.
$$

(46)

By equating the maximum deflections and solving for the static equivalent bending stiffness $D_{\text{app}}^{\text{stat}}$, we obtain

$$
D_{\text{app}}^{\text{stat}} = \frac{D_{\text{tot}}}{1 + \frac{q_s D_{\text{tot}}}{q_b L^3 C_c h_c}}.
$$

(47)

It is clear that the apparent bending stiffness will converge to a value lower than $D_{\text{tot}}$, for finite values of the shear modulus and beam length.

3.1.3 Method of equating eigenfrequencies

The most intuitive approach to obtaining a boundary condition dependent estimate is to consider eigenfrequencies. These can be obtained in a relatively straight-forward way from the 6th order method, see Eq. (19). The eigenfrequencies of a Bernoulli-Euler beam is given, for different boundary conditions, by

$$
\omega_m^{\text{BE}} = \kappa_m^2 \sqrt{\frac{D}{\mu}},
$$

(48)

where $\kappa_m^2$ depends on boundary conditions, see Table 2.

\(^{2}\) The eigenvalues $\kappa_m$, $m \in \mathbb{N}$, should not be confused with the continus $\kappa_1$, $\kappa_2$ and $\kappa_3$ solutions to the 6th order dispersion relation.
<table>
<thead>
<tr>
<th>$m$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>$\geq 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simply supported</td>
<td>$\pi$</td>
<td>$2\pi$</td>
<td>$3\pi$</td>
<td>$m\pi$</td>
</tr>
<tr>
<td>Free-free or clamped-clamped</td>
<td>4.730</td>
<td>7.853</td>
<td>10.996</td>
<td>$\frac{2m+1}{2}\pi$</td>
</tr>
<tr>
<td>Clamped-simply supported</td>
<td>3.927</td>
<td>7.069</td>
<td>10.210</td>
<td>$\frac{4m+1}{4}\pi$</td>
</tr>
<tr>
<td>Clamped-free</td>
<td>1.875</td>
<td>4.694</td>
<td>7.855</td>
<td>$\frac{2m-1}{2}\pi$</td>
</tr>
</tbody>
</table>

Table 2
Approximate values of $\kappa_mL$ for some simple boundary conditions.

Fig. 7. Different values of the apparent bending stiffness of the sandwich beam specified in Table 3, depending on boundary conditions. o, simply supported ends; □, clamped ends; Δ, free ends; ⋯, $\kappa_1$-method.

Now, by calculating the eigenfrequencies $\omega_m$ of a given sandwich beam configuration, a discrete estimate of $D_{app}$ can be obtained from Eq. (48) as

$$D_{app}^{(m)} = \begin{cases} D_{app}^{stat}, & m = 0, \\ \frac{\omega_m^2}{\kappa_m}, & m > 0. \end{cases}$$

(49)

This is consistent with earlier results, see equation (23). In order to obtain a continuous estimate, some type of interpolation scheme can be used. In Fig. 6 piecewise cubic interpolation has been utilized.

In Figure 7, the influence on the apparent bending stiffness of different boundary conditions is presented.
3.2 Apparent loss factor

By using complex valued elasticity and shear moduli the loss factors of the laminates and the core are implemented in the full model. For the Bernoulli-Euler model, however, a frequency dependent total loss factor $\eta_{app}$ must be used. An estimate for this entity can be obtained from the dispersion relation by using the complex valued material parameters. It is intuitive to assume that the apparent loss factor is proportional to the imaginary part of the main wavenumber, as is done in [7], i.e.

$$\kappa_1 = \Re(\kappa_1) \left(1 - \frac{i \eta_{app}}{4}\right) \rightarrow \eta_{app} = -4 \frac{\Im(\kappa_1)}{\Re(\kappa_1)}. \ (50)$$

The result is represented by the solid curve in Figure 8. Other possible ways of obtaining estimates of the apparent loss factor include analysis of measured or calculated transfer functions or reverberation time – the half-power bandwidth method was used to obtain estimates of the apparent loss factors for different boundary conditions, see Figure 8. As could be expected, clamped end conditions seem to imply a higher apparent loss factor than the simply supported case, and free end conditions seem to imply a lower loss factor.

The dashed curve represents the loss factor estimate given by Eq. (23), as the ratio of the imaginary part to the real part. The prediction agrees well with the higher order theory, for simply supported beams. As mentioned earlier, the denominator of each mode amplitude can be written as $\omega_n^2 - \omega^2$, where $\omega_n$ is obtained from Eq. (21) or indirectly Eq. (23). Using complex material
parameters, the imaginary part of \( \omega_n^2 \) will be proportional to the effective loss factor \( \eta_{\text{app}} \). Hence, it seems justified to use Eq. (23) with complex-valued material parameters to describe the forced motion of lightly damped, simply supported sandwich beams.

Losses can now be implemented in the simplified model by assuming a complex apparent bending stiffness \( D_{\text{app}} = \Re(D_{\text{app}})(1 + i\eta_{\text{app}}) \). The loss factors of the aluminium laminates were assumed to be approximately equal to \( 10^{-3} \) while for the plastic foam core a value of 0.04 was used. The latter value was obtained by comparing the results of the full model to measured data.

### 3.3 Implementing the apparent bending stiffness

Obtaining transfer function and mode shape estimates using the 6th order theory involves complicated numerical procedures. In contrast, the “equivalent” Bernoulli-Euler theory is simple to implement. For example, the deflection of a beam subject to some force distribution \( F' \) per unit length is given by [3]

\[
w(x, t) = e^{i\omega t} \sum_{m=0}^{\infty} \frac{\langle F', \phi_m \rangle \phi_m}{b(D\kappa_m^4 - i\mu\omega^2)\langle \phi_m, \phi_m \rangle},
\]

where \( \phi_m \) are the eigenfunctions and \( \kappa_m \) the eigenvalues corresponding to the particular boundary conditions, \( b \) is the width of the beam and \( \langle \cdot, \cdot \rangle \) is the inner product operator. For a sandwich beam element, the bending stiffness \( D \) is replaced by the frequency dependent \( D_{\text{app}}(\omega) \) or the discrete \( D_{\text{app}}^{(m)} \). The latter option is preferable in a mechanical viewpoint, since using a constant \( D_{\text{app}} \) for each mode implies a better description of the width of the resonance peaks. However, the frequency dependent estimate may be simpler to implement in various applications.

It should be noted that using a constant value of the bending stiffness for each mode is not the same thing as to let all modes share the same frequency dependent bending stiffness. A close approximation to the mode-discrete case could be achieved by implementing a piecewise constant bending stiffness curve, where the bending stiffness is constant in a region around each eigenfrequency, allowing the width of the resonance peaks to be correctly described locally. This could be what the error-minimizing numerical algorithm is trying to show in Fig. 6. However, implementing a discontinuous bending stiffness would also imply discontinuous mobility curves, phase velocities and so on.
4 An equivalent Timoshenko beam approach

As stated in the previous section, the greatest disadvantage of the equivalent Bernoulli-Euler beam is the need for a boundary condition-dependent estimate of the apparent bending stiffness. By instead utilizing an equivalent Timoshenko beam, where the $\kappa_3$ wavenumber is discarded, it is possible to find frequency dependent bending stiffness and shear modulus parameters such that the Timoshenko wavenumbers coincide with $\kappa_1$ and $\kappa_2$. This effectively means dividing the $6^{th}$ order sandwich dispersion relation by $k^2 + \kappa_3^2$, which yields a reduced $4^{th}$ order dispersion polynomial. This approach assumes that the effects of the nearfields associated with $\kappa_3$ are negligible.

Consider a $4^{th}$ order polynomial of even powers in $k$, with roots equal to the wavenumbers $\pm \kappa_1$ and $\pm i \kappa_2$ of the $6^{th}$ order sandwich theory (see Fig. 3 for frequencies well below $\omega_{\text{tot}}$):

\begin{equation}
(k - \kappa_1)(k + \kappa_1)(k - i \kappa_2)(k + i \kappa_2) = k^4 + (\kappa_2^2 - \kappa_1^2)k^2 - \kappa_1^2 \kappa_2^2. \tag{52}
\end{equation}

This expression is compared to the dispersion relation of a Timoshenko beam, which is obtained from Eq. (13) by setting $D_1 = D_2 = 0$. Assuming the rotational inertia $I_{\text{tot}}$ is negligible, we obtain

\begin{equation}
k^4 - \frac{\mu_{\text{tot}} \omega^2}{G_{\text{app}} T} k^2 - \frac{\mu_{\text{tot}} \omega^2}{D_{\text{app}} T} = 0, \tag{53}
\end{equation}

where $D_{\text{app}}^T$ and $G_{\text{app}}^T$ are the apparent bending stiffness and shear modulus. Now, identifying the coefficients of the polynomials yields

\begin{equation}
D_{\text{app}}^T(\omega) = \frac{\mu_{\text{tot}} \omega^2}{\kappa_1 \kappa_2}, \tag{54}
\end{equation}

\begin{equation}
G_{\text{app}}^T(\omega) = \frac{\mu_{\text{tot}} \omega^2}{h_c (\kappa_1^2 - \kappa_2^2)}. \tag{55}
\end{equation}

Thus, by calculating $\kappa_1$ and $\kappa_2$ from Eq. (13) with $I_{\text{tot}} = 0$, we can obtain the equivalent bending stiffness and shear modulus parameters for a Timoshenko beam with arbitrary boundary conditions. As can be seen in Fig. 9, the equivalent shear modulus $G_{\text{app}}^T$ is approximately constant and equal to $G_c$ for low frequencies, while at the high frequency end of the spectrum it increases rapidly. This is due to the fact that in ordinary Timoshenko theory, the high frequency asymptote of the main propagating wavenumber denotes a shear wave, not a flexural wave as is the case for the $6^{th}$ order sandwich theory. It follows that the dynamic shear modulus will yield an apparent $\kappa_1$ wavenumber equal to that of the higher order theory.

Analogously, the high frequency asymptote of the nearfield wavenumber is a constant and depends on the ratio of $G_{\text{app}}^T$ to $D_{\text{app}}^T$. Thus, the dynamic
Fig. 9. Equivalent normalized Timoshenko bending stiffness and shear modulus for the sandwich beam specified in Table 3. The dotted line represents $f = f_{\text{mid}}$. 

bending stiffness $D_{\text{app}}^T$ will display the same high-frequency behaviour as $G_{\text{app}}^T$, see Fig. 9. The low frequency asymptote converges towards the static pure bending stiffness $D_{\text{tot}}$. 

Asymptotically, the ratio of $G_{\text{app}}^T$ to $D_{\text{app}}^T$ can be shown to be approximately equal to $G_c/D_{\text{tot}}$ when $\omega \to 0$ as well as when $\omega \to \infty$. In the mid-frequency range, the ratio depends on the behaviour of $\kappa_1$ and $\kappa_2$. An approximate explicit expression for the normalized apparent bending stiffness and shear modulus can be given as 

$$\frac{D_{\text{app}}^T(\omega)}{D_{\text{tot}}} \approx \frac{G_{\text{app}}^T(\omega)}{G_c} \approx \begin{cases} 1, & \omega \ll \omega_{\text{mid}} \\ \sqrt{\frac{\mu_{\text{tot}}(D_1 + D_2)}{G_c h_c}} \omega, & \omega \gg \omega_{\text{mid}} \end{cases}$$  \quad (56) 

where $\omega_{\text{mid}}$ is a limit frequency defined by the intersection of the asymptotes, 

$$\omega_{\text{mid}} = \frac{G_c h_c}{\sqrt{\mu_{\text{tot}} (D_1 + D_2)}}$$  \quad (57) 

Eigenfrequencies can be calculated from standard Timoshenko beam formulae using iterative methods, see for example [14]. The results are shown in Figs. 10 and 11, from which it is clear that the ordinary constant-parameter Timoshenko formula deviates from the value yielded by the higher order theory, while the agreement between the dynamic parameter model and the higher order theory is satisfactory over the entire considered range. For the simply supported beam, the resonance frequencies corresponding to the 6th order theory were obtained from Eq. (21). In the case of the cantilever beam, the resonance frequencies were found by calculating the roots of the determinant Eq. (19). In the latter case, numerical instability occurred above the 48th mode, which is why the curve is truncated.
Fig. 10. Eigenfrequencies versus mode number for a simply supported beam – comparison between different models. ——, 6th order model; - - ordinary Timoshenko theory; - - , modified Timoshenko theory; · · ·, Bernoulli-Euler theory.

Fig. 11. Eigenfrequencies versus mode number for a clamped-free (cantilever) beam – comparison between different models. ——, 6th order model; - - ordinary Timoshenko theory; - - , modified Timoshenko theory; · · ·, Bernoulli-Euler theory.

5 Measurements and validation

In order to validate the models presented in the previous sections, measurements on an asymmetric sandwich beam with both ends free were performed. Material parameters and dimensions are given in Table 3. Using an accelerometer and an impact hammer connected to a two-channel signal analyzer, transfer accelerance functions were obtained for a number of different positions of the accelerometer and excitation point. These results were then compared to the predictions of the different models.
## Table 3

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L$</td>
<td>1.2 m</td>
</tr>
<tr>
<td>$h_1$</td>
<td>0.75 mm</td>
</tr>
<tr>
<td>$h_2$</td>
<td>2 mm</td>
</tr>
<tr>
<td>$h_c$</td>
<td>10.2 mm</td>
</tr>
<tr>
<td>$E_1, E_2$</td>
<td>70 GPa</td>
</tr>
<tr>
<td>$E_c$</td>
<td>130 MPa</td>
</tr>
<tr>
<td>$G_c$</td>
<td>45 MPa</td>
</tr>
<tr>
<td>$\eta_1, \eta_2$</td>
<td>$10^{-3}$</td>
</tr>
<tr>
<td>$\eta_c$</td>
<td>$4 \times 10^{-2}$</td>
</tr>
<tr>
<td>$\rho_1, \rho_2$</td>
<td>2700 kg/m$^3$</td>
</tr>
<tr>
<td>$\rho_c$</td>
<td>74 kg/m$^3$</td>
</tr>
</tbody>
</table>

In addition, comparisons between the 6th order model (referred to as "Full model") and the equivalent Bernoulli-Euler model (referred to as "BE-model") were made for simply supported and clamped boundary conditions. For these end conditions, no measured data was obtained due to practical considerations.

### 5.1 Measurement setup

The measurement setup consisted of a Brüel & Kjær type 4393 accelerometer (weighting 2.4 grams) and excitation hammer connected via signal amplifiers to a Hewlett-Packard 3562A signal analyzer, and an asymmetric sandwich beam (see Table 3) suspended in one end by means of an elastic rubber band, imposing approximate free-free boundary conditions (see Fig. 12). The sandwich beam consisted of two aluminium laminates bonded to a plastic foam core. In the HP 3562A analyzer, the 2048-point time record is Fourier transformed into a frequency resolution of 801 lines, from 2 Hz to 1252 Hz with a stepsize of 1.56 Hz. An exponential window was applied to the captured signals. The measurements were performed in the MWL laboratory at KTH, at normal indoor conditions.
5.2 Error sources

The presence of different types of measurement error may influence the validity of the results. For a detailed analysis of this problem, see for example [15] and [16]. Some of the possible error sources include the influence of the added mass of the accelerometer, fluid loading due to surrounding air, low signal-to-noise ratio due to low excitation force impulse or if the accelerometer is close to a node point, possible nonlinear behaviour due to high excitation force impulse and non-ideal boundary conditions. Some of these errors are easily detected; for example, the coherence function – which is obtained for each measurement – will indicate poor quality if the signal-to-noise ratio is low or if nonlinear phenomena are present. The influence of the added mass of the attached accelerometer could potentially be important, especially close to resonance frequencies since the apparent mass of the beam is very low in these regions. It is assumed that the influence of the utilized low-weight accelerometer is negligible in the considered frequency range.

Possible effects of the rubber band supporting the beam are assumed to be negligible. Only free-free boundary conditions were implemented due to the difficulty of achieving other types of configurations, i.e. clamped, simply supported etc.
Fig. 13. Transfer acceleration of a free-free beam, $x_1 = 0.19$ m and $x_2 = 0.1$ m. $---$, measured data; $- -$, 6th order model; $\cdots$, modified Bernoulli-Euler theory.

5.3 Results

Complex transfer acceleration functions were obtained for a number of different combinations of excitation / response coordinates (denoted by $x_1$ and $x_2$). The magnitudes of the transfer functions were then plotted together with the predictions yielded by the 6th order theory and the apparent bending stiffness model. As can be seen in Figure 13, both models provide reasonably accurate predictions of the transfer acceleration of a free-free sandwich beam. The higher order theory provides near excellent agreement in most of the frequency region considered, while the Bernoulli-Euler theory often fails between the resonances. However, in an acoustic viewpoint this is of no great concern since the resonance peaks contain most of the kinetic energy and thus are of greater importance. The measurement quality is indicated in the coherence plot shown in Figure 14.

6 Model comparisons

Due to the previously mentioned problems of obtaining results for other than free end conditions during measurements, the special cases of simply supported and clamped conditions were analyzed by comparing the results of the 6th order model and the modified Bernoulli-Euler model. As can be seen in the following figures the models agree well, indicting that the simple equivalent Bernoulli-Euler beam theory could be implemented also for these types of boundary conditions. The loss factor estimate utilized in the calculation of the acceleration of the clamped beam was obtained from the main propagat-
Fig. 14. Coherence of transfer acceleration measurement, $x_1 = 0.19$ m and $x_2 = 0.1$ m.

Fig. 15. Predicted transfer acceleration, simply supported ends, $x_1 = 0.19$ m, $x_2 = 0.1$ m. The peak levels in the high-frequency region are better described using the loss factor estimate obtained from Eq. (23). —, 6th order model; - -, modified Bernoulli-Euler theory, loss factor from Eq. (23); ---, modified Bernoulli-Euler theory, loss factor from $\kappa_1$.

If the wavenumber of the higher-order dispersion relation, as described earlier. It could be argued that some other estimate should be used; a lower value of $\eta_{app}$ in the upper half of the considered frequency region — as indicated in Fig. 8 — would improve the agreement of the models. In Fig. 15, curves obtained using the two different estimates of $\eta_{app}$ are displayed. Note that the peaks of the curve obtained from Eq. (23), using the frequency dependent bending stiffness $D_{app}(\omega)$, are narrower than those of the higher order theory — see Figure 16; this phenomenon can be understood by considering the frequency response of a simple mass-spring system, where the spring “constant” is frequency de-
Fig. 16. Detail of Figure 15, showing the 8th resonance peak. —, 6th order model; - - , modified Bernoulli-Euler theory, loss factor from Eq. (23); ..., modified Bernoulli-Euler theory, loss factor from $\kappa_1$.

Fig. 17. Predicted transfer acceleration, clamped ends, $x_1 = 0.19$ m, $x_2 = 0.1$ m. —, 6th order model; - - , modified Bernoulli-Euler theory

dependent. By instead using the discrete $D_{\text{app}}^{(m)}$ set of bending stiffnesses – each corresponding to a mode – the problem with narrow peaks is avoided.

The band-average mobility levels – defined in Eq. (58) – of a simply supported sandwich beam configuration are shown in Fig. 18, indicating again good agreement between the higher order model and the modified Bernoulli-Euler model. The apparent bending stiffness estimate was obtained from Eq. (23). The band-average mobility levels are given by

$$L_Y^{(m)} = 10 \log_{10} \left\{ \frac{1}{\omega_m^{(m)} - \omega_m^{(m+1)}} \int_{\omega_m}^{\omega_m^{(m+1)}} \frac{Y^2(\omega) d\omega}{Y_{\text{ref}}^2} \right\},$$

(58)
Fig. 18. Predicted mobility levels, simply supported ends, $x_1 = 0.19$ m, $x_2 = 0.1$ m. 
---, 6th order model; - - -, modified Bernoulli-Euler method using Eq. (23).

where $m$ is the band number, $\omega_m$ is the lower limit frequency of band $m$ and $Y_{ref} = 10^{-3}$ m/Ns is a reference mobility value.

7 Conclusions

A 6th order model has been derived which shows excellent agreement with measured data in the frequency range considered. This model, obtained by applying Hamilton’s principle, takes into account the shear deformation of the core and the effects of rotational inertia. Obtaining eigenfrequencies, mode shapes and deformations due to external exciting forces using this method implies solving large matrix systems for each frequency line considered. The calculations were performed on a standard PC using the numerical software package Matlab.

In addition, the possibility of using modified lower order methods – such as the Bernoulli-Euler or Timoshenko beam theories, in combination with frequency dependent parameters – to calculate the flexural response of sandwich beams subject to different loading and end conditions has been evaluated. The models have been verified by transfer accelerance measurements on a freely suspended asymmetric sandwich beam with aluminium laminates and a plastic foam core, indicating good agreement.
7.1 Conditions for obtaining satisfactory results using the apparent bending stiffness approach

The following items should be carefully considered in order to obtain satisfactory results using the apparent bending stiffness approach:

- The influence of rotational waves should be considered negligible. This condition is satisfied if \( \omega \ll \omega_{\text{rot}} \), where \( \omega_{\text{rot}} \) is defined in Eq. (14).
- The laminates should move in phase, i.e., the frequency should be far below the mass-spring-mass frequency of the sandwich beam.
- The laminates should be thin in comparison to the flexural wavelength, so that thin-beam theory is applicable. This condition is satisfied if \( h_j \kappa_1 \ll 1 \), where \( j = 1, 2 \) and \( \kappa_1 \) is the wavenumber corresponding to propagating flexural waves.
- In general, there might be problems close to junctions and discontinuities, due to the description of the nearfields.
- As the modes share the same frequency dependent bending stiffness, the issue of modal overlap could arise when consecutive eigenfrequencies become too close. This could be a complicating factor, especially when modelling plates using the apparent bending stiffness technique.
- In order to avoid narrowing of the resonance peaks – and thus an underestimation of the kinetic energy of the structure – the gradient \( \frac{dD_{\text{app}}}{d\omega} \) of the apparent bending stiffness should be limited.

References


[12] D. Backström: Modelling the Flexural Dynamics of Sandwich Beams using Bernoulli-Euler or Timoshenko Theory with Frequency Dependent Parameters, TRITA-AVE 2004:45, ISSN 1651-7660


31
Estimation of the material parameters of a sandwich beam from measured eigenfrequencies

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Abstract

A method for the estimation of selected material properties of a sandwich beam using measured eigenfrequencies has been developed. The approach utilizes 6th order theory in order to fit the input variables – the laminate elasticity modulus and the core shear modulus – to measured eigenfrequencies for free-free boundary conditions. The geometric properties of the beam are assumed to be known, as well as the total mass density. The commercial numerical software package Matlab is used to minimize the defined error function, although any program capable of multi-dimensional, unconstrained nonlinear optimization should be applicable.

Key words: sandwich, beam, material parameters, estimation, eigenfrequencies

1 Introduction

Prior to analyzing the dynamic behaviour of any given system, the mechanical properties of the system need to be identified. In the case of flexural vibration of thin, homogenous and prismatic beams, the necessary parameters are the bending stiffness, or indirectly the elasticity modulus of the beam material, and the mass density of the beam. For thick beams, also the effective shear modulus and sometimes the mass moment of inertia are needed. In comparison, sandwich structures comprise a three-layered composite where the thick and relatively soft core is bonded on each side to thin and stiff laminates.

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This inhomogenous structure needs more material parameters in order to correctly describe its dynamic behaviour when subject to flexural deformation. The faces, being relatively thin and stiff, could be modelled using beam or plate theory such as Bernoulli-Euler or the corresponding Kirchhoff theory, or, in case shear and rotational inertia effects need to be included in the calculations, by Timoshenko or Mindlin theory. The thick and soft core must in the extreme case be modelled using the generalized wave equation, which is capable of capturing all the mechanisms of deformation in the linear range, see for example [1]. This is however a cumbersome approach involving large amounts of fairly advanced mathematics and numerics, see [2]. As an alternative, models derived using Hamilton’s principle provide relatively simple yet in most aspects accurate means of predicting the dynamic response of sandwich structures. In this article, a 6th order model previously derived in [3] is implemented in order to compute the response – i.e. eigenfrequencies – of symmetric sandwich beams. By defining an error function to be minimized, the problem of finding the set of material parameters which yields the best prediction can be solved utilizing some unconstrained nonlinear optimization software such as Matlab. This is similar to what has been done in for example [4] and [5]. The approach is possible provided the boundary conditions of the beam are known – localized analysis utilizing directly the governing differential equations is an alternative, see [6].

For vibro-acoustical purposes, the frequency range of interest is typically up to a few kilohertz. In this range, most sandwich structures deform uniformly over the cross-section, implying higher order theory capable of resolving anti-phase motion is not necessary – an important exception is fluid-loaded plates found in shipbuilding applications. Thus, the material parameters needed for the full description of the vibration of symmetric beams and symmetric, non-orthotropic plates are the in-plane elasticity modulus and effective shear modulus of the core, and the elasticity modulus of the faces. Often, the elasticity modulus of the core is not important in comparison with the other parameters, reducing the minimization problem to two independent variables.
Symbol | Description
--- | ---
$L$ | the length of the sandwich beam [m]
$h_c$ | the thickness of the core [m]
$h_1, h_2, h_{\text{lam}}$ | the thicknesses of the laminates [m]
$d$ | distance between the centroids of the laminates [m]
$\rho_c$ | the core density [kg/m$^3$]
$\rho_1, \rho_2$ | the laminate densities [kg/m$^3$]
$\mu, \mu_{\text{tot}}$ | the mass p. u. a. of the beam [kg/m$^2$]
$G_c$ | the core shear modulus [Pa]
$\eta_c$ | the core loss factor [-]
$E_c$ | the core elasticity modulus [Pa]
$E_1, E_2, E_{\text{lam}}$ | the laminate elasticity moduli [Pa]
$D_{\text{tot}}$ | the total bending stiffness p. u. w. of the beam [Nm]
$D_1, D_2, D_{\text{lam}}$ | the bending stiffnesses p. u. w. of the laminates [Nm]
$\omega$ | angular frequency [Hz]
$f$ | period frequency [Hz]
$\gamma_n$ | eigenvalues [-]
$\alpha_n$ | correction factors [-]

Table 1
Definition of some of the most important parameters used in this article. p. u. w. is the abbreviation of per unit width and p. u. a. is per unit area.

![Diagram of a sandwich beam](image)

Fig. 1. An asymmetric sandwich beam.

2 Utilized sandwich model

2.1 Governing equations

As already mentioned, a 6th order model previously derived in [3] constitutes the foundation of the material parameter-finding algorithm. Consider a sandwich
beam of unit width and length \(L\), with laminate and core thicknesses \(h_j\) and \(h_c\), Young’s modulii \(E_j\) and \(E_c\), mass per unit area \(\mu_{\text{tot}}\) and with a core shear modulus \(G_c\), as shown in Fig. 1. The governing Newtonian equations, derived by applying Hamilton’s principle, are

\[
(D_1 + D_2) \left( \frac{\partial^4 w}{\partial x^4} - \frac{\partial^3 \beta}{\partial x^3} \right) - G_c h_c \left( \frac{\partial^2 w}{\partial x^2} - \frac{\partial \beta}{\partial x} \right) + \mu_{\text{tot}} \frac{\partial^2 w}{\partial t^2} = \left( D_1 + D_2 \right) \left( \frac{\partial^3 w}{\partial x^3} - \frac{\partial^2 \beta}{\partial x^2} \right) - G_c h_c \left( \frac{\partial w}{\partial x} - \beta \right) \tag{1}
\]

\[- p = 0, \]

\[- D_{\text{tot}} \frac{\partial^2 \beta}{\partial x^2} + (D_1 + D_2) \left( \frac{\partial^3 w}{\partial x^3} - \frac{\partial^2 \beta}{\partial x^2} \right) - G_c h_c \left( \frac{\partial w}{\partial x} - \beta \right) \tag{2}
\]

\[+ I_{\text{tot}} \frac{\partial^2 \beta}{\partial t^2} = 0, \]

where \(w\) is the total lateral deflection of the beam, oriented along the \(x\)-axis, \(\beta\) is the angle of pure bending, \(D_{\text{tot}}\) is the total bending stiffness per unit width of the sandwich beam, \(D_1\) and \(D_2\) are the bending stiffnesses per unit width of the faces, \(G_c\) is the effective core shear modulus and \(I_{\text{tot}}\) is the mass moment of inertia per unit width. The angle of pure shear deformation \(\gamma\) can be obtained from the geometrical condition \(u_x = \beta + \gamma\).

The effects of rotational inertia are negligible for many sandwich applications as pointed out in [7], hence \(I_{\text{tot}}\) can be set to zero without introducing any significant error. Note that if \(D_1, D_2 \to 0\) the model converges towards Timoshenko theory. The total bending stiffness \(D_{\text{tot}}\) of the sandwich beam is

\[D_{\text{tot}} = c_1 E_1 + c_2 E_2 + c_c E_c, \tag{3}\]

where the \(c\) coefficients are given by

\[c_1 = - y_0 h_1^2 + y_0^2 h_1 + \frac{1}{3} h_1^3, \]

\[c_2 = \frac{1}{3} \left( \left( h_1 + h_c + h_2 \right)^3 - \left( h_1 + h_c \right)^3 \right) + y_0^2 h_2 - y_0 \left( h_1 + h_c + h_2 \right)^2 - \left( h_1 + h_c \right)^2, \]

\[c_c = - y_0 \left( \left( h_1 + h_c \right)^2 - h_1^2 \right) + y_0^2 h_c + \frac{1}{3} \left( h_1 + h_c \right)^3 - h_1^3, \]

and the neutral layer coordinate \(y_0\) is

\[y_0 = \frac{h_1^2 E_1 + \left( 2h_1 + h_c \right) E_c h_c + \left( 2h_1 + 2h_c + h_2 \right) E_2 h_2}{2 \left( E_1 h_1 + E_c h_c + E_2 h_2 \right)}. \]

The following approximation for symmetric beams \((D_1 = D_2 = D_{\text{lam}})\) can be made if the laminates are thin and the core weak, see [8]:

\[D_{\text{tot}} \approx \frac{E_{\text{lam}} h_{\text{lam}}^2 d^2}{2}. \tag{4}\]
Here, $E_{\text{lam}}$ and $h_{\text{lam}}$ are the elasticity modulus and thickness of the laminates, respectively, and $d$ is the distance between the centroids of the laminates.

2.2 Boundary conditions

The boundary conditions, also obtained by applying Hamilton's principle, are

$$F = -D_{\text{tot}} \frac{\partial^2 \beta}{\partial x^2} + I_{\text{tot}} \frac{\partial^2 \beta}{\partial t^2} \quad \text{or} \quad w = 0, \quad (5)$$

$$M = -D_{\text{tot}} \frac{\partial \beta}{\partial x} \quad \text{or} \quad \beta = 0, \quad (6)$$

and

$$M_s = -(D_1 + D_2) \left( \frac{\partial^2 w}{\partial x^2} - \frac{\partial \beta}{\partial x} \right) \quad \text{or} \quad \frac{\partial w}{\partial x} = 0, \quad (7)$$

where $F$ is the shear force, $M$ is the total bending moment and $M_s$ is the bending moment acting on the laminates due to shear deformation of the core, all per unit width. Now, the dispersion equation is obtained by inserting the ansatzes $w = A e^{i(\omega t - kx)}$ and $\beta = B e^{i(\omega t - kx)}$, where $\omega$ is the angular frequency and $k$ is the wavenumber, into the governing equations and setting the system determinant equal to zero. This yields

$$(D_1 + D_2)D_{\text{tot}} k^6 + \left( G_c h_c D_{\text{tot}} - \omega^2 I_{\text{tot}} (D_1 + D_2) \right) k^4$$

$$- \omega^2 \left( \mu_{\text{tot}} (D_1 + D_2 + D_{\text{tot}}) + G_c h_c I_{\text{tot}} \right) k^2 + \mu_{\text{tot}} \omega^2$$

$$\times \left( \omega^2 I_{\text{tot}} - G_c h_c \right) = 0. \quad (8)$$

There are 3 independent solutions for $k$ which satisfies Eq. (8), denoted $k = \pm \kappa_1$, $\pm i\kappa_2$ and $\pm i\kappa_3$. The $\kappa$ quantities are positive and real valued for loss free systems below the cut-on frequency for rotational waves. If $I_{\text{tot}}$ is set to zero, this frequency limit becomes infinite, implying $\kappa_j \in \mathbb{R}$ for all frequencies.

2.3 General solution

The homogenous solutions to the governing equations are, using complex notation,

$$w(x, t) = \{ A_1 \sin \kappa_1 x + A_2 \cos \kappa_1 x + A_3 e^{\kappa_2 x} + A_4 e^{\kappa_5(x-L)} + A_5 e^{\kappa_3 x} + A_6 e^{\kappa_5(x-L)} \} e^{i\omega t}, \quad (9)$$

$$\beta(x, t) = \{ B_1 \sin \kappa_1 x + B_2 \cos \kappa_1 x + B_3 e^{\kappa_2 x} + B_4 e^{\kappa_5(x-L)} + B_5 e^{\kappa_3 x} + B_6 e^{\kappa_5(x-L)} \} e^{i\omega t}, \quad (10)$$
where the \( A_j \) and \( B_j \) coefficients depend on the boundary conditions of the beam. By inserting these solutions into Eq. (2) and equating the coefficients of \( x \)-depending functions to zero, a relationship between \( A \) and \( B \) is obtained as

\[
B_1 = X_2 A_2, \quad B_2 = X_1 A_1, \quad B_3 = X_3 A_3,
\]

\[
B_4 = X_4 A_4, \quad B_5 = X_5 A_5, \quad B_6 = X_6 A_6,
\]

where

\[
X_1 = -X_2 = \frac{\kappa_1 \{(D_1 + D_2)\kappa_1^2 + G_c h_c\}}{(D_{tot} + D_1 + D_2)\kappa_1^2 + G_c h_c - \omega^2 I_{tot}},
\]

\[
X_3 = -X_4 = -\frac{\kappa_2 \{(D_1 + D_2)\kappa_2^2 + G_c h_c\}}{(D_{tot} + D_1 + D_2)\kappa_2^2 + G_c h_c - \omega^2 I_{tot}},
\]

\[
X_5 = -X_6 = -\frac{\kappa_3 \{(D_1 + D_2)\kappa_3^2 + G_c h_c\}}{(D_{tot} + D_1 + D_2)\kappa_3^2 + G_c h_c - \omega^2 I_{tot}}.
\]

Thus, there are for each unloaded beam element 6 independent coefficients which need to be established from the boundary conditions.

2.4 Eigenfrequencies of a free-free sandwich beam, exact method

Although the eigenfrequencies of a simply supported sandwich beam can be obtained as an explicit expression, in general more elaborate methods need to be used. For a free beam, the following set of boundary conditions are applied to each beam end:

\[
F = -D_{tot} \frac{\partial^2 \beta}{\partial x^2} + I_{tot} \frac{\partial^2 \beta}{\partial \ell^2} = 0,
\]

\[
M = -D_{tot} \frac{\partial \beta}{\partial x} = 0,
\]

\[
M_s = -(D_1 + D_2) \left( \frac{\partial^2 w}{\partial x^2} - \frac{\partial \beta}{\partial x} \right) = 0,
\]

where the two first conditions correspond to zero applied shear force and bending moment. This results in an equation on the form

\[
\mathbf{M}(\omega) \bar{\mathbf{A}} = \bar{\mathbf{0}},
\]

where \( \mathbf{M}(\omega) \) is a 6 by 6 frequency dependent matrix, \( \bar{\mathbf{A}} \) is a column vector containing the \( A_j \) coefficients, and \( \bar{\mathbf{0}} \) is the zero vector. The only non-trivial solutions to this equation is obtained when \( \det \mathbf{M}(\omega) = 0 \), which is the eigenfrequency equation, satisfied by \( \omega = \omega_n \). In the case of free-free boundary
conditions, the system matrix with $I_{\text{tot}}$ set to zero is

$$M(\omega) = \begin{pmatrix}
    0 & -\kappa_1^2 & \kappa_2^2 \\
    -\kappa_1^2 \sin \kappa_1 L & -\kappa_1^2 \cos \kappa_1 L & \kappa_2^2 e^{-\kappa_2 L} \\
    0 & X_2 \kappa_1 & -X_3 \kappa_2 \\
    -X_1 \kappa_1 \sin \kappa_1 L & X_2 \kappa_1 \cos \kappa_1 L & -X_3 \kappa_2 e^{-\kappa_2 L} \\
    -X_1 D_{\text{tot}} \kappa_1^2 & 0 & X_3 D_{\text{tot}} \kappa_2^2 \\
    \kappa_2^2 e^{-\kappa_2 L} & \kappa_3^2 & \kappa_3^2 e^{-\kappa_3 L} \\
    \kappa_2^2 & \kappa_3^2 e^{-\kappa_3 L} & \kappa_3^2 \\
    X_4 \kappa_2 e^{-\kappa_2 L} & -X_5 \kappa_3 & X_6 \kappa_3 e^{-\kappa_3 L} \\
    X_4 \kappa_2 & -X_5 \kappa_3 e^{-\kappa_3 L} & X_6 \kappa_3 \\
    X_4 D_{\text{tot}} \kappa_2^2 & X_5 D_{\text{tot}} \kappa_3^2 & X_6 e^{-\kappa_3 L} D_{\text{tot}} \kappa_3^2 \\
    X_4 D_{\text{tot}} \kappa_2^2 & X_5 D_{\text{tot}} \kappa_3^2 e^{-\kappa_3 L} & X_6 D_{\text{tot}} \kappa_3^2 
\end{pmatrix}$$

(15)

By utilizing some numerical root-finding algorithm like Newton's method, the eigenfrequencies $\omega_n$ of the free beam configuration can be approximated. For reference, this iterative method is reproduced here; see for example [9] for further detail:

$$\omega_n^{m+1} = \omega_n^m - \frac{g(\omega_n^m)}{g'(\omega_n^m)}, \quad g(\omega_n^m) = \det M(\omega_n^m),$$

(16)

where $m$ is the iteration index. The prime denotes differentiation with respect to $\omega_n^m$ and can be estimated from central difference approximations, i.e.

$$g'(\omega_n^m) \approx \frac{g(\omega_n^m + \Delta \omega) - g(\omega_n^m - \Delta \omega)}{2 \Delta \omega},$$

(17)

where $\Delta \omega$ is some well-chosen stepsize, typically $\Delta \omega = 10^{-3}$ Hz. Some initial estimates for the desired eigenfrequencies could be obtained by simply defining in Matlab a vector containing equidistant frequency values, and observing when the sign of $g$ changes. Naturally, the distance between two consecutive values should be small enough so that no roots are lost in the process; for example, if in a given frequency range one could expect to find in the order of magnitude $n$ eigenfrequencies, the number of frequency values in the vector should be at least say $10n$.  

7
An approximate semi-empirical formula for the calculation of the eigenfrequencies of a free sandwich beam can be obtained by assuming a simple relationship for the wavenumber $k$ at resonance, as is done in classical thin-beam theory. Bernoulli-Euler theory results in the following values of the wavenumber at resonance:

$$kL = \gamma_n,$$  (18)

where $\gamma_n$ depends on boundary conditions. In the simplest case, for simply supported beam ends, $\gamma_n = n\pi$, which can be shown to be true also for the 6th order theory. In the general case, however, this approach is not applicable to 6th order theory since the two other wavenumbers cannot be neglected. Still, it can be used to obtain approximations to the eigenfrequencies without the need for defining the large matrix system described earlier. By assuming Eq. (18) is a valid approximation for sandwich beams for large $n$, the following expression yielding reasonably good estimates of the eigenfrequencies can be derived from the Timoshenko dispersion relation:

$$\omega_n^2 \approx \left(\frac{\gamma_n}{L}\right)^4 \frac{D_{\text{tot}}}{\mu_{\text{tot}}} \times \frac{Gc h c L^2}{\frac{\gamma_n^2}{2} D_{\text{tot}} + Gc h c L^2},$$  (19)

where

$$\gamma_n \approx \left(n + \frac{1}{2}\right)\pi$$  (20)

when both ends of the beam are free. It should be stated that Eq. (19) is approximate and valid in the Timoshenko frequency range, i.e., when the effects of bending of the faces are negligible compared to those of the total bending and shear deformations, see [3]. This frequency range is given by

$$0 \leq \omega \ll \frac{Gc h c}{\sqrt{\mu_{\text{tot}} (D_1 + D_2)}},$$  (21)

where the upper limit is denoted $\omega_{\text{Timo}}$. It can be noticed that $\omega_{\text{Timo}} = 2\pi f_{\text{Timo}}$ approximately gives the frequency limit above which bending of the laminates becomes the dominating mechanism of deformation. See Fig. 3 in the next section.

The above results will be further analyzed in Section 3.4 with the purpose of finding simple expressions for the estimation of the beam material parameters.
3 Identification of material parameters

The sought-for material parameters of the sandwich beam – see Eqs. (1) and (2) – are the elasticity modulus or bending stiffness of the identical laminates, i.e. \( D_1 = D_2 = D_{\text{lam}} \), and the effective shear modulus of the core, all other parameters are assumed to be known or of negligible importance. As mentioned earlier, the elasticity modulus of the core is not of direct interest as it does not have a significant influence on the dynamics of typical beams, and is assumed to be related to the core shear modulus by the isotropic relation \( E_c = 2(1 + \nu)G_c \), where \( \nu \) is Poisson’s ratio. Since the eigenfrequencies of the beam are not sensitive the value of \( \nu \) as \( G_c \) is known directly, it is set to a “default” value of \( \nu = 0.3 \). Thus, by defining an error function describing the proximity of the calculated set of eigenfrequencies to the measured, the optimal values of \( E_{\text{lam}} \) and \( G_c \) are found by minimization. Since the calculations involve the determinant of Eq. (15), a numerical approach is utilized. Any stable optimization algorithm for unconstrained multidimensional nonlinear problems should be capable of finding the optimal parameters, and the authors have utilized the built-in Matlab function \textit{fminsearch} for this purpose, see [10]. This particular function is based on the Nelder-Mead method; however, since the error function seems to behave nicely in the considered range, some simpler algorithm like \textit{steepest descent} or the \textit{conjugated gradient method} could probably be used without encountering major problems.

3.1 Defining the error function

A \( q \)-weighted error function is given by the following expression,

\[
e_q (E_{\text{lam}}, G_c) = \left\{ \frac{1}{N} \sum_{n=1}^{N} \left| \frac{\omega_n - \hat{\omega}_n}{\hat{\omega}_n} \right|^q \right\}^{\frac{1}{q}},
\]

where \( \omega_n \) are the calculated eigenfrequencies and \( \hat{\omega}_n \) are the measured eigenfrequencies. Hence, Eq. (22) is a magnitude-average of the relative error. Naturally, instead of angular frequencies, ordinary period frequencies can be used without altering the equation. Also, the question of weighting should be addressed. A value of \( q = 1 \) has been chosen, although this may not be the optimal choice. A value of \( q = 2 \) is often seen in mathematical litterature, i.e. the \textit{least squares method}, but the analytical advantage of continuous derivatives is assuming irrelevant when utilizing computational methods. Also, with \( q = 2 \), erroneous data could have a disproportionately large influence on
the final result. Thus, we simply let

$$
\epsilon = \frac{1}{N} \sum_{n=1}^{N} \left| \frac{\omega_n - \bar{\omega}_n}{\bar{\omega}_n} \right| = \frac{1}{N} \sum_{n=1}^{N} \left| \frac{f_n - \bar{f}_n}{\bar{f}_n} \right|
$$

(23)

where \( f \) are period frequencies. Dividing by \( N \) adds intuitive meaning to the error function (as the average) but is unnecessary in terms of results.

3.2 Stability of the method

The described method of error function minimization could be characterized as an inverse eigenfrequency relation – from a set of eigenfrequencies, the material parameters of the sandwich beam are obtained. The stability of the method is an important property since in practical applications, the data, i.e. the measured eigenfrequencies, would be subject to random and systematic errors. In order to test the stability, 2 typical sandwich structures are defined, their eigenfrequencies numerically calculated from the 6th order theory and then “contaminated” by an artificial error, and finally the inverse method is applied. This approach will produce a measure of the sensitivity of the method to measurement errors. The test sandwiches are defined in Table 2, and correspond to a unit-length beam with mm thick aluminium laminates and a mm thick core with a shear modulus of MPa and MPa, respectively. The mass per unit area of the beams is assumed to be kg/m². Both beam structures have a total bending stiffness per unit width of approximately Nm, and the laminate stiffnesses are Nm. The upper frequency limit of the Timoshenko range is approximately 1.9 kHz for beam 1 and 9.5 kHz for beam 2. Frequencies up to 2 kHz are considered, implying \( N = 23 \) captured resonances for beam 1 and \( N = 13 \) for beam 2. The measurement error is modelled as a zero-average normally distributed absolute error with a given standard deviation. The main error mechanism is assumed to originate from the finite frequency resolution of the utilized analyzer, but also other random processes could potentially be of importance, such as manual excitation of the beam using an impact hammer. Naturally, also bias errors are present during measurements, but these are not considered in the testing of the proposed method since they should have no impact on its stability. Thus, implementing the built-in Matlab finitesearch function with a maximum of 10 iterations, and running the test loop 1,000 times using the original values of \( G_c \) and \( E_{lam} \) as initial guesses and adding a normally distributed error with a 5 Hz standard deviation to the original eigenfrequencies, the “shot groups” shown in Figure 2 were obtained. The dashed lines indicate the desired “bull’s eyes”. Naturally, using another value of the standard deviation would have an impact on the obtained results; however, \( \sigma = 5 \) Hz is a reasonable assumption implying a typical error of in the order of magnitude a few Herz.
<table>
<thead>
<tr>
<th>Parameter</th>
<th>beam 1</th>
<th>beam 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_{\text{lam}}$</td>
<td>70 GPa</td>
<td></td>
</tr>
<tr>
<td>$E_c$</td>
<td>26 MPa</td>
<td>130 MPa</td>
</tr>
<tr>
<td>$G_c$</td>
<td>10 MPa</td>
<td>50 MPa</td>
</tr>
<tr>
<td>$\mu$</td>
<td>6 kg/m$^2$</td>
<td></td>
</tr>
<tr>
<td>$h_{\text{lam}}$</td>
<td>1 mm</td>
<td></td>
</tr>
<tr>
<td>$h_c$</td>
<td>10 mm</td>
<td></td>
</tr>
<tr>
<td>$L$</td>
<td>1 m</td>
<td></td>
</tr>
<tr>
<td>$f_{\text{BE}}$</td>
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<td>498 Hz</td>
</tr>
<tr>
<td>$f_{\text{imo}}$</td>
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<td>9.51 kHz</td>
</tr>
<tr>
<td>$N$</td>
<td>23</td>
<td>13</td>
</tr>
<tr>
<td>$f_1$</td>
<td>77 Hz</td>
<td>90 Hz</td>
</tr>
</tbody>
</table>

Table 2
Test sandwich parameters. $N$ is the number of eigenfrequencies within the considered frequency range, and $f_1$ represents the first eigenfrequency.

Fig. 2. “Shot groups” for test loop with 1,000 runs and a standard deviation of 5 Hz. Eigenfrequencies up to 2000 Hz were considered.

The frequency analyzer used by the authors in the measurement described in section 3.5 has a resolution of 801 frequency lines, which implies a maximum error below three Herz if the range 0–2 kHz is considered. The chosen error distribution can therefore in some sense be considered a “worst case scenario”.

The distribution of the calculated material parameters indicate that the elasticity modulus of the laminates is more difficult to estimate than the core shear modulus. This phenomenon can be explained by inspecting the frequency distribution of the potential energies corresponding to the various mechanisms of deformation. These energy densities were defined in [3] in order to derive the
governing equations briefly described in the previous section. These entities are defined by

\[
U_{\text{bend}} = \frac{1}{2} D_{\text{tot}} \left( \frac{\partial \beta}{\partial x} \right)^2, \\
U_{\text{shear}} = \frac{1}{2} G_c h_c \gamma^2, \\
U_{\text{lam}} = \frac{1}{2} (D_1 + D_2) \left( \frac{\partial \gamma}{\partial x} \right)^2.
\]

(24)

\( U_{\text{bend}}, U_{\text{shear}} \) and \( U_{\text{lam}} \) correspond to pure bending energy of the element, core shear deformation energy and pure bending energy of the laminates due to shear deformation \( \gamma \) of the core, respectively, all per unit length and width. By imposing a unit amplitude displacement wave \( w = 1 e^{-ikx} \) and a corresponding bending angle \( \beta = -ik_Y \), where \( Y \) is an amplitude factor similar to the \( X \) factors introduced earlier, and utilizing geometrical relationship between the deformation variables \( w, \beta \) and \( \gamma \), the time averages of these energy densities can be expressed as functions of frequency as

\[
U_{\text{bend}} = \frac{\kappa_1^4}{4} D_{\text{tot}} Y^2, \\
U_{\text{shear}} = \frac{\kappa_2^4 G_c h_c}{4} (1 - Y)^2, \\
U_{\text{lam}} = \frac{\kappa_3^4}{4} (D_1 + D_2) (1 - Y)^2,
\]

(25)

where the \( Y \) parameter is obtained from inserting the ansatzes for \( w \) and \( \beta \) into Eq. (2), yielding (neglecting the effects of rotational inertia)

\[
Y = \frac{(D_1 + D_2) \kappa_2^4 + G_c h_c}{(D_{\text{tot}} + D_1 + D_2) \kappa_3^4 + G_c h_c}, \quad Y \in [0, 1].
\]

(26)

In Figure 3 these energy densities, normalized with respect to the total potential energy, are plotted versus frequency for the two beam configurations. As can be seen, shear represents the dominating deformation mechanism in the frequency range of interest – the main part of the eigenfrequencies are located in the range dominated by shear energy. This could explain why the shear modulus is smaller to estimate compared to the bending stiffness of the laminates. As was suggested in the previous section, \( f_{\text{Tim}} \) could be used as an upper frequency limit for the shear range. For the lower limit, a simple explicit estimate can be obtained by comparing Bernoulli-Euler and Timoshenko theory. By finding the intersection between the Bernoulli-Euler wavenumber and the high frequency asymptotic of the main Timoshenko wavenumber, the
Fig. 3. Normalized energy densities for beams 1 and 2. —, \( \bar{U}_{\text{bend}} \), bending of the entire section; ••, \( \bar{U}_{\text{shear}} \), shear of the core; •••, \( \bar{U}_{\text{lam}} \), bending of the laminates.

The following expression is obtained:

\[
\left( \frac{\mu \varepsilon_{\text{BE}}^2}{D_{\text{tot}}} \right)^{\frac{1}{2}} = \sqrt{\frac{\mu}{G_c h_c}} \omega_{\text{BE}} \Rightarrow \omega_{\text{BE}} = \frac{G_c h_c}{\sqrt{\mu D_{\text{tot}}}}
\]  

(27)

where \( \omega_{\text{BE}} = 2\pi f_{\text{BE}} \) could be seen as an upper frequency limit for Bernoulli-Euler theory (note the similarities between this expression and that given for \( f_{\text{Timo}} \)). Now, denote the frequency region dominated by shear deformation \( \Delta_\omega \approx [f_{\text{BE}}, f_{\text{Timo}}] \). With \( f_{\text{BE}} = 100 \text{ Hz} \) for beam 1 and \( 498 \text{ Hz} \) for beam 2, we can see from Fig. 3 that these limits approximately indicate were the region dominated by shear energy is located. Further, the geometric average between \( f_{\text{BE}} \) and \( f_{\text{Timo}} \) indicates with great precision the location of the shear peak. For future reference this frequency is denoted \( f_\gamma \) and is

\[
f_\gamma = \sqrt{f_{\text{BE}} f_{\text{Timo}}}
\]

(28)

By introducing the ratio \( s \) of the number of eigenfrequencies within \( \Delta_\omega \) to the total number of captured eigenfrequencies, it is concluded that most of the eigenfrequencies belong in the shear range, see Tables 3 and 4.

The negative slopes of the data clusters in Fig. 2 indicate that if the laminate elasticity modulus is over-estimated, the core shear modulus will be slightly under-estimated, and vice-versa – this effect is also intuitive since the error in one parameter is compensated by an opposite error in the other in order to keep the resultant total stiffness of the structure constant.

The relative error distributions of the estimated parameters are shown in Figures 4 and 5. For the core shear modulus, the 95 % confidence interval \( c_G \) is approximately ± 2.4 % for beam 1 and ± 1.9 % for beam 2. Analogously, the confidence intervals \( c_E \) for the face elasticity modulus are ± 6.6 % and ± 9.1 %, respectively.
Fig. 4. Probability distribution for the relative error in the prediction of $E_{lam}$ and $G_C$, beam 1.

Fig. 5. Probability distribution for the relative error in the prediction of $E_{lam}$ and $G_C$, beam 2.

In order to evaluate the stability of the method when a more limited number of eigenfrequencies is available, the procedure is repeated with an upper frequency limit of 500 Hz instead of 2 kHz, i.e., only eigenfrequencies up to 500 Hz are considered. The number of captured eigenfrequencies thus decreases to $N = 6$ and $N = 3$ for the respective beam configuration, resulting in an expected severe loss of precision. The confidence intervals for the two beams are presented in Tables 3 and 4.

3.3 A note on the estimation of the loss factor $\eta_c$ of the core

The proposed method relies on eigenfrequency data only and can thus not be used directly to estimate the loss factors of the layers. In order to accomplish this, the error function defined in Eq. (22) needs to be extended with information on the amplitudes of the resonance peaks or their corresponding loss factors. If measured transfer or point acceleration data $\hat{A}$ is available, an
\[
\begin{array}{|c|c|c|}
\hline
f_{\text{max}} & 500 \text{ Hz} & 2000 \text{ Hz} \\
\hline
N & 6 & 23 \\
s & 0.83 & 0.91 \\
c_G & \pm 8.3 \% & \pm 2.4 \% \\
c_E & \pm 40.5 \% & \pm 6.6 \% \\
\hline
\end{array}
\]

Table 3
95\% confidence intervals, beam 1. \( s \) is the ratio of captured eigenfrequencies located within the shear range.

\[
\begin{array}{|c|c|c|}
\hline
f_{\text{max}} & 500 \text{ Hz} & 2000 \text{ Hz} \\
\hline
N & 3 & 13 \\
s & 0.67 & 0.85 \\
c_G & \pm 34.3 \% & \pm 1.9 \% \\
c_E & \pm 25.0 \% & \pm 9.1 \% \\
\hline
\end{array}
\]

Table 4
95\% confidence intervals, beam 2. \( s \) is the ratio of captured eigenfrequencies located within the shear range.

intuitive formulation consistent with Eq. (23) could be

\[
\epsilon = \frac{1}{N} \sum_{n=1}^{N} \left| \left( \frac{f_n/j_n - 1}{A_n/A_n - 1} \right) \right|_2,
\]

(29)

where \( | \cdot |_2 \) denotes the \( l^2 \) norm. As an alternative, measured loss factors obtained from for example the half-bandwidth method could be an utilized if the peak amplitudes are “chopped off” by the limited frequency resolution of the analyzer.

Since for most sandwich structures the core losses dominate, and since the eigenfrequencies of the structure are virtually independent of the losses, the prescribed eigenfrequency-method could be applied to obtain the elasticity modulus of the laminates (assuming the laminate loss factors are negligible) and the real part of the core shear modulus. Then, by letting

\[
G_c = G_c(1 + i\eta_c),
\]

where \( G_c \) is the obtained shear modulus estimate, finding \( \eta_c \) from forced-response data (i.e. measured acceleration functions) should be relatively straightforward; one would only need to find the value of \( \eta_c \) which would make the resonance peak levels of the model coincide with the measured data. The forced response problem is more complicated and involves solving a \( 12 \times 12 \) matrix system for each frequency line considered, see [3]. If however the losses of the
<table>
<thead>
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<th>Parameter</th>
<th>Interval</th>
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</thead>
<tbody>
<tr>
<td>$E_1, E_2$</td>
<td>[50, 200] GPa</td>
</tr>
<tr>
<td>$E_c$</td>
<td>[50, 200] MPa</td>
</tr>
<tr>
<td>$\rho_1, \rho_2$</td>
<td>[2500, 10000] kg/m$^2$</td>
</tr>
<tr>
<td>$\rho_c$</td>
<td>[50, 200] kg/m$^2$</td>
</tr>
<tr>
<td>$G_c$</td>
<td>obtained from $E_c$</td>
</tr>
<tr>
<td>$h_1, h_2$</td>
<td>[0.5, 2] mm</td>
</tr>
<tr>
<td>$h_c$</td>
<td>[10, 50] mm</td>
</tr>
<tr>
<td>$L$</td>
<td>[1, 3] m</td>
</tr>
</tbody>
</table>

Table 5
Material parameter intervals for random sandwich beam generation.

laminates cannot be neglected, the full minimization problem involving four parameters – the real and imaginary parts of the elasticity and shear moduli – needs to be addressed.

3.4 A semi-empirical approach

A quick estimate of say the shear modulus of the core can be obtained from Eq. (19), assuming the elasticity modulus of the faces is known. If the first few eigenfrequencies of a beam are known from experiments, an estimate of the shear modulus is given by solving for $G_c$:

$$G_c \approx G_1 = \frac{\omega_n^2 \gamma_n^2 L^2 \mu_{tot} D_{tot}}{h_c (\gamma_n^4 D_{tot} - \omega_n^4 L^4 \mu_{tot})}, \quad n \geq 3,$$

(30)

where $G_1$ is a first estimate of $G_c$. The estimate obtained from Eq. (30) should be regarded as an engineering approximation and is subject to the restrictions mentioned in section 2.5. In fact, by comparing the results given by Eq. (30) to the original shear modulus, one can notice a systematic overestimation. In order to establish a correction factor for this bias error, 10,000 random sandwich configurations were generated and tested by means of a matlab script. The material parameters and geometry of these beams were picked using rectangular probability distributions with ranges given in Table 5. It was found that for $n = 3$, multiplying the result given by Eq. (30) by 0.81 a better estimate would generally be obtained, see Fig. 6. Similarly, for $n = 4$ the result should be multiplied by 0.85. Thus, a better estimate for the sandwich beam shear modulus $G_c$ is given by $G_2$,

$$G_c \approx G_2 = \alpha_n G_1, \quad n \geq 3,$$

(31)
<table>
<thead>
<tr>
<th>( n )</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_n )</td>
<td>0.81</td>
<td>0.85</td>
<td>0.87</td>
<td>0.89</td>
<td>0.89</td>
</tr>
</tbody>
</table>

Table 6
Approximate correction factors for the estimation of the shear modulus of sandwich beams.

![Correction factor distribution, \( n = 3 \)](image)

Fig. 6. Histogram showing the statistical distribution of the ratio of \( G_c \) to \( G_1 \), for \( n = 3 \). 10,000 random sandwich configurations were used to establish the average correction factor, \( \alpha_3 \approx 0.81 \).

![Intersection of iso-\( \omega \) curves](image)

Fig. 7. Intersection of “iso-\( \omega \)-curves” for the beam specified in section 3.5; —, \( n = 3 \); - - - , \( n = 4 \). The thin dashed lines indicate the values of \( D_{\text{tot}} \) and \( G_c \) given by the full inverse calculation.

where \( \alpha_n \) are correction factors given in Table 6. One obvious application for the formula is as an initial-guess generator for the more complex inverse method. From two eigenfrequencies both the shear modulus \( G_c \) and the total bending stiffness \( D_{\text{tot}} \) and thus indirectly the elasticity modulus of the faces can be estimated by plotting Eq. (31) as a function of \( D_{\text{tot}} \) and finding the intersection of the curves, as shown in Figure 7. Zooming in reveals that the curves do not intersect in the “target” point, but this is expected due to the approximate nature of the method. If the third and fourth eigenfrequencies are known, the material parameters can be estimated by the following explicit
<table>
<thead>
<tr>
<th>Method</th>
<th>Full inverse</th>
<th>Eq. (32)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_c$ [MPa]</td>
<td>37</td>
<td>38</td>
</tr>
<tr>
<td>$E_{\text{lam}}$ [GPa]</td>
<td>62</td>
<td>59</td>
</tr>
<tr>
<td>$\epsilon_m$ [%]</td>
<td>0.34</td>
<td>0.76</td>
</tr>
</tbody>
</table>

Table 7
Predicted material parameters for the beam in section 3.5. $\epsilon_m$ is the mean deviation of the relative error of the 9 calculated eigenfrequencies.

formulae:

\[
G_c \approx \frac{\omega_1^2 \omega_4^2 L^2 \mu_{\text{tot}} (\alpha_3 \gamma_3^2 - \alpha_4 \gamma_4^2)}{h_c (\gamma_3^4 \omega_3^2 - \gamma_4^4 \omega_4^2)}, \\
D_{\text{tot}} \approx \frac{\omega_3^2 \omega_1^2 L^4 \mu_{\text{tot}} (\alpha_3 \gamma_3^2 - \alpha_4 \gamma_4^2)}{\gamma_3^3 \gamma_1^3 (\alpha_3 \gamma_3^2 \omega_3^2 - \alpha_4 \gamma_4^2 \omega_4^2)}. \tag{32}
\]

Taking the inverse of Eq. (4), the elasticity modulus $E_{\text{lam}}$ of the laminates is obtained as

\[
E_{\text{lam}} \approx \frac{2 D_{\text{tot}}}{h_{\text{lam}} d}. \tag{33}
\]

3.5 A real-world example – estimation of material parameters of a honeycomb beam

The first 9 eigenfrequencies of a sandwich beam are estimated from measured transfer accelerance graphs, using the simple peak-picking technique. The symmetric beam consists of two 1 mm thick faces bonded to each side of an 11 mm thick honeycomb paper core. The length of the beam is $L = 0.8 \text{ m}$ and the mass per unit area $\mu_{\text{tot}} = 4.75 \text{ kg/m}^2$. The measured eigenfrequencies are

\[
f_n = \{152, 350, 562, 772, 972, 1172, 1370, 1575, 1770\} \text{ Hz.}
\]

Plotting the error function clearly shows the well-defined minimum, see Fig. 8. The estimated material parameters and the accuracy of the predicted eigenfrequencies are given in Table 7. For this particular beam configuration, Eq. (32) gives a very accurate estimate with an average relative error magnitude of less than 1% for the 9 calculated eigenfrequencies.
Fig. 8. The error function corresponding to the honeycomb beam in section 3.5.

<table>
<thead>
<tr>
<th>Measured [Hz]</th>
<th>62</th>
<th>161</th>
<th>288</th>
<th>428</th>
<th>578</th>
<th>726</th>
<th>876</th>
<th>1026</th>
<th>1172</th>
<th>1321</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shi et al [Hz]</td>
<td>63</td>
<td>161</td>
<td>287</td>
<td>428</td>
<td>572</td>
<td>723</td>
<td>871</td>
<td>1018</td>
<td>1144</td>
<td>1309</td>
</tr>
<tr>
<td>Full inverse [Hz]</td>
<td>62</td>
<td>161</td>
<td>288</td>
<td>430</td>
<td>578</td>
<td>728</td>
<td>878</td>
<td>1026</td>
<td>1174</td>
<td>1320</td>
</tr>
<tr>
<td>Eq. (32)</td>
<td>64</td>
<td>164</td>
<td>291</td>
<td>432</td>
<td>577</td>
<td>723</td>
<td>869</td>
<td>1013</td>
<td>1156</td>
<td>1298</td>
</tr>
</tbody>
</table>

Table 8
Measured and predicted eigenfrequencies of the beam defined in section 3.6.

3.6 Comparing with the Shi-Sol-Hua method

In [5], the core shear modulus $G_c$ and the laminate elasticity modulus $E_{lam}$ of a symmetric PVC-GFRP\(^1\) sandwich beam originally studied in [2] is investigated by means of an inverse approach utilizing a finite element model. The known properties of the beam are the length $L = 1.65$ m, skin and core thicknesses $h_{lam} = 2.5$ mm and $h_c = 50$ mm, and the mass per unit area $\mu_{tot} = 12.95$ kg/m\(^3\). By using the first five eigenfrequencies, Shi et al estimate the core shear modulus to $G_c \approx 45$ Mpa and the laminate elasticity modulus to $E_{lam} \approx 9.2$ GPa. Using the same known data, the proposed inverse method predicts $G_c \approx 51$ Mpa and $E_{lam} \approx 8.6$ GPa. The eigenfrequencies obtained from the different methods are shown in Table 8. If the approximate Eqs. (32) and (33) are used, the obtained estimates become $G_c \approx 48$ Mpa and $E_{lam} \approx 9.2$ GPa. In Table 9 the accuracy of the methods is indicated using the mean deviation, indicating that the method provides a powerful tool for material parameter estimation.

The reasons for the improved results of the proposed inverse method in the mentioned case could be several; first, Shi et al utilize a FEM model, which could potentially introduce numerical error. Further, the error (or objective) functions are defined differently, with the proposed method minimizing the mean of the relative error magnitude (mean deviation) and not the mean of

\(^1\) The sandwich face sheets are made of glass fibre reinforced plastic, while the core is made of PVC.
<table>
<thead>
<tr>
<th>Method</th>
<th>Shi et al</th>
<th>Full inverse</th>
<th>Eq. (32)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_c$ [MPa]</td>
<td>45</td>
<td>51</td>
<td>48</td>
</tr>
<tr>
<td>$E_{\text{lam}}$ [GPa]</td>
<td>9.2</td>
<td>8.6</td>
<td>9.2</td>
</tr>
<tr>
<td>$\epsilon_m$ [%]</td>
<td>0.76</td>
<td>0.20</td>
<td>1.3</td>
</tr>
</tbody>
</table>

Table 9
Predicted material parameters for the beam in section 3.6. $\epsilon_m$ is the mean deviation of the relative error of the 10 calculated eigenfrequencies.

the squares (standard deviation). The latter option could imply that errors in the measured eigenfrequencies have a larger influence on the final results, i.e. the method might be less robust.

4 Conclusions

The possibility of utilizing the 6th order sandwich beam theory presented in [3] in order to estimate the material parameters from measured eigenfrequencies has been investigated. The inverse problem is formulated in terms of the minimization of an error function, which can be achieved by means of computational methods. In this paper, the built-in Matlab fminsearch function was implemented with successful results, although any numerical software capable of unconstrained nonlinear and multidimensional optimization could be used. The core of the error function, the eigenfrequency relation, is on the form of a determinant of a 6 by 6 frequency dependent boundary condition matrix. The roots of this determinant yield the eigenfrequencies, and can be obtained by implementing some simple root-finding algorithm like Newton’s method.

Only symmetric beams can be analyzed using the proposed method, since in the governing equations of the 6th order theory the bending stiffnesses of the face sheets appear in couples as $D_1 + D_2$. Higher order theory would in general be needed to resolve the different material parameters of the faces of an asymmetric beam. However, if the faces are of different thickness but consist of the same material, or if the properties are related in some other known way, the proposed method could be utilized.

As was expected, the elasticity modulus $E_{\text{lam}}$ of the faces of a symmetric sandwich beam was subject to larger error than the shear modulus $G_c$. This is probably due to the fact that the bulk of the captured eigenfrequencies are located in the shear range below $f_{\text{limo}}$. Since measurements are by nature limited in frequency range, and especially when using accelerometers, the resolution of $E_{\text{lam}}$ is typically lower than that of $G_c$.

Also a semi-empirical approach was considered. If the total bending stiffness
of a beam is known, the shear modulus can be estimated from eigenfrequencies – otherwise estimates for both parameters are obtained by finding the intersection of curves corresponding to different eigenfrequencies. This is a graphical approach which can be easily implemented in graphic calculators, while the more complex 6th order inverse method requires advanced numerical calculations.

References


[3] D. Backström: Modelling the Flexural Dynamics of Sandwich Beams using Bernoulli-Euler or Timoshenko Theory with Frequency Dependent Parameters, TRITA-AVE 2004:45, ISSN 1651-7660


Vibration transmission through sandwich composite beam junctions

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Abstract

Energy transmission over various types of junctions where two or more beams are connected is a well-known problem within the field of vibroacoustics, and thin beam-solutions for the attenuation over L-, T- and X-junctions have been published in the literature. The purpose of the present study is to calculate the corresponding properties for sandwich beams. One important application for transmission coefficients is Statistical Energy Analysis, SEA.

Key words: sandwich, beam, energy flow, transmission loss, junctions

1 Introduction

Since the last half-century, sandwich composite materials have played an increasingly important role in the industry. The need for low-weight yet stiff structures has led to the development of various types of composite plates and shells. The aircraft industry is of course a major driving force for the development of lightweight materials and methods for analyzing their behaviour, but also the vehicle and shipbuilding industries rely more and more on composites in order to increase speed, reduce weight and thereby also fuel consumption.

The vibroacoustic properties of sandwich composite materials is a rapidly growing field of research. As sound and vibration become more and more important aspects of product development, it is natural that more research is put into understanding the various acoustical properties of composite materials,

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in addition to the more explored fields of fracture mechanics. Besides from obvious properties, like the sound transmission loss (sound insulation) or modal densities of sandwich panels, also the transmission of vibration across various junctions between sandwich beams is of interest. The resulting transmission coefficients could then be implemented in for example SEA in order to model the acoustical properties of complex structures comprising sandwich elements.

2 Transmission through junctions – an introduction

In classical beam theory, transmission of flexural waves through various types of junctions is a well-known and explored subject investigated in for example [1], [2] and [3]. In the simplest case, a flexural wave is propagating towards an impedance discontinuity in a beam – this discontinuity could be due to a mass being attached to the beam, or to a change in beam properties – resulting in reflected and transmitted propagating and attenuating waves. In order to find the transmission loss, the wave fields to the left and right of the discontinuity are set up and the wave amplitudes are calculated so as to satisfy the basic coupling conditions. Imagine that a simply supported joint is applied to a sandwich beam. The joint allows rotation, but not translation of the beam, at its position. Here, the coupling conditions would imply zero deflection on both sides of the joint, continuity in angle and zero moment around the joint. These four conditions are then used to solve for the four unknown wave amplitudes, the amplitude of the incoming wave being assumed known. The magnitude-square ratio of the amplitude of the transmitted wave to the amplitude of the incoming wave then yields the transmission coefficient \( \tau \), defined as the ratio of transmitted to incoming flexural energy flow,

\[
\tau = \frac{\bar{\Pi}_{\text{trans}}}{\bar{\Pi}_{\text{in}}},
\]  

where the bars denote time averaged entities. The transmission loss, in dB, is in turn given by

\[
R = 10 \log \frac{1}{\tau}. 
\]  

For the mentioned problem, assuming the beam is thin so that Bernoulli-Euler theory is applicable, \( R \) is constant and approximately equal to 3 dB. This solution also approximates the transmission loss of a “free” 90° symmetric L-junction.

The flexural vibration of sandwich beams is described by the 6th order theory derived in [4]. This theory assumes a laterally incompressible core, which is a

\footnote{This is only true if the beam is assumed to have the same properties on both sides of the junction.}
Fig. 1. Homogenous and sandwich composite L-junctions. The solid arrows represent propagating flexural waves, and the dashed arrows represent attenuating flexural waves. The structures are semi-infinite.

valid assumption well below the cut-on frequency for anti-phase motion of the laminates, given by

$$\omega_{\text{rot}} = \sqrt{\frac{G_c h_c}{I_{\text{tot}}}},$$

(3)

where $G_c$ is the core shear modulus, $h_c$ the core thickness and $I_{\text{tot}}$ the mass moment of inertia per unit width of the beam. Further, in order to define the transmission loss of junctions, the energy flow must be expressed in terms of wave amplitudes. Finally, care is needed in order to establish the physically correct coupling conditions.

For junctions where the beams are not aligned, a flexural incoming wave generally results not only in a transmitted flexural wave, but also in a longitudinal wave. This coupling between flexural and longitudinal waves is not considered in this paper, i.e. for such junctions only rotation is allowed, the translational degrees of freedom of the junction itself being assumed "locked".

3 Sandwich beam theory

The mechanisms governing the flexural vibration of a sandwich composite beam are more complicated than those of an ordinary homogenous beam. In a sandwich beam, the shear deformation of the core is coupled to the bending of the laminates, whereas in a homogenous beam shear and bending deformations exist independent of each other. This added coupling term involving the derivative of the shear angle implies the governing differential equation will be of sixth order, see [4] for further details. The 6th order dispersion relation is given by inserting solutions with the time and space dependence given by $e^{i(\omega t - kr)}$ into the Newtonian equations of motion, given in [4], and computing
Fig. 2. Wavenumber magnitudes of a typical sandwich beam. —, \( \kappa_1 \), propagating; - -, \( \kappa_2 \), evanescent; - ·, \( \kappa_2 \), propagating; · · ·, \( \kappa_3 \), evanescent.

the determinant of the resulting algebraic system, yielding

\[
\begin{align*}
(D_1 + D_2)D_{tot}k^6 + (G_c h_c D_{tot} - \omega^2 I_{tot}(D_1 + D_2))k^4 \\
- \omega^2 (\mu_{tot}(D_1 + D_2 + D_{tot}) + G_c h_c I_{tot})k^2 + \mu_{tot}\omega^2 \\
\times (\omega^2 I_{tot} - G_c h_c) = 0,
\end{align*}
\]

where \( k \) is the wavenumber, \( D_1 \) and \( D_2 \) are the bending stiffnesses per unit width of the laminates, \( D_{tot} \) is the total bending stiffness per unit width of the beam, \( G_c \) is the effective shear modulus of the core, \( h_c \) is the core thickness, \( I_{tot} \) is the mass moment of inertia per unit width of the beam cross-section and \( \mu_{tot} \) is the mass per unit area of the beam. The wavenumber magnitudes of a typical beam are plotted in Figure 2. The solutions for the total displacement \( w \) and angle of pure bending \( \beta \) can be written in complex notation as

\[
w(x, t) = \{A_1 e^{-i\kappa_1 x} + A_2 e^{i\kappa_1 x} + A_3 e^{-\kappa_2 x} + A_4 e^{\kappa_2 x} \\
+ A_5 e^{-i\kappa_3 x} + A_6 e^{i\kappa_3 x}\} e^{i\omega t},
\]

and

\[
\beta(x, t) = \{B_1 e^{-i\kappa_1 x} + B_2 e^{i\kappa_1 x} + B_3 e^{-\kappa_2 x} + B_4 e^{\kappa_2 x} \\
+ B_5 e^{-i\kappa_3 x} + B_6 e^{i\kappa_3 x}\} e^{i\omega t},
\]

where \( \omega \) is the angular frequency and \( \kappa_i \) are wavenumber magnitudes, obtained from Eq. (4) as \( k = \pm \kappa_1, \pm i\kappa_2 \) and \( \pm i\kappa_3 \), and \( x \) and \( t \) are the spatial and time coordinates, respectively. The angle of pure shear deformation of the core, \( \gamma \), can be obtained from the geometrical relation

\[
\frac{\partial w}{\partial x} = \beta + \gamma.
\]
Physically, the terms in Eqs. (5) and (6) correspond to left- and right-going propagating and evanescent waves. It can be shown that the $B_i$ coefficients depend on $A_i$, see [4]. The relationship between the wave amplitudes are

$$
B_1 = -i\kappa_1 Y_1 A_1, \quad B_2 = i\kappa_1 Y_1 A_2,
$$
$$
B_3 = -\kappa_2 Y_2 A_3, \quad B_4 = \kappa_2 Y_2 A_4,
$$
$$
B_5 = -\kappa_3 Y_3 A_5, \quad B_6 = \kappa_3 Y_3 A_6,
$$

The $Y_j$ coefficients are given by

$$
Y_1 = \frac{(D_1 + D_2)\kappa_1^2 + G_c h_c}{(D_1 + D_2 + D_{tot})\kappa_1^2 + G_c h_c - \omega^2 I_{tot}},
$$
$$
Y_2 = \frac{(D_1 + D_2)\kappa_2^2 - G_c h_c}{(D_1 + D_2 + D_{tot})\kappa_2^2 - G_c h_c + \omega^2 I_{tot}},
$$
$$
Y_3 = \frac{(D_1 + D_2)\kappa_3^2 - G_c h_c}{(D_1 + D_2 + D_{tot})\kappa_3^2 - G_c h_c + \omega^2 I_{tot}}.
$$

The bending stiffnesses per unit width of the laminates are given by

$$
D_j = \frac{E_j h_j^3}{12},
$$

where $E_j$ is the elasticity modulus and $h_j$ the laminate thickness. The total bending stiffness per unit width of the beam can be written as

$$
D_{tot} = c_0 E_1 + c_1 E_2 + c_2 E_c,
$$

where $E_1$ and $E_2$ are the elasticity moduli of the laminates and $E_c$ is the modulus of the core material. The $c$ coefficients are given by

$$
c_0 = -y_0 h_1^2 + y_0^2 h_1 + \frac{1}{3} h_1^3,
$$
$$
c_1 = \frac{1}{3} \left( (h_1 + h_c + h_2)^3 - (h_1 + h_c)^3 \right) + y_0^2 h_2 - y_0 \left( (h_1 + h_c + h_2)^2 - (h_1 + h_c)^2 \right),
$$
$$
c_2 = -y_0 \left( (h_1 + h_c)^2 - h_1^2 \right) + y_0^2 h_c + \frac{1}{3} \left( (h_1 + h_c)^3 - h_1^3 \right),
$$

where $h_c$ is the thickness of the core. The neutral layer coordinate $y_0$ is

$$
y_0 = \frac{h_1^2 E_1 + (2h_1 + h_c) E_c h_c + (2h_1 + 2h_c + h_2) E_2 h_2}{2 (E_1 h_1 + E_c h_c + E_2 h_2)}.
$$

The mass moment of inertia per unit width of the beam is defined as

$$
I_{tot} = \int_0^H \rho(y)(y - y_0)^2 \, dy, \quad H = h_1 + h_c + h_2,
$$
where \( \rho(y) \) is the density distribution across the cross-section. In most applications, the influence of the mass moment of inertia is negligible and \( I_{\text{tot}} \) can thus often be set to zero. The boundary conditions are given by

\[
F = -D_{\text{tot}} \frac{\partial^2 \beta}{\partial x^2} + I_{\text{tot}} \frac{\partial^2 \beta}{\partial t^2} \quad \text{or} \quad w = 0, \tag{13}
\]

\[
M = -D_{\text{tot}} \frac{\partial \beta}{\partial x} \quad \text{or} \quad \beta = 0, \tag{14}
\]

and

\[
M_s = -(D_1 + D_2) \left( \frac{\partial^2 w}{\partial x^2} - \frac{\partial \beta}{\partial x} \right) = 0 \quad \text{or} \quad \frac{\partial w}{\partial x} = 0, \tag{15}
\]

where \( F \) is the shear force, \( M \) is the total bending moment and \( M_s \) is the bending moment acting on the laminates due to shear deformation of the core, all per unit width. The signs of these entities are defined positive as shown in Figure 3.

### 4 Energy flow in sandwich beams

In order to calculate the transmission loss over various types of junctions an expression for the energy flow through the beam is needed. In classical beam theory, the energy flow per unit width of a flexural wave is given by (see for example [2] or [5])

\[
\bar{\Pi}_{\text{BE}} = \omega D \kappa_{\text{BE}}^3 |A|^2, \tag{16}
\]

where \( D \) is the beam bending stiffness per unit width of the beam, \( \kappa_{\text{BE}} \) is the Bernoulli-Euler wavenumber magnitude and \( A \) is the amplitude of the propagating wave. The Bernoulli-Euler wavenumber magnitude is explicitly given by

\[
\kappa_{\text{BE}} = \sqrt[4]{\frac{\mu \omega^2}{D}}, \tag{17}
\]

with \( \kappa_{\text{BE}} \in \mathbb{R}^+ \) for loss-free systems. For sandwich beams, however, the mechanisms governing the energy flow are more complex. Consider a sandwich beam with negligible rotational inertia \( I_{\text{tot}} \), oriented along the \( x \)-axis. One way of obtaining the energy flow is to multiply the group velocity \( c_g \) with the time average of the total energy per unit length of the beam, \( \bar{E} \), i.e.

\[
\bar{\Pi} = c_g \bar{E}, \tag{18}
\]
Fig. 4. The group velocity of a typical sandwich beam normalized with the group velocity of a homogenous beam with bending stiffness $D_{\text{tot}}$. $f_{\text{Timo}}$ is the upper frequency limit for Timoshenko theory, see [4]. It can be shown that this frequency behaviour is also obtained for the ratio of energy flows.

see for example [1]. The group velocity is obtained from the dispersion relation by differentiating it with respect to $k$ and solving for $\frac{\partial \omega}{\partial k}$,

$$c_g = \frac{\kappa_1 \{3D_{\text{tot}}(D_1 + D_2)\kappa_1^4 + 2G_c h_c D_{\text{tot}} \kappa_1^2 - \mu_{\text{tot}} \omega^2 (D_1 + D_2 + D_{\text{tot}})\}}{\mu_{\text{tot}} \omega \{\kappa_1^2 (D_1 + D_2 + D_{\text{tot}}) + G_c h_c\}}. \quad (19)$$

The result is shown in Figure 4, where the frequency axis has been normalized with respect to the *Timoshenko frequency limit*,

$$f_{\text{Timo}} = \frac{G_c h_c}{2\pi \mu_{\text{tot}} (D_1 + D_2)}. \quad (20)$$

$f_{\text{Timo}}$ indicates the upper frequency limit for the applicability of ordinary Timoshenko theory to sandwich beam vibration, see [4].

The total energy per unit length is given by

$$E = \frac{1}{2} \left\{ D_{\text{tot}} \left( \frac{\partial \beta}{\partial x} \right)^2 + (D_1 + D_2) \left( \frac{\partial \gamma}{\partial x} \right)^2 + G_c h_c \gamma^2 + \mu_{\text{tot}} \left( \frac{\partial w}{\partial t} \right)^2 \right\}. \quad (21)$$

where the three first terms correspond to potential energy density and the last term represents the kinetic energy density. The first term is due to pure bending of the beam, the second term is due to the coupling between the shear deformation of the core and the bending of the laminates, and the third term represents the energy density due to shear deformation of the core. Neglecting the second term yields the energy density of a Timoshenko beam, while also neglecting the third term implies a degeneration of the model into classical Bernoulli-Euler theory.
Consider a right-going propagating wave with amplitude $A$. Omitting the time dependence, we have

$$ w = Ae^{-ik_1x}. $$  \hfill (22)

This wave is composed of two components, $\beta$ due to pure bending and $\gamma$ due to pure shear deformation of the core. From Eq. (8) we have

$$ \beta = -i\kappa_1 Y_1 Ae^{-ik_1x}, $$  \hfill (23)

and, from the geometric relation Eq. (7),

$$ \gamma = -i\kappa_1 (1 - Y_1) Ae^{-ik_1x}. $$  \hfill (24)

Using complex notation, the time average of the square of a harmonic variable is obtained by taking the square absolute value and dividing by two. Thus, we obtain

$$ \bar{E} = \frac{1}{4} \left\{ D_{tot} \left| \frac{\partial \beta}{\partial x} \right|^2 + (D_1 + D_2) \left| \frac{\partial \gamma}{\partial x} \right|^2 + G_c h_c |\gamma|^2 + \mu_{tot} \left| \frac{\partial w}{\partial t} \right|^2 \right\} $$

$$ = \frac{|A|^2}{4} \left\{ D_{tot} \kappa_1^4 Y_1^2 + (D_1 + D_2) \kappa_1^4 (1 - Y_1)^2 + G_c h_c \kappa_1^2 (1 - Y_1)^2 \right\} + \mu_{tot}\omega^2. $$  \hfill (25)

Now, by inserting $Y_1$ as given by Eq. (9) with $I_{tot} = 0$, it can be shown, by using the dispersion relation, that the above expression reduces to

$$ \bar{E} = \frac{\mu_{tot}\omega^2 |A|^2}{2}, $$  \hfill (26)

which is equal to the expression for the total energy per unit length of a Bernoulli-Euler beam, see [2]. It is however important to note that for a finite shear modulus $G_c$, the sum of the instantaneous potential and kinetic energy densities of the 6th order theory is not constant, as is the case for Bernoulli-Euler theory. This is due to the fact that the sandwich displacement wave is a composite wave comprising both shear and bending components. For such waves, the the instantaneous total energy density varies harmonically with $\omega t - kx$, as is seen in Figure 5. The dashed curve represents the total potential energy density $U$. For a displacement wave $w \propto \cos(\omega t - kx)$, the bending part of this energy density is fluctuating as $\cos^2(\omega t - kx)$ while the shear part is proportional to $\sin^2(\omega t - kx)$. The dotted curve represents the kinetic energy density of the beam, which is proportional to $\sin^2(\omega t - kx)$.

Eqs. (19) and (26) can now be used to obtain the energy flow from Eq. (18). This implies that the ratio of group velocities shown in Figure 4 also shows the ratio of energy flows.

A different approach can be used to obtain the same result. By instead considering the force and moment resultants, we can define the time average of
Fig. 5. Energy density components, at a given frequency, of a typical sandwich beam as a function of $\omega t - kx$, for a unit displacement wave. —, total energy density $E$; - - -, potential energy density $U$; · · ·, kinetic energy density $T$. The thin dashed line represents $\bar{E}$, given by Eq. (26).

the energy flow to be equal to

$$\bar{\Pi} = \frac{1}{2} \Re \left( -F \frac{\partial w^*}{\partial t} + M \frac{\partial \beta^*}{\partial t} + M_s \frac{\partial \gamma^*}{\partial t} \right),$$

(27)

where $F$ is the shear force, $M$ is the total bending moment due to pure bending of the beam and $M_s$ is the bending moment of the laminates due to shear deformation of the core, see Eqs. (13) to (15). Using the complex wave definitions given in Eqs. (22) to (24), we obtain

$$\bar{\Pi} = \frac{\omega \kappa_1^2 |A|^2}{2} \left\{ D_{\text{tot}} Y_1 (1 + Y_1) + (D_1 + D_2)(1 - Y_1)^2 \right\}.$$

(28)

From this expression it is easy to see that if we ignore shear deformation, i.e. if we let $Y_1 = 1$ so that the amplitude of $\gamma$ is zero in Eq. (24), we obtain the well-known expression for the energy flow in thin, homogenous beams as given by Eq. (16). Further, since $\kappa_1 \propto \sqrt{\omega}$ in both the low- and high frequency limits, we obtain

$$Y_1 \approx \begin{cases} 1, & \omega \to 0 \\ \frac{D_1 + D_2}{D_1 + D_2 + D_{\text{tot}}}, & \omega \to \infty, \end{cases}$$

(29)

and the low- and high frequency limits of the energy flow thus become

$$\bar{\Pi} \approx \begin{cases} \bar{\Pi}_{BE}|_{D=D_{\text{tot}}}, & \omega \to 0 \\ \bar{\Pi}_{BE}|_{D=D_{\infty}}, & \omega \to \infty, \end{cases}$$

(30)
Fig. 6. A simply supported joint connecting two semi-infinite beams. The solid arrows represent propagating waves while the dashed arrows denote nearfields. 

where $\Pi_{BE}$ is given by Eq. (16) and $D_\infty$ is the high-frequency asymptote of the apparent bending stiffness of the sandwich beam, i.e.

$$D_\infty|_{t_{\text{tot}}=0} = \lim_{\omega \to \infty} \frac{\mu_{\text{tot}} \omega^2}{\kappa_1^4} \approx D_1 + D_2.$$

(31)

Hence, in the low frequency region the energy flow is governed by pure bending of the beam, while in the high frequency region the energy flow is dominated by bending waves propagating in the laminates.

Either Eq. (18) or Eq. (28) can thus be used to calculate the energy flow due to a propagating displacement wave in a sandwich beam.

5 Transmission of energy over junctions

The transmission of mechanical waves through various junctions connecting Bernoulli-Euler beams has been extensively covered in the literature, see for example [2] and [1]. For sandwich beam theory the problem becomes more complex due to the introduction of shear deformation in the core and the resulting bending of the laminates.

As was mentioned in the introduction, the transmission loss $R$ is defined as the ratio of incident to transmitted energy flow, in dB. In Section 4, expressions for the time average of the energy flow in a sandwich beam were derived. In order to calculate the transmission loss of a given junction, the coupling conditions need to be defined.

5.1 The simply supported junction

The simplest beam coupling is a simply supported joint, i.e. a junction where lateral displacement is constrained, allowing only rotation, see Figure 6. For Bernoulli-Euler beams, this joint can be shown to have a frequency independent transmission loss. If the beams are equal, $R \approx 3$ dB. The intuitive
Fig. 7. The frequency independent transmission loss over a simply supported joint connecting two Bernoulli-Euler beams as a function of the thickness ratio. The beams are assumed to be made of the same material.

coupling or boundary conditions are, for Bernoulli-Euler theory,

- zero lateral displacement, i.e. \( w_a = w_b = 0 \),
- continuity in angle, i.e. \( w'_a = w'_b \),
- and zero total bending moment around the joint, i.e. \( -D_a w''_a = -D_b w''_b \),

where subscripts \( a \) and \( b \) denote beam \( a \) or \( b \), \( D \) is the bending stiffness, \( w \) is displacement and \( ' \) denotes differentiating with respect to \( x \). The junction is assumed to be located at \( x = 0 \). The above conditions result in four equations determining the four unknown coefficients of the beam displacement functions:

\[
\begin{align*}
    w_a &= 1e^{-\kappa_{BE}^a x} + A_1 e^{\kappa_{BE}^a x} + A_2 e^{\kappa_{BE}^a x}, \\
    w_b &= B_1 e^{-\kappa_{BE}^b x} + B_2 e^{-\kappa_{BE}^b x},
\end{align*}
\]

(32)

where \( \kappa_{BE}^a \) and \( \kappa_{BE}^b \) are the Bernoulli-Euler wavenumbers for flexural waves in the beams. Note that the incident wave is assumed to be of unit amplitude. By utilizing Eqs. (1), (2), (16) and (17), the transmission loss can now be calculated from the \( B_1 \) coefficient. In Figure 7, the transmission loss of a simply supported joint connecting two beams of different thickness is shown.

For a junction of sandwich beams, the following six coupling conditions are imposed at \( x = 0 \):

- zero lateral displacement, i.e. \( w_a = w_b = 0 \),
- continuity in total angle, i.e. \( w'_a = w'_b \),
- continuity in bending angle \( \beta \), i.e. \( \beta_a = \beta_b \),
- continuity in shear angle derivative, i.e. \( w''_a - \beta'_a = w''_b - \beta'_b \), which is the result of Eq. (15),
Transmission loss over a simply supported junction, sandwich beams

Fig. 8. A typical transmission loss for two identical sandwich beams connected by a simply supported junction. The dashed line indicates the approximate location of the maximum transmission loss, as given by Eq. (37).

- and zero total bending moment around the joint, i.e. \( M_a = M_b \), where \( M \) is given by Eq. (14).

These conditions are sufficient to solve the six unknown wave amplitudes, assuming as before a unit incident wave. The wave fields are

\[
\begin{align*}
    w_a &= 1e^{-i\kappa_a^a x} + A_1 e^{i\kappa_a^a x} + A_2 e^{i\kappa_a^b x} + A_3 e^{i\kappa_a^c x}, \\
    w_b &= B_1 e^{-i\kappa_b^a x} + B_2 e^{-i\kappa_b^b x} + B_3 e^{-i\kappa_b^c x},
\end{align*}
\]

(33)

where \( \kappa^j \) are the wavenumbers for the beams. Inserting the wave fields in Eq. (33) into the boundary conditions yields a set of algebraic equations in the unknown wave coefficients \( A_j \) and \( B_j \). The transmission coefficient \( \tau \) is then obtained from \( B_1 \) as the ratio of the transmitted to incident energy flow. If the beams are identical, this expression reduces to

\[
\tau = |B_1|^2. \tag{34}
\]

The frequency dependent transmission loss for a simply supported joint connecting two typical sandwich beams with identical properties is shown in Figure 8. The result obtained from the 6th order theory shows that in both the low and high frequency regions the transmission loss converges to ordinary beam behaviour, while in the mid frequency region there is a maximum corresponding to some interaction between the deformation mechanisms. This bandstop filtering behaviour cannot be obtained from 4th order theory. The \( B_1 \) coefficient of the transmitted propagating wave, explicitly expressed in terms of the \( Y_j \) factors, becomes

\[
B_1 = \frac{i (\kappa_1 (\kappa_2 - \kappa_3) Y_1 - \kappa_1 \kappa_2 Y_2 + \kappa_1 \kappa_3 Y_3)}{i (\kappa_1 \kappa_2 - \kappa_1 \kappa_3) Y_1 + (\kappa_2 \kappa_3 - i \kappa_1 \kappa_2) Y_2 + (i \kappa_1 \kappa_3 - \kappa_2 \kappa_3) Y_3}.
\]

(35)
The transmission loss maximum generally lies in the shear dominated frequency range $f \in [f_{BE}, f_{Timo}]$, where $f_{BE}$ is the Bernoulli-Euler frequency limit, given in [6] as

$$f_{BE} = \frac{G_c h_c}{2\pi \sqrt{\mu_{tot} D_{tot}}}. \quad (36)$$

For a wide range of beam configurations, the frequency at which maximum transmission loss is obtained is approximately

$$f_{max} \approx 0.9 f_{\gamma}^{0.9}, \quad \text{where} \quad f_{\gamma} = \sqrt{f_{BE} f_{Timo}}. \quad (37)$$

$f_{\gamma}$ was found in [6] to indicate the frequency at which the shear deformation energy density of an infinite beam peaks. The above estimate was obtained from correlation analysis of 1,000 random sandwich configurations, see Figure 9, where the beam properties were determined from rectangular probability distributions identical to those given in [6]. The 95% confidence interval for the relative error of Eq. (37) is approximately ±10%.

5.2 The $n$-junction

Consider a simply supported junction connecting $n$ beams, i.e. a generalized case of the simply supported junction described earlier. In one of the beams, say beam 1, an unit incident wave is propagating towards the junction, resulting in transmitted wave fields in all other beams. A total number of $3n$ unknown wave amplitudes need to be calculated from the coupling conditions. These become
Fig. 10. A junction of $n = 4$ identical beams hinged at $x_j = 0$.

- zero displacement, $w_j = 0$,
- all total angles equal, $w_1' = w_j'$,
- continuity in bending angle, $\beta_1 = \beta_j$,
- zero total bending moment around the joint, $M_1 = \sum_{j=2}^{n} M_j$,
- and zero net bending moment due to core shear, $M_s^{(1)} = \sum_{j=2}^{n} M_s^{(j)}$,

i.e. a total of $3n$ conditions. For the special case when all beams are identical, the sums in the moment expressions reduce to a multiplying factor of $(n - 1)$. The transmission coefficient $\tau_n$ can then be written in terms of $\tau_2$ as,

$$\tau_n = \frac{4}{n^2} \tau_2^2. \quad (38)$$

This implies that the transmission loss to any of the $n - 1$ outgoing beams can be written

$$R_n = R_2 + 20 \log_{10} \frac{n}{2}, \quad (39)$$

where $R_2$ is the transmission loss for two identical beams joined by means of a simply supported junction, see Figure 8. The result is shown in Figure 11.

5.3 The point mass junction

Consider an infinite beam with an attached mass per unit width $m$ located at $x = 0$, see Figure 12. The mass is assumed to have negligible moment of inertia, thus effectively being a point mass. If the beams are described by Bernoulli-Euler theory, the following coupling conditions are imposed at $x = 0$: 

$$\tau_n = \frac{4}{n^2} \tau_2^2. \quad (38)$$

This implies that the transmission loss to any of the $n - 1$ outgoing beams can be written

$$R_n = R_2 + 20 \log_{10} \frac{n}{2}, \quad (39)$$

where $R_2$ is the transmission loss for two identical beams joined by means of a simply supported junction, see Figure 8. The result is shown in Figure 11.
Fig. 11. The transmission loss over a simply supported junction connecting \( n \) identical beams. The dashed line indicates the approximate location of the maximum transmission loss, as given by Eq. (37).

![Transmission loss over a n-junction](image)

By inserting the Bernoulli-Euler wave solutions given in Eq. (32) into the above conditions, the unknown coefficients can be obtained. In the symmetric case when the beam properties on each side of the mass are identical, \( B_1 \) is given explicitly, in the case of Bernoulli-Euler theory, by

\[
B_1 = \frac{i(4D\kappa_{BE}^3 + m\omega^2)}{i(4D\kappa_{BE}^3 + m\omega^2) - m\omega^2}.
\]  

(40)

As expected, the transmission loss turns asymptotically to \( R \approx 3 \) dB when \( \omega \) or \( m \) goes to infinity as the joint approaches simply supported conditions. In general, however, \( R \) is frequency dependent and the junction acts as a low pass filter, see Figure 13.

Fig. 12. A point-mass on an infinite beam. The solid arrows represent propagating waves while the dashed arrows denote nearfields.

- equal lateral displacement, \( w_a = w_b \),
- continuity in angle, \( w'_a = w'_b \),
- zero total bending moment, \( -D_a w''_a = -D_b w''_b \),
- and lateral inertia equilibrium, \( -F_a + F_b = -m\omega^2 w_a \), where \( F = -Dw''' \).

By inserting the Bernoulli-Euler wave solutions given in Eq. (32) into the above conditions, the unknown coefficients can be obtained. In the symmetric case when the beam properties on each side of the mass are identical, \( B_1 \) is given explicitly, in the case of Bernoulli-Euler theory, by

\[
B_1 = \frac{i(4D\kappa_{BE}^3 + m\omega^2)}{i(4D\kappa_{BE}^3 + m\omega^2) - m\omega^2}.
\]  

(40)

As expected, the transmission loss turns asymptotically to \( R \approx 3 \) dB when \( \omega \) or \( m \) goes to infinity as the joint approaches simply supported conditions. In general, however, \( R \) is frequency dependent and the junction acts as a low pass filter, see Figure 13.
Fig. 13. The transmission loss for a typical Bernoulli-Euler beam loaded by a point mass. $m'$ is $m$ normalized with the mass per unit length and width of the sandwich beam.

Fig. 14. The transmission loss for a typical sandwich beam loaded by a point mass. $m'$ is $m$ normalized with the mass per unit length and width of the sandwich beam.

If instead $6^{th}$ order sandwich beam theory is utilized, with corresponding coupling conditions, a more complex frequency behaviour is obtained. In the limit of $m \to \infty$, the same bell shaped curve as for the simply supported junction is obtained, see Figure 14. The reason for this is that the infinite inertia of the point mass prevents lateral deflection of the beam at the junction, while it remains free to rotate. As $m \to 0$, the junction becomes transparent and $R \to 0$ dB.
Fig. 15. A contour plot of the transmission loss for a typical sandwich beam loaded by a point mass. $m'$ is $m$ normalized with the mass per unit width of one unit length of the sandwich beam.

![Contour Plot](image)

Fig. 16. An infinite beam supported on a spring located at $x = 0$.

5.4 The spring loaded junction

Yet a different coupling is obtained if the infinite sandwich beam is supported by an ideal spring located at $x = 0$. For the Bernoulli-Euler beam, the transmission loss is obtained directly from Eq. (40) by replacing the mass $m$ with $-K\omega^2$, where $K$ is the spring stiffness per unit width of the beam. The same operation can be done on the force coupling condition for the sandwich beam junction, i.e. the condition becomes

$$-F_a + F_b - Kw_a = 0.$$  \tag{41}

Eq. (40) also indicates that at a certain frequency $f_{R\infty}$, the spring loaded joint becomes effectively rigid, allowing no transmission of propagating waves to the region $x > 0$, while the reflected wave is equal in amplitude but opposite in
Fig. 17. The transmission loss of a typical sandwich beam loaded by a spring. \( K' \) is the spring stiffness normalized with the effective beam elasticity modulus. The dashed lines represents the approximate estimates of \( f_{R\infty} \).

Phase to the incident and therefore total cancellation is achieved in the region \( x < 0 \). For Bernoulli-Euler theory, this frequency is

\[
 f_{R\infty}^{BE} = \frac{1}{2\pi} \sqrt{\frac{D}{\mu}} \left( \frac{K}{4D} \right)^{\frac{3}{4}}, \tag{42}
\]

where \( \mu \) is the mass per unit length and width of the beam. Since \( K \) is a spring stiffness per unit width, it is natural to normalize it versus the total effective elasticity modulus of the beam, which is \( E_{\text{eff}} = 12D_{\text{vol}}/(h_1+h_c+h_2)^3 \) for a sandwich beam. The result is shown in Figure 17. As can be seen, the transmission loss asymptotically approaches the simply supported joint for \( K \to \infty \). The blocking frequency \( f_{R\infty} \) for a sandwich beam junction can be approximated by replacing \( D \) in Eq. (42), using the apparent bending stiffness as obtained from the main propagating wavenumber:

\[
 D_{\text{app}}(f_{R\infty}) = \frac{\mu(2\pi f_{R\infty})^2}{K_1^4}. \tag{43}
\]

The result is an implicit relation which yields a reasonably accurate estimate of \( f_{R\infty} \), at least for \( K' \) not much smaller than unity.

5.5 The mass-spring junction, type A

When the beam is loaded by both a mass and a spring, as shown in Figure 18, the force coupling condition is extended by the mass inertia term,

\[
 -F_a + F_b = -m\omega^2 w_a \tag{44}
\]
resulting in two phenomena. The same blocking of the transmission is obtained when the numerator of the $B_1$ coefficient vanishes, although $f_{R\infty}$ now also depends on $m$. In addition to this total reflection, total transmission follows at the mass-spring resonance $f_0^2 = K/m$, as the stiffness and inertia term cancel each other. Below $f_0$, the junction is dominated by the stiffness term, while the mass inertia governs the frequency range above. The transmission loss of a typical sandwich beam loaded by a mass-spring joint is shown in Figure 19. The blocking frequency is implicitly given by the relation

$$K - m(2\pi f_{R\infty})^2 - 4D\left\{\mu(2\pi f_{R\infty})^2\right\}^{\frac{3}{4}} = 0$$

for a Bernoulli-Euler beam, obtained from the numerator of $B_1$. By dividing by $m$ and identifying $f_0$, it can be shown that $f_{R\infty} < f_0$. By replacing $D$ by $D_{app}$ as given in Eq. (43), the blocking frequency of a sandwich beam junction can be estimated.

### 5.6 The mass-spring junction, type B

A different version of the mass-spring loaded beam is shown in Figure 20. It is necessary to include one more variable, the displacement $z$ of the mass, in order to define the system. The force coupling condition now becomes

$$-F_a + F_b - K(w_a - z) = 0,$$

which together with the governing equation for the mass,

$$K(w_a - z) = m\frac{\partial^2 z}{\partial t^2},$$

provides the seven system equations. The transmission loss is shown for a few selected values of $K$ and $m$ in Figure 21. In the low frequency region, the sys-
Transmission loss over a mass-spring loaded junction

Fig. 19. Some transmission loss curves of a typical sandwich beam loaded by a mass-spring joint, type A. The dashed curve represents \( f_0 \) for the chosen values of \( K' \) and \( m' \). Note that the frequency axis is linear in this plot.

Fig. 20. An infinite beam loaded by a mass-spring system located at \( x = 0 \), type B. The system behaves like a mass-loaded junction and the transmission loss approaches zero as \( f \to 0 \). Similarly, in the high frequency region the inertia of the mass implies \( z \to 0 \) and the junction asymptotically approaches the spring-loaded junction. For a junction of Bernoulli-Euler beams, the blocking frequency is given implicitly by

\[
4 \left\{ K - m(2\pi f_{R\infty})^2 \right\} D_1^\frac{1}{3} \left\{ \mu(2\pi f_{R\infty})^2 \right\}^\frac{2}{3} + Km(2\pi f_{R\infty})^2 = 0. \tag{48}
\]

By inserting the apparent bending stiffness from Eq. (43) into the above relation, the blocking frequency of the sandwich beam junction can be estimated.
Transmission loss over a mass-spring loaded junction

![Transmission loss graph]

Fig. 21. The transmission loss of a mass-spring loaded beam, type B. The dashed lines indicate the estimates of $f_{R\infty}$.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L$</td>
<td>2.5 m</td>
</tr>
<tr>
<td>$h_c$</td>
<td>12 mm</td>
</tr>
<tr>
<td>$h_{\text{lam}}$</td>
<td>1.25 mm</td>
</tr>
<tr>
<td>$b$ (beam width)</td>
<td>80 mm</td>
</tr>
<tr>
<td>$\mu$</td>
<td>5.35 kg/m$^2$</td>
</tr>
</tbody>
</table>

Table 1
Dimensions and mass of the sample beam used in the transmission loss measurements.

6 Measurements

6.1 Introduction

In order to verify parts of the theoretical analysis in the previous section, transmission loss measurements on a sandwich beam loaded by a mass were performed. First, the beam was assembled from two thin aluminium-plastic laminated sheets and a plastic foam core glued together using a strong two-component epoxy glue. Then, in order to obtain the core shear modulus, the free-free eigenfrequencies of the beam were measured by means of an accelerometer. During these measurements, the beam was suspended by rubber strings and excited with a hammer. The material parameters were established by minimizing an error function, as described in [6], using as known data the dimensions of the beam, see Table 1. The error function is plotted in Figure 22. The norm is minimized for $E_{\text{lam}} \approx 27$ GPa and $G_c \approx 35$ MPa.
6.2 Measurement theory

The transmission loss over a given junction is now measured using the following approach, analyzed in detail in [7]. The beam is horizontally suspended by means of strings and its ends embedded in sand in order to reduce reflections. The technique is capable of handling reflections, but in order to obtain the best results it is important to limit the amplitude of these as otherwise numerical problems might arise.

The beam is then excited by a shaker injecting white noise into the structure. The transfer accelerance between the point of the power input and four points symmetrically distributed around the junction is then measured by means of accelerometers. It is important that the distance between the measurement points and the junction is large enough so that the influence of nearfields is negligible. In these regions, the wave fields are determined by the propagating waves alone, reducing the problem to four unknowns – the amplitudes of the wave incident to the junction, the wave reflected from the junction, the transmitted wave and the wave reflected at the beam end, see Figure 23. The four measured accelerance function provide the data necessary in order to resolve the amplitudes as functions of frequency. Denoting the displacement wave fields \( w_1 \) and \( w_2 \) on the incident and transmitted sides of the junction, respectively, we have

\[
\begin{align*}
\text{w}_1 & \approx A_1 e^{-i\kappa_1 x} + A_2 e^{i\kappa_1 x}, \\
\text{w}_2 & \approx B_1 e^{-i\kappa_1 x} + B_2 e^{i\kappa_1 x},
\end{align*}
\]

where it has been assumed that the beam properties are identical on either side of the junction, which is located at \( x = 0 \). The wave components propagating away from the junction, corresponding to the \( A_2 \) and \( B_1 \) amplitudes, can
be decomposed into transmission and reflection ratios multiplying the wave amplitudes $A_1$ and $B_2$. Thus, we obtain

$$A_2 = rA_1 + tB_2$$

(51)

and

$$B_1 = tA_1 + rB_2.$$  

(52)

Solving the system for $r$ and $t$ yields these ratios as functions of the wave amplitudes, written in matrix form as

$$\begin{pmatrix} r \\ t \end{pmatrix} = \begin{pmatrix} A_1 & B_2 \\ B_2 & A_1 \end{pmatrix}^{-1} \begin{pmatrix} A_2 \\ B_1 \end{pmatrix}.$$  

(53)

The accelerometers are located symmetrically around the junction at $x_j = \pm\{d, d+s\}$. The measured transfer accelerance functions thus become

$$H_j \approx -\omega^2 w_j F,$$

(54)

where $w_j = w|_{x=x_j}$ and $F$ is the fourier transform of the force at the excitation point. It follows that the displacements at the accelerometer positions can be written

$$w_j \approx \frac{F}{\omega^2 H_j}.$$  

(55)

Now, the wave amplitudes $A_j$ and $B_j$ can be obtained from the four measured accelerance functions as functions of the force transform $F$. However, by inserting the coefficients into Eq. (53), the force term cancels out and the reflection and transmission ratios $r$ and $t$ are obtained as functions of frequency alone. The transmission coefficient $\tau$ is now obtained simply as

$$\tau = |t|^2.$$  

(56)
keeping in mind the beam symmetry. Besides from the requirement that the dominating nearfield should be negligible at the accelerometer positions, i.e. $\kappa_2 d$ should be larger than some limit value, it is also demonstrated that the value of $\kappa_1 s$ should lie between $0.1\pi$ and $0.8\pi$ in order to ensure numerical precision. If the distance $s$ between the accelerometers is chosen to be 3 cm, the latter condition is satisfied in the frequency range from approximately 300 to 4000 Hz.

### 6.3 Measurement setup

The beam ends are embedded in sand in order to reduce the amplitudes of the reflections, see Figure 24. The distance $d$ from the accelerometer positions 2 and 3 to the junction is chosen to be 30 cm, implying that the dominating nearfield is assumed negligible for $f$ greater than approximately 300 Hz as $\exp(-\kappa_2 d) < 0.1$. The distance $s$ between the accelerometer positions is set to 3 cm, assumingly resulting in stable numerical conditions in the frequency range between approximately 300 and 4000 Hz. By embedding 45 cm of the beam ends in sand reflections could be assumed substantially reduced for frequencies above 500 Hz when the sand covers more than one wavelength.

Other possible sources of error include the mass effect of the attached accelerometer, which is of type B&K 4393V and weighs 2.4 g, and the potential presence of cross modes in the high frequency range. The first cross mode is generated for $\kappa_1 b \approx 4.73$, which corresponds to a frequency of approximately 2.7 kHz.

The idealized case of a point mass junction is approximated by drilling a hole through the beam core and inserting a metal cylinder. The mass of the cylinder is 72.7 g, resulting in a mass per unit width of $m = 0.91$ kg/m (the width of the beam is 8 cm).
6.4 Results

There are a number of indicators which provide detailed information on the quality of the obtained results. First, the coherence of the measured transfer accelerance functions tell of the degree of linear relationship between the measured acceleration and force signals, see Figure 25. From this plot, it is evident that the frequency region between approximately a 100 Hz and 3 kHz is of sufficient quality.

Further, the condition number of the two system matrices that are used to resolve the wave amplitudes and the transmission and reflection coefficients could be of interest, see Figure 26. The condition numbers are plotted in the frequency region where the coherence is acceptable, below 3 kHz. The smooth solid curve represents the condition of the system matrix for decomposing the wave into its right and left going components, while the dashed curve corresponds to the matrix system in Eq. (53). Both estimates indicate that the systems are reasonably well conditioned in the considered frequency range, with the exception of very low frequencies. This is in accordance with the previous discussion on the value of $s$, which when set to 3 cm implies numerically stable conditions above 300 Hz (and below approximately 4 kHz).

Finally, the computed reflection coefficient of the embedded beam ends is plotted in Figure 27. In general, the reflection decreases with increasing frequency, which is expected as the effective length of the embeddings in terms of wavelengths increases.
Fig. 26. Condition numbers of the two system matrices. —, wave decomposition matrix; - - , RT-matrix, from Eq. (53). A value close to unit implies the system matrix is well-conditioned.

Fig. 27. The computed reflection coefficient (ratio of reflected to incident energy flows) of the sand-embedded beam ends. The straight line represents $|r|^2 = 1$.

The measured transmission loss is displayed in Figure 28 together with the predicted results obtained from the equations in section 5.3. The measured transmission loss is averaged into third-octave bands in order to reduce the presence of noise. As can be seen, the 6th order theory provides a better estimate of the transmission loss than the simple Bernoulli-Euler theory.
Fig. 28. Measured and predicted transmission loss over point mass junction, $m = 0.91 \text{ kg/m}$. – , measured (third-octave band averaged); - - , calculated, 6th order theory; · · · , calculated, Bernoulli-Euler theory.

7 Conclusions

The energy flow through a three-layered sandwich beam and the transmission loss over different junctions has been analyzed by means of 6th order theory developed in [4]. The purpose of the study is to provide a basis for the implementation of sandwich elements in Statistical Energy Analysis models, by means of coupling loss factors.

Some interesting properties of the energy flow in sandwich beams were found; the time average of the total energy density $E$ of a displacement wave with amplitude $A$ was shown to equal the instantaneous total energy of an ordinary Bernoulli-Euler (BE) beam, while its time dependence is of a more complicated nature. The low and high frequency asymptotes of the time average of the energy flow is equal to the energy flow in a BE-beam with a bending stiffness equal to $D_{tot}$ and $D_1 + D_2$, respectively.

The analysis of the energy flow across junctions also revealed interesting details; in all of the considered junctions a local (sometimes global) maximum transmission loss was obtained in the shear-dominated frequency range $f \in [f_{BE}, f_{Timo}]$. For the spring-loaded junctions a blocking phenomena similar to that obtained for ordinary thin beams occur. The frequency at which this total reflection occurs can be estimated by means of the apparent bending stiffness $D_{app}$.

In order to validate the theory, measurements on a mass-loaded beam was performed. The transmission coefficient was computed using a method originally developed for ordinary homogenous beams, and the results indicate that
the proposed method for the analysis of the transmission loss of sandwich beam junctions provide reasonably accurate results. The measurements could probably be further improved by using a longer beam specimen, as all the vital dimensions then could be increased. By covering an even larger portion of the beam ends in sand, reflections could be substantially decreased at lower frequencies. Further, the distance between the accelerometer positions and the field discontinuities – the embedded beam ends, the shaker position and the junction itself – could be increased, thereby decreasing the possible contamination of the data due to the presence of nearfields. As an alternative, a method involving additional measurements aiming at resolving also the amplitudes of the nearfields could potentially be utilized; however, as this involves larger matrix systems it is possible that the numerical conditions might introduce additional problems.

References


