DOCTORAL THESIS

Inequalities for Some Classes of Hardy Type Operators and Compactness in Weighted Lebesgue Spaces

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Mathematics

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Inequalities for some classes of Hardy type operators and compactness in weighted Lebesgue spaces

by

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To my parents and family

Abstract

This PhD thesis is devoted to investigate weighted differential Hardy inequalities and Hardy-type inequalities with kernel when the kernel has an integrable singularity, and also the additivity of the estimate of a Hardy type operator with a kernel.

The thesis consists of seven papers (Papers 1, 2, 3, 4, 5, 6, 7) and an introduction where a review on the subject of the thesis is given.

In Paper 1 weighted differential Hardy type inequalities are investigated on the set of compactly supported smooth functions, where necessary and sufficient conditions on the weight functions are established for which this inequality and two-sided estimates for the best constant hold.

In Papers 2, 3, 4 a more general class of α - order fractional integration operators are considered including the well-known classical Weyl, Riemann-Liouville, Erdelyi-Kober and Hadamard operators. Here $0 < \alpha < 1$.

In Papers 2 and 3 the boundedness and compactness of two classes of such operators are investigated namely of Weyl and Riemann-Liouville type, respectively, in weighted Lebesgue spaces for $1 < p \leq q < \infty$ and $0 < q < p < \infty$. As applications some new results for the fractional integration operators of Weyl, Riemann-Liouville, Erdelyi-Kober and Hadamard are given and discussed.

In Paper 4 the Riemann-Liouville type operator with variable upper limit is considered. The main results are proved by using a localization method equipped with the upper limit function and the kernel of the operator.

In Papers 5 and 6 the Hardy operator with kernel is considered, where the kernel has a logarithmic singularity. The criteria of the boundedness and compactness of the operator in weighted Lebesgue spaces are given for $1 < p \leq q < \infty$ and $0 < q < p < \infty$, respectively.

In Paper 7 we investigated the weighted additive estimates

$$\|u\mathbb{K}^{\pm} f\|_q \leq C\left(\|\rho f\|_p + \|vH^{\pm} f\|_p\right), \quad f \geq 0 \tag{$*$}$$

for integral operators $\mathbb{K}^+$ and $\mathbb{K}^-$ defined by

$$\mathbb{K}^+ f(x) := \int\limits_0^x K(x,s) f(s) ds, \quad \mathbb{K}^- f(x) := \int\limits_x^\infty K(x,s) f(s) ds.$$

It is assumed that the kernel $K = K(x,s)$ of the operator $\mathbb{K}^{\pm}$ belongs to the general Oinarov class. We derived the criteria for the validity of the inequality $(*)$ when $1 \leq p \leq q < \infty$.

Preface

This PhD thesis is mainly devoted to introduce and study weighted differential Hardy inequalities and new Hardy type integral inequalities involving Riemann-Liouville type operator and its conjugate Weyl type operator. Further we investigate boundedness and compactness of Hardy type operators with variable upper limit and integral operators with a logarithmic singularity in weighted Lebesgue spaces. Moreover, we have found additive estimates of a class of integral operators, which is much wider than previously studied. We also present some applications, which cover much wider classes of integral operators than studied before.

The thesis consists of an introduction and the following seven papers:

[1] A.M. Abylayeva, A.O. Baiarystanov and R. Oinarov, *A weighted differential Hardy inequality on* $\overset{\circ}{AC}(I)$, Siberian Math. J. **55** (2014), No.3, 387 - 401.

[2] A.M. Abylayeva, *Boundedness, compactness for a class of fractional integration operators of Weyl type*, Eurasian Math. J. **7** (2016), No.1, 9-27.

[3] A.M. Abylayeva, R. Oinarov, and L.-E. Persson, *Boundedness and compactness of a class of Hardy type operators*, Research report 2016 (submitted).

[4] A.M. Abylayeva, *Boundedness and compactness of the Hardy type operator with variable upper limit in weighted Lebesgue spaces*, Research report 2016-04, ISSN: 1400-4003, Department of Engineering Sciences and Mathematics, Luleå University of Technology, Sweden. Submitted to an International Journal.

[5] A.M. Abylayeva and L.-E. Persson, *Hardy type inequalities with logarithmic singularities*, Research report 2016-05, ISSN: 1400-4003, Department of Engineering Sciences and Mathematics, Luleå University of Technology, Sweden.

[6] A.M. Abylayeva, *Compactness of a class of integral operators with logarithmic singularities*, Research report 2016-06, ISSN: 1400-4003, Department of Engineering Sciences and Mathematics, Luleå University of Technology, Sweden.

[7] A.M. Abylayeva, A.O. Baiarystanov, L.-E. Persson and P. Wall, *Additive weighted* L_p *estimates of some classes of integral operators involving generalized Oinarov kernels*, J. Math. Inequal. (JMI), to appear 2016.

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Introduction

Integral operators are a wide class of linear operators that have applications in various fields of science, such as physics, economics, technical sciences and many others. Therefore the study of integral operators take an important place in modern mathematics.

In the last decades the issues of finding necessary and sufficient conditions for the weighted inequality

$$\|\mathbb{K}f\|_{q,u} \leq C\|f\|_{p,v} \tag{0.1}$$

and two-sided estimates for the best constant C in (0.1) are intensively studied for various integral operators $\mathbb{K}$, where

$$\|f\|_{p,v} := \left(\int\limits_0^\infty |f(x)|^p v(x)dx\right)^{\frac{1}{p}} < \infty.$$

In the case when one of the parameters p and q is equal to 1 or ∞, there is a general result ([28] Chapter XI, §1.5, Theorem 4, see also [18], Theorem 1.1) establishing the exact value of the best constants in (0.1). However, when $1 < p, q < \infty$ in the general case this problem remains open. Therefore a solution of this problem for various classes of integral operators is urgent.

In 1925 G.H.Hardy [24] obtained the inequality (0.1) when $p = q$ for the Hardy operator defined by

$$\mathbb{K}f(x) \equiv Hf(x) := \int_0^x f(t)dt$$

with the weighted functions $u(x) = x^{-p}$, $v \equiv 1$ with the exact value $C = \frac{p}{p-1}$ for the best constant C in (0.1), i.e. the inequality

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx, \quad f \geq 0, \tag{0.2}$$

holds which is called the classical Hardy inequality. In 1928 G.H.Hardy [25] proved the first weight modification of inequality (0.2), namely the inequality

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p x^\alpha dx \leq \left(\frac{p}{p-\alpha-1}\right)^p \int_0^\infty f^p(x)x^\alpha dx, \quad f \geq 0, \tag{0.3}$$

with the best constant $C = \left(\frac{p}{p-\alpha-1}\right)^p$, when $p > 1$ $\alpha > p-1$ (see [26], Theorem 330). It is nowadays known that the inequalities (0.2) and (0.3) are in a sense equivalent and also equivalent to some other power weighted variants of Hardy's inequality, see [56].

Since the middle of the last century the studing of a general weighted form of inequality (0.1) with the Hardy operator H i.e. the inequality

$$\left(\int_0^\infty u(x)\left|\int_0^x f(t)dt\right|^q dx\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty \left|f(t)\right|^p v(t)dt\right)^{\frac{1}{p}} \tag{0.4}$$

for $p = q$ was initiated (see for instance [8] by P.R. Beesack, [27] by J. Kadlec and A. Kufner, [57] by V.R. Portnov, [63] by V.N. Sedov and [76] by F.A. Sysoeva). However, for the case $p = q$ the necessary and sufficient condition for the validity of inequality (0.4) was first obtained, independently, in the works of G.Talenti [77] and G.Tomaselli [78]. In 1972 B.Muckenhoupt in [42] gave a simple excellent proof of this result, even in the more general case, when $u^q(x)dx$ and $v^p(t)dt$ were replaced by general Borel measures $d\mu(t)$ and $dv(t)$, respectively. A criterion for the inequality (0.4) to hold when $1 < p \leq q < \infty$ was given independently by J.Bradley [10], V.Kokilashvili [29] and B.Maz'ya [39]. And the case $1 < q < p < \infty$ was first described by B.Maz'ya and A.Rozin in the late seventies, see [38] and [39]. These results have been extended by G. Sinnamon [64] to the values of the parameters $0 < q < p < \infty$, $p > 1$, and the case $0 < q < p = 1$ has been described by G.Sinnamon and V.D.Stepanov [65]. G.Tomaselli [78] gave an alternative criterion for the weighted Hardy inequality (0.4) to hold when $p = q$, which V. Stepanov and L.-E. Persson generalized this result to the cases $1 < p \leq q < \infty$ and $1 < q < p < \infty$ in [54].

There are studies on the description of the inequalities in other terms [15] and [32], different from the above authors and also for negative values of the parameters p, q see e.g. [61].

Let us sum up some of the results above in the following Theorem:

Theorem A. *(i) If $1 \leq p \leq q < \infty$, then the inequality (0.4) holds for all measurable functions $f(x) \geq 0$ on (a,b) if and only if*

$$A_1 := \sup_{a<x<b}\left(\int_x^b u(t)dt\right)^{\frac{1}{q}}\left(\int_a^x v^{1-p'}(t)dt\right)^{\frac{1}{p'}} < \infty$$

or

$$A_{PS} := \sup_{t>0} \left(\int_0^t w(x) \left(\int_0^x v^{1-p'}(y)dy \right)^q dx \right)^{\frac{1}{q}} \left(\int_0^t v^{1-p'}(y)dy \right)^{-\frac{1}{p}} < \infty.$$

(ii) If $1 < q < p < \infty$*, then the inequality (0.4) holds if and only if*

$$A_2 := \left(\int_a^b \left(\int_x^b u(t)dt \right)^{\frac{r}{q}} \left(\int_a^x v^{1-p'}(t)dt \right)^{\frac{r}{q'}} v^{1-p'}(x)dx \right)^{\frac{1}{r}} < \infty$$

or

$$B_{PS} := \left(\int_0^\infty \left(\int_0^t w(x) \left(\int_0^x v^{1-p'}(y)dy \right)^q dx \right)^{\frac{r}{q}} \left(\int_0^t v^{1-p'}(y)dy \right)^{-\frac{r}{p}} v^{1-p'}(t)dt \right)^{-\frac{1}{r}} < \infty,$$

where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$.

(iii) If $0 < q < 1 < p < \infty$*, then the inequality (0.4) holds if and only if*

$$A_3 := \left(\int_a^b \left(\int_x^b u(t)dt \right)^{\frac{r}{p}} \left(\int_a^x v^{1-p'}(t)dt \right)^{\frac{r}{p'}} u(x)dx \right)^{\frac{1}{r}} < \infty.$$

(iv) If $0 < q < 1 = p$*, then the inequality (0.4) holds if and only if*

$$A_4 := \left(\int_a^b \left[\bar{v}(x) \int_x^b u(t)dt \right)^{\frac{q}{1-q}} u(x)dx \right)^{\frac{1}{q}-1} < \infty,$$

where $\bar{v}(x) = \operatorname{ess\,sup}_{a<t<x} \frac{1}{v(t)}$.

It is nowadays known that the conditions in (i)-(ii) in fact can be replaced by infinite many equivalent conditions, even by four different scales of conditions, see [15] (the case (i)), [55] (the case (ii)) and for even more information of this type the review article [34].

In connection with the investigation of operators in Lorentz spaces since 1990 the Hardy-type operators were actively studied on the class of monotone functions, see for example [18], [19], [20], [21], [22] and the references therein. Moreover, operators including the supremum, has began to be investigated recently, see for example [3], [16], [17], [53] and the references therein.

The inequality (0.4) and its dual inequality are equivalent to the differential inequality

$$||y||_{q,u} \leq C||y'||_{p,v} \tag{0.5}$$

respectively for $y(0) = 0$ and for $y(\infty) = 0$. We remark that P.Gurka [23] described the inequality (0.5) under the condition

$$y(0) = 0, \;\; y(\infty) = 0. \tag{0.6}$$

Historical background, a review of the research, the main results and their applications are given in the books [11], [12], [26], [31], [33], [41] and [51].

The inequality (0.5) with condition (0.6) was considered in [51], [31], but only in [51] an expanded version of the work of P. Gurka [23] was considered and two-sided estimates for the best constant C of (0.5) was stated.

The aim of this PhD thesis is to complement and extend several results in the area described above which is today called Hardy type inequalities and related boundedness and compactness results. Below we give a short description and motivation for these new contributions presented in this PhD thesis.

In Paper 1, using a new method, we obtained necessary and sufficient conditions for the validity of the inequality (0.5) with condition (0.6) for the cases $1 < p \leq q < \infty$ and $0 < q < p < \infty$, $p > 1$. We also derived two-sided estimates for the best constant C of (0.5), which are better than those in [51].

In 1979 O.D.Apyshev and M.Otelbaev [7] considered the inequality (0.5) for higher order derivative, namely the inequality

$$||y||_{q,u} \leq C||y^n||_{p,v}, \;\; n > 1 \tag{0.7}$$

$$y^{(i)}(0) = 0, \;\; i = 0, 1, ...n - 1. \tag{0.8}$$

But a criterion for the inequality (0.7) to hold was obtained only under certain restrictions on the weight functions. We mention that Chapter 4 of the book [31] is devoted only to such higher order Hardy type inequalities. We remark that the possible boundary values (of type (0.8)) are very crucial to make such investigations possible (see [31]).

The inequality (0.7) with the condition (0.8) is equivalent to the inequality (0.1), when the integral operator K is equal to the Riemann-Liouville operator I_α defined by

$$I_\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_0^x (x-y)^{(\alpha-1)} f(y)dy, \quad x > 0, \tag{0.9}$$

for $\alpha = n$, i.e.

$$\|I_\alpha f\|_{q,u} \leq C\|f\|_{p,v}. \tag{0.10}$$

A satisfactory criterion for the inequality (0.10) to hold for the Riemann-Liouville operator when $\alpha > 1$ was obtained in the papers [67], [70] and [69] of V.D.Stepanov.

An other generalization of (0.4) is a norm inequality of the form

$$\left(\int_0^\infty \left(\int_0^x k(x,y) f(y)dy \right)^q u(x)dx \right)^{\frac{1}{q}} \leq C \left(\int_0^\infty f^p(y)v(y)dy \right)^{\frac{1}{p}}, \quad f \geq 0, \tag{0.11}$$

for the Hardy-Volterra integral operator K given by

$$Kf(x) := \int_0^x k(x,y)f(y)dy, \quad x \geq 0, \tag{0.12}$$

with kernel $k(x,y)$, which is assumed to be non-negative and measurable on the triangle $\{(x,y) : 0 \leq y \leq x \leq \infty\}$. A number of authors have studied in their works several different classes of such operators. In [37] it was obtained a characterization of (0.11) in the case $1 < p \leq q < \infty$ with the special kernel $k(x,y) = \varphi(x/y)$, where $\varphi : (0,1) \to (0,\infty)$ is non-increasing and satisfying that $\varphi(ab) \leq D(\varphi(a) + \varphi(b))$ for all $0 < a, b < 1$. Moreover, a criterion of the $L_{p,v} \to L_{q,w}$ boundedness was given in [71] and [72] by V.D. Stepanov for the Volterra convolution operator (0.12) with $k(x,y) = k(x-y)$ for both the cases $1 < p \leq q < \infty$ and $1 < q < p < \infty$. An other class of studied operators of the type (0.12) has kernels satisfying some additional monotonicity and continuity conditions (see e.g. [9] by S. Bloom and R. Kerman). In the nineties it appeared some important works (see e.g. [45], [46] by R. Oinarov and [73], [74] by V.D. Stepanov) devoted to the class of the operators (0.12) with so called Oinarov kernels. A kernel $k(x,y) \geq 0$ satisfies the Oinarov condition if there is a constant $D \geq 1$ independent on x, y, z such that

$$D^{-1}k(x,y) \leq k(x,z) + k(z,y) \leq Dk(x,y), \quad 0 \leq y \leq z \leq x. \tag{0.13}$$

Let the kernel $k(x,y) \geq 0$ of the operator (0.12) satisfy the Oinarov condition (0.13). If

$$A_0(\alpha) := \sup_{t>0} \left(\int_t^\infty K^q(x,t) u(x) dx \right)^{\frac{1}{q}} \left(\int_0^t v^{1-p'}(y) dy \right)^{\frac{1}{p'}},$$

$$A_1(\alpha) := \sup_{t>0} \left(\int_t^\infty u(x) dx \right)^{\frac{1}{q}} \left(\int_0^t K^{p'}(t,y) v^{1-p'}(y) dy \right)^{\frac{1}{p'}},$$

$$B_0(\alpha) := \left(\int_0^\infty \left(\int_t^\infty K^q(x,t) u(x) dx \right)^{\frac{p}{p-q}} \left(\int_0^t v^{1-p'}(y) dy \right)^{\frac{p(q-1)}{p-q}} v^{1-p'}(t) dt \right)^{\frac{1}{q}-\frac{1}{p}},$$

and

$$B_1(\alpha) := \left(\int_0^\infty \left(\int_t^\infty u(x) dx \right)^{\frac{p}{p-q}} \left(\int_0^t K^{p'}(t,y) v^{1-p'}(y) dy \right)^{\frac{p(q-1)}{p-q}} u(t) dt \right)^{\frac{1}{q}-\frac{1}{p}},$$

then it is known that

$$||\mathbb{K}||_{L_{p,v} \to L_{q,u}} \approx A_0(\alpha) + A_1(\alpha), \;\; 1 < p \leq q < \infty, \tag{0.14}$$

and

$$||\mathbb{K}||_{L_{p,v} \to L_{q,u}} \approx B_0(\alpha) + B_1(\alpha), \;\; 1 < q < p < \infty. \tag{0.15}$$

Later on two-sided estimates of the types (0.14) and (0.15) were derived for more general operators and spaces, see e.g. [37], [35], [75], [31], [14], [12], [30], [47], [48] and [49].

The class of Oinarov kernels includes all above mentioned classes of kernels except Riemann-Liouville kernels for $0 < \alpha < 1$.

The Riemann-Liouville operator is a weakly singular integral operator when $0 < \alpha < 1$ and behaves very differently than when $\alpha > 1$.

For power weight function $v(x)$ and $u(y) \equiv 1$ the following classical result [26], Theorem 402, is well known:

If $p > 1$, $0 < \alpha < 1/p$, $p \leq q \leq p/(1-\alpha p)$ *or* $\alpha \geq 1/p$, $1 < p \leq q < \infty$, *then*

$$\left(\int_0^\infty x^{-\frac{1}{p}(p-q+pq\alpha)} (I_\alpha f)^q(x) dx \right)^{\frac{1}{q}} \leq C \left\| f \right\|_p . \tag{0.16}$$

The inequality (0.16) has been generalized in the following way in paper [6] of K.F.Andersen and E.T.Sawyer:

Let $0 < \alpha < \frac{1}{p}$ *and* $1 < p < q = \frac{p}{(1-\alpha p)}$. *Then*

$$\left\|uI_\alpha(uf)\right\|_q \leq C\left\|f\right\|_p$$

if and only if $K < \infty$, *where*

$$K := \sup_{0<h<\alpha}\left(\frac{1}{h}\int_a^{a+h} u^q(x)dx\right)^{\frac{1}{q}}\left(\frac{1}{h}\int_{a-h}^{a} u^{p'}(x)dx\right)^{\frac{1}{p'}}.$$

Moreover, in [59] D.V.Prokhorov and V.D.Stepanov proved the followings result:

Let $0 < \alpha < \frac{1}{p}$ and $1 < p < q = \frac{p}{(1-\alpha p)}$. Then

$$\left\|uI_\alpha f\right\|_q \leq C\left\|f\right\|_{p,v}, \tag{0.17}$$

if and only if

$$\|v\|_\infty < \infty.$$

When $\alpha \geq \frac{1}{2}$, $p = q = 2$ and $v \equiv 1$ the inequality (0.17) has been characterized by S. Newman and M. Solomyak within the spectral theory of pseudo-differential operators on the half-axis, see [44] and also references therein.

A criterion for the inequality (0.10) to hold for $1 < p \leq q < \infty$ was derived by M.Lorenti [36]. However, due to implicitness of the conditions the criteria in [36] make them difficult to verify. Therefore, we set a goal to derive explicit $L_{p,v} \to L_{q,u}$ criteria for the boundedness of the Riemann-Liouville operator in subsequent works.

In the case $0 < q < \infty$, $1 < p < \infty$, $\alpha > \frac{1}{p}$ and $v(\cdot) \equiv 1$ explicit criteria for $L_{p,v} \to L_{q,u}$ boundedness of the Riemann-Liouville and Weyl operators are obtained independently in works of A.Meskhi [40] and D.V.Prokhorov [58], see also [66]. A generalization of these results to the case when the function $u(\cdot)$ is not increasing was claimed in the paper [13] of S.M.Farsani. In the paper [59] of D.V.Prokhorov and V.D.Stepanov criteria for $L_{p,v} \to L_{q,u}$ boundedness and compactness of the Riemann-Liouville operator are given for $1 < p \leq q < \infty$ in the following cases:

a) $1 < \frac{q'}{p'} < \alpha \leq 1$ and the function v is not decreasing;

b) $1 < \frac{p}{q} < \alpha \leq 1$ and the function u is not increasing.

A generalization of these criteria for $L_p \to L_q$ boundedness of the Riemann-Liouville operator in the case of convolution type operator $\mathbf{K}$, defined by

$$\mathbf{K}f(x) := v(x)\int\limits_0^x K(x-s)u(s)f(s)ds, \quad x>0,$$

are given in the papers of N.A.Rautian [52] and R.Oinarov [50]. For the case when the kernel of the operator $\mathbf{K}$, defined by (0.12) is $k(x,y) = k(x-y)$ and the function $k(\cdot)$ has an integrable singularity in zero like the Riemann-Liouville operator the results in [52] were generalized by D.V.Prokhorov and V.D.Stepanov [59] in the case of inequality (0.11). Moreover, R.Oinarov [50] proved a general result of the type claimed by S.M.Farsani [13].

In addition to the Riemann-Liouville and Weyl operators the Erdey-Kober and Hadamard operators are important both in mathematics and for several applications.

One of the generalizations and unifications of these operators is the fractional integration operator I_g^α defined by:

$$I_g^\alpha\varphi(x) := \frac{1}{\Gamma(\alpha)}\int\limits_0^x \frac{\varphi(t)g'(t)dt}{[g(x)-g(t)]^{1-\alpha}}, \quad x>0, \ \ \alpha>0, \tag{0.18}$$

where $g(\cdot)$ is a local absolute continuous and increasing function on $I \equiv (0,\infty)$. In [62] the operator I_g^α is called a fractional integral of the function φ with respect to the function g of order α. In particular, in (0.18) when $g(x) = x$, $g(x) = x^\sigma, \sigma > 0$ and $g(x) = lnx$, we obtain the fractional integral Riemann-Liouville, Erdelyi-Kober type and a Hadamard operator, respectively.

In Papers 2 and 3 of this PhD thesis we consider the more general operators $K_{\alpha,\beta}$ and $T_{\alpha,\beta}$ defined as follows:

$$K_{\alpha,\beta}f(x) := \int\limits_x^b \frac{u(s)W^\beta(s)f(s)w(s)ds}{(W(s)-W(x))^{1-\alpha}}, \quad x\in I,$$

and

$$T_{\alpha,\beta}f(x) := \int\limits_a^x \frac{u(s)W^\beta(x)f(s)v(s)ds}{(W(s)-W(x))^{1-\alpha}}, \quad x\in I,$$

where $0 < \alpha < 1$, $\beta \in R$, $I = (a, b)$, $-\infty \leq a < b \leq \infty$ and $W(\cdot)$ is locally absolutely continuous and monotonically increasing function on I, $\frac{dW(t)}{dt} = w(x)$ and $u(\cdot)$ - non-negative measurable function in I.

In Paper 2 when $0 < \alpha < 1$, $p > \frac{1}{\alpha}$, $\beta \leq 0$ ($\beta < \frac{1}{p} - \alpha$, if $W(b) = \infty$) and $u \geq 0$ is a non-decreasing function we obtained necessary and sufficient conditions for the boundedness and compactness of the operator $\mathbb{K}_{\alpha,\beta}$ from $L_{p,w}$ into $L_{q,v}$, for the cases $\frac{1}{\alpha} < p \leq q < \infty$ and $0 < q < p < \infty$, when $b < \infty$ and for the case $1 < q < p < \infty$ when $b = \infty$.

Consequently, from these statements we obtain necessary and sufficient conditions for the boundedness and compactness of the weighted Weyl operator I_α^*, defined by

$$I_\alpha^* f(x) := w(x) \int_x^\infty \frac{u(s)s^\beta f(s)ds}{(s-x)^{1-\alpha}}, \quad x > 0, \quad 0 < \alpha < 1,$$

from L_p to L_q.

Note that from these results it seems that Theorems 3, 4, 7 and 8 of paper [13] are not true in general.

Similarly, in Paper 3 when $0 < \alpha < 1$, $p > \frac{1}{\alpha}$, $\beta \leq 0$ and u is a non-increasing function we derived necessary and sufficient conditions for the boundedness and compactness of the operator $\mathbb{T}_{\alpha,\beta}$ from $L_{p,w}$ into $L_{q,v}$, for the cases $\frac{1}{\alpha} < p \leq q < \infty$ and $0 < q < p < \infty$, when $b < \infty$ and for the case $1 < q < p < \infty$ when $b = \infty$.

Consequently, we obtained in particular necessary and sufficient conditions for the boundedness and compactness of the weighted Riemann-Liouville, Erdelyi-Kober and Hadamard operators from L_p into L_q, which generalize the well known results for these operators when $p > \frac{1}{\alpha}$.

In Paper 4 we considered the problem of boundedness and compactness of the operator $K_{\alpha,\varphi}$, defined in the following way

$$K_{\alpha,\varphi} f(x) := \int_a^{\varphi(x)} \frac{f(s)w(s)ds}{(W(x) - W(s))^{1-\alpha}}, \quad 0 < \alpha < 1,$$

from $L_{p,w}$ into $L_{q,v}$, where $\varphi(x)$ is a strictly increasing locally absolutely continuous function, which satisfies the following conditions

$$\lim_{x \to a^+} \varphi(x) = a, \quad \lim_{x \to b^-} \varphi(x) = b, \quad and \quad \varphi(x) \leq x.$$

Obviously, the results presented in this paper clearly generalizes the results in [1] and [4].

In Papers 5 and 6 we considered the operator K_γ with a logarithmic singularity defined by

$$\mathbb{K}_\gamma f(x) := v(x) \int_0^x u(s) s^{\gamma-1} \ln \frac{x}{x-s} f(s) ds, \quad x > 0.$$

When $\gamma = 0$, $v(\cdot) \equiv u(\cdot) \equiv 1$ this operator is called a fractional integration operator of infinitesimal order and it has wide applications in mathematical biology, see [43].

In Paper 5 we assumed that the function u is non-increasing and derived necessary and sufficient conditions for the boundedness of the operator $\mathbb{K}_\gamma$ from L_p into L_q, when $1 < p \leq q < \infty$ and $0 < q < p < \infty$, $p > 1$. Moreover, the compactness of the operator $\mathbb{K}_\gamma$ from L_p into L_q was proved in Paper 6 when $1 < p \leq q < \infty$.

We remark that the results in papers 5 and 6 clearly generalizes the main results in [5] and [2], respectively.

In Paper 7 we considered the weighted additive estimates

$$\|u\mathbb{K}^{\pm} f\|_q \leq C\left(\|\rho f\|_p + \|vH^{\pm} f\|_p\right), \;\; f \geq 0 \tag{0.19}$$

for the integral operators $\mathbb{K}^+$ and $\mathbb{K}^-$ defined by

$$\mathbb{K}^+ f(x) := \int_0^x K(x,s) f(s) ds, \quad \mathbb{K}^- f(x) := \int_x^\infty K(x,s) f(s) ds,$$

where the special cases H^+ and H^- are the usual Hardy operators defined by

$$H^+ f(x) := \int_0^x f(s) ds, \quad H^- f(x) := \int_x^\infty f(s) ds.$$

We assumed that kernel of the operators $\mathbb{K}^+$ and $\mathbb{K}^-$ belong to the generalised Oinarov class [48] and thus found exact criteria for the validity of the inequality (0.19) when $1 \leq p \leq q < \infty$ in much more general cases than previously known.

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Paper 1

A weighted differential Hardy inequality on $\overset{\circ}{AC}(I)$

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Remark: The text is the same but the format has been modified to fit the style in this PhD thesis.

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A WEIGHTED DIFFERENTIAL HARDY INEQUALITY ON $\overset{\circ}{AC}(I)$

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Abstract: A weighted differential Hardy inequality is examined on the set of locally absolutely continuous functions vanishing at the endpoints of an interval. Some generalizations of the available results and sharper estimates for the best constant are obtained.

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Keywords: weighted differential Hardy inequality, Lebesgue space, locally absolutely continuous functions

§1. Introduction

Assume that $I = (a, b)$, $-\infty \le a < b \le \infty$, $0 < p, q < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, ρ, v and $\rho^{1-p'} = \frac{1}{\rho^{p'-1}}$ are nonnegative locally summable functions on I and $v \neq 0$.

Let $0 < p < \infty$ and let $L_{p,\rho} \equiv L_{p,\rho}(I)$ be the space of measurable functions f on I such that the norm

$$\left\|f\right\|_{p,\rho} \equiv \left(\int\limits_a^b \rho(t)\left|f(t)\right|^p dt\right)^{\frac{1}{p}}$$

is finite. The symbol $W^1_{p,\rho} \equiv W^1_p(\rho, I)$, $p > 1$, stands for the collection of f locally absollutely continuous on I and having the norm

$$\left\|f\right\|_{W^1_{p,\rho}} = \left\|f'\right\|_{p,\rho} + \left|f(t_0)\right| \tag{20}$$

finite, where $t_0 \in I$ is a fixed point. Assume that $\lim_{t\to a+} f(t) \equiv f(a)$, $\lim_{t\to b-} f(t) \equiv f(b)$, and $\overset{\circ}{AC}_p(\rho, I) = \left\{f \in W^1_{p,\rho} : f(a) = f(b) = 0\right\}$, $AC_{p,l}(\rho, I) = \left\{f \in W^1_{p,\rho} : f(a) = 0\right\}$, $AC_{p,r}(\rho, I) = \left\{f \in W^1_{p,\rho} : f(b) = 0\right\}$.

The closures of $\overset{\circ}{AC}_p(\rho, I)$, $AC_{p,l}(\rho, I)$ and $AC_{p,r}(\rho, I)$ under (20) are denoted respectively by $\overset{\circ}{W}_p(\rho, I)$, $W^1_{p,l}(\rho, I)$ and $W^1_{p,r}(\rho, I)$.

We consider the weighted Hardy inequality in differential form on $\overset{\circ}{AC}_p(\rho, I)$ [1] :

$$\left(\int_a^b \upsilon(t)\left|f(t)\right|^q dt\right)^{\frac{1}{q}} \le C\left(\int_a^b \rho(t)\left|f'(t)\right|^p dt\right)^{\frac{1}{p}}. \tag{21}$$

Inequality (21) and its generalizations were the subject of investigations of many specialists in the last 50 years, and so these are studied well on $AC_{p,l}(\rho, I)$ and $AC_{p,r}(\rho, I)$. The history of the problem and the results can be found in [1, 2, 3]. In the recent years numerous equivalent criterions, ensuring this inequality, are obtained (for instance, see [4, 5]). But (21) is not studied adequately on $\overset{\circ}{AC}_p(\rho, I)$. Some results can be found in [1, 2] and only in the article [1] two-sided estimates for the best constant $C > 0$ of (21) are given.

Various applications of (21) in the qualitative theory of differential equations (see [6, 7, 8, 9]) necessitate studying it on $\overset{\circ}{AC}_p(\rho, I)$ with sharper estimates for the best constant.

In the present article by a method different from that in [1] we establish a more genaral result generalizing those in the above papers and give sharper two-sided estimates for the best constant $C > 0$ in (21).

§2. Necessary Notations and Statements

We study (21) on $\overset{\circ}{AC}_p(\rho, I)$ in dependence on the behavior of ρ at the endpoints of I. The weighted function ρ may vanish at the endpoints of I and thus we have

Theorem A. *Let* $1 < p < \infty$. *Then*

(i) if $\rho^{1-p'} \in L_1(I)$ *then, for every* $f \in W_p^1(\rho, I)$, *there exist* $\lim_{t\to a+} f(t) \equiv f(a)$, $\lim_{t\to b-} f(t) \equiv f(b)$, *and*

$$\overset{\circ}{W}_p(\rho, I) = \left\{f \in W_p^1(\rho, I) : f(a) = f(b) = 0\right\} \equiv \overset{\circ}{AC}_p(\rho, I);$$

(ii) if $\rho^{1-p'} \in L_1(a, c)$ *and* $\rho^{1-p'} \notin L_1(c, b), c \in I$, *then, for every* $f \in W_p^1(\rho, I)$, *there exist* $f(a)$ *and*

$$\overset{\circ}{W}_p(\rho, I) = W_{p,l}^1(\rho, I) = \left\{f \in W_p^1(\rho, I) : f(a) = 0\right\} \equiv AC_{p,l}(\rho, I);$$

(iii) if $\rho^{1-p'} \notin L_1(a, c)$ *and* $\rho^{1-p'} \in L_1(c, b), c \in I$, *then, for every* $f \in W_p^1(\rho, I)$, *there exist* $f(b)$ *and*

$$\overset{\circ}{W}_p(\rho, I) = W^1_{p,r}(\rho, I) = \left\{ f \in W^1_p(\rho, I) : f(b) = 0 \right\} \equiv AC_{p,r}(\rho, I_0)\,;$$

(iv) if $\rho^{1-p'} \notin L_1(a,c)$ *and* $\rho^{1-p'} \notin L_1(c,b), c \in I$, *then*

$$\overset{\circ}{W}_p(\rho, I) = W^1_{p,l}(\rho, I) = W^1_{p,r}(\rho, I) = f \in W^1_p(\rho, I)\,.$$

Generally speaking, the statements of Theorem A are known and they can be deduced from the results in [10, 11, 12]. We present the proof of (*ii*). The remaining statements are proven by analogy.

Assume that $\rho^{1-p'} \in L_1(a,c)$ and $\rho^{1-p'} \notin L_1(c,b), c \in I$. Then for $f \in W^1_p(\rho, I)$ we have

$$\int_a^c \left|f'(t)\right| dt \le \left(\int_a^c \rho^{1-p'} \right)^{\frac{1}{p'}} \left(\int_a^b \rho(t) \left|f'(t)\right|^p dt \right)^{\frac{1}{p}} < \infty.$$

Therefore, $f(a)$ is defined.

Let $f \in W^1_{p,l}(\rho, I)$. Then there exists a sequence $\{f_n\} \subset AC_{p,l}(\rho, I)$ such that $\left\|f - f_n\right\|_{W^1_{p,\rho}} \to 0$ as $n \to \infty$. Since

$$\left|f(t) - f_n(t)\right| \le \int_t^{t_0} \left|f'(s) - f'_n(s)\right| ds + \left|f(t_0) - f_n(t_0)\right|$$

for $a < t < t_0 < b$, the *Hölder* inequality yields

$$\left|f(t) - f_n(t)\right| \le \max\left\{ 1, \left(\int_a^{t_0} \rho^{1-p'} \right)^{\frac{1}{p'}} \right\} \left\|f - f_n\right\|_{W^1_{p,\rho}}.$$

Hence, $f(a) = 0$.

Let $a < \alpha \le t_0 < b$. In this case

$$\left|f(\alpha)\right| \le \left(\int_a^{\alpha} \rho^{1-p'} \right)^{\frac{1}{p'}} \left(\int_a^{\alpha} \rho(t) \left|f'\right|^p dt \right)^{\frac{1}{p}}$$

or

$$\left|f(\alpha)\right| \left(\int_a^{\alpha} \rho^{1-p'} \right)^{-\frac{1}{p'}} \le \left(\int_a^{\alpha} \rho(t) \left|f'\right|^p dt \right)^{\frac{1}{p}}$$

Let a point $\alpha^* = \alpha^*(a,\alpha) \in (a,\alpha)$ satisfy the relation

$$\int_{\alpha^*}^{\alpha} \rho^{1-p'} = \int_{a}^{\alpha^*} \rho^{1-p'}.$$

Introduce a function

$$f_\alpha(t) = \begin{cases} 0, & a < t \leq \alpha^*, \\ f(\alpha)\left(\int\limits_{\alpha^*}^{t} \rho^{1-p'}\right)\left(\int\limits_{\alpha^*}^{\alpha} \rho^{1-p'}\right)^{-1}, & \alpha^* \leq t \leq \alpha, \\ f(t), & \alpha \leq t < b. \end{cases}$$

Obviously, $f_\alpha \in AC_{p,l}(\rho, I)$. We have

$$\left\|f - f_\alpha\right\|_{W_p^1} = \left(\int_a^\alpha \rho\left|f' - f_\alpha'\right|^p\right)^{\frac{1}{p}}$$

$$\leq \left(\int_a^\alpha \rho\left|f'\right|^p\right)^{\frac{1}{p}} + \left|f(\alpha)\right|\left(\int_{\alpha^*}^{\alpha} \rho^{1-p'}\right)^{-\frac{1}{p'}} \leq \left(1 + 2^{\frac{1}{p'}}\right)\left(\int_a^\alpha \rho\left|f'\right|^p\right)^{\frac{1}{p}},$$

and so $\left\|f - f_\alpha\right\|_{W_p^1} \to 0$ as $\alpha \to 0$. Hence $f \in W_{p,l}^1(\rho, I)$ and $W_{p,l}^1(\rho, I) = \left\{f \in W_p^1(\rho, I) : f(a) = 0\right\}$.

Demonstrate that $\overset{\circ}{W}_p(\rho, I) = W_{p,l}^1(\rho, I)$. Since $\overset{\circ}{W}_p(\rho, I) \subset W_{p,l}^1(\rho, I)$, it suffices to establish that $\overset{\circ}{W}_p(\rho, I) \supset W_{p,l}^1(\rho, I)$. Let $f \in W_{p,l}^1(\rho, I)$ and $a < \alpha \leq t_0 < \beta < b$. Since $\int\limits_\beta^b \rho^{1-p'}ds = \infty$, for every $\beta \in I$ there exists a point $\beta^* = \beta^*(\beta, b) \in (\beta, b)$ such that

$$\left|f(\beta)\right|\left(\int_\beta^{\beta^*} \rho^{1-p'}\right)^{-\frac{1}{p'}} \leq \left(\int_\beta^b \rho(t)\left|f(t)\right|^p dt\right)^{\frac{1}{p}}.$$

Construct $f_{\alpha,\beta} \in \overset{\circ}{AC}_p(\rho, I)$ such that

$$f_{\alpha,\beta}(t) = \begin{cases} f_\alpha(t), & a < t \le \beta, \\ f(\beta)\left(\int\limits_\beta^{\beta^*} \rho^{1-p'}\right)^{-1} \int\limits_t^{\beta^*} \rho^{1-p'}, & \beta \le t \le \beta^*, \\ 0, & \beta^* \le t < b. \end{cases}$$

In this case

$$\left\|f - f_{\alpha,\beta}\right\|_{W^1_{p,\rho}} \le \left(\int\limits_a^\alpha \rho\left|f' - f'_\alpha\right|^p\right)^{\frac{1}{p}} + \left(\int\limits_\beta^{\beta^*} \rho\left|f' - f'_{\alpha,\beta}\right|^p\right)^{\frac{1}{p}} + \left(\int\limits_{\beta^*}^b \rho\left|f'\right|^p\right)^{\frac{1}{p}}$$

$$\le \left(1 + 2^{\frac{1}{p'}}\right)\left(\int\limits_a^\alpha \rho\left|f'\right|^p\right)^{\frac{1}{p}} + 2\left(\int\limits_\beta^b \rho\left|f'\right|^p\right)^{\frac{1}{p}} + \left|f(\beta)\right|\left(\int\limits_\beta^{\beta^*} \rho^{1-p'}\right)^{-\frac{1}{p'}}$$

$$\le \left(1 + 2^{\frac{1}{p'}}\right)\left(\int\limits_a^\alpha \rho\left|f'\right|^p\right)^{\frac{1}{p}} + 3\left(\int\limits_\beta^b \rho\left|f'\right|^p\right)^{\frac{1}{p}}.$$

Hence, $\left\|f - f_{\alpha,\beta}\right\|_{W^1_{p,\rho}} \to 0$ as $\alpha \to 0$ and $\beta \to b$. There fore, $f \in \overset{\circ}{W}_p(\rho, I)$. Theorem A is proven.

Let $a \le \alpha < \beta \le b$. Put

$$A_1(\alpha, \beta, x) = \left(\int\limits_\alpha^x \rho^{1-p'}\right)^{\frac{1}{p'}} \left(\int\limits_x^\beta \upsilon(t)dt\right)^{\frac{1}{q}},$$

$$A_2(\alpha, \beta, x) = \left(\int\limits_\alpha^x \rho^{1-p'}\right)^{-\frac{1}{p}} \left(\int\limits_\alpha^x \upsilon(t)\left(\int\limits_\alpha^t \rho^{1-p'}\right)^q dt\right)^{\frac{1}{q}},$$

$$A_1^*(\alpha, \beta, x) = \left(\int\limits_x^\beta \rho^{1-p'}\right)^{-\frac{1}{p'}} \left(\int\limits_\alpha^x \upsilon(t)dt\right)^{\frac{1}{q}},$$

$$A_2^*(\alpha,\beta,x) = \left(\int_x^\beta \rho^{1-p'}\right)^{-\frac{1}{p}} \left(\int_x^\beta v(t)\left(\int_t^\beta \rho^{1-p'}\right)^q dt\right)^{\frac{1}{q}}, \alpha < x < \beta;$$

$$A_i(\alpha,\beta) = \sup_{\alpha<x<\beta} A_i(\alpha,\beta,x),$$

$$A_i^*(\alpha,\beta) = \sup_{\alpha<x<\beta} A_i^*(\alpha,\beta,x), i = 1,2,$$

$$\gamma_1 = \min\left(p^{\frac{1}{q}}\left(p'\right)^{\frac{1}{p'}}, q^{\frac{1}{q}}\left(q'\right)^{\frac{1}{p'}}\right), \gamma_2 = p'.$$

The best constants C in (21) on $\overset{\circ}{AC}_p(\rho,(\alpha,\beta)), AC_{p,l}(\rho,(\alpha,\beta))$ and $AC_{p,r}(\rho,(\alpha,\beta))$ are denoted by $C = J_0(\alpha,\beta)$, $C = J_l(\alpha,\beta)$, and $C = J_r(\alpha,\beta)$, respectively.

In view of [3, 13], we can say that

Theorem B. *Let* $1 < p \leq q < \infty$. *Then*

$$\max\{A_1(\alpha,\beta), A_2(\alpha,\beta)\} \leq J_l(\alpha,\beta) \leq \min\{\gamma_1 A_1(\alpha,\beta), \gamma_2 A_2(\alpha,\beta)\}, \quad (22)$$

$$\max\left\{A_1^*(\alpha,\beta), A_2^*(\alpha,\beta)\right\} \leq J_r(\alpha,\beta) \leq \min\left\{\gamma_1 A_1^*(\alpha,\beta), \gamma_2 A_2^*(\alpha,\beta)\right\}. \quad (23)$$

Assume that

$$B(\alpha,\beta) = \left(\int_\alpha^\beta \left(\int_x^\beta v\right)^{\frac{p}{p-q}} \left(\int_\alpha^x \rho\right)^{\frac{p(q-1)}{p-q}} \rho(x)dx\right)^{\frac{p-q}{pq}},$$

$$B^*(\alpha,\beta) = \left(\int_\alpha^\beta \left(\int_\alpha^x v\right)^{\frac{p}{p-q}} \left(\int_x^\beta \rho\right)^{\frac{p(q-1)}{p-q}} \rho(x)dx\right)^{\frac{p-q}{pq}}.$$

Since $\rho^{1-p'}$ is locally summable on I, we by [3, 14] have (see [14], Remark)

Theorem C. *Let* $0 < q < p < \infty, p > 1$. *Then*

$$\mu^- B(\alpha,\beta) \leq J_l(\alpha,\beta) \leq \mu^+ B(\alpha,\beta), \quad \mu^- B^*(\alpha,\beta) \leq J_r(\alpha,\beta) \leq \mu^+ B^*(\alpha,\beta),$$

where $\mu^- = \left(\frac{p-q}{p}\right)^{\frac{1}{q'}}$, $\mu^+ = \left(p'\right)^{\frac{1}{pq'}} q^{\frac{1}{q}}$ *for* $1 < q < p < \infty$ *and* $\mu^- = q^{\frac{1}{q}}\left(p'\right)^{\frac{1}{q'}} \frac{p-q}{p}$, $\mu^+ = p^{\frac{1}{p}}\left(p'\right)^{\frac{1}{q'}}\left(\frac{p}{p-q}\right)^{\frac{p-q}{pq}}$ *for* $0 < q < 1 < p < \infty$.

§3. The Main Results

3.1. **The case of** $1 < p \le q < \infty$**.** Let

$$\int_a^b \rho^{1-p'}(s)ds < \infty. \tag{24}$$

Definition 1. A point $c_i \in I, i = 1,2,$ is called a *midpoint* for $\left(A_i, A_i^*\right)$ if $A_i(a,c_i) = A_i^*(c_i,b) \equiv T_{c_i}(a,b) < \infty, i = 1,2.$

Theorem 1. *Assume that* $1 < p \le q < \infty$ *and* (24) *holds. Then* (21) *is fulfilled on* $\overset{\circ}{A}C_p(\rho, I)$ *if and only if there exits a midpoint* $c_i \in I$ *for* $\left(A_i, A_i^*\right)$ *at least for one of the numbers* $i = 1,2$ *and the best constant* $J_0(a,b)$ *in* (21) *in this case satisfies the estimate*

$$2^{\frac{q-p}{pq}} max\left\{T_{c_1}(a,b), T_{c_2}(a,b)\right\} \le J_0(a,b) \le min\left\{\gamma_1 T_{c_1}(a,b), \gamma_2 T_{c_2}(a,b)\right\}. \tag{25}$$

Corollary 1 [9]. *In the case of* $p = q$, *we have*

$$max\left\{T_{c_1}(a,b), T_{c_2}(a,b)\right\} \le J_0(a,b) \le min\left\{p^{\frac{1}{p}}\left(p'\right)^{\frac{1}{p'}} T_{c_1}(a,b), p' T_{c_2}(a,b)\right\}.$$

To prove Theorem 1, we use

Lemma 1. *Let* $1 < p \le q < \infty$ *and assume that* (24) *holds. Then a midpoint for* $\left(A_i, A_i^*\right)$, *i=1,2, exists if and only if, for a given* $c \in I$, *there exist*

$$\limsup_{x\to a} A_i(a,c,x) < \infty, \quad \limsup_{x\to b} A_i^*(c,b,x) < \infty, \quad i = 1,2. \tag{26}$$

Proof of Lemma 1. *Sufficiency:* (26) yields

$$\lim_{c\to a} A_i(a,c) < \infty, \quad \lim_{c\to b} A_i^*(c,b) < \infty, \quad i = 1,2.$$

Demonstrate that

$$\lim_{c\to b} A_i(a,c) > \lim_{c\to b} A_i^*(c,b), \quad i = 1,2. \tag{27}$$

Indeed, if

$$\lim_{c\to b} A_i(a,c) \le \lim_{c\to b} A_i^*(c,b) < \infty \tag{28}$$

then (24) implies that

$$\int_c^b v(t)dt < \infty, \quad c \in I.$$

Hence,

$$\lim_{c \to b} A_i^* (c, b) = 0, \quad i = 1, 2. \tag{29}$$

For $i = 1$, (29) is obvious and, for $i = 2$, it follows from the inequality that

$$\left(\int_c^b \rho^{1-p'} \right)^{-\frac{1}{p}} \left(\int_c^b v(t) \left(\int_t^b \rho^{1-p'} \right)^q dt \right)^{\frac{1}{q}} \leq \left(\int_c^b \rho^{1-p'} \right)^{\frac{1}{p'}} \left(\int_c^b v(t)dt \right)^{\frac{1}{q}}.$$

Since $A_i(a, c)$ is a nonnegative nondecreasing continuous function in $c \in I$, from (28) and (29) it follows that $A_i(a, b), i = 1, 2$. Thus, $v(t) \equiv 0$ on I and the latter contradicts the conditions on v. Hence, (27) holds. In the same way, we justify the inequality

$$\lim_{c \to a} A_i^* (c, b) > \lim_{c \to b} A_i (a, c), \quad i = 1, 2. \tag{30}$$

In view of (27) and (30), the continuity and monotonicity of $A_i(a, c)$ and $A_i^*(c, b)$ in $c \in I$ imply the existence of points $c_i \in I$ such that $A_i(a, c_i) = A_i^*(c_i, b), \quad i = 1, 2$.

Necessity: Let a midpoint $c_i \in I$ for $\left(A_i, A_i^*\right)$, $i = 1, 2$, exist. The definition of c_i yields

$$A_i(a, c_i) = A_i^*(c_i, b) < \infty, \quad i = 1, 2.$$

If $c \geq c_1$ then (24) implies that

$$\limsup_{x \to a} A_1(a, c, x) = \lim_{t \to a} \sup_{a<x<t} \left(\int_a^x \rho^{1-p'} \right)^{\frac{1}{p'}} \left(\int_x^c v(t)dt \right)^{\frac{1}{q}}$$

$$\leq \sup_{a<x<c_1} \left(\int_a^x \rho^{1-p'} \right)^{\frac{1}{p'}} \left(\int_x^{c_1} v(t)dt \right)^{\frac{1}{q}} + \lim_{t \to a} \sup_{a<x<t} \left(\int_a^x \rho^{1-p'} \right)^{\frac{1}{p'}} \left(\int_{c_1}^c v(t)dt \right)^{\frac{1}{q}}$$

$$= A_1(a, c_1) + \lim_{t \to a} \left(\int_a^t \rho^{1-p'} \right)^{\frac{1}{p'}} \left(\int_{c_1}^c v(t)dt \right)^{\frac{1}{q}} = A_1(a, c_1) < \infty,$$

$$\lim_{x\to b}\sup A_1^*(c,b,x)=\lim_{t\to b}\sup_{t<x<b}\left(\int_x^b \rho^{1-p'}\right)^{\frac{1}{p'}}\left(\int_c^x v(t)dt\right)^{\frac{1}{q}}$$

$$\leq \sup_{c_1<x<b}\left(\int_x^b \rho^{1-p'}\right)^{\frac{1}{p'}}\left(\int_{c_1}^x v(t)dt\right)^{\frac{1}{q}}=A_1^*(c_1,b)<\infty.$$

In the case of $c<c_1$ we similarly have

$$\lim_{x\to a}\sup A_1(a,c,x)\leq A_1(a,c_1)<\infty,$$

$$\lim_{x\to b}\sup A_1^*(c,b,x)=\lim_{t\to b}\sup_{t<x<b}\left(\int_x^b \rho^{1-p'}\right)^{\frac{1}{p'}}\left(\int_c^x v(t)dt\right)^{\frac{1}{q}}$$

$$\leq A_1^*(c_1,b)+\lim_{t\to a}\left(\int_t^b \rho^{1-p'}\right)^{\frac{1}{p'}}\left(\int_c^{c_1} v(t)dt\right)^{\frac{1}{q}}=A_1^*(c_1,b)<\infty.$$

In the case of A_2 and A_2^* we have

$$\lim_{x\to a}\sup A_2(a,c,x)=\lim_{t\to a}\sup_{a<x<t}\left(\int_a^x \rho^{1-p'}\right)^{-\frac{1}{p}}\left(\int_a^x v(t)\left(\int_t^b \rho^{1-p'}\right)^q dt\right)^{\frac{1}{q}}$$

$$\leq \sup_{a<x<c_2}\left(\int_a^x \rho^{1-p'}\right)^{-\frac{1}{p}}\left(\int_a^x v(t)\left(\int_t^b \rho^{1-p'}\right)^q dt\right)^{\frac{1}{q}}=A_2(a,c_2)<\infty$$

for every $c\in I$ and similarly

$$\lim_{x\to b}\sup A_2^*(c,b,x)\leq A_2^*(c_2,b)<\infty.$$

Lemma 1 is proven.

Proof of Theorem 1. *Necessity:* Let (21) hold on $\overset{\circ}{AC}_p(\rho,I)$ with the best constant $C=J_0(a,b)$. Assume that $a<c^-<c<c^+<b$. Put

$$f_0(t) = \begin{cases} \left(\int\limits_a^{c^-} \rho^{1-p'}\right)^{-1} \int\limits_a^t \rho^{1-p'}, & a < t < c^-, \\ 1, & c^- \leq t \leq c^+, \\ \left(\int\limits_{c^+}^b \rho^{1-p'}\right)^{-1} \int\limits_t^b \rho^{1-p'}, & c^+ \leq t < b. \end{cases} \tag{31}$$

The function f_0 is locally absolutely continuous on I and

$$\int\limits_a^b \rho(s) \left|f_0'(s)\right|^p ds = \int\limits_a^{c^-} \rho(s) \left|f_0'(s)\right|^p ds + \int\limits_{c^-}^{c^+} \rho(s) \left|f_0'(s)\right|^p ds + \int\limits_{c^+}^b \rho(s) \left|f_0'(s)\right|^p ds$$

$$= \left(\int\limits_a^{c^-} \rho^{1-p'}\right)^{-p} \int\limits_a^{c^-} \rho\rho^{p(1-p')} + \left(\int\limits_{c^+}^b \rho^{1-p'}\right)^{-p} \int\limits_{c^+}^b \rho\rho^{p(1-p')}$$

$$= \left(\int\limits_a^{c^-} \rho^{1-p'}\right)^{1-p} + \left(\int\limits_{c^+}^b \rho^{1-p'}\right)^{1-p} < \infty. \tag{32}$$

Hence, $f_0 \in W_p^1(\rho, I)$ and

$$\lim_{t \to a+} f_0(t) \equiv f_0(a) = 0, \quad \lim_{t \to b-} f_0(t) \equiv f_0(b) = 0$$

by construction. In this case $f_0 \in \overset{\circ}{AC}_p(\rho, I)$. Inserting f_0 in (21), we have

$$J_0(a, b) \geq \frac{\left(\int\limits_a^b \upsilon(t) \left|f_0(t)\right|^q dt\right)^{\frac{1}{q}}}{\left(\int\limits_a^b \rho(t) \left|f_0'(t)\right|^p dt\right)^{\frac{1}{p}}} \tag{33}$$

The direct calculation yields

$$\int\limits_a^b \upsilon(t) \left|f_0(t)\right|^q dt = \int\limits_a^{c^-} \upsilon(t) \left|f_0(t)\right|^q dt + \int\limits_{c^-}^{c^+} \upsilon(t) \left|f_0(t)\right|^p dt + \int\limits_{c^+}^b \upsilon(t) \left|f_0(t)\right|^q dt$$

$$= \left(\int\limits_a^{c^-} \rho^{1-p'}\right)^{-q} \int\limits_a^{c^-} \upsilon(t) \left(\int\limits_a^t \rho^{1-p'}\right)^q dt$$

$$+\int\limits_{c^-}^{c^+} v(t)dt+\left(\int\limits_{c^+}^{b}\rho^{1-p'}\right)^{-q}\int\limits_{c^+}^{b} v(t)\left(\int\limits_{t}^{b}\rho^{1-p'}\right)^{q} dt. \tag{34}$$

By (32) - (34), we obtain the inequalities

$$J_0^q(a,b)\geq \frac{\left(\int\limits_a^{c^-}\rho^{1-p'}\right)^{-q}\int\limits_a^{c^-} v(t)\left(\int\limits_a^{t}\rho^{1-p'}\right)^{q}dt}{\left(\left(\int\limits_a^{c^-}\rho^{1-p'}\right)^{1-p}+\left(\int\limits_{c^+}^{b}\rho^{1-p'}\right)^{1-p}\right)^{\frac{q}{p}}}+\frac{\left(\int\limits_{c^+}^{b}\rho^{1-p'}\right)^{-q}\int\limits_{c^+}^{b} v(t)\left(\int\limits_t^{b}\rho^{1-p'}\right)^{q}dt}{\left(\left(\int\limits_a^{c^-}\rho^{1-p'}\right)^{1-p}+\left(\int\limits_{c^+}^{b}\rho^{1-p'}\right)^{1-p}\right)^{\frac{q}{p}}}, \tag{35}$$

$$J_0^q(a,b)\geq \frac{\int\limits_{c^-}^{c} v(t)dt+\int\limits_{c}^{c^+} v(t)dt}{\left(\left(\int\limits_a^{c^-}\rho^{1-p'}\right)^{1-p}+\left(\int\limits_{c^+}^{b}\rho^{1-p'}\right)^{1-p}\right)^{\frac{q}{p}}}. \tag{36}$$

Multiplying the numerator and denominator of the right-hand sides in (35) and (36) by $\left(\int\limits_a^{c^-}\rho^{1-p'}\right)^{\frac{q}{p'}}$, we derive

$$J_0^q(a,b)\geq \frac{\left(\int\limits_a^{c^-}\rho^{1-p'}\right)^{-\frac{q}{p}}\int\limits_a^{c^-} v(t)\left(\int\limits_a^{t}\rho^{1-p'}\right)^{q}dt}{\left(1+\left(\int\limits_a^{c^-}\rho^{1-p'}\right)^{p-1}\left(\int\limits_{c^+}^{b}\rho^{1-p'}\right)^{1-p}\right)^{\frac{q}{p}}}+\frac{\left(\int\limits_a^{c^-}\rho^{1-p'}\right)^{\frac{q}{p'}}\left(\int\limits_{c^+}^{b}\rho^{1-p'}\right)^{-q}\int\limits_{c^+}^{b} v(t)\left(\int\limits_t^{b}\rho^{1-p'}\right)^{q}dt}{\left(1+\left(\int\limits_a^{c^-}\rho^{1-p'}\right)^{p-1}\left(\int\limits_{c^+}^{b}\rho^{1-p'}\right)^{1-p}\right)^{\frac{q}{p}}}, \tag{37}$$

$$J_0^q(a,b)\geq \frac{\left(\int\limits_a^{c^-}\rho^{1-p'}\right)^{\frac{q}{p'}}\int\limits_{c^-}^{c} v(t)dt+\left(\int\limits_a^{c^-}\rho^{1-p'}\right)^{\frac{q}{p'}}\int\limits_{c}^{c^+} v(t)dt}{\left(1+\left(\int\limits_a^{c^-}\rho^{1-p'}\right)^{p-1}\left(\int\limits_{c^+}^{b}\rho^{1-p'}\right)^{1-p}\right)^{\frac{q}{p}}}. \tag{38}$$

Since the left-hand sides of (37) and (38) are independet of $c^-\in(a,c)$, passing to the limit as $c^-\to a$ on the right-hand sides, we infer

$$J_0^q(a,b) \geq \frac{\lim\limits_{x\to a}\sup\left(\int\limits_a^x \rho^{1-p'}\right)^{-\frac{q}{p}} \int\limits_a^x v(t)\left(\int\limits_a^t \rho^{1-p'}\right)^q dt}{\left(1+\lim\limits_{c^-\to a}\left(\int\limits_a^{c^-}\rho^{1-p'}\right)^{p-1}\left(\int\limits_{c^+}^b \rho^{1-p'}\right)^{1-p}\right)^{\frac{q}{p}}}$$

$$+\frac{\lim\limits_{c^-\to a}\left(\int\limits_a^{c^-}\rho^{1-p'}\right)^{\frac{q}{p'}}\left(\int\limits_{c^+}^b \rho^{1-p'}\right)^{-q}\int\limits_{c^+}^b v(t)\left(\int\limits_t^b \rho^{1-p'}\right)^q dt}{\left(1+\lim\limits_{c^-\to a}\left(\int\limits_a^{c^-}\rho^{1-p'}\right)^{p-1}\left(\int\limits_{c^+}^b \rho^{1-p'}\right)^{1-p}\right)^{\frac{q}{p}}}$$

$$=\lim_{x\to a}\sup\left(\int\limits_a^x \rho^{1-p'}\right)^{-\frac{q}{p'}}\int\limits_a^x v(t)\left(\int\limits_a^t \rho^{1-p'}\right)^q dt = \lim_{x\to a}\sup A_2^q(a,c,x)\,, \tag{39}$$

$$J_0^q(a,b) \geq \frac{\lim\limits_{x\to a}\sup\left(\int\limits_a^x \rho^{1-p'}\right)^{\frac{q}{p'}}\int\limits_x^c v(t)dt + \lim\limits_{c^-\to a}\left(\int\limits_a^{c^-}\rho^{1-p'}\right)^{\frac{q}{p'}} v(t)dt}{\left(1+\lim\limits_{c^-\to a}\left(\int\limits_a^{c^-}\rho^{1-p'}\right)^{p-1}\left(\int\limits_{c^+}^b \rho^{1-p'}\right)^{1-p}\right)^{\frac{q}{p}}}$$

$$=\lim_{x\to a}\sup\left(\int\limits_a^x \rho^{1-p'}\right)^{\frac{q}{p'}}\int\limits_x^c v(t)dt = \lim_{x\to a}\sup A_1^q(a,c,x)\,. \tag{40}$$

Multiplying the numerator and denominator of the right-hand sides in (35) and (36) by $\left(\int\limits_{c^+}^b \rho^{1-p'}\right)^{\frac{q}{p'}}$ and passing to the limit as $c^+ \to b$, we obtain

$$J_0^q(a,b) \geq \frac{\lim\limits_{c^+\to b}\left(\int\limits_{c^+}^b \rho^{1-p'}\right)^{\frac{q}{p'}}\left(\int\limits_a^{c^-}\rho^{1-p'}\right)^{-q}\int\limits_a^{c^-} v(t)\left(\int\limits_a^t \rho^{1-p'}\right)^q dt}{\left(\lim\limits_{c^+\to b}\left(\int\limits_{c^+}^b \rho^{1-p'}\right)^{p-1}\left(\int\limits_a^{c^-}\rho^{1-p'}\right)^{1-p}+1\right)^{\frac{q}{p}}}$$

$$+\frac{\lim\limits_{x\to b}\sup\left(\int\limits_x^b \rho^{1-p'}\right)^{-\frac{q}{p}}\int\limits_x^b v(t)\left(\int\limits_t^b \rho^{1-p'}\right)^q dt}{\left(\lim\limits_{c^+\to b}\left(\int\limits_{c^+}^b \rho^{1-p'}\right)^{p-1}\left(\int\limits_a^{c^-} \rho^{1-p'}\right)^{1-p}+1\right)^{\frac{q}{p}}}$$

$$=\lim_{x\to b}\sup\left(\int\limits_x^b \rho^{1-p'}\right)^{-\frac{q}{p}}\int\limits_x^b v(t)\left(\int\limits_t^b \rho^{1-p'}\right)^q dt=\lim_{x\to b}\sup\left(A_2^*(c,b,x)\right)^q, \quad (41)$$

$$J_0^q(a,b)\geq\frac{\lim\limits_{c^+\to b}\left(\int\limits_{c^+}^b \rho^{1-p'}\right)^{\frac{q}{p'}}\int\limits_{c^-}^c v(t)dt+\lim\limits_{x\to b}\sup\left(\int\limits_x^b \rho^{1-p'}\right)^{\frac{q}{p'}}\int\limits_c^x v(t)dt}{\left(\lim\limits_{c^+\to b}\left(\int\limits_{c^+}^b \rho^{1-p'}\right)^{p-1}\left(\int\limits_a^{c^-} \rho^{1-p'}\right)^{1-p}+1\right)^{\frac{q}{p}}}$$

$$=\lim_{x\to b}\sup\left(\int\limits_x^b \rho^{1-p'}\right)^{\frac{q}{p'}}\int\limits_c^x v(t)dt=\lim_{x\to b}\sup\left(A_1^*(c,b,x)\right)^q. \quad (42)$$

Relations (39) - (42) ensure (26). By Lemma 1, there exist midpoints $c_i\in I$ for $\left(A_i,A_i^*\right)$, i=1,2. Definition 1 yields the equality $A_i(a,c_i)=A_i^*(c_i,b)\equiv T_{c_i}(a,b)<\infty, i=1,2$.

Since $A_i(a,c_i,x)$ and $A_i^*(c_i,b,x)$ are continuous in x on $(a,c_i]$ and $[c_i,b)$, respectively, and

$$A_i(a,c_i)\geq\lim_{x\to a}\sup A_i(a,c_i,x), \quad A_i^*(c_i,b)\geq\lim_{x\to b}\sup A_i^*(c_i,b,x),$$

the two possibilities are open: If $A_i(a,c_i)=\lim_{x\to a}\sup A_i(a,c_i,x)$ or $A_i^*(c_i,b)=\lim_{x\to b}\sup A_i^*(c_i,b,x)$ then (39) - (42) imply the estimate $J_0(a,b)\geq T_{c_i}(a,b), i=1,2$, i.e., the left part of (25) holds. Otherwise, there exist points c_i^-,c_i^+, $a<c_i^-\leq c_i$, $c_i\leq c_i^+<b$, such that $c_1^-\neq c_1$, $c_1^+\neq c_1$, $A_i(a,c_i)=A_i\left(a,c_i,c_i^-\right)$, and $A_i^*(c_i,b)=A_i^*\left(c_i,b,c_i^+\right)$.

To justify the left estimate in (25), we estimate $T_{c_1}(a,b)$ and $T_{c_2}(a,b)$ separately. First, we examine $T_{c_2}(a,b)$.

Let $c^-=c_2^-$ in (35). Estimate (37) yields

$$J_0^q(a,b) \geq \frac{\left(\int\limits_a^{c_2^-} \rho^{1-p'}\right)^{-\frac{q}{p}} \int\limits_a^{c_2^-} v(t)\left(\int\limits_a^t \rho^{1-p'}\right)^q dt \left(\int\limits_{c_2^+}^b \rho^{1-p'}\right)^{\frac{q}{p'}}}{\left(\left(\int\limits_{c_2^+}^b \rho^{1-p'}\right)^{p-1} + \left(\int\limits_a^{c_2^-} \rho^{1-p'}\right)^{p-1}\right)^{\frac{q}{p}}}$$

$$+\frac{\left(\int\limits_{c_2^+}^b \rho^{1-p'}\right)^{-\frac{q}{p}} \int\limits_{c_2^+}^b v(t)\left(\int\limits_t^b \rho^{1-p'}\right)^q dt \left(\int\limits_a^{c_2^-} \rho^{1-p'}\right)^{\frac{q}{p'}}}{\left(\left(\int\limits_{c_2^+}^b \rho^{1-p'}\right)^{p-1} + \left(\int\limits_a^{c_2^-} \rho^{1-p'}\right)^{p-1}\right)^{\frac{q}{p}}}$$

(we take the expressions for $A_2\left(a, c_2, c_2^-\right)$ and $A_2^*\left(c_2, b, c_2^+\right)$ into account)

$$= \frac{\left(A_2\left(a, c_2, c_2^-\right)\right)^q \left(\int\limits_{c_2^+}^b \rho^{1-p'}\right)^{\frac{q}{p'}} + \left(A_2^*\left(c_2, b, c_2^+\right)\right)^q \left(\int\limits_a^{c_2^-} \rho^{1-p'}\right)^{\frac{q}{p'}}}{\left(\left(\int\limits_{c_2^+}^b \rho^{1-p'}\right)^{p-1} + \left(\int\limits_a^{c_2^-} \rho^{1-p'}\right)^{p-1}\right)^{\frac{q}{p}}}$$

(by the definition of c_2, we have $A_2(a, c_2) = A_2\left(a, c_2, c_2^-\right)$ and $A_2^*(a, c_2) = A_2^*\left(c_2, b, c_2^+\right)$)

$$= \frac{\left(A_2(a, c_2,)\right)^q \left(\int\limits_{c_2^+}^b \rho^{1-p'}\right)^{\frac{q}{p'}} + \left(A_2^*(c_2, b)\right)^q \left(\int\limits_a^{c_2^-} \rho^{1-p'}\right)^{\frac{q}{p'}}}{\left(\left(\int\limits_{c_2^+}^b \rho^{1-p'}\right)^{p-1} + \left(\int\limits_a^{c_2^-} \rho^{1-p'}\right)^{p-1}\right)^{\frac{q}{p}}}$$

(since c_2 is a midpoint for $\left(A_2, A_2^*\right)$)

$$= \left(T_{c_2}(a,b)\right)^q \frac{\left(\int\limits_{c_2^+}^{b} \rho^{1-p'}\right)^{\frac{q}{p'}} + \left(\int\limits_{a}^{c_2^-} \rho^{1-p'}\right)^{\frac{q}{p'}}}{\left(\left(\int\limits_{c_2^+}^{b} \rho^{1-p'}\right)^{p-1} + \left(\int\limits_{a}^{c_2^-} \rho^{1-p'}\right)^{p-1}\right)^{\frac{q}{p}}} \geq 2^{1-\frac{q}{p}} \left(T_{c_2}(a,b)\right)^q. \tag{43}$$

Estimate $T_{c_1}(a,b)$. Similarly, putting $c = c_1$, $c^- = c_1^-$, and $c^+ = c_2^+$ in (36), in view of (38) we obtain

$$J_0^q(a,b) \geq \frac{\left(\int\limits_{a}^{c_1^-} \rho^{1-p'}\right)^{\frac{q}{p'}} \int\limits_{c_1^-}^{c_1} v(t)dt \left(\int\limits_{c_1^+}^{b} \rho^{1-p'}\right)^{\frac{q}{p'}}}{\left(\left(\int\limits_{c_2^+}^{b} \rho^{1-p'}\right)^{p-1} + \left(\int\limits_{a}^{c_2^-} \rho^{1-p'}\right)^{p-1}\right)^{\frac{q}{p}}} + \frac{\left(\int\limits_{c_1^+}^{b} \rho^{1-p'}\right)^{\frac{q}{p'}} \int\limits_{c_1}^{c_1^+} v(t)dt \left(\int\limits_{a}^{c_1^-} \rho^{1-p'}\right)^{\frac{q}{p'}}}{\left(\left(\int\limits_{c_2^+}^{b} \rho^{1-p'}\right)^{p-1} + \left(\int\limits_{a}^{c_2^-} \rho^{1-p'}\right)^{p-1}\right)^{\frac{q}{p}}}$$

$$= \frac{\left(A_1\left(a, c_1, c_1^-\right)\right)^q \left(\int\limits_{c_1^+}^{b} \rho^{1-p'}\right)^{\frac{q}{p'}} + \left(A_1^*\left(c_1, b, c_1^+\right)\right)^q \left(\int\limits_{a}^{c_1^-} \rho^{1-p'}\right)^{\frac{q}{p'}}}{\left(\left(\int\limits_{c_2^+}^{b} \rho^{1-p'}\right)^{p-1} + \left(\int\limits_{a}^{c_2^-} \rho^{1-p'}\right)^{p-1}\right)^{\frac{q}{p}}}$$

$$= \frac{\left(A_1(a, c_1,)\right)^q \left(\int\limits_{c_1^+}^{b} \rho^{1-p'}\right)^{\frac{q}{p'}} + \left(A_1^*(c_1, b)\right)^q \left(\int\limits_{a}^{c_1^-} \rho^{1-p'}\right)^{\frac{q}{p'}}}{\left(\left(\int\limits_{c_2^+}^{b} \rho^{1-p'}\right)^{p-1} + \left(\int\limits_{a}^{c_2^-} \rho^{1-p'}\right)^{p-1}\right)^{\frac{q}{p}}}$$

$$= \left(T_{c_1}(a,b)\right)^q \frac{\left(\int\limits_{c_1^+}^{b} \rho^{1-p'}\right)^{\frac{q}{p'}} + \left(\int\limits_{a}^{c_1^-} \rho^{1-p'}\right)^{\frac{q}{p'}}}{\left(\left(\int\limits_{c_2^+}^{b} \rho^{1-p'}\right)^{p-1} + \left(\int\limits_{a}^{c_2^-} \rho^{1-p'}\right)^{p-1}\right)^{\frac{q}{p}}} \geq 2^{1-\frac{q}{p}} \left(T_{c_1}(a,b)\right)^q. \tag{44}$$

The left estimate in (25) results from (43) and (44). The necessity is proven.

Sufficiency: Assume the existence of a midpoint $c_i \in I$ for (A_i, A_i^*), $i = 1, 2$. In this case we have $A_i(a, c_i) = A_i^*(c_i, b) = T_{c_i}(a, b) < \infty$, $i = 1, 2$. Since $f(a) = f(b) = 0$ for $f \in \overset{\circ}{AC}_p(\rho, I)$, the restriction of $\overset{\circ}{AC}_p(\rho, I)$ to (a, c_i) and (c_i, b) belongs to $AC_{p,l}(\rho, (a, c_i))$ and $AC_{p,r}(\rho, (c_i, b))$, respectively.

Theorem 8 implies that

$$\int_a^b v(t)\left|f(t)\right|^q dt = \int_a^{c_i} v(t)\left|f(t)\right|^q dt + \int_{c_i}^b v(t)\left|f(t)\right|^q dt$$

$$\leq (\gamma_i A_i(a, c_i))^q \left(\int_a^{c_i} \rho(s)\left|f'(s)\right|^p ds\right)^{\frac{q}{p}} + \left(\gamma_i A_i^*(c_i, b)\right)^q \left(\int_{c_i}^b \rho(s)\left|f'(s)\right|^p ds\right)^{\frac{q}{p}}$$

$$\leq (\gamma_i T_{c_i}(a, b))^q \left(\int_a^{c_i} \rho(s)\left|f'(s)\right|^p ds + \int_{c_i}^b \rho(s)\left|f'(s)\right|^p ds\right)^{\frac{q}{p}}$$

$$= (\gamma_i T_{c_i}(a, c_i))^q \left(\int_a^b \rho(s)\left|f'(s)\right|^p ds\right)^{\frac{q}{p}},$$

i.e., (21) holds and the best constant $C = J_0(a, b)$ in (21) meets the estimate

$$J_0(a, b) \leq \min\left\{\gamma_1 T_{c_1}(a, b),\ \ \gamma_2 T_{c_2}(a, b)\right\},$$

which defines the right-hand side of (25). Theorem 1 is proven.

Remark 1. Theorem 1 improves the estimate for $J_0(a, b)$ in [1]. For example, in Theorem 8.8 of [1], under the assumption $A_1(a, a) = A_1^*(b, b) = 0$ (the latter is equivalent to the conditions $\lim_{x\to a} A_1(a, c, x) = 0$ and $\lim_{x\to b} A_1^*(c, b, x) = 0$), it is established that

$$2^{-\frac{1}{p}} A \leq J_0(a, b) \leq \left(1 + \frac{q}{p'}\right)^{\frac{1}{q}} \left(1 + \frac{p'}{q}\right)^{\frac{1}{p'}} A,$$

where $A = \inf_{a<c<b} \max\left\{A_1(a, c), A_1^*(c, b)\right\}$.

Under our conditions, it is easily seen that $A = T_1(a, b)$.

Let

$$\int_a^c \rho^{1-p'}(s) ds < \infty, \quad \int_c^b \rho^{1-p'}(s) ds = \infty, \quad c \in I, \tag{45}$$

or

$$\int_a^c \rho^{1-p'}(s)ds = \infty, \quad \int_c^b \rho^{1-p'}(s)ds < \infty, \quad c \in I. \tag{46}$$

Theorem 2. *Let* $1 < p \le q < \infty$. *If* (45) *or* (46) *holds then the best constant* $J_0(a,b)$ *in* (21) *meets the estimate*

$$\max\{A_1(a,b), A_2(a,b)\} \le J_0(a,b) \le \min\{\gamma_1 A_1(a,b), \gamma_2 A_2(a,b)\} \tag{47}$$

$$\max\left\{A_1^*(a,b), A_2^*(a,b)\right\} \le J_0(a,b) \le \min\left\{\gamma_1 A_1^*(a,b), \gamma_2 A_1^*(a,b)\right\} \tag{48}$$

PROOF OF THEOREM 2. Since $\overset{\circ}{AC}_p(\rho, I)$ is dense everywhere in $\overset{\circ}{W}_p(\rho, I)$,

$$J_0(a,b) = \sup_{f\in \overset{\circ}{AC}_p(\rho,I)} \frac{\left(\int_a^b v(t)\left|f(t)\right|^q dt\right)^{\frac{1}{q}}}{\left(\int_a^b \rho(t)\left|f'(t)\right|^p dt\right)^{\frac{1}{p}}} = \sup_{f\in \overset{\circ}{W}_p(\rho,I)} \frac{\left(\int_a^b v(t)\left|f(t)\right|^q dt\right)^{\frac{1}{q}}}{\left(\int_a^b \rho(t)\left|f'(t)\right|^p dt\right)^{\frac{1}{p}}}. \tag{49}$$

Let (45) hold. In view of item (*ii*) of Theorem *A*, $\overset{\circ}{W}_p(\rho, I) = \left\{f \in W_p^1(\rho, I) : f(a) = 0\right\} = AC_{p,l}(\rho, I)$. Hence, $J_0(I) = J_l(I)$ and (47) follows from Theorem B. By analogy we justify (48) in the case (46). Theorem 2 is proven.

Finally, let

$$\int_a^c \rho^{1-p'}(s)ds = \infty, \quad \int_c^b \rho^{1-p'}(s)ds = \infty, \quad c \in I. \tag{50}$$

Theorem 3. *Assume that* $1 < p \le q < \infty$ *and* (50) *is valid. Then* (21) *fails on the set* $\overset{\circ}{W}_p(\rho, I)$, *i.e.,* $J_0(a,b) = \infty$.

PROOF OF THEOREM 3. By condition, (50) holds. By Theorem A (item (iv)) $\overset{\circ}{W}_p(\rho, I) = W_p^1(\rho, I)$. Since $f(x) \equiv 1 \in W_p^1(\rho, I)$, (49) yields $J_0(a,b) = \infty$. Theorem 3 is proven.

3.2. The case of $0 < q < p < \infty$.

Definition 2. A point $c \in I$ is called a *midpoint* for (B, B^*) if $B(a,c) = B^*(c,b) \equiv T(a,b) < \infty$.

Obviously, for a midpoint for (B, B^*) to exist, it is necessary and sufficient that $B(a,\beta) < \infty$, $\beta \in I$, and $B^*(\alpha,b) < \infty$, $\alpha \in I$.

Theorem 4. *Assume that* $0 < q < p < \infty$, $p > 1$, *and* (24) *holds. Then* (21) *is fulfilled on* $\overset{\circ}{AC}_p(\rho, I)$ *if and only if there exists a midpoint* $c \in I$ *for* (B, B^*) *; in this case the best constant* $J_0(a,b)$ *in* (21) *satisfies the estimate*

$$q^{\frac{1}{q}} \left(\frac{p-q}{p-1}\right)^{\frac{1}{q'}} T(a,b) \le J_0(a,b) \le 2^{\frac{p-q}{pq}} \mu^+ T(a,b). \tag{51}$$

PROOF. *Necessity:* Assume that $0 < q < p < \infty$, $p > 1$, and (21) holds on $\overset{\circ}{AC}_p(\rho, I)$ with $C = J_0(a,b)$. Let $a < \alpha < \beta < b$. In view of the conditions on the weighted functions v and ρ, the quantities $B(\alpha,c)$, $c \in (\alpha,b)$, $B^*(c,\beta)$, and $c \in (a,\beta)$ are finite. Hence, there exists a midpoint $c = c(\alpha,\beta) \in (\alpha,\beta)$ for $B(\alpha,\beta)$ and $B^*(\alpha,\beta)$, i.e.,

$$\int_\alpha^c \left(\int_x^c v\right)^{\frac{p}{p-q}} \left(\int_\alpha^x \rho^{1-p'}\right)^{\frac{p(q-1)}{p-q}} \rho^{1-p'}(x)dx = \int_c^\beta \left(\int_c^x v\right)^{\frac{p}{p-q}} \left(\int_x^\beta \rho^{1-p'}\right)^{\frac{p(q-1)}{p-q}} \rho^{1-p'}(x)dx. \tag{52}$$

Introduce the function

$$f_0(t) = \begin{cases} 0, & a < t \le \alpha, \\ \frac{1}{b_1} \int_\alpha^t \left(\int_x^c v\right)^{\frac{1}{p-q}} \left(\int_\alpha^x \rho^{1-p'}\right)^{\frac{(q-1)}{p-q}} \rho^{1-p'}(x)dx, & \alpha \le t \le c, \\ \frac{1}{b_2} \int_t^\beta \left(\int_c^x v\right)^{\frac{1}{p-q}} \left(\int_x^\beta \rho^{1-p'}\right)^{\frac{(q-1)}{p-q}} \rho^{1-p'}(x)dx, & c \le t \le \beta, \\ 0, & \beta \le t \le b, \end{cases}$$

where

$$b_1 = \int_\alpha^c \left(\int_x^c v\right)^{\frac{1}{p-q}} \left(\int_\alpha^x \rho^{1-p'}\right)^{\frac{(q-1)}{p-q}} \rho^{1-p'}(x)dx,$$

$$b_2 = \int_c^\beta \left(\int_c^x v\right)^{\frac{1}{p-q}} \left(\int_x^\beta \rho^{1-p'}\right)^{\frac{(q-1)}{p-q}} \rho^{1-p'}(x)dx.$$

Obviously, $f_0 \in \overset{\circ}{AC}_p(\rho, I)$. For a function f_0 we have

$$\left(\int_a^b \rho(t)\left|f_0'(t)\right|^p dt\right)^{\frac{1}{p}} = \left(\frac{1}{b_1^p}(B(\alpha,c))^{\frac{qp}{p-q}} + \frac{1}{b_2^p}(B^*(c,\beta))^{\frac{qp}{p-q}}\right)^{\frac{1}{p}}$$

$$= (T(\alpha,\beta))^{\frac{q}{p-q}}\left(\frac{1}{b_1^p} + \frac{1}{b_2^p}\right)^{\frac{1}{p}}, \tag{53}$$

$$\int_a^b v(t)\left|f_0(t)\right|^q dt = \int_\alpha^c v(t)(f_0(t))^q dt + \int_c^\beta v(t)(f_0(t))^q dt$$

$$= q\int_\alpha^c f_0'(f_0(t))^{q-1}\int_t^c v(s)dsdt + q\int_c^\beta \left(-f_0'(t)\right)(f_0(t))^{q-1}\int_c^t v(s)dsdt. \tag{54}$$

Since

$$f_0(t) \geq \frac{1}{b_1}\left(\int_t^c v\right)^{\frac{1}{p-q}}\int_\alpha^t\left(\int_\alpha^x \rho^{1-p'}\right)^{\frac{q-1}{p-q}}\rho^{1-p'}(x)dx = \frac{1}{b_1}\frac{p-q}{p-1}\left(\int_t^c v\right)^{\frac{1}{p-q}}\left(\int_\alpha^x \rho^{1-p'}\right)^{\frac{p-1}{p-q}}$$

for $\alpha \leq t \leq c$ and similarly

$$f_0(t) \geq \frac{1}{b_1}\frac{p-q}{p-1}\left(\int_c^t v\right)^{\frac{1}{p-q}}\left(\int_t^\beta \rho^{1-p'}\right)^{\frac{p-1}{p-q}}$$

for $c \leq t \leq \beta$, we infer

$$\int_\alpha^c f_0'(t)(f_0(t))^{q-1}\int_t^c v(s)dsdt \geq \left(\frac{1}{b_1}\frac{p-q}{p-1}\right)^{q-1}(B(\alpha,c))^{\frac{pq}{p-q}},$$

$$\int_c^\beta \left(-f_0'(t)\right)(f_0(t))^{q-1}\int_c^t v(s)dsdt \geq \left(\frac{1}{b_2}\frac{p-q}{p-1}\right)^{q-1}(B^*(c,\beta))^{\frac{pq}{p-q}}.$$

Hence, (54) yields

$$\left(\int_a^b v(t)\left|f_0(t)\right|^q dt\right)^{\frac{1}{q}} \geq q^{\frac{1}{q}}\left(\frac{p-q}{p-1}\right)^{\frac{1}{q'}}\left(\frac{1}{b_1^q}(B(\alpha,c))^{\frac{pq}{p-q}} + \frac{1}{b_2^q}(B^*(c,\beta))^{\frac{pq}{p-q}}\right)^{\frac{1}{q}}$$

$$= q^{\frac{1}{q}} \left(\frac{p-q}{p-1}\right)^{\frac{1}{q'}} (T(\alpha,\beta))^{\frac{p}{p-q}} \left(\frac{1}{b_1^q} + \frac{1}{b_2^q}\right)^{\frac{1}{q}}.$$

Since $\frac{p}{q} > 1$, we have $\left[\left(\frac{1}{b_1^q} + \frac{1}{b_2^q}\right)^{\frac{p}{q}}\right]^{\frac{1}{p}} \geq \left(\frac{1}{b_1^p} + \frac{1}{b_2^p}\right)^{\frac{1}{p}}$. Hence,

$$\left(\int_a^b v(t) \left|f_0(t)\right|^q dt\right)^{\frac{1}{q}} \geq q^{\frac{1}{q}} \left(\frac{p-q}{p-1}\right)^{\frac{1}{q'}} (T(\alpha,\beta))^{\frac{p}{p-q}} \left(\frac{1}{b_1^p} + \frac{1}{b_2^p}\right)^{\frac{1}{p}}. \tag{55}$$

Relations (21), (53) and (55) imply that

$$q^{\frac{1}{q}} \left(\frac{p-q}{p-1}\right)^{\frac{1}{q'}} T(\alpha,\beta) \leq J_0(a,b). \tag{56}$$

The absolute continuity of the integral ensures the continuity of $T(\alpha,\beta)$ in α and β for $a \leq \alpha < \beta \leq b$. In view of the independence of the right-hand side (56) of α and β, $a < \alpha < \beta < b$, we have

$$q^{\frac{1}{q}} \left(\frac{p-q}{p-1}\right)^{\frac{1}{q'}} T(a,b) \leq J_0(a,b), \tag{57}$$

i.e., there exists a midpoint $c \in I$ for (B, B^*) and (57) is true.

Sufficiency: Let a midpoint $c \in I$ for (B, B^*) exist, i.e., $B(a,c) = B^*(c,b) = T(a,b) < \infty$. Arguing as in the sufficiency part of Theorem 1 and involing Theorem C, we derive that

$$\int_a^b v(t) \left|f(t)\right|^q dt = \int_\alpha^c v(t) \left|f(t)\right|^q dt + \int_c^\beta v(t) \left|f(t)\right|^q dt$$

$$\leq (\mu^+ B(a,c))^q \left(\int_a^c \rho(s) \left|f'(s)\right|^p ds\right)^{\frac{q}{p}} + (\mu^+ B^*(c,b))^q \left(\int_c^b \rho(s) \left|f'(s)\right|^p ds\right)^{\frac{q}{p}}$$

$$\leq (\mu^+ T(a,b))^q \, 2^{\frac{p-q}{p}} \left(\int_\alpha^c \rho(s) \left|f'(s)\right|^p ds + \int_c^\beta \rho(s) \left|f'(s)\right|^p ds\right)^{\frac{q}{p}},$$

i.e., (21) is fulfilled and $J_0(a,b) \leq \mu^+ 2^{\frac{p-q}{p}} T(a,b)$; the last estimate and (57) ensure (51). Theorem 4 is proven.

Remark 2. The comparision of (51) and the estimate

$$2^{-\frac{1}{p}} q^{\frac{1}{q}} \left(\frac{p-q}{p-1} \right)^{\frac{1}{q'}} \tilde{B} \le J_0(a,b) \le 2^{\frac{p-q}{pq}} (p')^{\frac{1}{pq'}} q^{\frac{1}{q}} \tilde{B},$$

where $\tilde{B} = \min_{a<c<b} \max \{B(a,c), B(b,c)\}$, obtained in the case of $1 \le q < p < \infty$ in Theorem 8.17 of [1], shows that the estimate in (51) is better than that of [1].

Theorem 5. *Let* $0 < q < p < \infty, p > 1$. *If* (26) *or* (27) *holds then the best constant* $J_0(a,b)$ *in* (21) *satisfies the estimate* $\mu^- B(a,b) \le J_0(a,b) \le \mu^+ B(a,b)$ *or* $\mu^- B^*(a,b) \le J_0(a,b) \le \mu^+ B^*(a,b)$, *respectively.*

Theorem 6. *Assume that* $0 < q < p < \infty, p > 1$, *and* (50) *holds. Then* (21) *fails on* $\overset{\circ}{AC}_p(\rho, I)$; *i.e.,* $J_0(a,b) = \infty$.

Theorems 5 and 6 are proven by analogy with Theorem 2 and 3.

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Paper 2

Boundedness, compactness for a class of fractional integration operators of Weyl type.

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BOUNDEDNESS, COMPACTNESS FOR A CLASS OF FRACTIONAL INTEGRATION OPERATORS OF WEYL TYPE

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Abstract. We establish criteria for the boundedness and compactness for a class of operators of fractional integration involving the Weyl operator.

1 Introduction

Let $I = (a,b)$, $0 \le a < b \le \infty$, $0 < q,p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$. Let u, v be almost everywhere positive and locally integrable functions on I. By $L_{p,u} \equiv L_p(u,I)$ we denote the set of all measurable functions f on I such that

$$\|f\|_{p,u} = \left(\int_a^b |f(x)|^p u(x) dx \right)^{\frac{1}{p}} < \infty.$$

In the case $u \equiv 1$ we write $L_p \equiv L_p(I)$. Let W be a positive strictly increasing and locally absolutely continuous function on I. Suppose $\frac{dW(x)}{dx} \equiv w(x)$ for almost everywhere $x \in I$.

Let $1 > \alpha > 0$. We consider the operator

$$K_{\alpha,\beta} f(x) = \int_x^b \frac{u(s)W^\beta(s)f(s)w(s)ds}{(W(s) - W(x))^{1-\alpha}}, \quad x \in I. \tag{1}$$

In the case $\beta = 0$, $u \equiv 1$ the dual operator to operator (1) has the form

$$K^*_{\alpha,\beta} f(x) = \int_a^x \frac{f(s)w(s)ds}{(W(x) - W(s))^{1-\alpha}}, \quad x \in I. \tag{2}$$

Operator (2) is called [15] the operator of fractional integration of the function f of the function W. Weighted estimates for operator (2) were previously considered in [12], [1].

When $W(x) = x$, $u \equiv 1$, $\beta = 0$ operator (1) is the Weyl operator

$$I_\alpha^* f(x) = \int_x^b \frac{f(s)ds}{(s-x)^{1-\alpha}}, \quad x \in I, \tag{3}$$

which is dual to the Riemann-Liouville operator

$$I_\alpha g(s) = \int_a^s \frac{g(x)dx}{(s-x)^{1-\alpha}}, \quad s \in I. \tag{4}$$

Operators (3) and (4) acting from the weighted space $L_{p,u}$ to the weighted space $L_{q,v}$ are investigated in papers [3], [3], [7], [15], [20], [21] and others, where necessary and sufficient conditions for their boundedness, compactness are obtained for various relations between the parameters α, p, q and under various assumptions regarding the weight functions u and v. Two-sided estimates of their norms are also obtained.

We investigate operator (1) acting from the space $L_{p,w}$ to $L_{q,v}$. From the obtained results new assertions follow, in simple terms, for operators (3) and (4), generalizing the results of [7], [15], [20].

The positivity and monotonicity of W implies the existence of the non-negative limit $\lim\limits_{x\to a^+} W(x) \equiv W(a)$. Futher, we assume $W(a) = 0$ and otherwise, we consider the operator $K_{\alpha,\beta}$ in the form, where function $W(x)$ is replaced by the function $W_0(x) = W(x) - W(a), x \in I$.

Further, the norm of the linear operator T from a normed space to another one is denoted briefly by $\|T\|$. Which spaces are meant will be clear from the context.

Throughout the paper the products of the form $0 \cdot \infty$ are supposed be equal to zero. Relations $A \ll B$, $A \gg B$ mean $A \leq cB$ with a constant c depending only on p, q, α which can be different in different places. If $A \ll B$ and $A \gg B$ then we write $A \approx B$. By $\mathbb{Z}$ we denote the set of all integer numbers, χ_E denotes the characteristic function of the set E.

2 Auxiliary assertions

To prove the main results we need some well-known assertions.

Along with operator (1) we consider the Hardy operator

$$H_{\alpha,\beta}f(x) = \int\limits_x^b u(s)W^{\beta+\alpha-1}(s)f(s)w(s)ds. \tag{1}$$

It is easy to see that for $f \geq 0$

$$K_{\alpha,\beta}f(x) \geq H_{\alpha,\beta}f(x), \quad \forall x \in I. \tag{2}$$

Issues of boundedness and compactness of operator (1) in weighted Lebesgue spaces were studied quite completely. A summary of the results can be found in [5]. The following Theorem A and Theorem B are corollaries of Theorem 5 and Theorem 6 in [5].

Theorem A. *Let* $1 < p \leq q < \infty$. *The operator* $H_{\alpha,\beta}$ *is bounded from* $L_{p,w}$ *to* $L_{q,v}$ *if and only if*

$$A_{\alpha,\beta} = \sup_{z\in I}\left(\int\limits_a^z v(x)dx\right)^{\frac{1}{q}}\left(\int\limits_z^b u^{p'}(s)W^{p'(\alpha+\beta-1)}(s)w(s)ds\right)^{\frac{1}{p'}} < \infty.$$

Moreover, $\|H_{\alpha,\beta}\| \approx A_{\alpha,\beta}$.

Theorem B. *Let* $0 < q < p < \infty$, $p > 1$. *The operator* $H_{\alpha,\beta}$ *is bounded from* $L_{p,w}$ *to* $L_{q,v}$ *if and only if*

$$B_{\alpha,\beta} = \left(\int\limits_a^b\left(\int\limits_z^b u^{p'}(s)W^{p'(\alpha+\beta-1)}w(s)ds\right)^{\frac{q(p-1)}{p-q}}\right.$$
$$\left.\times\left(\int_a^z v(x)dx\right)^{\frac{q}{p-q}} v(z)dz\right)^{\frac{p-q}{pq}} < \infty.$$

Moreover, $\|H_{\alpha,\beta}\| \approx B_{\alpha,\beta}$.

Remark 2.1. *In the case* $1 < q < p < \infty$, $p > 1$ *the value* $B_{\alpha,\beta}$ *is equivalent to the value*

$$\widetilde{B}_{\alpha,\beta}(a,b) = \left(\int\limits_a^b\left(\int\limits_z^b u^{p'}(s)W^{p'(\alpha+\beta-1)}(s)w(s)ds\right)^{\frac{p(q-1)}{p-q}}\right.$$
$$\left.\times\left(\int_a^z v(x)dx\right)^{\frac{p}{p-q}} u^{p'}(z)W^{p'(\alpha+\beta-1)}(z)w(z)dz\right)^{\frac{p-q}{pq}}.$$

Remark 2.2. *Note that a function u non-decreasing on I and such that $uW^{\beta+\alpha-1} \in L_{p',w}(z,b)$, for all $z \in I$, exists if and only if $W^{\beta+\alpha-1} \in L_{p',w}(z,b)$ for all $z \in I$.*

3 Boundedness of the operator $K_{\alpha,\beta}$

Theorem 3.1. *Let $0 < \alpha < 1$, $\frac{1}{\alpha} < p \le q < \infty$ and $\beta \le 0$ ($\beta < \frac{1}{p} - \alpha$ when $W(b) = \infty$). Let u be a non-decreasing function on I. Then the operator $K_{\alpha,\beta}$ is bounded from $L_{p,w}$ to $L_{q,v}$ if and only if $A_{\alpha,\beta} < \infty$. Moreover, $\|K_{\alpha,\beta}\| \approx A_{\alpha,\beta}$.*

Proof. Necessity. Let the operator $K_{\alpha,\beta}$ be bounded from $L_{p,w}$ to $L_{q,v}$. Then, in view of (2), the operator $H_{\alpha,\beta}$ is bounded from $L_{p,w}$ to $L_{q,v}$ and $\|K_{\alpha,\beta}\| \ge \|H_{\alpha,\beta}\|$, therefore by Theorem A the value $A_{\alpha,\beta} < \infty$ and

$$\|K_{\alpha,\beta}\| \gg A_{\alpha,\beta}. \tag{1}$$

Sufficiency. Since the function W is continuous and strictly increasing on I and $W(a) = 0$, then for any $k \in \mathbb{Z}$ the set $\left\{x : W(x) \le 2^k, x \in I\right\}$ is non-empty. Denoting $x_k = \sup\left\{x : W(x) \le 2^k, x \in I\right\}$ we obtain a sequence of points $\{x_k\}_{k\in\mathbb{Z}}$ such that $0 < x_k \le x_{k+1}$, $\forall k \in \mathbb{Z}$, and if $x_k < b$, then $W(x_k) = 2^k$, $2^k \le W(x) \le 2^{k+1}$ for $x_k \le x \le x_{k+1}$, $\int\limits_{x_{k-1}}^{x_k} w(s)ds = 2^{k-1}$, and if $x_{k+1} = b$, then $\int\limits_{x_k}^{x_{k+1}} w(s)ds \le 2^k$. These facts will be used below without reminders. We assume that $I_k = [x_k, x_{k+1})$, $k \in \mathbb{Z}$, $\mathbb{Z}_0 = \{k : k \in \mathbb{Z},\ I_k \ne \emptyset\}$. Then $\mathbb{Z}_0 \subseteq \mathbb{Z}$ and $I = \bigcup\limits_{k\in\mathbb{Z}} I_k = \bigcup\limits_{k\in\mathbb{Z}_0} I_k$. Since $I_k = \emptyset$, $\forall k \in \mathbb{Z} \setminus \mathbb{Z}_0$, and integrals over these intervals are equal to zero, then in the sequel, without loss of generality, we suppose that $\mathbb{Z} = \mathbb{Z}_0$.

Let $A_{\alpha,\beta} < \infty$. We need to prove that the inequality

$$\|T_{\alpha,\beta} f\|_{q,v} \ll A_{\alpha,\beta}\|f\|_{p,w}, \quad f \in L_{p,w}, \tag{2}$$

holds, which means $\|T_{\alpha,\beta}\| \ll A_{\alpha,\beta}$ and, together with (1), gives

$$\|T_{\alpha,\beta}\| \approx A_{\alpha,\beta}.$$

It suffices to prove inequality (2) for $f \ge 0$. So let $f \ge 0$. Using the relation $I = \bigcup\limits_k I_k$, we have

$$\|K_{\alpha,\beta} f\|_{q,v}^q = \sum_k \int\limits_{x_{k-1}}^{x_k} v(x) \left(\int\limits_x^b \frac{u(s)W^\beta(s) f(s) w(s) ds}{(W(s) - W(x))^{1-\alpha}} \right)^q dx$$

$$= \sum_k \int_{x_{k-1}}^{x_k} v(x) \left[\left(\int_x^{x_{k+1}} + \int_{x_{k+1}}^{b} \right) \frac{u(s)W^{\beta}(s)f(s)w(s)ds}{(W(s)-W(x))^{1-\alpha}} \right]^q dx$$

$$\ll \sum_k \int_{x_{k-1}}^{x_k} v(x) \left(\int_x^{x_{k+1}} \frac{u(s)W^{\beta}(s)f(s)w(s)ds}{(W(s)-W(x))^{1-\alpha}} \right)^q dx$$

$$+ \sum_k \int_{x_{k-1}}^{x_k} v(x) \left(\int_{x_{k+1}}^{b} \frac{u(s)W^{\beta}(s)f(s)w(s)ds}{(W(s)-W(x))^{1-\alpha}} \right)^q dx = J_1 + J_2. \tag{3}$$

We estimate the values J_1 and J_2 separately. Using Hölder's inequality, nondecreasing of the function u and $\beta \leq 0$ and in view of change of variables $W(s) = W(x)t$ we have

$$J_1 = \sum_k \int_{x_{k-1}}^{x_k} v(x) \left(\int_x^{x_{k+1}} \frac{u(s)W^{\beta}(s)f(s)w(s)ds}{(W(s)-W(x))^{1-\alpha}} \right)^q dx$$

$$\leq \sum_k \int_{x_{k-1}}^{x_k} v(x) \left(\int_x^{x_{k+1}} |f(s)|^p w(s)ds \right)^{\frac{q}{p}} \left(\int_x^{x_{k+1}} \frac{u^{p'}(s)W^{p'\beta}(s)w(s)ds}{(W(s)-W(x))^{(1-\alpha)p'}} \right)^{\frac{q}{p'}} dx$$

$$\leq \sum_k \left(\int_{x_{k-1}}^{x_{k+1}} |f(s)|^p w(s)ds \right)^{\frac{q}{p}} u^q(x_{k+1}) \int_{x_{k-1}}^{x_k} v(x) W^{\frac{q}{p'}(p'\beta + p'(\alpha-1))}(x)$$

$$\times W^{\frac{q}{p'}}(x) \left(\int_1^{\frac{W(x_{k+1})}{W(x_{k-1})}} t^{p'\beta}(t-1)^{p'(\alpha-1)}dt \right)^{\frac{q}{p'}} dx$$

$$\leq \sum_k \left(\int_{x_{k-1}}^{x_{k+1}} |f(s)|^p w(s)ds \right)^{\frac{q}{p}} u^q(x_{k+1})$$

$$\times 2^{\frac{q}{p'}(p'(\beta+\alpha-1))(k-1)} 2^{\frac{q}{p'}k} \left(\int_1^4 t^{p'\beta}(t-1)^{p'(\alpha-1)}dt \right)^{\frac{q}{p'}} \int_{x_{k-1}}^{x_k} v(x)dx. \tag{4}$$

By the assumptions of the theorem $\alpha > \frac{1}{p}$, therefore $\int\limits_1^4 t^{p'\beta}(t-1)^{p'(\alpha-1)}dt < \infty$.

The expression $F = u^q(x_{k+1})2^{q(\beta+\alpha-1)(k-1)}2^{\frac{q}{p'}k}$ is estimated as follows. Since $\beta+\alpha-1<0$ then

$$F = u^q(x_{k+1})2^{3q|\beta+\alpha-1|}2^{q(\beta+\alpha-1)(k+2)}2^{-\frac{q}{p'}}2^{\frac{q}{p'}(k+1)}$$

$$= 2^{3q|\beta+\alpha-1|-\frac{q}{p'}}u^q(x_{k+1})2^{q(\beta+\alpha-1)(k+2)}\left(\int\limits_{x_{k+1}}^{x_{k+2}} w(s)ds\right)^{\frac{q}{p'}}$$

$$\leq 2^{3q|\beta+\alpha-1|-\frac{q}{p'}}\left(\int\limits_{x_{k+1}}^{x_{k+2}} W^{p'(\beta+\alpha-1)}(s)u^{p'}(s)w(s)ds\right)^{\frac{q}{p'}}.$$

Substituting this estimate in (4) we obtain

$$J_1 \ll \sum_k \left(\int\limits_{x_{k-1}}^{x_{k+1}} |f(s)|^p w(s)ds\right)^{\frac{q}{p}} \int\limits_{x_{k-1}}^{x_k} v(x)dx$$

$$\times\left(\int\limits_{x_{k+1}}^{x_{k+2}} u^{p'}(s)W^{p'(\beta+\alpha-1)}w(s)ds\right)^{\frac{q}{p'}}$$

$$\leq A^q_{\alpha,\beta}\sum_k\left(\int\limits_{x_{k-1}}^{x_{k+1}} |f(s)|^p w(s)ds\right)^{\frac{q}{p}} \leq A^q_{\alpha,\beta}\left(\sum_k\int\limits_{x_{k-1}}^{x_{k+1}} |f(s)|^p w(s)ds\right)^{\frac{q}{p}}$$

$$\ll A^q_{\alpha,\beta}||f||^q_{p,w}. \tag{5}$$

Now, we estimate J_2.

$$J_2 = \sum_k \int\limits_{x_{k-1}}^{x_k} v(x)\left(\int\limits_{x_{k+1}}^{b} \frac{u(s)W^\beta(s)f(s)w(s)ds}{(W(s)-W(x))^{1-\alpha}}\right)^q dx$$

$$\leq \sum_k \int\limits_{x_{k-1}}^{x_k} v(x)\left(\int\limits_{x_{k+1}}^{b} \frac{u(s)W^\beta(s)f(s)w(s)ds}{(W(s)-W(x_k))^{1-\alpha}}\right)^q dx$$

$$\leq \sum_k \int_{x_{k-1}}^{x_k} v(x) \left(\int_{x_{k+1}}^{b} \frac{u(s)W^{\beta}(s)f(s)w(s)ds}{(W(s) - \frac{1}{2}W(x_{k+1}))^{1-\alpha}} \right)^q dx$$

$$\leq 2^{q(1-\alpha)} \sum_k \int_{x_{k-1}}^{x_k} v(x) \left(\int_{x_{k+1}}^{b} \frac{u(s)W^{\beta}(s)f(s)w(s)}{(W(s))^{1-\alpha}} \right)^q dx$$

$$\ll \int_a^b v(x) \left(\int_x^b u(s)W^{\beta+\alpha-1}(s)f(s)w(s)ds \right)^q dx = \|H_{\alpha,\beta}f\|^q_{q,v}. \tag{6}$$

Then, by Theorem A

$$J_2 \ll A^q_{\alpha,\beta}\|f\|^q_{p,w}. \tag{7}$$

Inequalities (3), (5) and (7) imply inequality (2). □

Theorem 3.2. *Let* $0 < \alpha < 1$, $0 < q < p < \infty$, $p > \frac{1}{\alpha}$ *and* $\beta \leq 0$ ($\beta < \frac{1}{p} - \alpha$ *in the case* $W(b) = \infty$). *Let u be a non-decreasing function on I. Then the operator* $K_{\alpha,\beta}$ *is bounded from* $L_{p,w}$ *to* $L_{q,v}$ *if and only if* $B_{\alpha,\beta} < \infty$. *Moreover,* $\|K_{\alpha,\beta}\| \approx B_{\alpha,\beta}$.

Proof. Necessity and the estimate

$$\|K_{\alpha,\beta}\| \gg B_{\alpha,\beta} \tag{8}$$

follows by relation (2) and Theorem B.
Sufficiency. Let $B_{\alpha,\beta} < \infty$. If the inequality

$$\|K_{\alpha,\beta}f\|_{q,v} \ll B_{\alpha,\beta}\|f\|_{p,w}, \tag{9}$$

holds then by (8) and (9) we obtain $\|K_{\alpha,\beta}\| \approx B_{\alpha,\beta}$.

To prove (9) we use relation (3) of Theorem 3.1. Estimate for J_2 directly follows by (6) and Theorem B:

$$J_2 \ll B^q_{\alpha,\beta}\|f\|^q_{p,w}. \tag{10}$$

By (5) we have

$$J_1 \ll \sum_k \left(\int_{x_{k-1}}^{x_{k+1}} |f(s)|^p w(s)ds \right)^{\frac{q}{p}} \int_{x_{k-1}}^{x_k} v(x)dx$$

$$\times \left(\int_{x_k}^{b} u^{p'}(s)W^{p'(\beta+\alpha-1)}(s)w(s)ds \right)^{\frac{q}{p'}}$$

(applying the Hölder inequality with the exponents $\frac{p}{q}$, $\frac{p}{p-q}$)

$$\leq \left(\sum_k \left(\int_{x_{k-1}}^{x_k} v(x)dx \right)^{\frac{p}{p-q}} \left(\int_{x_k}^{b} u^{p'}(s) W^{p'(\beta+\alpha-1)}(s) w(s) ds \right)^{\frac{q(p-1)}{p-q}} \right)^{\frac{p-q}{p}}$$

$$\times \left(\sum_k \int_{x_{k-1}}^{x_{k+1}} |f(s)|^p w(s) ds \right)^{\frac{q}{p}} \leq G \|f\|_{p,w}^q, \quad (11)$$

where

$$G = \left(\sum_k \left(\int_{x_{k-1}}^{x_k} v(x)dx \right)^{\frac{p}{p-q}} \left(\int_{x_k}^{b} u^{p'}(s) W^{p'(\beta+\alpha-1)}(s) w(s) ds \right)^{\frac{q(p-1)}{p-q}} \right)^{\frac{p-q}{p}}.$$

Using the relation

$$\left(\int_{x_{k-1}}^{x_k} v(x)dx \right)^{\frac{p}{p-q}} = \frac{p}{p-q} \int_{x_{k-1}}^{x_k} v(x) \left(\int_{x_{k-1}}^{x} v(t)dt \right)^{\frac{q}{p-q}} dx$$

we estimate G:

$$G \ll \left(\sum_k \int_{x_{k-1}}^{x_k} v(x) \left(\int_{x_{k-1}}^{x} v(t)dt \right)^{\frac{q}{p-q}} dx \right.$$

$$\left. \times \left(\int_{x_k}^{b} u^{p'}(s) W^{p'(\beta+\alpha-1)}(s) w(s) ds \right)^{\frac{q(p-1)}{p-q}} \right)^{\frac{p-q}{p}}$$

$$\leq \left(\sum_k \int_{x_{k-1}}^{x_k} \left(\int_{a}^{x} v(t)dt \right)^{\frac{q}{p-q}} \right.$$

$$\left. \times \left(\int_{x}^{b} u^{p'}(s) W^{p'(\beta+\alpha-1)}(s) w(s) ds \right)^{\frac{q(p-1)}{p-q}} v(x)dx \right)^{\frac{p-q}{p}}$$

$$\leq B^q_{\alpha,\beta}. \tag{12}$$

By (11) and (12) it follows that

$$J_1 \ll B^q_{\alpha,\beta}\|f\|^q_{p,w}. \tag{13}$$

Therefore, by (3), (10) and (13) it follows that inequality (9) holds. □

4 The compactness of the operator $K_{\alpha,\beta}$

Theorem 4.1. *Let $0 < \alpha < 1$, $\frac{1}{\alpha} < p \leq q < \infty$ and $\beta \leq 0$ ($\beta < \frac{1}{p} - \alpha$ if $W(b) = \infty$). Let u be a non-decreasing function on I. Then the operator $K_{\alpha,\beta}$ is compact from $L_{p,w}$ to $L_{q,v}$ if and only if $A_{\alpha,\beta} < \infty$ and*

$$\lim_{z\to a^+} A_{\alpha,\beta}(z) = \lim_{z\to b^-} A_{\alpha,\beta}(z) = 0,$$

where

$$A_{\alpha,\beta}(z) = \left(\int_a^z v(x)dx\right)^{\frac{1}{q}} \left(\int_z^b u^{p'}(s)W^{p'(\beta+\alpha-1)}(s)w(s)ds\right)^{\frac{1}{p'}}.$$

Proof. Necessity. Let the operator $K_{\alpha,\beta}$ be compact from $L_{p,w}$ to $L_{q,v}$. Then the operator is bounded and therefore, by Theorem 3.1, $A_{\alpha,\beta} < \infty$. First, we prove that $\lim_{z\to b^-} A_{\alpha,\beta}(z) = 0$.

Let $F(t) = \int_t^b u^{p'}(s)W^{p'(\beta+\alpha-1)}(s)w(s)ds$. Since $A_{\alpha,\beta} < \infty$ and function u non-decreasing then $0 < F(t) < \infty$ for $t \in I$. Consider the family of functions $\{f_t\}_{t\in I}$, where

$$f_t(x) = \chi_{(t,b)}(x)u^{p'-1}(x)W^{(p'-1)(\beta+\alpha-1)}(x)(F(t))^{-\frac{1}{p}}. \tag{1}$$

Then

$$\int_a^b |f_t(x)|^p w(x)dx = (F(t))^{-1}\int_t^b u^{p'}(x)W^{p'(\beta+\alpha-1)}(x)w(x)dx \equiv 1. \tag{2}$$

We show that the family of functions $\{f_t\}$ weakly converges to zero in $L_{p,w}$. Let $g \in L_{p',w^{1-p'}} = (L_{p,w})^*$.

Applying the Holder inequality and using (2) we have

$$\int_a^b f_t(x)g(x)dx \leq \left(\int_t^b |f_t(x)|^p w(x)dx\right)^{\frac{1}{p}} \left(\int_t^b |g(x)|^{p'} w^{1-p'}(x)dx\right)^{\frac{1}{p'}}$$

$$= \left(\int_t^b |g(x)|^{p'} w^{1-p'}(x)dx \right)^{\frac{1}{p'}}.$$

Since $g \in L_{p',w^{1-p'}}$ then the last integral converges to zero as $t \to b$, which means the weak convergence to zero the family of function $\{f_t\}$. Then, by the compactness of the operator $K_{\alpha,\beta}$ from $L_{p,w}$ to $L_{q,v}$

$$\lim_{z \to b^-} ||K_{\alpha,\beta} f_t||_{q,v} = 0. \tag{3}$$

We have

$$||K_{\alpha,\beta} f_t||_{q,v}^q = \int_a^b v(x) \left(\int_x^b \frac{u(s)W^\beta(s) f_t(s) w(s) ds}{(W(s) - W(x))^{1-\alpha}} \right)^q dx$$

$$\geq \int_a^t v(x) \left(\int_t^b \frac{u(s)W^\beta(s) f_t(s) w(s) ds}{(W(s) - W(x))^{1-\alpha}} \right)^q dx$$

$$\geq \int_a^t v(x)dx \left(\int_t^b u(s) W^{\beta+\alpha-1}(s) f_t(s) w(s) ds \right)^q$$

$$= (F(t))^{-\frac{q}{p}} \left(\int_t^b u^{p'}(s) W^{p'(\beta+\alpha-1)}(s) w(s) ds \right)^q \int_a^t v(x)dx = \left(A_{\alpha,\beta}(t) \right)^q. \tag{4}$$

By (3) and (4) we obtain that $\lim\limits_{t \to b^-} A_{\alpha,\beta}(t) = 0$.

Now, we show $\lim\limits_{t \to a^+} A_{\alpha,\beta}(t) = 0$.

The compactness of the operator $K_{\alpha,\beta} : L_{p,w} \to L_{q,v}$ implies the compactness of the adjoint operator

$$K^*_{\alpha,\beta} g(x) = u(s) W^\beta(s) w(s) \int_a^s \frac{g(x)dx}{(W(s) - W(x))^{1-\alpha}}$$

from $L_{q',v^{1-q'}}$ to $L_{p',w^{1-p'}}$.

We introduce the family of functions $\{g_t\}_{t \in I}$, where

$$g_t(x) = \chi_{(a,t)}(x) \left(\int_a^t v(x)dx \right)^{-\frac{1}{q'}} v(x).$$

Since almost everywhere $v > 0$ and $A_{\alpha,\beta} < \infty$ then the function g_t is well defined.

In view of the equality

$$\int_a^b |g_t(x)|^{q'} v^{1-q'}(x)dx = \left(\int_a^t v(x)dx\right)^{-1} \left(\int_a^t v(x)dx\right) = 1$$

for $f \in L_{q,v} = (L_{q',v^{1-q'}})^*$ we have

$$\int_a^b f(x)g_t(x)dx \leq \left(\int_a^t |f(x)|^q v(x)dx\right)^{\frac{1}{q}} \left(\int_a^t |g_t(x)|^{q'} v^{1-q'}(x)dx\right)^{\frac{1}{q'}}$$

$$= \left(\int_a^t |f(x)|^q v(x)dx\right)^{\frac{1}{q}}.$$

Consequently $\lim\limits_{t\to a^+} \int\limits_a^b f(x)g_t(x)dx = 0$ for any $f \in L_{q,v}$, which means the weak convergence to zero the family of functions g_t. Then by the compactness of the operator $K^*_{\alpha,\beta}$ from $L_{q',v^{1-q'}}$ to $L_{p',w^{1-p'}}$

$$\lim_{t\to a^+} ||K^*_{\alpha,\beta} g_t||_{p',w^{1-p'}} = 0. \tag{5}$$

We have

$$||K^*_{\alpha,\beta} g_t||^{p'}_{p',w^{1-p'}} \geq \int_t^b |u(s)W^\beta(s)w(s)|^{p'} \left(\int_a^t \frac{g_t(x)dx}{(W(s)-W(x))^{1-\alpha}}\right)^{p'} w^{1-p'}(s)ds$$

$$\geq \int_t^b u^{p'}(s)W^{p'(\beta+\alpha-1)}(s)w(s)ds \left(\int_a^t v(x)dx\right)^{-\frac{p'}{q'}} \left(\int_a^t v(x)dx\right)^{p'} = \left(A_{\alpha,\beta}(t)\right)^{p'}. \tag{6}$$

By (5) and (6) it follows that $\lim\limits_{t\to a^+} A_{\alpha,\beta}(t) = 0$. The necessity is proved.

Sufficiency. Let $A_{\alpha,\beta} < \infty$ and $\lim\limits_{z\to a^+} A_{\alpha,\beta}(z) = \lim\limits_{z\to b^-} A_{\alpha,\beta}(z) = 0$.

Yet for $a < c < d < b$

$$P_c f = \chi_{(a,c]} f, \;\; P_{cd} f = \chi_{(c,d]} f, \;\; Q_d f = \chi_{(d,b)} f.$$

Then $f = P_c f + P_{cd} f + Q_d f$ and by the equalities $P_{cd}K_{\alpha,\beta}Q_d \equiv 0$, $P_{cd}K_{\alpha,\beta}P_c \equiv 0$, $Q_d K_{\alpha,\beta}P_c \equiv 0$, we have

$$K_{\alpha,\beta} f = P_{cd}K_{\alpha,\beta}P_{cd} f + Q_d K_{\alpha,\beta}Q_d f + P_{cd}K_{\alpha,\beta}Q_d f + P_c K_{\alpha,\beta} f. \tag{7}$$

We show that the operator $P_{cd}K_{\alpha,\beta}P_{cd}$ is compact from $L_{p,w}$ to $L_{q,v}$. Since $P_{cd}K_{\alpha,\beta}P_{cd} f(x) = 0$ when $x \in I \backslash (c, d]$ then it suffices to show that the operator $P_{cd}K_{\alpha,\beta}P_{cd}$ is compact from $L_{p,w}(c, d)$ to $L_{q,v}(c, d)$ and this is equivalent to the compactness of the operator $Kf(x) = \int_c^d K(x, s) f(s) ds$ with the kernel

$$K(x, s) = \frac{u(s) W^{\beta}(s) v^{\frac{1}{q}}(x) \chi_{(c,d)}(s - x) w^{\frac{1}{p'}}(s)}{(W(s) - W(x))^{1-\alpha}}$$

from L_p to L_q.

Let $\{x_k\}_{k \in \mathbb{Z}}$ be a sequence of points introduces in the proof of Theorem 3.1. There are the points x_{i-1}, x_n, $x_{i-1} < x_n$ such that $x_{i-1} \leq c < x_i$, $x_{n-1} < d \leq x_n$. We assume that the number c, d are chosen so that $x_i < x_{n-1}$. Similarly to obtaining estimates of J_1, J_2 in Theorem 3.1, we have

$$\int_c^d \left(\int_c^d |K(x,s)|^{p'} ds \right)^{\frac{q}{p'}} dx = \int_c^d v(x) \left(\int_x^d \frac{u^{p'}(s) W^{p'\beta}(s) w(s) ds}{(W(s) - W(x))^{p'(1-\alpha)}} \right)^{\frac{q}{p'}} dx$$

$$\leq \sum_{k=i}^{n} \int_{x_{k-1}}^{x_k} v(x) \left[\left(\int_{x_{k+1}}^{b} + \int_x^{x_{k+1}} \right) \frac{u^{p'}(s) W^{p'\beta}(s) w(s) ds}{(W(s) - W(x))^{p'(1-\alpha)}} \right]^{\frac{q}{p'}} dx$$

$$\leq \mu(n - i + 1) A^q_{\alpha,\beta} < \infty,$$

where the constant μ does not depend on i, n. Therefore, on the basis of the theorem in Kantorovich and Akilov [2] (page 420), the operator K is compact from $L_p(c, d)$ to $L_q(c, d)$, which is equivalent to the compactness of the operator $P_{cd}K_{\alpha,\beta}P_{cd}$ from $L_{p,w}$ to $L_{q,v}$.

By (7) we have

$$||K_{\alpha,\beta} - P_{cd}K_{\alpha,\beta}P_{cd}|| \leq ||Q_d K_{\alpha,\beta}Q_d|| + ||P_{cd}K_{\alpha,\beta}Q_d|| + ||P_c K_{\alpha,\beta}||. \tag{8}$$

We shall show that the right-hand side of (8) tends to zero as $c \to a^+$, $d \to b^-$. This will imply that the operator $K_{\alpha,\beta}$ being a uniform limit of compact operators, is compact from $L_{p,w}$ to $L_{q,v}$.

On the basis of Theorem 3.1, we have:

$$\|Q_dK_{\alpha,\beta}Q_df\|_{q,v} = \left(\int_d^b v(x)\left(\int_x^b \frac{u(s)W^\beta(s)f(s)w(s)ds}{(W(s)-W(x))^{1-\alpha}}\right)^q dx\right)^{\frac{1}{q}}$$

$$\ll \sup_{d<z<b}\left(\int_d^z v(x)dx\right)^{\frac{1}{q}}\left(\int_z^b u^{p'}(s)W^{p'(\beta+\alpha-1)}(s)w(s)ds\right)^{\frac{1}{p'}}\|f\|_{p,w}$$

$$\leq \sup_{d<z<b} A_{\alpha,\beta}(z)\|f\|_{p,w}.$$

Hence

$$\lim_{d\to b^-}\|Q_dK_{\alpha,\beta}Q_df\| \ll \lim_{d\to b^-}\sup_{d<z<b} A_{\alpha,\beta}(z) = \lim_{z\to b^-} A_{\alpha,\beta}(z) = 0; \tag{9}$$

Let $1 > \varepsilon > 0$. To estimate $\|P_{cd}K_{\alpha,\beta}Q_df\|_{q,v}$ we introduce the functions v_ε, u_ε defined by $v_\varepsilon(x) = v(x)$ for $x \in (a,d]$ and $v_\varepsilon(x) = \varepsilon^q v(x)$ for $x \in I \setminus (a,d]$, $u_\varepsilon(s) = u(s)$ for $s \in (d,b)$ and $u_\varepsilon(s) = \varepsilon u(s)$ for $s \in I \setminus (d,b)$. Obviously, the function u_ε is non-decreasing on I. Then by Theorem 3.1

$$\|P_{cd}K_{\alpha,\beta}Q_df\|_{q,v} = \left(\int_c^d v(x)\left(\int_d^b \frac{u(s)W^\beta(s)f(s)w(s)ds}{(W(s)-W(x))^{1-\alpha}}\right)^q dx\right)^{\frac{1}{q}}$$

$$\leq \left(\int_a^b v_\varepsilon(x)\left(\int_x^b \frac{u_\varepsilon(s)W^\beta(s)f(s)w(s)ds}{(W(s)-W(x))^{1-\alpha}}\right)^q dx\right)^{\frac{1}{q}} \ll A^\varepsilon_{\alpha,\beta}\|f\|_{p,w}, \tag{10}$$

where

$$A^\varepsilon_{\alpha,\beta} = \sup_{z\in I}\left(\int_a^z v_\varepsilon(x)dx\right)^{\frac{1}{q}}\left(\int_z^b u_\varepsilon^{p'}(s)W^{p'\beta}(s)w(s)ds\right)^{\frac{1}{p'}}.$$

We estimate $A^\varepsilon_{\alpha,\beta}$.

$$A^\varepsilon_{\alpha,\beta} \leq \sup_{a<z<d}\left(\int_a^z v(x)dx\right)^{\frac{1}{q}}\left(\varepsilon^{p'}\int_z^d u^{p'}(s)W^{p'\beta}(s)w(s)ds\right.$$

$$\left. + \int_d^b u^{p'}(s)W^{p'\beta}(s)w(s)ds\right)^{\frac{1}{p'}}$$

$$+\sup_{d<z<b}\left(\int_a^d v(x)dx+\varepsilon^q\int_d^z v(x)dx\right)^{\frac{1}{q}}\left(\int_z^b u^{p'}(s)W^{p'\beta}(s)w(s)ds\right)^{\frac{1}{p'}}$$

$$\ll 2(\varepsilon A_{\alpha,\beta}+A_{\alpha,\beta}(d)).$$

Hence, by (10) we have

$$\|P_{cd}K_{\alpha,\beta}Q_d f\|_{q,v}\ll(\varepsilon A_{\alpha,\beta}+A_{\alpha,\beta}(d))\|f\|_{p,w}. \tag{11}$$

Where, due to the independence of the left-hand side of (28) of $\varepsilon>0$, by letting $\varepsilon\to 0^+$, we obtain

$$\|P_{cd}K_{\alpha,\beta}Q_d f\|_{q,v}\ll A_{\alpha,\beta}(d)\|f\|_{p,\omega}.$$

Then

$$\lim_{d\to b^-}\|P_{cd}K_{\alpha,\beta}Q_d\|\ll\lim_{d\to b^-}A_{\alpha,\beta}(d)=0. \tag{12}$$

Similarly, we obtain

$$\|P_cK_{\alpha,\beta}\|_{q,v}=\left(\int_a^c v(x)\left(\int_x^b\frac{u(s)W^\beta(s)f(s)w(s)ds}{(W(s)-W(x))^{1-\alpha}}\right)^q dx\right)^{\frac{1}{q}}$$

$$\ll\sup_{a<z<c}A_{\alpha,\beta}(z)\|f\|_{p,w}.$$

Therefore

$$\lim_{c\to a^+}\|P_cK_{\alpha,\beta}f\|\ll\lim_{c\to a^+}\sup_{a<z<c}A_{\alpha,\beta}(z)=\lim_{z\to a^+}A_{\alpha,\beta}(z)=0. \tag{13}$$

By (8), (9), (12) and (13) it follows that $\lim\limits_{c\to a^+,d\to b^-}\|K_{\alpha,\beta}-P_{cd}K_{\alpha,\beta}P_{cd}\|=0.$ □

Theorem 4.2. *Let $0<\alpha<1$, $p>\frac{1}{\alpha}$ and $\beta\le 0$ ($\beta<\frac{1}{p}-\alpha$ in the case $W(b)=\infty$). Let u be a non-decreasing function on I. If $b<\infty$ and $0<q<p<\infty$ or $a=0, b=\infty$ and $1<q<p<\infty$, then the operator $K_{\alpha,\beta}$ is compact from $L_{p,w}$ to $L_{q,v}$ if and only if $B_{\alpha,\beta}<\infty$.*

Proof. In the case $b<\infty$ and $0<q<p<\infty$ the statement of Theorem 4.2 follows by Ando Theorem and its generalizations [10]. Therefore, we prove Theorem 4.2 in the case $a=0, b=\infty$ and $1<q<p<\infty$.
Necessity. Let the operator $K_{\alpha,\beta}$ be compact from $L_{p,w}$ to $L_{q,v}$. Then the operator is bounded. Hence, by Theorem 3.2 $B_{\alpha,\beta}<\infty$.

Sufficiency. Let $B_{\alpha,\beta} < \infty$. Here $K_{\alpha,\beta}f = P_d K_{\alpha,\beta} P_d f + P_d K_{\alpha,\beta} Q_d F + Q_d K_{\alpha,\beta} f$. Therefore

$$\|K_{\alpha,\beta} - P_d K_{\alpha,\beta} P_d\| \leq \|P_d K_{\alpha,\beta} Q_d\| + \|Q_d K_{\alpha,\beta}\|. \tag{14}$$

Since $d < \infty$ then the operator $P_d K_{\alpha,\beta} P_d$ is compact from $L_{p,w}(0,d)$ to $L_{q,v}(0,d)$, which is equivalent to its compactness from $L_{p,w}$ to $L_{q,v}$. We show that the right-hand side of (12) tends to zero as $d \to \infty$. Then the operator $K_{\alpha,\beta}$ is compact from $L_{p,w}$ to $L_{q,v}$ as the uniform limit of compact operators. On the basis of Theorem 3.2

$$\|Q_d K_{\alpha,\beta}\| \leq \left(\int_d^\infty \left(\int_z^\infty u^{p'}(s) W^{p'(\alpha+\beta-1)} w(s) ds \right)^{\frac{q(p-1)}{p-q}} \times \left(\int_d^z v(x)dx \right)^{\frac{q}{p-q}} v(z)dz \right)^{\frac{(p-q)}{pq}}.$$

Hence, since $B_{\alpha,\beta} < \infty$, it follows that

$$\lim_{d\to\infty} \|Q_d K_{\alpha,\beta}\| = 0. \tag{15}$$

Let $1 > \varepsilon > 0$. To estimate $\|P_d K_{\alpha,\beta} Q_d f\|$ we suppose as above, that $v_\varepsilon(x) = v(x)$ for $x \in (0,d]$ and $v_\varepsilon(x) = \varepsilon^q v(x)$ for $x \in (d,\infty)$, $u_\varepsilon(s) = u(s)$ for $s \in (d,\infty)$ and $u_\varepsilon(s) = \varepsilon u(s)$ for $s \in (0,d]$. Obviously, the function u_ε is non-decreasing on $I = (0,\infty)$. Now, by Theorem 3.2, estimating the norm $\|P_d K_{\alpha,\beta} Q_d\|$ as in (10), and then passing to the limit as $\varepsilon \to 0^+$, we obtain

$$\|P_d K_{\alpha,\beta} Q_d\| \ll \left(\int_0^d v(x)dx \right)^{\frac{1}{q}} \left(\int_d^\infty u^{p'}(s) W^{p'(\alpha+\beta-1)}(s) w(s) ds \right)^{\frac{1}{p'}} = A_{\alpha,\beta}(d). \tag{16}$$

By Remark 1 $B_{\alpha,\beta} \approx \widetilde{B}_{\alpha,\beta}(0,\infty)$. Since $A_{\alpha,\beta}(d) \ll \widetilde{B}_{\alpha,\beta}(d,\infty)$ then by (14) it follows that $\lim\limits_{d\to\infty} \|P_d K_{\alpha,\beta} Q_d\| = 0$. Hence by (13) it follows that the right-hand side of (12) tends to zero as $d \to \infty$. □

5 Dual case

We consider the operator

$$T_{\alpha,\beta} f(x) = u(x) W^\beta(x) \int_a^x \frac{v(s) f(s) ds}{(W(x) - W(s))^{1-\alpha}}$$

acting from $L_{p,v}$ to $L_{q,w}$.

Assume that

$$A^*_{\alpha,\beta}(z) = \left(\int_a^z v(x)dx \right)^{\frac{1}{p'}} \left(\int_z^b u^q(x) W^{q(\beta+\alpha-1)}(x) w(x) dx \right)^{\frac{1}{q}},$$

$$A^*_{\alpha,\beta} = \sup_{z \in I} A^*_{\alpha,\beta}(z).$$

Theorem 5.1. *Let* $0 < \alpha < 1$, $1 < p \leq q < \frac{1}{1-\alpha}$ *and* $\beta \leq 0$ ($\beta < 1 - \frac{1}{q} - \alpha$ *in the case* $W(b) = \infty$). *Let* u *be a non-decreasing function on* I. *Then the operator* $T_{\alpha,\beta}$

i) *is bounded from* $L_{p,v}$ *to* $L_{q,w}$ *if and only if* $A^*_{\alpha,\beta}(z) < \infty$, *moreover*, $\|T_{\alpha,\beta}\| \approx A^*_{\alpha,\beta}$,

ii)*is compact from* $L_{p,v}$ *to* $L_{q,w}$ *if and only if* $A^*_{\alpha,\beta}(z) < \infty$ *and*

$$\lim_{z \to a} A^*_{\alpha,\beta}(z) = \lim_{z \to b} A^*_{\alpha,\beta}(z) = 0.$$

Proof. The operator $T_{\alpha,\beta}$ acting from $L_{p,v}$ to $L_{q,w}$ is adjoint to the operator

$$\widetilde{K}_{\alpha,\beta} f(x) = v(x) \int_x^b \frac{u(s) W^{\beta}(s) f(s) ds}{(W(x) - W(s))^{1-\alpha}}$$

acting from $L_{q',w^{1-q'}}$ to $L_{p',v^{1-p'}}$, which is equivalent to the action of the operator $K_{\alpha,\beta}$ from $L_{q',\omega}$ to $L_{p',v}$. Consequently, the operator $T_{\alpha,\beta}$ is bounded and compact from $L_{p,v}$ to $L_{q,\omega}$ if and only if the operator $K_{\alpha,\beta}$ is bounded and compact from $L_{q',\omega}$ to $L_{p',v}$ respectively. Since by the assumptions of Theorem 5.1 it follows that $\frac{1}{\alpha} < q' \leq p' < \infty$ then on the basis of Theorems 3.1 and 4.1 the validity of the Statements i) and ii) of Theorem 5.1 follows. □

Similarly, on the basis of Theorem 4.2, we have

Theorem 5.2. *Yet* $0 < \alpha < 1$, $1 < q < \min\{p, \frac{1}{1-\alpha}\}$, $p > 1$ *and* $\beta \leq 0$ ($\beta < 1 - \frac{1}{q} - \alpha$ *in the case* $W(b) = \infty$). *Let* u *be a non-decreasing function on* I. *Then the operator* $T_{\alpha,\beta}$ *is bounded and compact from* $L_{p,v}$ *to* $L_{q,w}$ *if and only if* $B^*_{\alpha,\beta}(z) < \infty$, *where*

$$B^*_{\alpha,\beta} = \left(\int_a^b \left(\int_a^x v(x)dx \right)^{\frac{p(q-1)}{p-q}} \left(\int_x^b u^q(s) W^{q(\beta+\alpha-1)} w(s) ds \right)^{\frac{p}{p-q}} v(x)dx \right)^{\frac{p-q}{p}}.$$

6 Applications

We consider the weighted Weyl operator

$$\widetilde{I}^*_\alpha g(s) = \omega(s) \int\limits_s^\infty \frac{\rho(x) g(x) dx}{(x-s)^{1-\alpha}}, \ \ s > 0$$

and the weighted Riemann-Liouville operator

$$\widetilde{I}_\alpha f(x) = \rho(x) \int\limits_0^x \frac{\omega(s) f(s) ds}{(x-s)^{1-\alpha}}, \ \ x > 0$$

acting from L_p to L_q, where the weight functions ρ and ω are almost everywhere positive and locally integrable on $I = (0, \infty)$. The actions of the operator $K_{\alpha,\beta}$ from $L_{p,\omega}$ to $L_{q,v}$ and the operator $T_{\alpha,\beta}$ from $L_{p,v}$ to $L_{q,\omega}$ are equivalent to the actions of the operators

$$\widetilde{K}_{\alpha,\beta} g(s) = v^{\frac{1}{q}}(s) \int\limits_s^b \frac{u(x) W^\beta(x) w^{\frac{1}{p'}}(x) g(x) dx}{(W(x) - W(s))^{1-\alpha}}, \tag{1}$$

$$\widetilde{T}_{\alpha,\beta} f(x) = w^{\frac{1}{q}}(x) u(x) W^\beta(x) \int\limits_a^x \frac{v^{\frac{1}{p'}}(s) f(s) ds}{(W(x) - W(s))^{1-\alpha}}, \tag{2}$$

from L_p to L_q, respectively.

Let $\omega(s) = v^{\frac{1}{q}}(s)$ in (1) and $\omega(s) = v^{\frac{1}{p'}}(s)$ in (2). If $W(x) = x, a = 0, b = \infty$ and $\rho(x) = u(x) x^\beta$ then the operators (1) and (2) coincide with the operators I^*_α and I_α, respectively. Therefore, by Theorems 3.1- 4.2 we have

Corollary 6.1. *Let $0 < \alpha < 1$, $\beta < \frac{1}{p} - \alpha$ and $\rho(x) = u(x) x^\beta$, where u is a non-decreasing function on $I = (0, \infty)$. Then the operator $\widetilde{I}^*_\alpha$*

i) for $\frac{1}{\alpha} < p \le q < \infty$ is bounded from L_p to L_q if and only if $\widetilde{A}_\alpha < \infty$, moreover, $\|\widetilde{I}^*_\alpha\| \approx \widetilde{A}_\alpha$, and is compact from L_p to L_q if and only if $\widetilde{A}_\alpha < \infty$ and $\lim\limits_{z \to 0^+} \widetilde{A}_\alpha(z) = \lim\limits_{z \to \infty} \widetilde{A}_\alpha(z) = 0$, where

$$\widetilde{A}_\alpha(z) = \left(\int\limits_z^\infty \rho^{p'}(x) x^{p'(\alpha-1)} dx \right)^{\frac{1}{p'}} \left(\int\limits_0^z \omega^q(s) ds \right)^{\frac{1}{q}}, \ \widetilde{A}_\alpha = \sup_{z \in I} \widetilde{A}_\alpha(z);$$

ii) for $0 < \max\{q, \frac{1}{\alpha}\} < p < \infty$ *is bounded (compact) from* L_p *to* L_q *if* $b < \infty$ *(for* $1 < \max\{q, \frac{1}{\alpha}\} < p < \infty$ *if* $b = \infty$*) if and only if* $\widetilde{B}_\alpha < \infty$*, moreover,* $\|\widetilde{I}^*_\alpha\| \approx \widetilde{B}_\alpha$*, where*

$$\widetilde{B}_\alpha = \left(\int_0^\infty \left(\int_0^z \omega^q(s)ds \right)^{\frac{q}{p-q}} \left(\int_z^\infty \rho^{p'}(x)x^{p'(\alpha-1)}dx \right)^{\frac{q(p-1)}{p-q}} \omega^q(z)dz \right)^{\frac{p-q}{qp}} .$$

Remark 6.2. In the case $\beta = 0$, $0 < \max\left\{q, \frac{1}{\alpha}\right\} < p < \infty$ the boundedness and compactness of the operator $\widetilde{I}^*_\alpha$ fram L_p to L_q was also studied in [7]. However, the assertions of Theorems 4.1 and 4.2 and Theorems 7 and 8 in [7] are not correct, because the given there criteria involves the integral $\int_{2t}^\infty \left(u(t)t^{\alpha-1}\right)^{p'} dt$ for non-decreasing functions u which for $\frac{1}{p} < \alpha$ diverges for any such function.

Theorem 5.1 and 5.2 imply

Corollary 6.3. *Let* $0 < \alpha < 1$, $\beta < 1 - \frac{1}{q} - \alpha$ *and* $\rho(x) = u(x)x^\beta$*, where* u *is a non-decreasing function on* $I = (0, \infty)$*. Then the operator* $\widetilde{I}_\alpha$

i) for $1 < p \le q < \frac{1}{1-\alpha}$ *is bounded from* L_p *to* L_q *if and only if* $\widetilde{A}^*_\alpha < \infty$*, moreover,* $\|\widetilde{I}_\alpha\| \approx \widetilde{A}^*_\alpha$*, and compact from* L_p *to* L_q *if and only if* $\widetilde{A}^*_\alpha < \infty$ *and* $\lim_{z\to 0^+} \widetilde{A}^*_\alpha(z) = \lim_{z\to\infty} \widetilde{A}^*_\alpha(z) = 0$*, where*

$$\widetilde{A}^*_\alpha(z) = \left(\int_z^\infty \rho^q(x)x^{q(\alpha-1)}dx \right)^{\frac{1}{q}} \left(\int_0^z \omega^{p'}(s)ds \right)^{\frac{1}{p'}} , \quad \widetilde{A}^*_\alpha = \sup_{z\in I} \widetilde{A}^*_\alpha(z);$$

ii) for $1 < q < \min\{p, \frac{1}{1-\alpha}\} < \infty$, $p > 1$ *is bounded(compact) if and only if* $\widetilde{B}^*_\alpha < \infty$*, moreover,* $\|\widetilde{I}_\alpha\| \approx \widetilde{B}^*_\alpha$*, where*

$$\widetilde{B}^*_\alpha = \left(\int_0^\infty \left(\int_z^\infty \rho^q(x)x^{q(\alpha-1)}dx \right)^{\frac{p}{p-q}} \left(\int_0^z \omega^{p'}(s)ds \right)^{\frac{p(q-1)}{p-q}} \omega^{p'}(z)dz \right)^{\frac{p-q}{pq}} .$$

From (2) for $W(x) = x^\sigma$, $\sigma > 0$ $a = 0$, $b = \infty$, $\rho(x) = u(x)x^{\sigma\beta+\frac{\sigma-1}{q}}$, $v^{\frac{1}{p'}}(s) = \omega(s)s^{\sigma\gamma+\sigma-1}$ *we obtain the weighted Erdelyi-Kober operator*

$$E_{\alpha,\gamma}f(x) = \rho(x) \int_0^x \frac{\omega(s)s^{\sigma\gamma+\sigma-1}f(s)ds}{(x^\sigma - s^\sigma)^{1-\alpha}},$$

where γ is a real number, and for $W(x) = \ln\frac{x}{a}$, $a > 0$, $\rho(x) = u(x)x^{-\frac{1}{q}}(\ln\frac{x}{a})^{\beta}$, $v^{\frac{1}{p'}}(s) = \omega(s)\frac{1}{s}$ we obtain the weighted Hadamar operator

$$\mathcal{H}_\alpha f(x) = \rho(x)\int_a^x \frac{\omega(s)f(s)ds}{s(\ln\frac{x}{s})^{1-\alpha}}.$$

Corollary 6.4. *Let $0 < \alpha < 1$, $\beta < 1 - \frac{1}{q} - \alpha$ and $\rho(x) = u(x)x^{\sigma\beta + \frac{\sigma-1}{q}}$, where u is a non-decreasing function on $I = (0,\infty)$. Then the operator $E_{\alpha,\gamma}$*

i) for $1 < p \le q < \frac{1}{1-\alpha}$ is bounded from L_p to L_q if and only if $A^\circ_{\alpha,\gamma} < \infty$, moreover, $\|E_{\alpha,\gamma}\| \approx A^\circ_{\alpha,\gamma}$ and compact from L_p to L_q if and only if $A^\circ_{\alpha,\gamma} < \infty$ and $\lim\limits_{z\to 0^+} A^\circ_{\alpha,\gamma}(z) = \lim\limits_{z\to\infty} A^\circ_{\alpha,\gamma}(z) = 0$, where $A^\circ_{\alpha,\gamma} = \sup\limits_{z\in I} A^\circ_{\alpha,\gamma}(z)$,

$$A^\circ_{\alpha,\gamma}(z) = \left(\int_z^\infty \rho^q(x)x^{q\sigma(\alpha-1)}dx\right)^{\frac{1}{q}}\left(\int_0^z \omega^{p'}(s)s^{p'(\sigma\gamma+\sigma-1)}ds\right)^{\frac{1}{p'}}.$$

ii) for $1 < q < \min\{p, \frac{1}{1-\alpha}\} < \infty$, $p > 1$ is bounded (compact) from L_p to L_q if and only if $B^\circ_{\alpha,\gamma} < \infty$, moreover, $\|E_{\alpha,\gamma}\| \approx B^\circ_{\alpha,\gamma}$, where

$$B^\circ_{\alpha,\gamma} = \left(\int_0^\infty \left(\int_z^\infty \rho^q(x)x^{q\sigma(\alpha-1)}dx\right)^{\frac{p}{p-q}} \left(\int_0^z \omega^{p'}(s)s^{p'(\sigma\gamma+\sigma-1)}ds\right)^{\frac{p(q-1)}{p-q}} \omega^{p'}(z)z^{p'(\sigma\gamma+\sigma-1)}dz\right)^{\frac{p-q}{pq}}.$$

Corollary 6.5. *Let $a > 0, 0 < \alpha < 1$, $\beta \le 0$ ($\beta < 1 - \frac{1}{q} - \alpha$ in the case $b = \infty$) and $\rho(x) = u(x)x^{-\frac{1}{q}}(ln\frac{x}{a})^{\beta}$, where u is non-decreasing function on $I = (a,b)$. Then the operator $\mathcal{H}_\alpha$*

i) for $1 < p \le q < \frac{1}{1-\alpha}$ is bounded from L_p to L_q if and only if $A^1_\alpha < \infty$, moreover, $\|H_\alpha\| \approx A^1_\alpha$, and compact from L_p to L_q if and only if $A^1_\alpha < \infty$ and $\lim\limits_{z\to a^+} A^1_\alpha(z) = \lim\limits_{z\to b^-} A^1_\alpha(z) = 0$, where $A^1_\alpha = \sup\limits_{z\in I} A^1_\alpha(z)$,

$$A^1_\alpha(z) = \left(\int_z^b \rho^q(x)\left(\ln\frac{x}{a}\right)^{q(\alpha-1)}dx\right)^{\frac{1}{q}}\left(\int_a^z \omega^{p'}(s)s^{-p'}ds\right)^{\frac{1}{p'}};$$

ii) for $1 < q < \min\{p, \frac{1}{1-\alpha}\} < \infty,\ p > 1$ *is bounded (compact) from* L_p *to* L_q *if and only if* $B^1_\alpha < \infty$*, moreover,* $\|H_\alpha\| \approx B^1_\alpha$*, where*

$$B^1_\alpha = \left(\int_a^b \left(\int_z^b \rho^q(x) \left(ln\frac{x}{a} \right)^{q(\alpha-1)} dx \right)^{\frac{p}{p-q}} \left(\int_a^z \omega^{p'}(s) s^{-p'} ds \right)^{\frac{p(q-1)}{p-q}} \omega^{p'}(z) z^{-p'} dz \right)^{\frac{p-q}{pq}}.$$

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Boundedness and compactness of a class of Hardy type operators

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BOUNDEDNESS AND COMPACTNESS OF A CLASS OF HARDY TYPE OPERATORS

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Abstract. We establish characterizations both of boundedness and compactness of a general class of fractional integration operators involving the Riemann-Liouville, Hadamard and Erdelyi-Kober operators. In particular, these results imply new results in the theory of Hardy type inequalities. As applications both new and well-known results are pointed out.

1 Introduction

Let $I = (a, b)$, $0 \le a < b \le \infty$. Let v and u be almost everywhere positive functions, which are locally integrable on the interval I.

Let $0 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Denote by $L_{p,v} \equiv L_p(v, I)$ the set of all functions f measurable on I such that $\|f\|_{p,v} := \left(\int_a^b |f(x)|^p v(x)dx \right)^{\frac{1}{p}} < \infty$.

Let W be a non-negative, strictly increasing and locally absolutely continuous function on I. Suppose that $\frac{dW(x)}{dx} = w(x)$, a.e. $x \in I$.

We consider the Hardy type operator $T_{\alpha,\beta}$ defined by

$$T_{\alpha,\beta}f(x) := \int_a^x \frac{u(s)W^\beta(s)f(s)w(s)ds}{(W(x) - W(s))^{1-\alpha}}, \quad x \in I. \tag{1}$$

When $u \equiv 1$ and $\beta = 0$ the operator $T_{\alpha,\beta}$ is called the fractional integration operator of a function f with respect to a function W ([15], p.248). When $u \equiv 1$ and $W(x) = x$ the operator (1) becomes the Riemann-Liouville operator I_α defined by

$$I_\alpha f(x) := \int_a^x \frac{f(s)ds}{(x - s)^{1-\alpha}}. \tag{2}$$

When $u \equiv 1$ and $W(x) \equiv ln\frac{x}{a}$, $a > 0$, this operator is the Hadamard operator $\mathcal{H}_\alpha$ defined by

$$\mathcal{H}_\alpha f(x) := \int_a^x \frac{f(s)ds}{s\left(ln\frac{x}{s}\right)^{1-\alpha}}.$$

Moreover, when $u \equiv 1$ and $W(x) = x^\sigma$, $\sigma > 0$, we get the operator $E_{\alpha,\beta}$ of Erdelyi-Kober type ([15], p.246) defined by

$$E_{\alpha,\beta} f(x) := \int_a^x \frac{f(s)s^{\sigma\beta+\sigma-1}ds}{(x^\sigma - s^\sigma)^{1-\alpha}}.$$

There are a lot of works devoted to mapping properties of the Riemann-Liouville operator I_α. Two-weighted estimates of the operator I_α of the order $\alpha > 1$ in weighted Lebesgue spaces were first obtained in the papers [17] and [18]. The singular case $0 < \alpha < 1$ was studied with different restrictions in [3], [7], [9], [15], [5], [21] and some others. The most general results among them are given in [7] and [21] under the assumption that one of the weight functions is increasing or decreasing.

In this work we investigate the problems of boundedness and compactness of the operator $T_{\alpha,\beta}$ defined by (1) from $L_{p,w}$ to $L_{q,v}$ when $0 < \alpha < 1$. When $\alpha > 1$ the results follow from the results in [11].

The operator $T_{\alpha,\beta}$ was studied in [1] and [12] when $u \equiv 1$, $\beta = 0$ and $u \equiv 1$, $\beta > -\frac{1}{p'}$, respectively .

Due to non-negativity and monotone increase of the function W the limit $\lim\limits_{x\to a^+} W(x) \equiv W(a) \geq 0$ exists.

We also consider the Hardy type operator $T^0_{\alpha,\beta}$ defined by

$$T^0_{\alpha,\beta} f(x) := \int_a^x \frac{u(s)W_0^\beta(s)f(s)w(s)ds}{(W_0(x) - W_0(s))^{1-\alpha}}, \quad x \in I,$$

where $W_0(x) = W(x) - W(a)$.

Since we also suppose that $\beta \geq 0$, then for $f \geq 0$ we have $T_{\alpha,\beta}f(x) \approx T^0_{\alpha,\beta}f(x) + W(a)T^0_{\alpha,0}f(x)$, where the equivalence constants do not depend on x and f. Therefore, without loss of generality, we can assume that $W(a) = 0$. For short writing we denote by $\|K\|$ the norm of a linear operator K acting from one normalized space to another, since from the context we shall in each case clearly see which spaces the operator is acting between.

The paper this organized as follows: In order not to disturb our discussions later on some auxiliary statement are given in Section 2. The main results concerning the boundedness of operator $T_{\alpha,\beta}$, including the corresponding Hardy type inequalities, can be found in Section 3. The main results about the compactness are presented in Section 4. Moreover, in Section 5 some similar results for the dual operator $T^0_{\alpha,\beta}$ are given. Finally, Section 6 is reserved for some applications (both new and well-known results).

Conventions: the indeterminate form $0 \cdot \infty$ is assumed to be zero. The relations $A \ll B$ and $A \gg B$ respectively mean $A \le cB$ and $A \ge cB$, where a positive constant c can be dependent only on the parameters p, q, α and β. The relation $A \approx B$ is interpreted as $A \ll B \ll A$. The set of all integers is denoted by Z. Moreover, $\chi_{(c,a)}(\cdot)$ is the characteristic function of the interval $(c, a) \subset I$.

2 Auxiliary statements

To prove the main results we shall need some auxiliary results from the standard literature on Hardy type inequalities (see [5] and [4]).

Together with the operator (1) we consider the following Hardy type operator $H_{\alpha,\beta}$ defined by

$$H_{\alpha,\beta}f(x) = \frac{1}{W^{1-\alpha}(x)} \int_a^x u(s)W^{\beta}(s)f(s)w(s)ds. \tag{1}$$

It is easy to see that for $f \ge 0$ we have

$$T_{\alpha,\beta}f(x) \ge H_{\alpha,\beta}f(x), \quad \forall x \in I. \tag{2}$$

The problem of boundedness of operators of the form (1) in weighted Lebesgue spaces have been very well studied. The history and development of Hardy type inequalities with relevant references can be found in [5].

In view [16] the following statements are consequences of Theorem 5 of [5]:

Lemma 2.1. *Let $1 < p \le q < \infty$ and let the operator $H_{\alpha,\beta}$ be defined by* (1). *Then the inequality*

$$\left(\int_a^b \left(H_{\alpha,\beta}f(x)\right)^q v(x)dx \right)^{\frac{1}{q}} \le C \left(\int_a^b (f(x))^p \, w(x)dx \right)^{\frac{1}{p}} \tag{3}$$

holds if and only if

$$A_{\alpha,\beta} = \sup_{z\in I}\left(\int_a^z u^{p'}(s)W^{p'\beta}(s)w(s)ds\right)^{\frac{1}{p'}}\left(\int_z^b W^{q(\alpha-1)}(x)v(x)dx\right)^{\frac{1}{q}} < \infty.$$

Moreover, $C \approx A_{\alpha,\beta}$.

Lemma 2.2. *Let* $0 < q < p < \infty$, $p > 1$ *and let the operator* $H_{\alpha,\beta}$ *be defined by* (1). *Then the inequality* (3) *holds if and only if*

$$B_{\alpha,\beta} = \left(\int_a^b\left(\int_z^b W^{q(\alpha-1)}(x)v(x)dx\right)^{\frac{p}{p-q}}\right.$$

$$\left.\times\left(\int_a^z u^{p'}(s)W^{p'\beta}(s)w(s)ds\right)^{\frac{p(q-1)}{p-q}} u^{p'}(z)W^{p'\beta}(z)w(z)dz\right)^{\frac{p-q}{pq}} < \infty.$$

Moreover, $C \approx B_{\alpha,\beta}$.

Remark 2.3. *In the case* $1 < q < p < \infty$, $p > 1$ *it is well known and easy to prove that the value* $B_{\alpha,\beta}$ *is equivalent to the value*

$$\widetilde{B}_{\alpha,\beta} = \left(\int_a^b\left(\int_z^b W^{q(\alpha-1)}(x)v(x)dx\right)^{\frac{q}{p-q}}\right.$$

$$\left.\times\left(\int_a^z u^{p'}(s)W^{p'\beta}(s)w(s)ds\right)^{\frac{q(p-1)}{p-q}} W^{q(\alpha-1)}(z)v(z)dz\right)^{\frac{p-q}{pq}}.$$

3 Boundedness of the operator $T_{\alpha,\beta}$

The main results in this Section reads:

Theorem 3.1. *Let* $0 < \alpha < 1$, $1 < p \le q < \infty$ *and* $\beta \ge 0$. *Let* u *be a non-increasing function on* I. *Then the operator* $T_{\alpha,\beta}$ *defined by* (1) *is bounded from* $L_{p,w}$ *to* $L_{q,v}$ *if and only if* $A_{\alpha,\beta} < \infty$. *Moreover,* $\|T_{\alpha,\beta}\| \approx A_{\alpha,\beta}$.

Theorem 3.2. *Let $0 < \alpha < 1$, $0 < q < p < \infty$, $p > \frac{1}{\alpha}$ and $\beta \geq 0$. Let u be a non-increasing function on I. Then the operator $T_{\alpha,\beta}$ is bounded from $L_{p,w}$ to $L_{q,v}$ if and only if $B_{\alpha,\beta} < \infty$. Moreover, $\|T_{\alpha,\beta}\| \approx B_{\alpha,\beta}$.*

These two theorems can be reformulated as the following new information in the theory of Hardy type inequalities:

Theorem 3.3. *Let $0 < \alpha < 1$, $\beta \geq 0$ and u be a non-increasing function on I. Then the inequality*

$$\left(\int_a^b \left(T_{\alpha,\beta}f(x)\right)^q v(x)dx\right)^{\frac{1}{q}} \leq C\left(\int_a^b (f(x))^p w(x)dx\right)^{\frac{1}{p}} \tag{1}$$

holds if and only if

a) $A_{\alpha,\beta} < \infty$ for the case $1 < p \leq q < \infty$,

b) $B_{\alpha,\beta} < \infty$ for the case $0 < q < p < \infty$, $p > \frac{1}{\alpha}$.

Moreover, for the best constant C in (1) it yields that $C \approx A_{\alpha,\beta}$ in case a) and $C \approx B_{\alpha,\beta}$ in case b).

Proof of Theorem 3.1. Necessity. Let the operator $T_{\alpha,\beta}$ be bounded from $L_{p,w}$ to $L_{q,v}$. Then, in view of (2), the operator $H_{\alpha,\beta}$ is bounded from $L_{p,w}$ to $L_{q,v}$, and $\|T_{\alpha,\beta}\| \geq \|H_{\alpha,\beta}\|$. Consequently, by Lemma 2.1 we have that $A_{\alpha,\beta} < \infty$ and

$$\|T_{\alpha,\beta}\| \gg A_{\alpha,\beta}. \tag{2}$$

Sufficiency. Since the function W is continuous and strictly increasing on I and $W(a) = 0$, then for any $k \in Z$ we can define $x_k := \sup\left\{x : W(x) \leq 2^k, x \in I\right\}$. We obtain a sequence of points $\{x_k\}_{k>-\infty}$ such that $0 < x_k \leq x_{k+1}$, $\forall k \in Z$, and if $x_k < b$, then $W(x_k) = 2^k$, $2^k \leq W(x) \leq 2^{k+1}$ for $x_k \leq x \leq x_{k+1}$, $\int_{x_{k-1}}^{x_k} w(s)ds = 2^{k-1}$, and if $x_{k+1} = b$, then $\int_{x_k}^{x_{k+1}} w(s)ds \leq 2^k$. These facts will be used below without reminders. We assume that $I_k = [x_k, x_{k+1})$, $k \in Z$, $Z_0 = \{k : k \in Z,\ I_k \neq \emptyset\}$. Then $Z_0 \subseteq Z$ and $I = \bigcup_{k\in Z} I_k = \bigcup_{k\in Z_0} I_k$. Since $I_k = \emptyset$, $\forall k \in Z \setminus Z_0$, and integrals over these intervals are equal to zero, then in the sequel, without loss of generality, we can suppose that $Z = Z_0$.

Let $A_{\alpha,\beta} < \infty$. We need to prove that the inequality

$$\|T_{\alpha,\beta}f\|_{q,v} \ll A_{\alpha,\beta}\|f\|_{p,w}, \quad f \in L_{p,w}, \tag{3}$$

holds, which means $\|T_{\alpha,\beta}\| \ll A_{\alpha,\beta}$ and, together with (2), this gives that

$$\|T_{\alpha,\beta}\| \approx A_{\alpha,\beta}.$$

Let $f \geq 0$. Using the relation $I = \bigcup\limits_k I_k$, we have that

$$\|T_{\alpha,\beta} f\|_{q,v}^q = \sum_k \int_{x_k}^{x_{k+1}} v(x) \left(\int_a^x \frac{u(s)W^\beta(s)f(s)w(s)ds}{(W(x)-W(s))^{1-\alpha}} \right)^q dx$$

$$= \sum_k \int_{x_k}^{x_{k+1}} v(x) \left[\left(\int_a^{x_{k-1}} + \int_{x_{k-1}}^x \right) \frac{u(s)W^\beta(s)f(s)w(s)ds}{(W(x)-W(s))^{1-\alpha}} \right]^q dx$$

$$\ll \sum_k \int_{x_k}^{x_{k+1}} v(x) \left(\int_a^{x_{k-1}} \frac{u(s)W^\beta(s)f(s)w(s)ds}{(W(x)-W(s))^{1-\alpha}} \right)^q dx$$

$$+ \sum_k \int_{x_k}^{x_{k+1}} v(x) \left(\int_{x_{k-1}}^x \frac{u(s)W^\beta(s)f(s)w(s)ds}{(W(x)-W(s))^{1-\alpha}} \right)^q dx := J_1 + J_2. \tag{4}$$

We now estimate J_1 and J_2 separately. Using the monotonicity of W we find that

$$J_1 = \sum_k \int_{x_k}^{x_{k+1}} v(x) \left(\int_a^{x_{k-1}} \frac{u(s)W^\beta(s)f(s)w(s)ds}{(W(x)-W(s))^{1-\alpha}} \right)^q dx$$

$$\leq \sum_k \int_{x_k}^{x_{k+1}} v(x) \left(\int_a^{x_{k-1}} \frac{u(s)W^\beta(s)f(s)w(s)ds}{(W(x_k)-W(x_{k-1}))^{1-\alpha}} \right)^q dx$$

$$= 2^{2q(1-\alpha)} \sum_k \int_{x_k}^{x_{k+1}} v(x) \left(\frac{1}{2^{k+1}} \right)^{q(1-\alpha)} \left(\int_a^{x_{k-1}} u(s)W^\beta(s)f(s)w(s)ds) \right)^q dx$$

$$\ll \sum_k \int_{x_k}^{x_{k+1}} v(x)W^{q(\alpha-1)}(x) \left(\int_a^x u(s)W^\beta(s)f(s)w(s)ds) \right)^q dx \leq \|H_{\alpha,\beta} f\|_{q,v}^q.$$

Hence, by Lemma 2.1 we get that

$$J_1 \ll A_{\alpha,\beta}^q \|f\|_{p,w}^q. \tag{5}$$

Moreover, by using Hölder's inequality and that the function u is increasing, we obtain that

$$J_2 = \sum_k \int_{x_k}^{x_{k+1}} v(x) \left(\int_{x_{k-1}}^x \frac{u(s)W^\beta(s)f(s)w(s)ds}{(W(x)-W(s))^{1-\alpha}} \right)^q dx$$

$$\leq \sum_k \int_{x_k}^{x_{k+1}} v(x) \left(\int_{x_{k-1}}^{x} f^p(s)w(s)ds \right)^{\frac{q}{p}} \left(\int_{x_{k-1}}^{x} \frac{u^{p'}(s)W^{p'\beta}(s)w(s)ds}{(W(x)-W(s))^{p'(1-\alpha)}} \right)^{\frac{q}{p'}} dx$$

$$\leq \sum_k \left(\int_{x_{k-1}}^{x_{k+1}} f^p(s)w(s)ds \right)^{\frac{q}{p}} u^q(x_{k-1}) \int_{x_k}^{x_{k+1}} v(x) \left(\int_a^x \frac{W^{p'\beta}(s)w(s)ds}{(W(x)-W(s))^{p'(1-\alpha)}} \right)^{\frac{q}{p'}} dx. \quad (6)$$

A change of variables $W(s) = W(x)t$ in the last integral, implies that

$$\int_a^x \frac{W^{p'\beta}(s)w(s)ds}{(W(x)-W(s))^{p'(1-\alpha)}} = \frac{W^{p'\beta+1}(x)}{W^{p'(1-\alpha)}(x)} \int_0^1 t^{p'\beta}(1-t)^{p'(\alpha-1)}dt. \quad (7)$$

Since $\beta \geq 0$, $\alpha > \frac{1}{p}$, then the Euler beta function $\int_0^1 t^{p'\beta}(1-t)^{p'(\alpha-1)}dt$ converges. Consequently, from (6) and (7) it follows that

$$J_2 \ll \sum_k \left(\int_{x_{k-1}}^{x_{k+1}} f^p(s)w(s)ds \right)^{\frac{q}{p}} u^q(x_{k-1}) \int_{x_k}^{x_{k+1}} v(x) \frac{W^{\frac{q}{p'}(p'\beta+1)}dx}{W^{q(1-\alpha)}(x)}$$

$$\leq \sum_k \left(\int_{x_{k-1}}^{x_{k+1}} f^p(s)w(s)ds \right)^{\frac{q}{p}} u^q(x_{k-1}) W^{\frac{q}{p'}(p'\beta+1)}(x_{k+1}) \int_{x_k}^{x_{k+1}} v(x)W^{q(\alpha-1)}(x)dx$$

$$= 2^{2(q\beta+\frac{q}{p'})} \sum_k \left(\int_{x_{k-1}}^{x_{k+1}} f^p(s)w(s)ds \right)^{\frac{q}{p}}$$

$$\times u^q(x_{k-1}) W^{\frac{q}{p'}(p'\beta+1)}(x_{k-1}) \int_{x_k}^{x_{k+1}} v(x)W^{q(\alpha-1)}(x)dx$$

$$\ll \sum_k \left(\int_{x_{k-1}}^{x_{k+1}} f^p(s)w(s)ds \right)^{\frac{q}{p}}$$

$$\times u^q(x_{k-1}) \left(\int_a^{x_{k-1}} W^{p'\beta}(s)w(s)ds \right)^{\frac{q}{p'}} \int_{x_k}^{x_{k+1}} v(x)W^{q(\alpha-1)}(x)dx$$

$$\leq \sum_k \left(\int_{x_{k-1}}^{x_{k+1}} f^p(s)w(s)ds \right)^{\frac{q}{p}}$$

$$\times \left(\int_a^{x_k} u^{p'}(s)W^{p'\beta}(s)w(s)ds \right)^{\frac{q}{p'}} \int_{x_k}^{b} v(x)W^{q(\alpha-1)}(x)dx$$

$$\leq A^q_{\alpha,\beta} \sum_k \left(\int_{x_{k-1}}^{x_{k+1}} f^p(s)w(s)ds \right)^{\frac{q}{p}} \leq A^q_{\alpha,\beta} \left(\sum_k \int_{x_{k-1}}^{x_{k+1}} f^p(s)w(s)ds \right)^{\frac{q}{p}}$$

$$\ll A^q_{\alpha,\beta} \|f\|^q_{p,w}. \tag{8}$$

By combining (4), (5) and (8) we obtain (3). The proof is complete.

Proof of Theorem 3.2. Necessity. Similarly as in the proof of Theorem 3.1 and the estimate

$$\|T_{\alpha,\beta}\| \gg B_{\alpha,\beta}, \tag{9}$$

follow from (2) and Lemma 2.2.

Sufficiency. Let $B_{\alpha,\beta} < \infty$. If we show that $\|T_{\alpha,\beta}\| \ll B_{\alpha,\beta}$, then this fact and (9) imply that $\|T_{\alpha,\beta}\| \approx B_{\alpha,\beta}$. Next, we use relation (4). For the estimate J_1 we have obtained $J_1 \ll \|H_{\alpha,\beta} f\|^q_{q,v}$. Hence, by Lemma 2.2 we obtain that

$$J_1 \ll B^q_{\alpha,\beta} \|f\|^q_{p,w}. \tag{10}$$

Moreover, from Theorem 3.1, obvious estimates and Hölder's inequality it follows that

$$J_2 \ll \sum_k \left(\int_{x_{k-1}}^{x_{k+1}} f^p(s)w(s)ds \right)^{\frac{q}{p}}$$

$$\times u^q(x_{k-1})W^{\frac{q}{p'}(p'\beta+1)}(x_{k+1}) \int_{x_k}^{x_{k+1}} v(x)W^{q(\alpha-1)}(x)dx$$

$$= 2^{3(q\beta+\frac{q}{p'})}(2^{p'\beta+1}-1)^{\frac{q}{p'}} \sum_k \left(\int_{x_{k-1}}^{x_{k+1}} f^p(s)w(s)ds \right)^{\frac{q}{p}} \times u^q(x_{k-1}) \left(2^{(p'\beta+1)(k-1)} - 2^{(p'\beta+1)(k-2)} \right)^{\frac{q}{p'}} \int_{x_k}^{x_{k+1}} v(x)W^{q(\alpha-1)}(x)dx$$

$$\ll \sum_k \left(\int_{x_{k-1}}^{x_{k+1}} f^p(s)w(s)ds \right)^{\frac{q}{p}} \times u^q(x_{k-1}) \left(\int_{x_{k-2}}^{x_{k-1}} W^{p'\beta}(s)w(s)ds \right)^{\frac{q}{p'}} \int_{x_k}^{x_{k+1}} v(x)W^{q(\alpha-1)}(x)dx$$

$$\ll \sum_k \left(\int_{x_{k-1}}^{x_{k+1}} f^p(s)w(s)ds \right)^{\frac{q}{p}} \times \left(\int_{x_{k-2}}^{x_{k-1}} u^{p'}(s)W^{p'\beta}(s)w(s)ds \right)^{\frac{q}{p'}} \int_{x_k}^{x_{k+1}} v(x)W^{q(\alpha-1)}(x)dx$$

(we apply Hölder's inequality with the conjugate exponents $\frac{p}{q}$, $\frac{p}{p-q}$)

$$\leq J_{21}^{\frac{p-q}{p}} \left(\sum_k \int_{x_{k-1}}^{x_{k+1}} f^p(s)w(s)ds \right)^{\frac{q}{p}} \ll J_{21}^{\frac{p-q}{p}} \|f\|_{p,w}^q, \tag{11}$$

where

$$J_{21} = \sum_k \left(\int_{x_{k-2}}^{x_{k-1}} u^{p'}(s)W^{p'\beta}(s)w(s)ds \right)^{\frac{q(p-1)}{p-q}} \left(\int_{x_k}^{x_{k+1}} v(x)W^{q(\alpha-1)}(x)dx \right)^{\frac{p}{p-q}}.$$

To estimate J_{21} we use the relation

$$\left(\int_{x_{k-2}}^{x_{k-1}} u^{p'}(s)W^{p'\beta}(s)w(s)ds\right)^{\frac{q(p-1)}{p-q}}$$
$$\ll \int_{x_{k-2}}^{x_{k-1}}\left(\int_{x_{k-2}}^{t} u^{p'}(s)W^{p'\beta}(s)w(s)ds\right)^{\frac{p(q-1)}{p-q}} u^{p'}(t)W^{p'\beta}(t)w(t)dt.$$

Then

$$J_{21} \ll \sum_k \int_{x_{k-2}}^{x_{k-1}}\left(\int_{x_{k-2}}^{t} u^{p'}(s)W^{p'\beta}(s)w(s)ds\right)^{\frac{p(q-1)}{p-q}}$$
$$\times u^{p'}(t)W^{p'\beta}(t)w(t)dt\left(\int_{x_k}^{x_{k+1}} v(x)W^{q(\alpha-1)}(x)dx\right)^{\frac{p}{p-q}}$$

$$\leq \sum_k \int_{x_{k-2}}^{x_{k-1}}\left(\int_{a}^{t} u^{p'}(s)W^{p'\beta}(s)w(s)ds\right)^{\frac{p(q-1)}{p-q}}$$
$$\times\left(\int_{t}^{b} v(x)W^{q(\alpha-1)}(x)dx\right)^{\frac{p}{p-q}} u^{p'}(t)W^{p'\beta}(t)w(t)dt$$
$$\leq B_{\alpha,\beta}^{\frac{qp}{p-q}}. \tag{12}$$

By substitution of (12) in (11) we obtain that

$$J_2 \ll B_{\alpha,\beta}^q ||f||_{p,w}^q. \tag{13}$$

Now, by combining (4), (10) and (13) we obtain that

$$||T_{\alpha,\beta}f||_{q,v} \ll B_{\alpha,\beta}||f||_{p,w}.$$

Consequently, $||T_{\alpha,\beta}||_{q,v} \ll B_{\alpha,\beta}$. The proof is complete.

4 Compactness of the operator $T_{\alpha,\beta}$

The main results in this Section reads:

Theorem 4.1. *Let* $0 < \alpha < 1$, $\frac{1}{\alpha} < p \le q < \infty$ *and* $\beta \ge 0$. *Let* u *be a non-increasing function on* I. *Then the operator* $T_{\alpha,\beta}$ *is compact from* $L_{p,w}$ *to* $L_{q,v}$ *if and only if* $A_{\alpha,\beta} < \infty$ *and*

$$\lim_{z\to a^+} A_{\alpha,\beta}(z) = \lim_{z\to b^-} A_{\alpha,\beta}(z) = 0,$$

where

$$A_{\alpha,\beta}(z) = \left(\int_a^z u^{p'}(s)W^{p'\beta}(s)w(s)ds\right)^{\frac{1}{p'}} \left(\int_z^b W^{q(\alpha-1)}(x)v(x)dx\right)^{\frac{1}{q}}.$$

Theorem 4.2. *Let* $0 < \alpha < 1$, $p > \frac{1}{\alpha}$ *and* $\beta \ge 0$. *Let* u *be a non-increasing function on* I. *If* $b < \infty$ *and* $0 < q < p < \infty$ *or* $b = \infty$ *and* $1 < q < p < \infty$, *then the operator* $T_{\alpha,\beta}$ *is compact from* $L_{p,w}$ *to* $L_{q,v}$ *if and only if* $B_{\alpha,\beta} < \infty$.

Proof of Theorem 4.1. Necessity. Let the operator $T_{\alpha,\beta}$ be compact from $L_{p,w}$ to $L_{q,v}$. Then it is bounded and consequently, by Theorem 3.1, we have that $A_{\alpha,\beta} < \infty$. First we need to show that $\lim\limits_{z\to a^+} A_{\alpha,\beta}(z) = 0$. Consider the family of functions $\{f_t\}_{t\in I}$, where

$$f_t(x) = \chi_{(a,t)}(x)u^{p'-1}(x)W^{(p'-1)\beta}(x)\left(\int_a^t u^{p'}(s)W^{p'\beta}(s)w(s)ds\right)^{-\frac{1}{p}}, \quad x \in I. \tag{1}$$

We note that

$$\left(\int_a^b |f_t(x)|^p w(x)dx\right)^{\frac{1}{p}} = \left(\int_a^t |f_t(x)|^p w(x)dx\right)^{\frac{1}{p}}$$

$$= \left(\int_a^t u^{p'}(s)W^{p'\beta}(s)w(s)ds\right)^{-\frac{1}{p}} \left(\int_a^t u^{p'}(s)W^{p'\beta}(s)w(s)ds\right)^{\frac{1}{p}} = 1. \tag{2}$$

Next we show that the family of functions $\{f_t\}_{t\in I}$ defined by (1) converges weakly to zero in $L_{p,w}$. Let $g \in L_{p',w^{1-p'}} = \left(L_{p,w}\right)^*$. Then, by Hölder's inequality and (2), we find that

$$\int_a^b f_t(x)g(x)dx \le \left(\int_a^t |f_t(x)|^p w(x)dx\right)^{\frac{1}{p}} \left(\int_a^t |g(s)|^{p'} w^{1-p'}(s)ds\right)^{\frac{1}{p'}}$$

$$= \left(\int_a^t |g(s)|^{p'} w^{1-p'}(s)ds\right)^{\frac{1}{p'}}. \tag{3}$$

Since $g \in L_{p',w^{1-p'}}$, then the last integral in (3) converges to zero as $t \to a^+$, which means weak convergence of the family of functions $\{f_t\}$ to zero as $t \to a^+$. Therefore, from the compactness of the operator $T_{\alpha,\beta}$ from $L_{p,w}$ to $L_{q,v}$ it follow that

$$\lim_{t\to a^+} \|T_{\alpha,\beta} f_t\|_{q,v} = 0. \tag{4}$$

Moreover,

$$\|T_{\alpha,\beta} f_t\|_{q,v}^q = \int_a^b v(x)\left(\int_a^x \frac{u(s)W^\beta(s)f_t(s)w(s)ds}{(W(x)-W(s))^{1-\alpha}}\right)^q dx$$

$$\ge \int_t^b v(x)\left(\int_a^t \frac{u(s)W^\beta(s)f_t(s)w(s)ds}{(W(x)-W(s))^{1-\alpha}}\right)^q dx$$

$$\ge \int_t^b \frac{v(x)dx}{W^{q(1-\alpha)}(x)}\left(\int_a^t u^{p'}(s)W^{p'\beta}(s)w(s)ds\right)^{-\frac{q}{p}} \left(\int_a^t u^{p'}(s)W^{p'\beta}(s)w(s)ds\right)^q$$

$$= A_{\alpha,\beta}^q(t). \tag{5}$$

From (4) and (5) it follows that $\lim_{t\to a^+} A_{\alpha,\beta}(t) = 0$.

Now, we show that $\lim_{t\to b^-} A_{\alpha,\beta}(t) = 0$.

From the compactness of the operator $T_{\alpha,\beta}$ from $L_{p,w}$ to $L_{q,v}$ it follows compactness of the conjugate operator

$$T^*_{\alpha,\beta} g(s) = u(s)W^p(s)w(s)\int_s^b \frac{g(x)dx}{(W(x)-W(s))^{1-\alpha}}$$

from $L_{q',v^{1-q'}}$ to $L_{p',w^{1-p'}}$.

For $t \in I$ we introduce the family $\{g_t\}_{t\in I}$ of functions:

$$g_t(x) = \chi_{[t,b)}(x)\left(\int_t^b W^{q(\alpha-1)}(x)v(x)dx\right)^{-\frac{1}{q'}} W^{(q-1)(\alpha-1)}(x)v(x). \tag{6}$$

The family $\{g_t\}_{t\in I}$ of functions defined by (6) is correctly defined, since due to condition $A_{\alpha,\beta} < \infty$ the involving integrals are finite. We show that for all $t \in I$ the functions $g_t \in L_{q',v^{1-q'}}$ converges weakly to zero as $t \to b^-$.

Indeed,

$$\|g_t\|_{q',v^{1-q'}} = \left(\int_t^b |g_t(x)|^{q'} v^{1-q'}(x)dx\right)^{\frac{1}{q'}}$$

$$= \left(\int_t^b W^{q(\alpha-1)}(x)v(x)dx\right)^{-\frac{1}{q'}} \left(\int_t^b |W^{(q-1)(\alpha-1)}(x)v(x)|^{q'} v^{1-q'}(x)dx\right)^{\frac{1}{q'}}$$

$$= \left(\int_t^b W^{q(\alpha-1)}(x)v(x)dx\right)^{-\frac{1}{q'}} \left(\int_t^b W^{q(\alpha-1)}(x)v(x)dx\right)^{\frac{1}{q'}} = 1. \tag{7}$$

By using (7) with $f \in L_{q,v} = \left(L_{q',v^{1-q'}}\right)^*$ we obtain that

$$\int_a^b g_s(x)f(x)dx \le \left(\int_t^b |g_t(x)|^{q'} v^{-\frac{q'}{q}}(x)dx\right)^{\frac{1}{q'}} \left(\int_t^b |f(x)|^q v(x)dx\right)^{\frac{1}{q}}$$

$$\le \|g_t\|_{q',v^{1-q'}} \left(\int_t^b |f(x)|^q v(x)dx\right)^{\frac{1}{q}} = \left(\int_t^b |f(x)|^q v(x)dx\right)^{\frac{1}{q}}.$$

Since $f \in L_{q,v}$, then the last integral tends to zero as $t \to b^-$, that gives the weak convergence to zero of $\{g_t\}_{t\in I}$ in $L_{q',v^{1-q'}}$ as $t \to b^-$. By compactness of $T^*_{\alpha,\beta} : L_{q',v^{1-q'}} \to L_{p',w^{1-p'}}$ it follows that

$$\lim_{s\to b^-} \|T^*_{\alpha,\beta} g_t\|_{p',w^{1-p'}} = 0. \tag{8}$$

Furthermore, we note that

$$\|T^*_{\alpha,\beta}g_t\|_{p',w^{1-p'}} = \left(\int_a^b |u(s)W^{\beta}(s)w(s)|^{p'}\right.$$

$$\left.\left(\int_s^b \frac{g_t(x)dx}{(W(x)-W(s))^{1-\alpha}}\right)^{p'} w^{1-p'}(s)ds\right)^{\frac{1}{p'}}$$

$$\geq \left(\int_a^t u^{p'}(s)W^{p'\beta}(s)w(s)\left(\int_t^b \frac{g_t(x)dx}{(W(x)-W(s))^{1-\alpha}}\right)^{p'} ds\right)^{\frac{1}{p'}}$$

$$\geq \left(\int_a^t u^{p'}(s)W^{p'\beta}(s)w(s)ds\right)^{\frac{1}{p'}} \left(\int_t^b W^{q(\alpha-1)}(x)v(x)dx\right)^{-\frac{1}{q'}}$$

$$\left(\int_t^b \frac{W^{(q-1)(\alpha-1)}(x)v(x)dx}{W^{1-\alpha}(x)}\right)^{q} = A_{\alpha,\beta}(t).$$

Hence, according to (8) we have that $\lim\limits_{s\to b^-} A_{\alpha,\beta}(s) = 0$. The proof of the necessity is complete.

Sufficiency. For $a < c < d < b$ we define

$$P_c f := \chi_{(a,c]} f, \ \ P_{cd} f := \chi_{(c,d]} f, \ \ Q_d f := \chi_{(d,b)} f.$$

Then

$$f = P_c f + P_{cd} f + Q_d f$$

and since $P_c T_{\alpha,\beta} P_{cd} \equiv 0$, $P_c T_{\alpha,\beta} Q_d \equiv 0$, $P_{cd} T_{\alpha,\beta} Q_d \equiv 0$, we have that

$$T_{\alpha,\beta} f = P_{cd} T_{\alpha,\beta} P_{cd} f + P_c T_{\alpha,\beta} P_c f + P_{cd} T_{\alpha,\beta} P_c f + Q_d T_{\alpha,\beta} f. \tag{9}$$

We show that the operator $P_{cd}T_{\alpha,\beta}P_{cd}$ is compact from $L_{p,w}$ to $L_{q,v}$. Since $P_{cd}T_{\alpha,\beta}P_{cd}f(x) = 0$ for $x \in I\setminus(c,d)$, then it is enough to show that the operator $P_{cd}T_{\alpha,\beta}P_{cd}$ is compact from $L_{p,w}(c,d)$ to $L_{q,v}(c,d)$. This, in turn, is equivalent to compactness of the operator

$$Tf(x) = \int_c^d K(x,s)f(s)ds$$

from $L_p(c,d)$ to $L_q(c,d)$ with the kernel

$$K(x,s) = \frac{u(s)W^{\beta}(s)v^{\frac{1}{q}}(x)\chi_{(c,d)}(x-s)w^{\frac{1}{p'}}(s)}{(W(x)-W(s))^{1-\alpha}}.$$

Let $\{x_k\}_{k\in\mathbb{Z}}$ be the sequence of points defined in the proof of Theorem 3.1. There are points x_i, x_{n+1}, $x_i < x_{n+1}$ such that $x_i \le c < x_{i+1}$, $x_n < d \le x_{n+1}$. We assume that the numbers c, d are chosen so that $x_{i+1} < x_n$. Similarly to obtaining estimates of J_1 and J_2 in Theorem 3.1, we have that

$$\int_c^d \left(\int_c^d |K(x,s)|^{p'} ds\right)^{\frac{q}{p'}} dx = \int_c^d v(x)\left(\int_c^x \frac{u^{p'}(s)W^{p'\beta}(s)w(s)ds}{(W(x)-W(s))^{p'(1-\alpha)}}\right)^{\frac{q}{p'}} dx$$

$$\le \sum_{k=i}^{n}\int_{x_k}^{x_{k+1}} v(x)\left[\left(\int_a^{x_{k-1}} + \int_{x_{k-1}}^{x}\right)\frac{u^{p'}(s)W^{p'\beta}(s)w(s)ds}{(W(x)-W(s))^{p'(1-\alpha)}}\right]^{\frac{q}{p'}} dx$$

$$\le \mu(n-i+1)A^q_{\alpha,\beta} < \infty,$$

where the constant μ does not depend on i, n.

Therefore, on the basis of Kantarovich condition ([2], p.420), the operator T is compact from $L_p(c,d)$ to $L_q(c,d)$, which is equivalent to compactness of the operator $P_{cd}T_{\alpha,\beta}P_{cd}$ from $L_{p,w}$ to $L_{q,v}$.

From (9) it follows that

$$\|T_{\alpha,\beta} - P_{cd}T_{\alpha,\beta}P_{cd}\| \le \|P_cT_{\alpha,\beta}P_c\| + \|P_{cd}T_{\alpha,\beta}P_c\| + \|Q_dT_{\alpha,\beta}\|. \qquad (10)$$

We will show that the right hand side of (10) tends to zero at $c \to a$ and $d \to b$. Then the operator $T_{\alpha,\beta}$ as the uniform limit of compact operators is compact from $L_{p,w}$ to $L_{q,v}$.

By using Theorem 3.1 we find that

$$\|P_cT_{\alpha,\beta}P_cf\|_{q,v} = \left(\int_a^c v(x)\left|\int_a^x \frac{u(s)W^{\beta}(s)f(s)w(s)ds}{(W(x)-W(s))^{1-\alpha}}\right|^q dx\right)^{\frac{1}{q}}$$

$$\ll \sup_{a<z<c}\left(\int_a^z u^{p'}(s)W^{p'\beta}(s)w(s)ds\right)^{\frac{1}{p'}}\left(\int_z^c v(x)W^{q(\alpha-1)}(x)dx\right)^{\frac{1}{q}}\|f\|_{p,w}$$

$$\leq \sup_{a<z<c} A_{\alpha,\beta}(z)\|f\|_{p,w}.$$

Consequently, $\|P_cT_{\alpha,\beta}P_c\| \ll \sup\limits_{a<z<c} A_{\alpha,\beta}(z)$. Hence,

$$\lim_{c\to a^+}\|P_cT_{\alpha,\beta}P_c\| \ll \lim_{c\to a^+}\sup_{a<z<c} A_{\alpha,\beta}(z) = \lim_{z\to a^+} A_{\alpha,\beta}(z) = 0. \tag{11}$$

To estimate $\|P_{cd}T_{\alpha,\beta}P_c\|$ we assume that $v_\varepsilon(x) = v(x)$ for $x \in (c,d]$ and $v_\varepsilon(x) = \varepsilon^q v(x)$ for $x \in (a,c]$, $u_\varepsilon(s) = u(s)$ for $s \in (a,c]$ and $u_\varepsilon(s) = \varepsilon u(s)$ for $s \in (c,d]$, where $1 > \varepsilon > 0$. Obviously, the function u_ε is non-increasing on I. Then, according to Theorem 3.1 we obtain that

$$\begin{aligned}\|P_{cd}T_{\alpha,\beta}P_c\|_{q,v} &= \left(\int_c^d v(x)\left|\int_a^c \frac{u(s)W^\beta(s)f(s)w(s)ds}{(W(x)-W(s))^{1-\alpha}}\right|^q dx\right)^{\frac{1}{q}}\\ &\leq \left(\int_a^d v_\varepsilon(x)\left|\int_a^x \frac{u_\varepsilon(s)W^\beta(s)f(s)w(s)ds}{(W(x)-W(s))^{1-\alpha}}\right|^q dx\right)^{\frac{1}{q}}\\ &\ll A^\varepsilon_{\alpha,\beta}\|f\|_{p,w},\end{aligned} \tag{12}$$

where

$$A^\varepsilon_{\alpha,\beta} = \sup_{a<z<d}\left(\int_z^d W^{q(\alpha-1)}(x)v_\varepsilon(x)dx\right)^{\frac{1}{q}}\left(\int_a^z u_\varepsilon^{p'}(s)W^{p'\beta}(s)w(s)ds\right)^{\frac{1}{p'}}.$$

We estimate the expression $A^\varepsilon_{\alpha,\beta}$ from above as follows:

$$\begin{aligned}A^\varepsilon_{\alpha,\beta} &\leq \sup_{a<z<c}\left(\int_c^d W^{q(\alpha-1)}(x)v(x)dx + \varepsilon^q\int_z^c W^{q(\alpha-1)}(x)v(x)dx\right)^{\frac{1}{q}}\\ &\qquad\left(\int_a^z u^{p'}(s)W^{p'\beta}(s)w(s)ds\right)^{\frac{1}{p'}}\\ &+ \sup_{c<z<d}\left(\int_z^d W^{q(\alpha-1)}(x)v(x)dx\right)^{\frac{1}{q}}\\ &\qquad\left(\int_a^c u^{p'}(s)W^{p'\beta}(s)w(s)ds + \varepsilon^{p'}\int_c^z u^{p'}(s)W^{p'\beta}(s)w(s)ds\right)^{\frac{1}{p'}}\end{aligned}$$

$$\leq \sup_{a<z<c}\left(\int_c^d W^{q(\alpha-1)}(x)v(x)dx\right)\left(\int_a^z u^{p'}(s)W^{p'\beta}(s)w(s)ds\right) + \varepsilon A_{\alpha,\beta}$$

$$+ \sup_{c<z<d}\left(\int_z^d W^{q(\alpha-1)}(x)v(x)dx\right)^{\frac{1}{q}}\left(\int_a^c u^{p'}(s)W^{p'\beta}(s)w(s)ds\right)^{\frac{1}{p'}} + \varepsilon A_{\alpha,\beta}$$

$$\leq 2\left(A_{\alpha,\beta}(c) + \varepsilon A_{\alpha,\beta}\right). \tag{13}$$

Since the left side of (12) does not depend on $\varepsilon > 0$, then substituting (13) in (12) and letting $\varepsilon \to 0$, we get that

$$\|P_{cd}T_{\alpha,\beta}P_c f\| \ll A_{\alpha,\beta}(c)\|f\|_{p,w}.$$

Therefore $\|P_{cd}T_{\alpha,\beta}P_c\| \ll A_{\alpha,\beta}(c)$ and we conclude that

$$\lim_{c\to a^+}\|P_{cd}T_{\alpha,\beta}P_c\| \ll \lim_{c\to a^+} A_{\alpha,\beta}(c) = 0. \tag{14}$$

Next, arguing as above we find that

$$\|Q_d T_{\alpha,\beta} f\|_{q,v} = \left(\int_d^b v(x)\left|\int_a^x \frac{u(s)W^\beta(s)f(s)w(s)ds}{(W(x)-W(s))^{1-\alpha}}\right|^q dx\right)^{\frac{1}{q}}$$

$$\ll \sup_{d<z<b} A_{\alpha,\beta}(z)\|f\|_{p,w}.$$

Consequently,

$$\lim_{d\to b^-}\|Q_d T_{\alpha,\beta}\| \leq \lim_{d\to b^-}\sup_{d<z<b} A_{\alpha,\beta}(z) = \lim_{z\to b^-} A_{\alpha,\beta}(z) = 0. \tag{15}$$

From (11), (14) and (15) it follows that the right hand side of (10) tends to zero as $c \to a^+$ and $d \to b^-$. The proof is complete.

Proof of Theorem 4.2. In the case $b < \infty$ and $0 < q < p < \infty$ the statement of Theorem 4.2 follows from the Ando Theorem and its generalizations [10]. Therefore, we only need to prove Theorem 4.2 in the case $a = 0, b = \infty$ and $1 < q < p < \infty$.

Necessity. Let the operator $T_{\alpha,\beta}$ be compact from $L_{p,w}$ to $L_{q,v}$. Then the operator is bounded. Hence, by Theorem 3.2, $B_{\alpha,\beta} < \infty$.

Sufficiency. Let $B_{\alpha,\beta} < \infty$. Here $T_{\alpha,\beta}f = P_d T_{\alpha,\beta} P_d f + Q_d T_{\alpha,\beta} f$. Therefore

$$\|T_{\alpha,\beta} - P_d T_{\alpha,\beta} P_d\| \leq \|Q_d T_{\alpha,\beta}\|. \tag{16}$$

Since $d < \infty$, then the operator $P_d T_{\alpha,\beta} P_d$ is compact from $L_{p,w}(0,d)$ to $L_{q,v}(0,d)$, which is equivalent to its compactness from $L_{p,w}$ to $L_{q,v}$. We show that the right-hand side of (16) tends to zero as $d \to \infty$. Then the operator $T_{\alpha,\beta}$ is compact from $L_{p,w}$ to $L_{q,v}$ as the uniform limit of compact operators.

Let $1 > \varepsilon > 0$. To estimate $\|Q_d T_{\alpha,\beta} f\|$ we suppose that $v_\varepsilon(x) = v(x)$ for $x \in [d,\infty)$ and $v_\varepsilon(x) = \varepsilon^q v(x)$ for $x \in (0,d)$. Using the relations $B_{\alpha,\beta} \approx \widetilde{B}_{\alpha,\beta}$ (see Remark 2.3), in view of Theorem 3.2, we have that

$$\|Q_d T_{\alpha,\beta} f\| \leq \left(\int_a^\infty v_\varepsilon(x) \left| \int_a^x \frac{u(s)W^\beta(s)f(s)w(s)ds}{(W(x)-W(s))^{1-\alpha}} \right|^q dx \right)^{\frac{1}{q}}$$

$$\ll \widetilde{B}^\varepsilon_{\alpha,\beta} \|f\|_{p,w}$$

or

$$\|Q_d T_{\alpha,\beta}\| \ll \widetilde{B}^\varepsilon_{\alpha,\beta}, \tag{17}$$

where

$$\widetilde{B}^\varepsilon_{\alpha,\beta} = \left(\int_a^\infty \left(\int_z^\infty W^{q(\alpha-1)}(x) v_\varepsilon(x) dx \right)^{\frac{q}{p-q}} \right.$$

$$\left. \times \left(\int_a^z u^{p'}(s) W^{p'\beta}(s) w(s) ds \right)^{\frac{q(p-1)}{p-q}} W^{q(\alpha-1)}(z) v_\varepsilon(z) dz \right)^{\frac{p-q}{pq}}.$$

Passing to the limit $\varepsilon \to 0^+$, from (17) it follows that

$$\|Q_d T_{\alpha,\beta}\| \ll \left(\int_d^\infty \left(\int_z^\infty W^{q(\alpha-1)}(x) v(x) dx \right)^{\frac{q}{p-q}} \right.$$

$$\left. \times \left(\int_a^z u^{p'}(s) W^{p'\beta}(s) w(s) ds \right)^{\frac{q(p-1)}{p-q}} W^{q(\alpha-1)}(z) v(z) dz \right)^{\frac{p-q}{pq}}.$$

Hence,

$$\lim_{d\to\infty} \|Q_d T_{\alpha,\beta}\| = 0. \tag{18}$$

Obviously, (18) implies that the right-hand side of (16) tends to zero as $d \to \infty$. The proof is complete.

5 Some dual results

Here we consider the dual operator $K^*_{\alpha,\beta}$ defined by

$$K^*_{\alpha,\beta}g(s) = \int_s^b \frac{u(s)W^\beta(s)g(x)v(x)dx}{(W(x)-W(s))^{1-\alpha}} \tag{1}$$

and its mapping properties from $L_{p,v}$ to $L_{q,w}$.

We define

$$A^*_{\alpha,\beta}(z) := \left(\int_a^z u^q(s)W^{q\beta}(s)w(s)ds\right)^{\frac{1}{q}} \left(\int_z^b W^{p'(\alpha-1)}(x)v(x)dx\right)^{\frac{1}{p'}},$$

$$A^*_{\alpha,\beta} = \sup_{z\in I} A^*_{\alpha,\beta}(z).$$

Our first main result here reads:

Theorem 5.1. *Let $0<\alpha<1$, $1<p\le q<\frac{1}{1-\alpha}$ and $\beta\ge 0$. Let u be a non-increasing function on I. Then the operator $K^*_{\alpha,\beta}$ defined by (1)*

i) is bounded from $L_{p,v}$ to $L_{q,w}$ if and only if $A^*_{\alpha,\beta}<\infty$ and moreover, $\|K^*_{\alpha,\beta}\| \approx A^*_{\alpha,\beta}$;

ii) is compact from $L_{p,v}$ to $L_{q,w}$ if and only if $A^*_{\alpha,\beta}<\infty$ and

$$\lim_{z\to a^+} A^*_{\alpha,\beta}(z) = \lim_{z\to b^-} A^*_{\alpha,\beta}(z) = 0.$$

Proof. The operator $K^*_{\alpha,\beta}$ acting from $L_{p,v}$ to $L_{q,w}$ is conjugate to the operator

$$K_{\alpha,\beta}f(x) = v(x)\int_a^x \frac{u(s)W^\beta(s)f(s)ds}{(W(x)-W(s))^{1-\alpha}}$$

acting from $L_{q',w^{1-q'}}$ to $L_{p',v^{1-p'}}$, which is equivalent to the action of the operator $T_{\alpha,\beta}$ from $L_{q',w}$ to $L_{p',v}$. Consequently, the operator $K^*_{\alpha,\beta}$ is bounded and compact from $L_{p,v}$ to $L_{q,w}$ if and only if the operator $T_{\alpha,\beta}$ is respectively bounded and compact from $L_{q',w}$ to $L_{p',v}$. Moreover, $\|K^*_{\alpha,\beta}\| = \|T_{\alpha,\beta}\|$. Since, by the conditions of Theorem 5.1 we have $\frac{1}{\alpha}<q'\le p'<\infty$, then the statements i) and ii) in Theorem 5.1 follows directly from Theorem 3.1 and Theorem 4.1. The proof is complete. □

Similarly, in view of Theorem 3.2 we have the following:

Theorem 5.2. *Let $0 < \alpha < 1$, $1 < q < \min\{p, \frac{1}{\alpha-1}\}$, $p > 1$ and $\beta \geq 0$. Let u be a non-increasing function on I. Then the operator $K^*_{\alpha,\beta}$ defined by (1) is bounded and compact from $L_{p,v}$ to $L_{q,w}$ if and only if $B^*_{\alpha,\beta} < \infty$, where*

$$B^*_{\alpha,\beta} = \left(\int_a^b \left(\int_z^b W^{p'(\alpha-1)}(x)v(x)dx \right)^{\frac{q(p-1)}{p-q}} \left(\int_a^z u^q(s)W^{q\beta}(s)w(s)ds \right)^{\frac{q}{p-q}} \times u^q(s)W^{q\beta}(s)w(s)ds \right)^{\frac{p-q}{pq}} .$$

Theorems 5.1 and 5.2 implies especially the following new information in the theory of Hardy type inequalities:

Theorem 5.3. *Let $0 < \alpha < 1$, $\beta \geq 0$ and u be a non-increasing function on I. Then*

$$\left(\int_a^b \left(K^*_{\alpha,\beta} f(x) \right)^q w(x)dx \right)^{\frac{1}{q}} \leq C \left(\int_a^b (f(x))^p v(x)dx \right)^{\frac{1}{p}} \tag{2}$$

holds if and only if

a) $A^*_{\alpha,\beta} < \infty$ for the case $1 < p \leq q \leq \frac{1}{1-\alpha}$,

b) $B^*_{\alpha,\beta} < \infty$ for the case $1 < q < min(p, \frac{1}{\alpha-1})$, $p > 1$.

Moreover, for the best constant C in (2) it yields that $C \approx A^*_{\alpha,\beta}$ in case a) and $C \approx B^*_{\alpha,\beta}$ in case b).

Theorem 5.3 supplements the results of [2].

6 Applications

By applying our results in special cases we obtain both new and well-known results. Here we just consider the Riemann-Liouville, Erdelyi-Kober and Hadamard operators mentioned in our introduction. We use the weight functions ρ and ω and consider these operators on the forms $\widetilde{I}_\alpha$, $\widetilde{E}_{\alpha,\gamma}$ and $\widetilde{\mathcal{H}}_\alpha$ defined by

$$\widetilde{I}_\alpha f(x) := \rho(x)\left[I_\alpha(f\omega)\right](x),$$

$$\widetilde{E}_{\alpha,\gamma} f(x) := \rho(x)\left[E_{\alpha,\gamma}(f\omega)\right](x),$$

$$\widetilde{\mathcal{H}}_\alpha f(x) := \rho(x)\left[\mathcal{H}_\alpha(f\omega)\right](x),$$

where ρ and ω are almost everywhere positive functions locally summable on I with degrees q and p', respectively.

The action of the operator $T_{\alpha,\beta}$ from $L_{p,v}$ to $L_{q,w}$ is equivalent to the action of the operator

$$\widetilde{T}_{\alpha,\beta}f(x) = v^{\frac{1}{q}}(x)\int_a^x \frac{u(s)W^{\beta}(s)w^{\frac{1}{p'}}(s)f(s)ds}{(W(x)-W(s))^{1-\alpha}}$$

from L_p to L_q. Therefore, in the case $W(x) = x$ we have $\rho(x) = v^{\frac{1}{q}}(x)$, $\omega(x) = u(x)x^{\beta}$ and

$$\widetilde{I}_{\alpha}f(x) = \rho(x)\int_a^x \frac{\omega(s)f(s)ds}{(x-s)^{1-\alpha}}.$$

If $W(x) = x^{\sigma}$, $\sigma > 0$, then $u(s)W^{\beta}(s)w^{\frac{1}{p'}}(s) = u(s)s^{\sigma\beta-\frac{\sigma-1}{p'}} = u(s)s^{\sigma\gamma+\sigma-1}$, where $\gamma = \beta - \frac{\sigma-1}{\sigma p}$. Consequently, $\rho(x) = v^{\frac{1}{q}}(x)$, $\omega(s) = u(s)$ and

$$\widetilde{E}_{\alpha,\gamma}f(x) = \rho(x)\int_a^x \frac{\omega(s)s^{\sigma\gamma+\sigma-1}f(s)ds}{(x^{\sigma}-s^{\sigma})^{1-\alpha}}.$$

Now, we assume that $a > 0$ and $W(x) = \ln\frac{x}{a}$. Then $u(s)W^{\beta}(s)w^{\frac{1}{p'}}(s) = u(s)\left(\ln\frac{s}{a}\right)^{\beta}\left(\frac{a}{s}\right)^{\frac{1}{p'}} = a^{\frac{1}{p'}}u(s)s^{\frac{1}{p}}\left(\ln\frac{s}{a}\right)^{\beta}\frac{1}{s}$. In this case $\rho(x) = v^{\frac{1}{q}}(x)$, $\omega(s) = u(s)s^{\frac{1}{p}}\left(\ln\frac{s}{a}\right)^{\beta}$ and

$$\widetilde{\mathcal{H}}_{\alpha}f(x) = \rho(x)\int_a^x \frac{\omega(s)f(s)ds}{s\left(\ln\frac{x}{s}\right)^{1-\alpha}}.$$

Below we present statements for boundedness and compactness of the operators $\widetilde{I}_{\alpha}$, $\widetilde{E}_{\alpha,\gamma}$ and $\widetilde{\mathcal{H}}_{\alpha}$ from L_p to L_q. These statements are consequences of Theorems 3.1, 3.2, 4.1 and 4.2.

We define

$$A^1_{\alpha}(z) := \left(\int_z^b \left(\rho(x)x^{\alpha-1}\right)^q dx\right)^{\frac{1}{q}}\left(\int_a^z \omega^{p'}(s)ds\right)^{\frac{1}{p'}},\quad A^1_{\alpha} := \sup_{z\in I} A^1_{\alpha}(z),$$

$$B^1_{\alpha} := \left(\int_a^b \left(\int_z^b |\rho(x)x^{\alpha-1}|^q dx\right)^{\frac{p}{p-q}}\left(\int_a^z \omega^{p'}(s)ds\right)^{\frac{p(q-1)}{p-q}}\omega^{p'}(z)dz\right)^{\frac{p-q}{pq}}.$$

Corollary 6.1. *Let $0 < \alpha < 1$, $\beta \geq 0$ and $\omega(s) = u(s)s^{\beta}$. Let u be a non-increasing function on I. Then*

i) for $\frac{1}{\alpha} < p \leq q < \infty$ the operator $\widetilde{I_\alpha}$ is bounded from L_p to L_q if and only if $A^1_\alpha < \infty$ and, moreover, $\|\widetilde{I_\alpha}\| \approx A^1_\alpha$. It is compact from L_p to L_q if and only if $A^1_\alpha < \infty$ and $\lim\limits_{z\to a^+} A^1_\alpha(z) = \lim\limits_{z\to b^-} A^1_\alpha(z) = 0$;

ii) for $0 < q < p < \infty$ and $p > \frac{1}{\alpha}$ the operator $\widetilde{I_\alpha}$ is bounded (compact if $b < \infty$ or $b = \infty$ and $1 \leq q < p < \infty$) from L_p to L_q if and only if $B^1_\alpha < \infty$.

Remark 6.2. *Corollary 6.1 generalizes the results of Theorems 1 and 2, 5 and 6 in [7], where the case $\beta = 0$ was considered. Even in this case the results of Corollary 6.1 are different (and in a sense simpler to use) than those in [7], because in [7] the statements are given in terms of two expressions while here we only need one condition.*

We define

$$A^2_{\alpha,\gamma}(z) := \left(\int_z^b |\rho(x)x^{\sigma(\alpha-1)}|^q dx\right)^{\frac{1}{q}} \left(\int_a^z |\omega(s)s^{\sigma\gamma+\sigma-1}|^{p'} ds\right)^{\frac{1}{p'}},$$

$$A^2_{\alpha,\gamma} := \sup_{z\in I} A^2_{\alpha,\gamma}(z),$$

$$B^2_{\alpha,\gamma} := \left(\int_a^b \left(\int_z^b |\rho(x)x^{\sigma(\alpha-1)}|^q dx\right)^{\frac{p}{p-q}} \times \left(\int_a^z |\omega(s)s^{\sigma\gamma+\sigma-1}|^{p'} ds\right)^{\frac{p(p-1)}{p-q}} |\omega(z)z^{\sigma\gamma+\sigma-1}|^{p'} dz\right)^{\frac{p-q}{pq}}.$$

Corollary 6.2. *Let $0 < \alpha < 1$, $\sigma > 0$, $\beta \geq 0$ and $\gamma = \beta - \frac{\sigma-1}{\sigma p}$. Let ω be a non-increasing function on I. Then*

i) for $\frac{1}{\alpha} < p \leq q < \infty$ the operator $\widetilde{E}_{\alpha,\gamma}$ is bounded from L_p to L_q if and only if $A^2_{\alpha,\gamma} < \infty$ and, moreover, $\|\widetilde{E}_{\alpha,\gamma}\| \approx A^2_{\alpha,\gamma}$. It is compact from L_p to L_q if and only if $A^2_{\alpha,\gamma} < \infty$ and $\lim\limits_{z\to a^+} A^2_{\alpha,\gamma}(z) = \lim\limits_{z\to b^-} A^2_{\alpha,\gamma}(z) = 0$;

ii) for $0 < q < p < \infty$ and $p > \frac{1}{\alpha}$ the operator $\widetilde{E}_{\alpha,\gamma}$ is bounded (compact if $b < \infty$ or $b = \infty$ and $1 \leq q < p < \infty$) from L_p to L_q if and only if $B^2_{\alpha,\gamma} < \infty$.

To formulate statements corresponding to the operator $\widetilde{\mathcal{H}}_\alpha$ we define

$$A_\alpha^3(z) := \left(\int_z^b \left| \rho(x) \left(\ln \frac{x}{a} \right)^{\alpha-1} \right|^q dx \right)^{\frac{1}{q}} \left(\int_a^z \omega^{p'}(s) ds \right)^{\frac{1}{p'}}, \quad A_\alpha^3 := \sup_{z \in I} A_\alpha^3(z),$$

$$B_\alpha^3 := \left(\int_a^b \left(\int_z^b \left| \rho(x) \left(\ln \frac{x}{a} \right)^{\alpha-1} \right|^q dx \right)^{\frac{p}{p-q}} \left(\int_a^z \omega^{p'}(s) ds \right)^{\frac{p(q-1)}{p-q}} \omega^{p'}(z) dz \right)^{\frac{p-q}{pq}}.$$

Corollary 6.3. *Let $a > 0$, $0 < \alpha < 1$, $\beta \geq 0$ and $\omega(s) = u(s) s^{\frac{1}{p}} \left(\ln \frac{s}{a} \right)^\beta$. Let u be a non-increasing function on I. Then*

i) for $\frac{1}{\alpha} < p \leq q < \infty$ the operator $\widetilde{\mathcal{H}}_\alpha$ is bounded from L_p to L_q if and only if $A_\alpha^3 < \infty$ and, moreover, $\|\widetilde{\mathcal{H}}_\alpha\| \approx A_\alpha^3$. It is compact from L_p to L_q if and only if $A_\alpha^3 < \infty$ and $\lim\limits_{z \to a^+} A_\alpha^3(z) = \lim\limits_{z \to b^-} A_\alpha^3(z) = 0$;

ii) for $0 < q < p < \infty$ and $p > \frac{1}{\alpha}$ the operator $\widetilde{\mathcal{H}}_\alpha$ is bounded (compact if $b < \infty$ or $b = \infty$ and $1 \leq q < p < \infty$) from L_p to L_q if and only if $B_\alpha^3 < \infty$.

Finally, we consider the operator $\widetilde{I_\alpha^*} g(s) = \rho(s)[I_\alpha^*(g\omega)](s)$, $s \in I$, acting from L_p to L_q, where I_α^* is the Weil operator

$$I_\alpha^* g(s) = \int_s^b \frac{g(x) dx}{(x-s)^{1-\alpha}}.$$

The action of the operator $K_{\alpha,\beta}^*$ from $L_{p,v}$ to $L_{q,w}$ is equivalent to the action of the operator

$$\widetilde{K_{\alpha,\beta}^*} g(s) = w^{\frac{1}{q}}(s) u(s) W^\beta(s) \int_s^b \frac{v^{\frac{1}{p'}}(x) g(x) dx}{(W(x) - W(s))^{1-\alpha}}$$

from L_p to L_q. Therefore, when $W(x) = x$ we have

$$\rho(s) = u(s) s^\beta, \quad \omega(x) = v^{\frac{1}{p'}}(x),$$

$$\widetilde{I_\alpha^*} g(s) = \rho(s) \int_s^b \frac{\omega(x) g(x) dx}{(x-s)^{1-\alpha}}.$$

We define

$$A_\alpha^*(z) := \left(\int_a^z \rho^q(s)ds\right)^{\frac{1}{q}} \left(\int_z^b |\omega(x)x^{\alpha-1}|^{p'}dx\right)^{\frac{1}{p'}}, \quad A_\alpha^* := \sup_{z\in I} A_\alpha^*(z),$$

$$B_\alpha^* := \left(\int_a^b \left(\int_z^b |\omega(x)x^{\alpha-1}|^{p'}dx\right)^{\frac{q(p-1)}{p-q}} \left(\int_a^z \rho^q(s)ds\right)^{\frac{q}{p-q}} \rho^q(z)dz\right)^{\frac{p-q}{pq}}.$$

From Theorems 5.1 and 5.2 we have the following result:

Corollary 6.4. *Let* $0 < \alpha < 1$, $\beta \geq 0$ *and* $\rho(s) = u(s)s^\beta$. *Let* u *be a non-increasing function on* I. *Then*

i) for $1 < p \leq q < \frac{1}{1-\alpha}$ *the operator* $\widetilde{I_\alpha^*}$ *is bounded from* L_p *to* L_q *if and only if* $A_\alpha^* < \infty$ *and, moreover,* $\|\widetilde{I_\alpha}\| \approx A_\alpha^*$. *It is compact from* L_p *to* L_q *if and only if* $A_\alpha^* < \infty$ *and* $\lim_{z\to a^+} A_\alpha^*(z) = \lim_{z\to b^-} A_\alpha^*(z) = 0$;

ii) for $1 < q < \{\min(p, \frac{1}{1-\alpha})\} < \infty$ *and* $p > 1$ *the operator* $\widetilde{I_\alpha^*}$ *is bounded (compact) from* L_p *to* L_q *if and only if* $B_\alpha^* < \infty$.

Remark 6.6. *From the results in Corollary 6.1 - 6.3 follows some corresponding Hardy type inequalities, which seem to be new even it they are special cases of our Theorems 3.3 and 5.3.*

Authors' contributions

All the authors contributed equally and significantly in writing this paper. All the authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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Paper 4

Boundedness and compactness of the Hardy type operator with variable upper limit in weighted Lebesgue spaces

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Boundedness and compactness of the Hardy type operator with variable upper limit in weighted Lebesgue spaces

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Abstract: Let $0 < \alpha < 1$. The operator of the form

$$K_{\alpha,\varphi}f(x) = \int_a^{\varphi(x)} \frac{f(t)w(t)dt}{(W(x) - W(t))^{(1-\alpha)}}, \quad x > 0,$$

is considered, where the real weight functions $v(x)$ and $w(x)$ are locally integrable on $I := (a, b)$, $0 \le a < b \le \infty$ and $\frac{dW(x)}{dx} \equiv w(x)$.

In this paper we derive criteria for the operator $K_{\alpha,\varphi}$, $0 < \alpha < 1$, $0 < p; q < \infty$, $p > \frac{1}{\alpha}$ to be bounded and compact from the spaces $L_{p,w}$ to the spaces $L_{q,v}$.

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1 Introduction

Let $0 < p, q < \infty$, $I = (a, b)$, $0 \le a < b \le \infty$, $0 < \alpha < 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Let $W : I \to R$ be a strictly increasing and locally absolutely continuous function on I. Suppose that $\frac{dW(x)}{dx} \equiv w(x)$ almost every $x \in I$ and $W(a) = \lim\limits_{t \to a^+} W(t) > -\infty$.

Let $v : I \to I$ be a non-negative locally integrable function on I and $\varphi : I \to I$ be a strictly increasing locally absolutely continuous function with the property:

$$\lim_{x \to a^+} \varphi(x) = a, \ \lim_{x \to b^-} \varphi(x) = b, \ \varphi(x) \le x, \quad \forall x \in I.$$

Consider the operator in the form

$$K_{\alpha,\varphi} f(x) = \int\limits_a^{\varphi(x)} \frac{f(s) w(s) ds}{(W(x) - W(s))^{1-\alpha}}, \quad x \in I, \tag{1}$$

from $L_{p,w} = L_{p,w}(I)$ to $L_{q,v} = L_{q,v}(I)$, where $L_{p,w}$ is the space of measurable functions $f : I \to R$ for which the functional

$$\|f\|_{p,w} = \left(\int\limits_a^b |f(x)|^p w(x) dx \right)^{\frac{1}{p}}, \quad 0 < p < \infty,$$

is finite. Let

$$W_0(x) = W(x) - W(a). \tag{2}$$

Then $W_0(x) \ge 0$, $W_0(a) = 0$, and the operator (1) can be written as

$$K_{\alpha,\varphi} f(x) = \int\limits_a^{\varphi(x)} \frac{f(s) w(s) ds}{(W_0(x) - W_0(s))^{1-\alpha}}, \quad x \in I.$$

Therefore, unless otherwise stated, further on we will assume that in (1) $W(\cdot) \ge 0$ and $W(a) = 0$.

In the case $\varphi(x) \equiv x$ the operator (1) is studied in the papers [1, 2], and in the case $\varphi(x) \equiv x$, $W(x) = x$ the operator (1) is the Riemann-Liouville operator and its various aspects are considered in many papers and books, for example in [3, 4, 5, 6, 6].

Together with operator (1) we consider the operator

$$K'_{\alpha,\varphi} g(s) = \int\limits_{\varphi^{-1}(s)}^{b} \frac{g(x) v(x) dx}{(W(x) - W(s))^{1-\alpha}}, \quad s \in I \tag{3}$$

from $L_{p,w}$ to $L_{q,v}$, where φ^{-1} is an inverse function to φ.

Throughout this paper expressions of the form $\frac{0}{0}$, $0 \cdot \infty$ are supposed be equal to zero. The relation $A \ll B$ ($A \gg B$) means that $A \le CB$ ($B \le CA$) with a constant C depending only on p, q, α which can be different in different places. If $A \ll B$ and $A \gg B$, then we write $A \approx B$. By Z we denote the set of all integer numbers and χ_E denotes the characteristic function of the set E.

2 Auxiliary results.

Besides the operator (1) we also consider the operator

$$H_\varphi f(x) = \frac{1}{W^{1-\alpha}(x)} \int_a^{\varphi(x)} f(s)w(s)ds, \quad x \in I. \tag{1}$$

From (1), (1) it is easy to see that

$$K_{\alpha,\varphi} f \geq H_\varphi f \tag{2}$$

for $f \geq 0$.

In assumptions about the function φ the boundedness of the operator (1) from $L_{p,w}$ to $L_{q,v}$ is equivalent (see [8]) to the boundedness of the Hardy type operator

$$Hf(x) = \frac{1}{W^{1-\alpha}(\varphi^{-1}(x))} \int_a^x f(s)w(s)ds, \quad x \in I,$$

from $L_{p,w}$ to $L_{q,\widetilde{v}}$, where $\widetilde{v}(t) = v(\varphi^{-1}(t))(\varphi^{-1}(t))'$. Therefore, from the results of the study the Hardy inequality (see, for example, [9]), we have

Lemma 2.1. *Let $1 < p \leq q < \infty$. Then the operator (1) is bounded from $L_{p,w}$ to $L_{q,v}$ if and only if $A = \sup_{t\in I} A(t) < \infty$, where*

$$A(t) = \left(\int_t^b W^{q(\alpha-1)}(x)v(x)dx \right)^{\frac{1}{q}} W^{\frac{1}{p'}}(\varphi(t)).$$

Moreover, $\|H_\varphi\| \approx A$.

Remark 2.2. *Here and below $\|T\|$ denotes the norm of the operator $T : L_{p,w} \to L_{q,v}$, where the operator T either $T = H_\varphi$ or $T = K_{\alpha,\varphi}$.*

Lemma 2.3. *Let $0 < q < p < \infty, \quad p > 1$. Then the operator (1) is bounded from $L_{p,w}$ to $L_{q,v}$ if and only if*

$$B = \left(\int_a^b \left(\int_t^b W^{q(\alpha-1)}(x)v(x)dx \right)^{\frac{q}{p-q}} W^{\frac{q(p-1)}{p-q}}(\varphi(t)) \frac{v(t)dt}{W^{q(1-\alpha)}(t)} \right)^{\frac{p-q}{pq}} < \infty.$$

Moreover, $\|H_\varphi\| \approx B$.

We also need the following Lemma:

Lemma 2.4. *Let $0 < \beta < 1$ and the function $\gamma(\cdot)$ defined on I, such that $0 < \gamma(x) \leq 1, \quad \forall x \in I$. Then*

$$\int_0^{\gamma(x)} \frac{dz}{(1-z)^{1-\beta}} \leq \frac{\gamma(x)}{\beta}, \quad \forall x \in I.$$

Indeed, using the inequality $(1-\gamma(x))^{\beta} \geq 1-\gamma(x)$, we have

$$\int_0^{\gamma(x)} \frac{dz}{(1-z)^{1-\beta}} = \frac{1}{\beta}[1-(1-\gamma(x))^{\beta}] \leq \frac{1}{\beta}[1-(1-\gamma(x))] = \frac{\gamma(x)}{\beta}.$$

3 The main results.

Our first main result reads:

Theorem 3.1. *Let $1 < p \leq q < \infty$, $\frac{1}{p} < \alpha < 1$ and A be defined as in Lemma 2.1. Then the operator* (1) *is bounded from $L_{p,w}$ to $L_{q,v}$ if and only if $A < \infty$. Moreover,*

$$\|K_{\alpha,\varphi}\| \approx A. \tag{1}$$

Our next main result reads:

Theorem 3.2. *Let $0 < q < p < \infty$, $p > \frac{1}{\alpha}$, $0 < \alpha < 1$ and B be defined as in Lemma 2.3. Then the operator* (1) *is bounded from $L_{p,w}$ to $L_{q,v}$ if and only if $B < \infty$. Moreover,*

$$\|K_{\alpha,\varphi}\| \approx B. \tag{2}$$

In the case $0 \neq W(a) > -\infty$, in accordance with Remark 2.2 the following theorems follows from Theorems 3.1 and 3.2, respectively:

Corollary 3.1. *Let $1 < p \leq q < \infty$, $\frac{1}{p} < \alpha < 1$ and W_0 be defined by* (2)*. Then the operator* (1) *is bounded from $L_{p,w}$ to $L_{q,v}$ if and only if*

$$A_0 = \sup_{a<z<b} \left(\int_z^b W_0^{q(\alpha-1)}(x) v(x) dx \right)^{\frac{1}{q}} W_0^{\frac{1}{p'}}(\varphi(z)) < \infty.$$

Moreover, $\|K_{\alpha,\varphi}\| \approx A_0$.

Corollary 3.2. *Let $0 < q < p < \infty$, $p > \frac{1}{\alpha}$, $0 < \alpha < 1$ and W_0 be defined by* (2)*. Then the operator* (1) *is bounded from $L_{p,w}$ to $L_{q,v}$ if and only if*

$$B_0 = \left(\int_a^b \left(\int_t^b W_0^{q(\alpha-1)}(x) v(x) dx \right)^{\frac{q}{p-q}} W_0^{\frac{q(p-1)}{p-q}}(\varphi(t)) \frac{v(t)dt}{W_0^{q(1-\alpha)}(t)} \right)^{\frac{p-q}{pq}} < \infty.$$

Moreover, $\|K_{\alpha,\varphi}\| \approx B_0$.

For the operator (3) we have the following results:

Theorem 3.3. *Let $1 < p \le q < \frac{1}{1-\alpha}$, $0 < \alpha < 1$ and W_0 be defined by* (2)*. Let $W(a) > -\infty$. Then the operator $K'_{\alpha,\varphi}$ defined by* (3) *is bounded from $L_{p,w}$ to $L_{q,v}$ if and only if*

$$A' = \sup_{a<z<b}\left(\int\limits_z^b W_0^{p'(\alpha-1)}(x)v(x)dx\right)^{\frac{1}{p'}} W_0^{\frac{1}{q}}(\varphi(z)) < \infty.$$

Moreover, $\|K'_{\alpha,\varphi}\| \approx A'$.

Theorem 3.4. *Let $1 < q < min\{p, \frac{1}{1-\alpha}\}$, $0 < \alpha < 1$ and W_0 be defined by* (2)*. Let $W(a) > -\infty$. Then the operator $K'_{\alpha,\varphi}$ defined by* (3) *is bounded from $L_{p,w}$ to $L_{q,v}$ if and only if*

$$B' = \left(\int\limits_a^b\left(\int\limits_t^b W_0^{p'(\alpha-1)}(x)v(x)dx\right)^{\frac{p(q-1)}{p-q}} W_0^{\frac{p}{p-q}}(\varphi(t))\frac{v(t)dt}{W_0^{p'(1-\alpha)}(t)}\right)^{\frac{p-q}{pq}} < \infty.$$

Moreover, $\|K'_{\alpha,\varphi}\| \approx B'$.

The boundedness of the operator (1) from $L_{p,w}$ to $L_{q,v}$ is equivalent to the boundedness of the adjoint operator

$$K^*_{\alpha,\varphi}g(s) = w(s)\int\limits_{\varphi^{-1}(s)}^b \frac{g(x)dx}{(W(x)-W(s))^{1-\alpha}}, \quad s \in I$$

from $L_{q',v^{1-q'}}$ to $L_{p',w^{1-p'}}$, which in turn is equivalent to the boundedness of the operator $K'_{\alpha,\varphi}$ defined by (3) from $L_{q',w}$ to $L_{p',v}$. Therefore, making by replacing q' and p' by p and q, respectively in Theorems 3.3 and 3.4, we obtain the assertions of Corollaries 3.1 and 3.2, respectively.

Our main result concerning compactness of the operator $K_{\alpha,\varphi}$ reads:

Theorem 3.5. *Let $0 < \alpha < 1$ and $\frac{1}{\alpha} < p \le q < \infty$. Then the following statements are equivalent:*

i) $K_{\alpha,\varphi} : L_{p,w} \to L_{q,v}$ *is compact;*

ii) $A < \infty$ *and* $\lim\limits_{t\to a^+} A(t) = \lim\limits_{t\to b^-} A(t) = 0.$

Theorem 3.6. *Let $b < \infty$, $0 < \alpha < 1$, $0 < q < p < \infty$ and $p > \frac{1}{\alpha}$. Then the operator $K_{\alpha,\varphi}$ is compact from $L_{p,w}$ to $L_{q,v}$ if and only if $B < \infty$ holds.*

4 Proofs of the main results.

Proof of Theorem 3.1. Necessity. Let the operator (1) be bounded from $L_{p,w}$ to $L_{q,v}$. Then from (1), (1), (6) it follows that the operator H_φ boundedly maps from $L_{p,w}$ to $L_{q,v}$ and $\|K_{\alpha,\varphi}\| \ge \|H_\varphi\|$. Consequently, by virtue of Lemma 2.1,

$$\|K_{\alpha,\varphi}\| \gg A. \tag{1}$$

Sufficiency. Let $A < \infty$. Consider the function $W(\varphi(x))$. In view of the conditions imposed on the function φ and W we have that the function $W(\varphi(x))$ is continuous, strictly increasing and $W(\varphi(a)) = W(a) = 0$.

For any $k \in Z$ we define $x_k = \sup\{x \in I : W(\varphi(x)) \le 2^k\}$. Hence, $a < x_k \le x_{k+1} \le b$ for any $k \in Z$ and $W(\varphi(x_k)) \equiv \lim\limits_{x \to x_k} W(\varphi(x)) \le 2^k$, but if $x_k < b$, then $x_{k-1} < x_k$ and $W(\varphi(x_k)) = 2^k$.

Assume that $\varphi(x_k) = t_k$, $\;I_k = [x_k, x_{k+1})$, $\;J_k = [t_k, t_{k+1})$ and $Z_0 = \{k \in Z : I_k \neq \emptyset\}$. Then

$$I = \bigcup_{k \in Z_0} I_k = \bigcup_{k \in Z_0} J_k, \tag{2}$$

$$W(\varphi(x_k)) = W(t_k) = 2^k, \quad k \in Z_0, \tag{3}$$

$$2^k \le W(\varphi(x)) < 2^{k+1}, \quad \text{for} \quad x \in I_k, \quad k \in Z_0. \tag{4}$$

Let $f \in L_{p,w}$. By using (2) and the relation $\varphi(x_{k-1}) \le x_{k-1} < x_k$, $k \in Z_0$ we have

$$\int_a^b v(x)|K_{\alpha,\varphi} f(x)|^q dx \le \sum_k \int_{x_k}^{x_{k+1}} v(x) \left(\int_a^{\varphi(x)} \frac{|f(s)|w(s)ds}{(W(x)-W(s))^{1-\alpha}} \right)^q dx$$

$$\le 2^{q-1} \left(\sum_k \int_{x_k}^{x_{k+1}} v(x) \left(\int_{\varphi(x_{k-1})}^{\varphi(x)} \frac{|f(s)|w(s)ds}{(W(x)-W(s))^{1-\alpha}} \right)^q dx \right.$$

$$\left. + \sum_k \int_{x_k}^{x_{k+1}} v(x) \left(\int_a^{\varphi(x_{k-1})} \frac{|f(s)|w(s)ds}{(W(x)-W(s))^{1-\alpha}} \right)^q dx \right) = 2^{q-1}(F_1 + F_2). \tag{5}$$

Here and in the sequal, the summation is taken over the set Z_0 with respect to index k.

We estimate the expressions F_1 and F_2 separately. Applying Hölder's inequality, we obtain

$$F_1 = \sum_k \int_{x_k}^{x_{k+1}} v(x) \left(\int_{\varphi(x_{k-1})}^{\varphi(x)} \frac{|f(s)|w(s)ds}{(W(x)-W(s))^{1-\alpha}} \right)^q dx$$

$$\le \sum_k \int_{x_k}^{x_{k+1}} v(x) \left(\int_{\varphi(x_{k-1})}^{\varphi(x)} |f(s)|^p w(s)ds \right)^{\frac{q}{p}} \left(\int_{\varphi(x_{k-1})}^{\varphi(x)} \frac{w(s)ds}{(W(x)-W(s))^{p'(1-\alpha)}} \right)^{\frac{q}{p'}} dx$$

$$\le \sum_k \left(\int_{\varphi(x_{k-1})}^{\varphi(x_{k+1})} |f(s)|^p w(s)ds \right)^{\frac{q}{p}} \int_{x_k}^{x_{k+1}} v(x) \left(\int_a^{\varphi(x)} \frac{w(s)ds}{(W(x)-W(s))^{p'(1-\alpha)}} \right)^{\frac{q}{p'}} dx. \tag{6}$$

Making the change of the variable $W(s) = W(x)z$ in the last integral and applying Lemma 2.4, we find that

$$\int_a^{\varphi(x)} \frac{w(s)ds}{(W(x)-W(s))^{p'(1-\alpha)}} \le \frac{W(x)}{W^{p'(1-\alpha)}(x)} \int_0^{\frac{W(\varphi(x))}{W(x)}} \frac{dz}{(1-z)^{1-p'(\alpha-\frac{1}{p})}}$$

$$\leq \frac{1}{p'(\alpha-\frac{1}{p})}\frac{W(\varphi(x))}{W^{p'(1-\alpha)}(x)}.$$

Substituting this in (6) and using (2) - (4), we obtain that:

$$F_1 \ll \sum_k \left(\int_{t_{k-1}}^{t_{k+1}} |f(s)|^p w(s)ds\right)^{\frac{q}{p}} \int_{x_k}^{x_{k+1}} W^{q(\alpha-1)}(x)v(x)W^{\frac{q}{p'}}(\varphi(x))dx$$

$$\leq \sum_k \left(\int_{t_{k-1}}^{t_{k+1}} |f(s)|^p w(s)ds\right)^{\frac{q}{p}} 2^{\frac{q}{p'}(k+1)} \int_{x_k}^{x_{k+1}} W^{q(\alpha-1)}(x)v(x)dx$$

$$\ll \sum_k \left(\int_{t_{k-1}}^{t_{k+1}} |f(s)|^p w(s)ds\right)^{\frac{q}{p}} W^{\frac{q}{p'}}(\varphi(x_k)) \int_{x_k}^{x_{k+1}} W^{q(\alpha-1)}(x)v(x)dx \quad (7)$$

$$\ll A^q \sum_k \left(\int_{t_{k-1}}^{t_{k+1}} |f(s)|^p w(s)ds\right)^{\frac{q}{p}} \ll A^q \left(\sum_k \int_{t_{k-1}}^{t_{k+1}} |f(s)|^p w(s)ds\right)^{\frac{q}{p}}$$

$$\ll A^q \|f\|_{p,w}^q. \quad (8)$$

In order to estimate F_2 we use (2), (3) and the estimate $W(x) \geq W(\varphi(x))$, $\quad x \in I$, to deduce that

$$F_2 := \sum_k \int_{x_k}^{x_{k+1}} v(x)\left(\int_a^{\varphi(x_{k-1})} \frac{f(s)w(s)ds}{(W(x)-W(s))^{1-\alpha}}\right)^q dx$$

$$\leq \sum_k \int_{x_k}^{x_{k+1}} v(x)\left(\int_a^{\varphi(x_{k-1})} \frac{f(s)w(s)ds}{(W(x)-W(\varphi(x_{k-1})))^{1-\alpha}}\right)^q dx$$

$$\leq \sum_k \int_{x_k}^{x_{k+1}} \frac{v(x)dx}{(W(x)-W(\varphi(x_{k-1})))^{q(1-\alpha)}}\left(\int_a^{\varphi(x_{k-1})} f(s)w(s)ds\right)^q.$$

Taking the following estimates

$$W(x)-W(\varphi(x_{k-1})) = W(x)-\frac{1}{2}\cdot 2^k = W(x)-\frac{1}{2}W(\varphi(x_k))$$

$$\geq W(x)-\frac{1}{2}W(x_k) \geq W(x)-\frac{1}{2}W(x) = \frac{1}{2}W(x),$$

for $x_k \leq x \leq x_{k+1}$, into account, we obtain that

$$F_2 \leq 2^{q(1-\alpha)} \sum_k \int_{x_k}^{x_{k+1}} \frac{v(x)}{W^{q(1-\alpha)}(x)}\left(\int_0^{\varphi(x_{k-1})} f(s)w(s)ds\right)^q dx$$

$$\ll \sum_k \int\limits_{x_k}^{x_{k+1}} v(x) \left(\frac{1}{W^{1-\alpha}(x)} \int\limits_a^{\varphi(x)} f(s)w(s)ds \right)^q dx \le \|H_\varphi f\|_{q,v}^q. \tag{9}$$

Hence, on the basis of Lemma 2.1,

$$F_2 \ll A^q \|f\|_{p,w}^q. \tag{10}$$

From (5), (8) and (10) it follows that the operator (1) is bounded from $L_{p,w}$ to $L_{q,v}$, Moreover, $\|K_{\alpha,\varphi}\| \ll A$, which together with (5) gives (1). The proof is complete.

Proof of Theorem 3.2. Necessity. Let the operator (1) be bounded from $L_{p,w}$ to $L_{q,v}$. Then, as in Theorem 3.1, from (6) and from Lemma 2.3, we have

$$\|K_{\alpha,\varphi}\| \gg B. \tag{11}$$

Sufficiency. Let $B < \infty$. To estimate the norm of the operator (1), we proceed from the relation (5). By virtue of (9) and Lemma 2.3, we have

$$F_2 \ll B^q \|f\|_{p,w}^q. \tag{12}$$

Estimating F_1 in a similar way as in Theorem 3.1, we obtain the relation (7) and applying Hölder's inequality with exponents $\frac{p}{q}$ and $\frac{p}{p-q}$, we have

$$F_1 \ll \sum_k \left(\int\limits_{t_{k-1}}^{t_{k+1}} |f(s)|^p w(s)ds \right)^{\frac{q}{p}} W^{\frac{q}{p'}}(\varphi(x_k)) \int\limits_{x_k}^{x_{k+1}} W^{q(\alpha-1)}(x)v(x)dx$$

$$\le \left(\sum_k \int\limits_{t_{k-1}}^{t_{k+1}} |f(s)|^p w(s)ds \right)^{\frac{q}{p}}$$

$$\times \left(\sum_k W^{\frac{q(p-1)}{p-q}}(\varphi(x_k)) \left(\int\limits_{x_k}^{x_{k+1}} W^{q(\alpha-1)}(x)v(x)dx \right)^{\frac{p}{p-q}} \right)^{\frac{p-q}{p}}$$

$$\le 2^{\frac{q}{p}} \|f\|_{p,w}^q \left(\frac{p}{p-q} \sum_k W^{\frac{q(p-1)}{p-q}}(\varphi(x_k)) \right.$$

$$\left. \times \int\limits_{x_k}^{x_{k+1}} \left(\int\limits_t^{x_{k+1}} W^{q(\alpha-1)}(x)v(x)dx \right)^{\frac{q}{p-q}} W^{q(\alpha-1)}(t)v(t)dt \right)^{\frac{p-q}{p}}$$

$$\ll \left(\sum_k \int\limits_{x_k}^{x_{k+1}} \left(\int\limits_t^b W^{q(\alpha-1)}(x)v(x)dx \right)^{\frac{q}{p-q}} W^{\frac{q(p-1)}{p-q}}(\varphi(t)) \frac{v(t)dt}{W^{q(1-\alpha)}(t)} \right)^{\frac{p-q}{p}} \|f\|_{p,w}^q$$

$$\le B^q \|f\|_{p,w}^q. \tag{13}$$

From (5), (12) and (13) it follows that the operator (1) is bounded from $L_{p,w}$ to $L_{q,v}$ and, moreover, $\|K_{\alpha,\varphi}\| \ll B$, which together with (11) gives (2). The proof is complete.

Proofs of Theorems 3.3 and 3.4: The proof are similar to those of Theorems 3.1 and 3.2, respectively, so we omit the details.

Proof of Theorem 3.5. Necessity. Suppose that the operator (1) is compact from $L_{p,w}(I)$ to $L_{q,v}(I)$. We show that (ii) is true.

Since the operator $K_{\alpha,\varphi}$ is compact we get that the operator (1) is bounded. Then, from Theorem 3.1 its follows that $A < \infty$.

To prove $\lim\limits_{t\to a^+} A(t) = \lim\limits_{t\to b^-} A(t) = 0$ we use the well known fact that a compact operator maps a weakly convergent sequence into a strongly convergent one. For $a < s < b$ consider the family of functions

$$f_s(x) = \chi_{(a,\varphi(s)]}(x) W^{-\frac{1}{p}}(\varphi(s)), \ x \in I. \tag{14}$$

It is easy to see that $\{f_s\}_{s\in(a,b)} \in L_{p,w}$.

Indeed,

$$\|f_s\|_{p,w} = \left(\int_a^b |f_s(x)|^p w(x)dx \right)^{\frac{1}{p}} = W^{-\frac{1}{p}}(\varphi(s)) \left(\int_a^{\varphi(s)} w(x)dx \right)^{\frac{1}{p}} = 1. \tag{15}$$

We show that the family of functions (14) converges weakly to zero in $L_{p,w}$.

By using properties of $\varphi(x)$ and the Hӧlder inequality together with (15) we find that

$$\int_a^b f_s(x)g(x)dx = \int_a^{\varphi(s)} f_s(x)g(x)dx$$

$$\le \left(\int_a^b |f_s(x)|^p w(x)dx \right)^{\frac{1}{p}} \left(\int_a^s |g(x)|^{p'} w^{1-p'}(x)dx \right)^{\frac{1}{p'}}$$

$$= \left(\int_a^s |g(x)|^{p'} w^{1-p'}(x)dx \right)^{\frac{1}{p'}} \tag{16}$$

for all $g \in L_{p',w^{1-p'}}$.

Since $g \in L_{p',w^{1-p'}}$, then last integral in (16) tends to zero when $s \to a^+$, which means weak convergence $f_s \to 0$ at $s \to a^+$. Since a compact operator in a Banach space every weakly convergent sequence translates into a strongly convergent one, then we get that

$$\lim_{s\to a^+} \|K_{\alpha,\varphi} f_s\|_{q,v} = 0. \tag{17}$$

On the other hand, by using properties of functions $W(x)$ and $\varphi(x)$ we have

$$\|K_{\alpha,\varphi} f_s\|_{q,v} = \left(\int_a^b v(x) \left| \int_a^{\varphi(x)} \frac{f_s(t)w(t)dt}{(W(x) - W(t))^{1-\alpha}} \right|^q dx \right)^{\frac{1}{q}}$$

$$\geq \left(\int_s^b v(x) \left| \int_a^{\varphi(s)} \frac{W^{-\frac{1}{p}}(\varphi(s)) w(t) dt}{(W(x) - W(t))^{1-\alpha}} \right|^q dx \right)^{\frac{1}{q}}$$

$$\geq W^{-\frac{1}{p}}(\varphi(s)) \left(\int_s^b v(x) W^{q(\alpha-1)}(x) dx \right)^{\frac{1}{q}} \int_a^{\varphi(s)} w(t) dt$$

$$= W^{\frac{1}{p'}}(\varphi(s)) \left(\int_s^b v(x) W^{q(\alpha-1)}(x) dx \right)^{\frac{1}{q}} = A(s). \tag{18}$$

By combining (17) and (18) we find that $\lim_{s \to a^+} A(s) = 0$.

Next we show that $\lim_{t \to b^-} A(t) = 0$. The compactness of the operator $K_{\alpha,\varphi}$ implies compactness of the dual operator

$$K^*_{\alpha,\varphi} g(t) = w(t) \int_{\varphi^{-1}(t)}^b \frac{g(x) dx}{(W(x) - W(t))^{1-\alpha}}, \quad t \in I, \tag{19}$$

from $L_{q',v^{1-q'}}$ to $L_{p',w^{1-p'}}$.

For $a < s < b$ we consider the family of functions

$$g_s(x) = \chi_{[s,b)}(x) \left(\int_s^b v(t) W^{q(\alpha-1)}(t) dt \right)^{-\frac{1}{q'}} W^{(q-1)(\alpha-1)}(x) v(x), \quad x \in I. \tag{20}$$

These functions are properly defined, since the integrals in the definition of the functions $g_s(x)$, are finite because $A < \infty$.

In addition, $g_s \in L_{q',v^{1-q'}}$, for any $s \in (a, b)$. Indeed,

$$\|g_s\|_{q',v^{1-q'}} = \left(\int_a^b |g_s(x)|^{q'} v^{1-q'}(x) dx \right)^{\frac{1}{q'}}$$

$$= \left(\int_s^b W^{q(\alpha-1)}(t) v(t) dt \right)^{-\frac{1}{q'}} \left(\int_s^b |W^{(q-1)(\alpha-1)}(x) v(x)|^{q'} v^{1-q'}(x) dx \right)^{\frac{1}{q'}}$$

$$= \left(\int_s^b W^{q(\alpha-1)}(t) v(t) dt \right)^{-\frac{1}{q'}} \left(\int_s^b W^{q(\alpha-1)}(t) v(t) dt \right)^{\frac{1}{q'}} = 1. \tag{21}$$

From (21) it follows that

$$\int_a^b g_s(x) f(x) dx = \int_s^b g_s(x) f(x) dx$$

$$\leq \left(\int_s^b |g_s(x)|^q v^{-\frac{q'}{q}}(x)dx\right)^{\frac{1}{q'}} \left(\int_s^b |f(x)|^q v(x)dx\right)^{\frac{1}{q}}$$

$$\leq \left(\int_s^b |f(x)|^q v(x)dx\right)^{\frac{1}{q}} \|g_s\|_{q',v^{1-q'}} = \left(\int_s^b |f(x)|^q v(x)dx\right)^{\frac{1}{q}}$$

for all $f \in L_{q,v}$.

Since $f \in L_{q,v}$, the last integral tends to zero at $s \to b^-$. Hence, the family of functions $\{g_s\}_{s\in(a,b)}$ converge weakly to zero in $L_{q',v^{1-q'}}$ when $s \to b^-$.

The dual operator $K^*_{\alpha,\varphi}$ is compact from $L_{q',v^{1-q'}}$ to $L_{p',w^{1-p'}}$. Therefore,

$$\lim_{s\to b^-} \|K^*_{\alpha,\varphi} g_s\|_{p',w^{1-p'}} = 0. \tag{22}$$

However, the following estimate holds:

$$\|K^*_{\alpha,\varphi} g_s\|_{p',w^{1-p'}} = \left(\int_a^b w(t) \left| \int_{\varphi^{-1}(t)}^b \frac{g_s(x)dx}{(W(x)-W(t))^{1-\alpha}} \right|^{p'} dt\right)^{\frac{1}{p'}}$$

$$\geq \left(\int_a^{\varphi(s)} w(t) \left| \int_{\varphi^{-1}(t)}^b \frac{g_s(x)dx}{(W(x)-W(t))^{1-\alpha}} \right|^{p'} dt\right)^{\frac{1}{p'}}$$

$$\geq \left(\int_a^{\varphi(s)} w(t) \left| \int_s^b \frac{W^{(q-1)(\alpha-1)}(x)v(x)dx}{(W(x))^{1-\alpha}} \right|^{p'} dt\right)^{\frac{1}{p'}} \left(\int_s^b W^{q(\alpha-1)}(t)v(t)dt\right)^{-\frac{1}{q'}}$$

$$= \left(\int_s^b W^{q(\alpha-1)}(t)v(t)dt\right)^{-\frac{1}{q'}} \int_s^b W^{q(\alpha-1)}(t)v(t)dt \left(\int_a^{\varphi(s)} w(t)dt\right)^{\frac{1}{p'}} = A(s).$$

Consequently, by using (22) we have that $\lim_{s\to b^-} A(s) = 0$. Thus, the implication (i) $\Rightarrow$ (ii) holds.

Sufficiency. Now we will prove (ii)$\Rightarrow$ (i).

Let $a < c < d < b$. We take d such that $\varphi(d) > c$ and put $P_c f = \chi_{(a,c]} f$, $P_{cd} f = \chi_{(c,d]} f$, $Q_d f = \chi_{(d,b)} f$.

Then $f = \chi_{(a,c]} f + \chi_{(c,d]} f + \chi_{(d,b)} f = P_c f + P_{cd} f + Q_d f$.

We find that

$$K_{\alpha,\varphi} f = (P_c + P_{cd} + Q_d) K_{\alpha,\varphi} f = (P_c + P_{cd}) K_{\alpha,\varphi} (P_c + P_{cd} + Q_d) f + Q_d K_{\alpha,\varphi} f$$

$$= P_c K_{\alpha,\varphi} P_c f + P_c K_{\alpha,\varphi} P_{cd} f + P_c K_{\alpha,\varphi} Q_d f + P_{cd} K_{\alpha,\varphi} P_c f$$

$$+ P_{cd} K_{\alpha,\varphi} P_{cd} f + P_{cd} K_{\alpha,\varphi} Q_d f + Q_d K_{\alpha,\varphi} f.$$

Thus, since $P_cK_{\alpha,\varphi}P_{cd} \equiv 0$, $P_cK_{\alpha,\varphi}Q_d \equiv 0$, $P_{cd}K_{\alpha,\varphi}Q_d \equiv 0$ we can conclude that

$$K_{\alpha,\varphi}f = P_cK_{\alpha,\varphi}P_cf + P_{cd}K_{\alpha,\varphi}P_cf + P_{cd}K_{\alpha,\varphi}P_{cd}f + Q_dK_{\alpha,\varphi}f. \tag{23}$$

We show that the operator $P_{cd}K_{\alpha,\varphi}P_{cd}$ is compact from $L_{p,w}(I)$ to $L_{q,v}(I)$. Since $P_{cd}K_{\alpha,\varphi}P_{cd}f(x) = 0$ when $x \in I\backslash(c,d]$, then it suffices to show that the operator $P_{cd}K_{\alpha,\varphi}P_{cd}$ is compact from $L_{p,w}(c,d)$ to $L_{q,v}(c,d)$ and this is equivalent to the compactness from $L_{p,w}(c,d)$ to $L_{q,v}(c,d)$ of the operator $Kf(x) = \int_c^d K(x,s)f(s)ds$ with the kernel

$$K(x,t) = \frac{v^{\frac{1}{q}}(x)\chi_{(c,d]}(t)\theta(\varphi(x)-t)w^{\frac{1}{p'}}(t)}{(W(x)-W(t))^{(1-\alpha)}},$$

where $\theta(z)$ is Heaviside's unit step function, (that is, $\theta(z) = 1$ for $z \geq 0$ and $\theta(z) = 0$ for $z < 0$).

From the proof of the Theorem 3.1 there are points x_k, x_i such that $k - i = m \geq 1$, $x_k \geq d$ and $c \geq x_i$. Therefore, making the change of the variable $W(s) = W(x)z$ in the integral below and applying Lemma 2.4, we have that

$$\int_c^d\left(\int_c^d |K(x,t)|^{p'}dt\right)^{\frac{q}{p'}}dx = \int_c^d v(x)\left(\int_c^{\varphi(x)} \frac{\chi_{(c,d]}(t)w(t)dt}{(W(x)-W(t))^{p'(1-\alpha)}}\right)^{\frac{q}{p'}}dx$$

$$\leq \int_c^d v(x)\left(\int_a^{\varphi(x)} \frac{w(t)dt}{(W(x)-W(t))^{p'(1-\alpha)}}\right)^{\frac{q}{p'}}dx$$

$$\ll \int_{x_i}^{x_k} v(x)W^{q(\alpha-1)}(x)v(x)W^{\frac{q}{p'}}(\varphi(x))dx$$

$$\leq W^{\frac{q}{p'}}(\varphi(x_k))\int_{x_i}^{x_k} v(x)W^{q(\alpha-1)}(x)dx$$

$$\ll W^{\frac{q}{p'}}(\varphi(x_i))\int_{x_i}^{b} v(x)W^{q(\alpha-1)}(x)dx \leq A^q < \infty.$$

Therefore, on the basis of the theorem in Kantorovich and Akilov (see [10], page 420), the operator K is compact from $L_p(c,d)$ to $L_q(c,d)$, which is equivalent to the compactness of the operator $P_{cd}K_{\alpha,\varphi}P_{cd}$ from $L_{p,w}(I)$ to $L_{q,v}(I)$.

By using (23) we find that

$$\|K_{\alpha,\varphi} - P_{cd}K_{\alpha,\varphi}\| \leq \|P_cK_{\alpha,\varphi}\| + \|Q_dK_{\alpha,\varphi}\| + \|P_{cd}K_{\alpha,\varphi}P_c\|. \tag{24}$$

We will show that the right-hand side of (24) tends to zero as $c \to a^+$ and $d \to b^-$. This will imply that the operator $K_{\alpha,\varphi}$ being a uniform limit of compact operators, is compact from $L_{p,w}(I)$ to $L_{q,v}(I)$.

Consider each of the operators in (24) separately. By Theorem 3.1 we have

$$\|P_cK_{\alpha,\varphi}P_cf\|_{q,v} = \left(\int_a^c v(x)\left|\int_a^{\varphi(x)} \frac{f(t)w(t)dt}{(W(x)-W(t))^{(1-\alpha)}}\right|^q dx\right)^{\frac{1}{q}}$$

$$\ll \sup_{a<t<c} W^{\frac{1}{p'}}(\varphi(t))\left(\int_t^c W^{q(\alpha-1)}(x)v(x)dx\right)^{\frac{1}{q}}\|f\|_{p,w}$$

$$\leq \sup_{a<t<c} A(t)\|f\|_{p,w}.$$

Hence, $\|P_cK_{\alpha,\varphi}P_c\| \ll \sup_{a<t<c} A(t)$. Then

$$\lim_{c\to a^+}\|P_cK_{\alpha,\varphi}P_c\| \ll \lim_{t\to a^+} A(t) = 0. \tag{25}$$

Let $v_d = Q_dv$. Then, by Theorem 3.1 we obtain that

$$\|Q_bK_{\alpha,\varphi}f\|_{q,v} = \|K_{\alpha,\varphi}f\|_{q,v_d}$$

$$\ll \sup_{a<t<b} W^{\frac{1}{p'}}(\varphi(t))\left(\int_t^b W^{q(\alpha-1)}(x)v_d(x)dx\right)^{\frac{1}{q}}\|f\|_{p,w}$$

$$= \sup_{d<t<b} W^{\frac{1}{p'}}(\varphi(t))\left(\int_t^b W^{q(\alpha-1)}(x)v(x)dx\right)^{\frac{1}{q}}\|f\|_{p,w} = \sup_{d<t<b} A(t)\|f\|_{p,w}.$$

Consequently,

$$\lim_{d\to b^-}\|Q_dK_{\alpha,\varphi}\| \ll \lim_{t\to b^-} A(t) = 0. \tag{26}$$

Now we will prove that

$$\lim_{c\to a^+}\|P_{cd}K_{\alpha,\varphi}P_c\| = 0. \tag{27}$$

Since $\varphi(d) > c$ and the function $\varphi(x)$ is continuous then there exists a point $z \in (c,d)$ such that $\varphi(z) = c$. Since $\varphi(x)$ is a strictly increasing function, then $z = \varphi^{-1}(c)$.

We have that

$$\|P_{cd}K_{\alpha,\varphi}P_cf\|_{q,v}^q = \int_c^{\varphi^{-1}(c)} v(x)\left|\int_a^{\varphi(x)} \frac{\chi_{(a,c]}(t)f(t)w(t)dt}{(W(x)-W(t))^{(1-\alpha)}}\right|^q dx$$

$$+\int_{\varphi^{-1}(c)}^d v(x)\left|\int_a^{\varphi(x)} \frac{\chi_{(a,c]}(t)f(t)w(t)dt}{(W(x)-W(t))^{(1-\alpha)}}\right|^q dx = J_1 + J_2. \tag{28}$$

By Theorem 3.1, we get that

$$J_1 \leq \int_a^{\varphi^{-1}(c)} v(x)\left|\int_a^{\varphi(x)} \frac{f(t)w(t)dt}{(W(x)-W(t))^{(1-\alpha)}}\right|^q dx$$

$$\ll \sup_{a<t<\varphi^{-1}(c)} A^q(t)\|f\|_{p,w}^q. \tag{29}$$

Making the change of the variable $W(t) = W(x)s$ in the integral below and applying Hölder's inequality and Lemma 2.1 we obtain that

$$J_2 = \int_{\varphi^{-1}(c)}^{d} v(x)\left(\int_a^c \frac{f(t)w(t)dt}{(W(x)-W(t))^{(1-\alpha)}}\right)^q dx$$

$$\leq \int_{\varphi^{-1}(c)}^{d} v(x)\left(\int_a^c \frac{w(t)dt}{(W(x)-W(t))^{p'(1-\alpha)}}\right)^{\frac{q}{p'}} dx\,\|f\|_{p,w}^q$$

$$= \int_{\varphi^{-1}(c)}^{d} v(x)\frac{(W(x))^{\frac{q}{p'}}}{(W(x))^{q(1-\alpha)}}\left(\int_a^{\frac{W(c)}{W(x)}} \frac{ds}{(1-s)^{p'(1-\alpha)}}\right)^{\frac{q}{p'}} dx\,\|f\|_{p,w}^q$$

$$\ll \int_{\varphi^{-1}(c)}^{d} v(x)\frac{(W(x))^{\frac{q}{p'}}}{(W(x))^{q(1-\alpha)}}\left(\frac{W(c)}{W(x)}\right)^{\frac{q}{p'}} dx\,\|f\|_{p,w}^q$$

$$= W^{\frac{q}{p'}}(c)\int_{\varphi^{-1}(c)}^{d} v(x)(W(x))^{q(1-\alpha)}dx\,\|f\|_{p,w}^q$$

$$= A^q(\varphi^{-1}(c))\,\|f\|_{p,w}^q. \tag{30}$$

Since $\varphi^{-1}(c) \to a^+$ at $c \to a^+$, then from (29), (30) and (28) we have (27).

From (25), (26) and (27) it follows that the right side of (24) tends to zero with $c \to a^+$ and $d \to b^-$. The proof is complete. □

Proof of Theorem 3.6. The statement of Theorem 3.6 follows by Ando Theorem and its generalizations [11].

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Paper 5

Hardy type inequalities with logarithmic singularities

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Hardy type inequalities with logarithmic singularities

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A. Abylayeva and L.-E. Persson. Hardy type inequalities with logarithmic singularities, Luleå University of Technology, Department of Engineering Sciences and Mathematics, Research Report 5 (2016).

Abstract: We establish criteria for boundedness for some classes of integral operators with logarithmic singularities in weighted Lebesgue spaces for cases $1 < p \leq q < \infty$ and $1 < q < p < \infty$. As corollaries some corresponding new Hardy inequalities are pointed out.

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1 Introduction

Let $0 < q < \infty$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $R_+ = (0, \infty)$. Moreover, let $u : R_+ \to R$ and $v : R_+ \to R$ be weight functions, i.e. non-negative measurable functions on R_+.

Since the 70-s of the last century weighted estimates of the form

$$\|vKf\|_q \le C\|uf\|_p \tag{1}$$

are intensively studied in the literature for different classes of the operators K, where $\|\cdot\|_p$ is the usual norm of the space $L_p \equiv L_p(R_+)$.

Here the operator K is defined by

$$Kf(x) = \int_0^x \mathcal{K}(x,s)f(s)ds, \tag{2}$$

where $\mathcal{K}(x,s)$ is a kernel i.e. a measurable function on $R_+ \times R_+$. To characterise all weights so that inequalities of the type (2) hold are very important questions in the theory of what today are called Hardy type inequalities. To characterise (1) without restrictions of the kernel $\mathcal{K}(x,s)$ is still an open question.

Review of research in the period 1970 – 1982, where estimates of the form (1) are given, can be found in [5]. Some directions of research of the estimate (1) until 2009 for integral operators are summarized in the books [6, 5, 4, 3, 12]. Estimates of the form (1) are considered not only in Lebesgue spaces but also in other function spaces (see. e.g. [7, 8, 6] and Chapter 11 of the book [5]). Moreover, in [8] a sequence of classes of measurable kernels $\mathcal{K}(x,s)$ was considered and a full description of weights v and u was given so that the estimate (1) holds for the operator K defined by (2). However, these results do not include operators in the form of (2), when the kernel $\mathcal{K}(\cdot,\cdot)$ has a singularity, for example the Riemann-Liouville operator

$$R_\alpha f(x) = \int_0^x \frac{f(s)ds}{(x-s)^{1-\alpha}}, \tag{3}$$

when $0 < \alpha < 1$. The estimate of the form (1) remains open for the operator (3) in the general case. However, the following cases are studied: $v \equiv u$ in [3], $u \equiv 1$ in [15, 20] and u is non-decreasing in [7] and when one of the weighted functions v, u is non-increasing in [21] .

The estimate (1) for a singular operator in a form

$$Kf(x) = \int_0^x s^{\gamma-1} \ln \frac{x}{x-s} f(s)ds, \tag{4}$$

is equivalent to an estimate

$$\|K_\gamma f\|_q \le C\|f\|_p \tag{5}$$

for the operator

$$K_\gamma f(x) = v(x)\int_0^x u(s)s^{\gamma-1} \ln \frac{x}{x-s} f(s)ds. \tag{6}$$

The estimate (5) is equivalent to the boundedness of the operator (6) from L_p to L_q with the norm $\|K_\gamma\| = C$, where C is the best constant in (5). The operator (4) in the case $\gamma = 0$ is called a fractional integration operator of infinitesimal order [16].

The operator

$$K_\gamma^* f(s) = u(s)s^{\gamma-1} \int_s^\infty v(x) \ln \frac{x}{x-s} f(x)dx, \quad s > 0, \tag{7}$$

is dual to the operator K_γ with respect to the scalar product $\int_0^\infty f(x)g(x)dx$.

The main purpose of this paper is to establish the boundedness of the operator (6) and the dual operator (7) from L_p to L_q.

In the case $u(x) \equiv 1$ of boundedness from L_p to L_q of the operator (6) was studied in [1].

The main results (Theorems 1-4) are presented in Section 3. As corollaries some corresponding new Hardy type inequalities (Corollaries 1-4) are pointed out. The detailed proofs are given in Section 4 and in order not to disturb the argumentations in these proofs some auxiliary results are collected in Section 2.

Conventions: Uncertainties of the type $0 \cdot \infty$, $\frac{0}{0}$, $\frac{\infty}{\infty}$ are assumed to be zero. The inequality of the form $A \leq \beta B$ is written in the form $A \ll B$, where the positive constant β may be dependent on the parameters p, q, γ, and the relation $A \approx B$ means that $A \ll B \ll A$. $\chi_{(a,b)}(\cdot)$ denotes a characteristic function of the interval (a, b), Z is the set of integer numbers. The notations $\sum_k$, $\sup_k$ mean $\sum_{k\in Z}$, $\sup_{k\in Z}$, respectively.

2 Auxiliary results

Since

$$\ln \frac{x}{x-s} = \int_0^s \frac{dt}{x-t} \quad \text{for } x > s \geq 0, \tag{1}$$

the following inequalities

$$\frac{s}{x-s} > \ln \frac{x}{x-s} > \frac{s}{x}, \quad x > s > 0 \tag{2}$$

hold. The function $\ln \frac{x}{x-s}$ decreases with respect to x and increases with respects to s when $x > s \geq 0$, and from the inequality (2) it follows that the functions $x \ln \frac{x}{x-s}$, $\frac{1}{s} \ln \frac{x}{x-s}$ also decreases with respect to x and increases with respects to s when $x > s > 0$. Indeed,

$$\frac{\partial}{\partial x}\left(x \ln \frac{x}{x-s}\right) = \ln \frac{x}{x-s} - \frac{s}{x-s} < 0,$$

and

$$\frac{\partial}{\partial s}\left(\frac{1}{s} \ln \frac{x}{x-s}\right) = \frac{1}{s^2}\left(\frac{s}{x-s} - \ln \frac{x}{x-s}\right) > 0$$

for $x > s > 0$.

From (1) we have

$$\int_0^x \ln\frac{x}{x-s}f(s)ds = \int_0^x\int_0^s \frac{dt}{x-t}f(s)ds = \int_0^x \frac{1}{x-t}\int_t^x f(s)dsdt. \tag{3}$$

In the case when the function u is positive a.e. in R_+ we put $u(s)s^{\gamma-1}f(s) = g'(s)$. Then from (3) and (6) it follows that the inequality (5) is equivalent to the inequality

$$\left(\int_0^\infty \left|v(x)\int_0^x \frac{g(x)-g(s)}{x-s}ds\right|^q dx\right)^{\frac{1}{q}} \le C\left(\int_0^\infty |g'(x)u^{-1}(x)x^{1-\gamma}|^p dx\right)^{\frac{1}{p}} \tag{4}$$

for differentiable functions g.

Similarly, if the function v is positive a.e. in R_+, then the inequality (5) for the operator (7) is equivalent to the inequality

$$\left(\int_0^\infty \left|u(s)s^\gamma\int_s^\infty \frac{f(x)-f(s)}{x-s}\frac{dx}{x}\right|^q ds\right)^{\frac{1}{q}} \le C\left(\int_0^\infty |f'(x)v^{-1}(x)|^p dx\right)^{\frac{1}{p}} \tag{5}$$

for any differentiable functions f. In this case we have that

$$\int_s^\infty \ln\frac{x}{x-s}f(x)dx = \int_s^\infty f(x)\int_x^\infty \frac{sdt}{t(t-s)}dx = s\int_s^\infty \frac{1}{t-s}\int_t^x f(s)ds\frac{dt}{t}.$$

Along with the operator K_γ defined by (6) we consider the operator H_γ defined by

$$H_\gamma f(x) = \frac{v(x)}{x}\int_0^x u(s)s^\gamma f(s)ds, \quad x>0.$$

It easy to see that

$$K_\gamma f \ge H_\gamma f \tag{6}$$

for $f \ge 0$. Let

$$A(x) = \left(\int_0^x u^{p'}(s)s^{\gamma p'}ds\right)^{\frac{1}{p'}}\left(\int_x^\infty \frac{v^q(t)}{t^q}dt\right)^{\frac{1}{q}}, \quad A = \sup_{x>0} A(x).$$

For the operator H_γ the following theorem holds [5, 4, 12]:

Theorem A. *Let $1 < p \le q < \infty$. Then the operator H_γ is bounded from L_p to L_q if and only if $A < \infty$. Moreover, $\|H_\gamma\| \approx A$.*

Remark 2.1. *Here and below for any operator T the value $\|T\|$ denotes the norm of the operator T from L_p to L_q.*

The corresponding result for the case $q < p$ reads:

Theorem B. *Let $0 < q < p < \infty$, $p > 1$. The operator H_γ is bounded from L_p to L_q if and only if*

$$B = \left(\int_0^\infty \left(\int_x^\infty \frac{v^q(t)}{t^q} dt \right)^{\frac{p}{p-q}} \left(\int_0^x u^{p'}(s) s^{p'\gamma} ds \right)^{\frac{p(q-1)}{p-q}} u^{p'}(x) x^{p'\gamma} dx \right)^{\frac{p-q}{pq}} < \infty.$$

Moreover, $\|H_\gamma\| \approx B$.

Remark 2.2. *In the case $1 < q < p < \infty$, the constant B is equivalent to the constant*

$$\widetilde{B} = \left(\int_0^\infty \left(\int_x^\infty \frac{v^q(t)}{t^q} dt \right)^{\frac{q}{p-q}} \left(\int_0^x u^{p'}(s) s^{p'\gamma} ds \right)^{\frac{q(p-1)}{p-q}} \frac{v^q(x)}{x^q} dx \right)^{\frac{p-q}{pq}}.$$

3 The main results

Our first main result reads:

Theorem 3.1. *Let $1 < p \le q < \infty$, $\gamma > \frac{1}{p}$, and $u(x)$ be a non-increasing function. Then the operator K_γ defined by* (6) *is bounded from L_p to L_q if and only if $A < \infty$ and, moreover, $\|K_\gamma\| \approx A$.*

Corollary 3.2. *Let the function u be positive a.e. on R_+ and the conditions of Theorem 3.1 be fulfilled. Then the Hardy type inequality* (4) *holds if and only if $A < \infty$. Moreover, $A \approx C$, where C is the best constant in* (4).

The corresponding result for the case $q < p$ reads:

Theorem 3.3. *Let $p > 1$, $0 < q < p < \infty$ and $\gamma > \frac{1}{p}$. Let u be a non-increasing function on R_+. Then the operator K_γ defined by* (6) *is bounded from L_p to L_q if and only if $B < \infty$ and, moreover, $\|K_\gamma\| \approx B$.*

Corollary 3.4. *Let $0 < q < p < \infty$. Let the function u be positive a.e. in R_+ and the conditions of Theorem 3.3 be fulfilled. Then the Hardy type inequality* (4) *holds if and only if $B < \infty$. Moreover, $B \approx C$ for the best constant C in* (4).

We define

$$A^*(x) = \left(\int_x^\infty \frac{v^{p'}(t)}{t^{p'}} \right)^{\frac{1}{p'}} \left(\int_o^x s^{q\gamma} u^q(s) ds \right)^{\frac{1}{q}}, \quad A^* = \sup_{x>0} A^*(x),$$

and

$$B^* = \left(\int_0^\infty \left(\int_x^\infty \frac{v^q(t)}{t^{p'}} \right)^{\frac{q(p-1)}{p-q}} \left(\int_0^x s^{q\gamma} u^q(s) \right)^{\frac{q}{p-q}} x^{q\gamma} u^q(x) dx \right)^{\frac{p-q}{pq}}.$$

We consider the operator K_γ^* (defined by (7)) from L_p to L_q. If $1 < p, q < \infty$, then the operator K_γ^* is bounded from L_p to L_q if and only if the operator K_γ is bounded from $L_{q'}$ to $L_{p'}$. In this case the conditions $1 < p \le q < \infty$ and $1 < q < p < \infty$ are equivalent to the conditions $1 < q' \le p' < \infty$ and $1 < p' < q' < \infty$, respectively. Therefore from Theorems 3.1 and 3.3, we have the following:

Theorem 3.5. *Let $1 < p \le q < \infty$ and $\gamma > \frac{1}{p}$. Then the operator K_γ^* defined by* (7) *is bounded from L_p to L_q if only if $A^* < \infty$ and, moreover, $\|K_\gamma^*\| \approx A^*$.*

Corollary 3.6. *Let the function v be positive a.e. on R_+ and the conditions of Theorem 3.5 be fulfilled. Then the Hardy type inequality* (5) *holds if and only if $A^* < \infty$. Moreover, $A^* \approx C$, where C is the best constant in* (5).

Theorem 3.7. *Let $1 < q < p < \infty$ and $\gamma > \frac{1}{p}$. Then the operator K_γ^* defined by* (7) *is bounded from L_p to L_q if only if $B^* < \infty$ and, moreover, $\|K_\gamma^*\| \approx B^*$.*

Corollary 3.8. *Let the function v be positive a.e. on R_+ and the conditions of Theorem 3.7 be fulfilled. Then the Hardy type inequality* (5) *holds if and only if $B^* < \infty$. Moreover, $B^* \approx C$ for the best constant C in* (5).

4 Proofs of the main results

Proof of Theorem 3.1. Necessity. Let the operator (6) be bounded from L_p to L_q. Then, in view of (6), the operator H_γ is bounded from L_p to L_q and $\|K_\gamma\| \ge \|H_\gamma\|$. Therefore, by Theorem A the value $A < \infty$ and

$$\|K_\gamma\| \gg A. \tag{1}$$

Sufficiency. Let $A < \infty$. Since $\ln \frac{x}{x-s} \ge 0$ when $x > s \ge 0$, then it is enough to prove the inequality (5) for $f \ge 0$. Let $0 \le f \in L_p$. Then we have

$$\begin{aligned} \|K_\gamma f\|_q^q &= \sum_k \int_{2^k}^{2^{k+1}} v^q(x) \left(\int_0^x u(s) s^{\gamma-1} \ln \frac{x}{x-s} f(s) ds \right)^q dx \\ &\ll \sum_k \int_{2^k}^{2^{k+1}} v^q(x) \left(\int_0^{2^{k-1}} u(s) s^{\gamma-1} \ln \frac{x}{x-s} f(s) ds \right)^q dx \\ &+ \sum_k \int_{2^k}^{2^{k+1}} v^q(x) \left(\int_{2^{k-1}}^x u(s) s^{\gamma-1} \ln \frac{x}{x-s} f(s) ds \right)^q dx := I_1 + I_2. \end{aligned} \tag{2}$$

We estimate I_1 and I_2 separately. Using the monotonicity of the function $\frac{1}{s}\ln\frac{x}{x-s}$ with respect to the variables x and s, we obtain that for $x > s \geq 0$

$$I_1 \leq \sum_k \int_{2^k}^{2^{k+1}} v^q(x) \left(\int_0^{2^{k-1}} u(s)s^\gamma \frac{1}{2^{k-1}} \ln \frac{2^k}{2^k - 2^{k-1}} f(s)ds \right)^q dx$$

$$\leq (\ln 2)^q \sum_k \int_{2^k}^{2^{k+1}} \frac{v^q(x)}{(2^{k-1})^q} \left(\int_0^{2^{k-1}} u(s)s^\gamma f(s)ds \right)^q dx$$

$$\ll \int_0^\infty \frac{{}^q(x)}{x^q} \left(\int_0^x u(s)s^\gamma f(s)ds \right)^q dx = \|H_\gamma f\|_q^q. \tag{3}$$

In view of Theorem A from (3) it follows that

$$I_1 \ll A^q \|f\|_q^q. \tag{4}$$

By now using the fact that the function u is increasing, applying Hölder's and Jensen's inequalities and making the change of the variable $s = xt$ in the integral below, we have

$$I_2 \leq \sum_k u^q(2^{k-1}) \int_{2^k}^{2^{k+1}} v^q(x) \left(\int_0^x s^{p'(\gamma-1)} \ln^{p'} \frac{x}{x-s} ds \right)^{\frac{q}{p'}} dx \left(\int_{2^{k-1}}^{2^{k+1}} f^p(t)dt \right)^{\frac{q}{p}}$$

$$\leq \sum_k \left(\int_{2^{k-1}}^{2^{k+1}} f^p(t)dt \right)^{\frac{q}{p}} u^q(2^{k-1}) \int_{2^k}^{2^{k+1}} v^q(x)x^{q(\gamma-1)} \left(\int_0^x \ln^{p'} \frac{x}{x-s} ds \right)^{\frac{q}{p'}} dx$$

$$= \beta^{\frac{q}{p'}} \sum_k \left(\int_{2^{k-1}}^{2^{k+1}} f^p(t)dt \right)^{\frac{q}{p}} u^q(2^{k-1}) \int_{2^k}^{2^{k+1}} v^q(x)x^{q(\gamma-1)+\frac{q}{p'}} dx$$

$$\ll \sum_k \left(\int_{2^{k-1}}^{2^{k+1}} f^p(t)dt \right)^{\frac{q}{p}} u^q(2^{k-1}) \left[2^{(k-1)(\gamma+\frac{1}{p'})} \left(\int_{2^k}^{2^{k+1}} \frac{v^q(x)}{x^q} dx \right)^{\frac{1}{q}} \right]^q$$

$$\ll \sum_k \left(\int_{2^{k-1}}^{2^{k+1}} f^p(t)dt \right)^{\frac{q}{p}} \left[u(2^{k-1}) \left(\int_0^{2^{k-1}} s^{p'\gamma} ds \right)^{\frac{1}{p'}} \left(\int_{2^k}^{2^{k+1}} \frac{v^q(x)}{x^q} dx \right)^{\frac{1}{q}} \right]^q$$

$$\ll \sum_k \left(\int_{2^{k-1}}^{2^{k+1}} f^p(t)dt \right)^{\frac{q}{p}} \left[\left(\int_0^{2^{k-1}} s^{p'\gamma} u^{p'}(s)ds \right)^{\frac{1}{p'}} \left(\int_{2^k}^{2^{k+1}} \frac{v^q(x)}{x^q} dx \right)^{\frac{1}{q}} \right]^q$$

$$\leq A^q \left(\sum_k \int_{2^{k-1}}^{2^{k+1}} f^p(t)dt \right)^{\frac{q}{p}} \ll A^q \|f\|_p^q, \tag{5}$$

where $\beta = \int_0^1 t^{p'(\gamma-1)} \ln^{p'} \frac{1}{1-t} dt$. The finiteness of β follows from the estimate

$$\beta \leq \ln^{p'} 2 \int_0^{\frac{1}{2}} s^{p'(\gamma-1)} ds + \max\{1, 2^{-p'(\gamma-1)}\} \int_{\ln 2}^{\infty} t^{p'} e^{-t} dt$$

and from the condition $\gamma > \frac{1}{p}$.

From (2), (4) and (5) it follows that

$$\|K_\gamma f\|_q \ll A\|f\|_p.$$

Hence, $\|K_\gamma\| \ll A$. This relation together with (1) gives $\|K_\gamma\| \approx A$. The proof is complete. □

Proof of Theorem 3.3. Necessity. Let the operator (6) be bounded from L_p to L_q. Then, in view of (6), the operator H_γ is bounded from L_p to L_q and $\|K_\gamma\| \geq \|H_\gamma\|$. Therefore, by Theorem B the value $B < \infty$ and

$$\|K_\gamma\| \gg B. \tag{6}$$

Sufficiency. Let $B < \infty$. We have the estimate (2) for $0 \leq f \in L_p$. In view of Theorem B and from (3) we have that

$$I_1 \ll B^q \|f\|_q^q. \tag{7}$$

Moreover, from the estimate I_2 in the proof of Theorem 3.1 it follows that

$$I_2 \ll \sum_k \left(\int_{2^{k-1}}^{2^{k+1}} f^p(t)dt \right)^{\frac{q}{p}} u^q(2^{k-1}) 2^{k\frac{q}{p'}(p'\gamma+1)} \int_{2^k}^{2^{k+1}} \frac{v^q(x)}{x^q} dx$$

$$\ll \sum_k \left(\int_{2^{k-1}}^{2^{k+1}} f^p(t)dt \right)^{\frac{q}{p}} \left(u^{p'}(2^{k-1}) \int_{2^{k-2}}^{2^{k-1}} t^{p'\gamma} dt \right)^{\frac{q}{p'}} \int_{2^k}^{2^{k+1}} \frac{v^q(x)}{x^q} dx$$

$$\leq \sum_k \left(\int_{2^{k-1}}^{2^{k+1}} f^p(t)dt \right)^{\frac{q}{p}} \left(\int_{2^{k-2}}^{2^{k-1}} u^{p'}(t) t^{p'\gamma} dt \right)^{\frac{q}{p'}} \int_{2^k}^{2^{k+1}} \frac{v^q(x)}{x^q} dx. \tag{8}$$

By now using the Hölder inequality with exponents $\frac{p}{q}$, $\frac{p}{p-q}$ and the estimate

$$\left(\int_{2^{k-2}}^{2^{k-1}} u^{p'}(t) t^{\gamma p'} dt \right)^{\frac{q(p-1)}{p-q}} \ll \int_{2^{k-2}}^{2^{k-1}} \left(\int_{2^{k-2}}^{x} u^{p'}(s) s^{\gamma p'} ds \right)^{\frac{p(q-1)}{p-q}} u^{p'}(x) x^{\gamma p'} dx$$

in (8) we find that

$$I_2 \ll \left(\sum_k \left(\int_{2^{k-2}}^{2^{k-1}} u^{p'}(t) t^{\gamma p'} dt \right)^{\frac{q(p-1)}{p-q}} \left(\int_{2^k}^{2^{k+1}} \frac{v^q(x)}{x^q} dx \right)^{\frac{p}{p-q}} \right)^{\frac{p-q}{q}}$$

$$\times \left(\sum_k \int_{2^{k-1}}^{2^{k+1}} f^p(t) dt \right)^{\frac{q}{p}}$$

$$\ll \left(\sum_k \int_{2^{k-2}}^{2^{k-1}} \left(\int_0^x u^{p'}(s) s^{\gamma p'} ds \right)^{\frac{p(q-1)}{p-q}} \left(\int_x^\infty \frac{v^q(t)}{t^q} dt \right)^{\frac{p}{p-q}} u^{p'}(x) x^{\gamma p'} dx \right)^{\frac{p-q}{q}}$$

$$\times \left(\sum_k \int_{2^{k-1}}^{2^{k+1}} f^p(t) dt \right)^{\frac{q}{p}}$$

$$\ll B^q \|f\|_p^q. \tag{9}$$

From (3), (7) and (9) we obtain the estimate

$$\|K_\gamma f\|_q \ll B \|f\|_p \, ,$$

which together with (6) gives $\|K_\gamma\| \approx B$. The proof is complete. □

As mentioned before the proofs of Theorem 3.5 and 3.7 follows by using Theorems 3.1 and 3.3, respectively, and a standard duality argument.

We finalize this paper with the following remarks:

Remark 4.1. *This paper is an essentially improved and enlarged version of the paper [1] (in Russian).*

Remark 4.2. *The current status of the mentioned open question to characterize the Hardy type inequality* (1) - (2) *without restriction on the kernel* $\mathcal{K}(x,s)$ *was recently described in [13]. However, the cases considered in this paper are new and can not be found there.*

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Paper 6

Compactness of a class of integral operators with logarithmic singularities

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Remark: The text is the same but the format has been modified to fit the style in this PhD thesis

Compactness of a class of integral operators with logarithmic singularities

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Abstract: We establish criteria for compactness for some classes of integral operators with logarithmic singularities in weighted Lebesgue spaces for cases $1 < p \leq q < \infty$ and $1 < q < p < \infty$.

AMS Mathematics Subject Classification (MSC 2010): 26A33, 26D10, 47G10

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1 Introduction

Let $0 < q < \infty$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $R_+ = (0, \infty)$. Moreover, let $u : R_+ \to R$ and $v : R_+ \to R$ be weight functions, i.e. non-negative measurable functions on R_+.

Weighted estimates of the form

$$\|vKf\|_q \leq C\|uf\|_p \tag{1}$$

are intensively studied in the literature for different classes of the operators K, where $\|\cdot\|_p$ is the usual norm of the space $L_p \equiv L_p(R_+)$. We refer to [5] and the books [6, 5, 4, 3, 12] when K is defined by

$$Kf(x) = \int_0^x \mathcal{K}(x,s)f(s)ds, \tag{2}$$

where $\mathcal{K}(x,s)$ is a kernel i.e. measurable function on $R_+ \times R_+$.

Estimates of the form (1) are considered not only in Lebesgue spaces but also in other function spaces (see. e.g. [7, 8, 6] and Chapter 11 of the book [5]). We also refer to [8] and the recent review article [13].

However, all of these results do not include operators of the form of (2), when the kernel $\mathcal{K}(\cdot,\cdot)$ has a singularity, for example, the Riemann-Liouville operator

$$R_\alpha f(x) = \int_0^x \frac{f(s)ds}{(x-s)^{1-\alpha}}, \tag{3}$$

when $0 < \alpha < 1$. Some special cases are studied in [3, 7, 15, 20, 21].

The estimate (1) for a singular operator in a form

$$Kf(x) = \int_0^x s^{\gamma-1} \ln \frac{x}{x-s} f(s)ds, \tag{4}$$

is equivalent to an estimate

$$\|K_\gamma f\|_q \leq C\|f\|_p \tag{5}$$

for the operator

$$K_\gamma f(x) = v(x)\int_0^x u(s)s^{\gamma-1} \ln \frac{x}{x-s} f(s)ds. \tag{6}$$

The estimate (5) is equivalent to the boundedness of the operator (6) from L_p to L_q with the norm $\|K_\gamma\| = C$, where C is the best constant in (5). The operator (4) in the case $\gamma = 0$ is called a fractional integration operator of infinitesimal order [16].

The operator

$$K_\gamma^* f(s) = u(s)s^{\gamma-1}\int_s^\infty v(x) \ln \frac{x}{x-s} f(x)dx, \quad s > 0 \tag{7}$$

is dual to the operator K_γ with respect to the scalar product $\int_0^\infty f(x)g(x)dx$.

When the function u is non- increasing, criterion of boundedness of the operator (6) and the dual operator (7) from L_p to L_q are obtained in [2].

Recently, some new characterizations of (5) for the operators K_γ and K_γ^*, defined by (5) and (6), respectively, are proved in [2]. In this paper we complement these results by establishing the exact compactness criteria of the operators K_γ and K_γ^* from L_p to L_q.

In the case $u(x) \equiv 1$ of compactness from L_p to L_q of the operator (6) was studied in [1].

The main results (Theorems 1-4) are presented in Section 3. The detailed proofs are given in Section 4 and in order not to disturb the argumentations in these proofs some auxiliary results are collected in Section 2.

Conventions: Uncertainties of the type $0 \cdot \infty$, $\frac{0}{0}$, $\frac{\infty}{\infty}$ are assumed to be zero. The inequality of the form $A \le \beta B$ is written in the form $A \ll B$, where the positive constant β may be dependent on the parameters p, q, γ, and the relation $A \approx B$ means that $A \ll B \ll A$. $\chi_{(a,b)}(\cdot)$ denotes a characteristic function of the interval (a, b), Z is a set of integer numbers. The notations $\sum\limits_k$, $\sup\limits_k$ mean $\sum\limits_{k\in Z}$, $\sup\limits_{k\in Z}$, respectively.

2 Auxiliary results.

Since

$$\ln \frac{x}{x-s} = \int_0^s \frac{dt}{x-t} \quad \text{for } x > s \ge 0, \tag{1}$$

the following inequalities

$$\frac{s}{x-s} > \ln \frac{x}{x-s} > \frac{s}{x}, \quad x > s > 0 \tag{2}$$

hold. The function $\ln \frac{x}{x-s}$ decreases with respect to x and increases with respects to s when $x > s \ge 0$, and from the inequality (2) it follows that the functions $x \ln \frac{x}{x-s}$, $\frac{1}{s} \ln \frac{x}{x-s}$ also decreases with respect to x and increases with respects to s when $x > s > 0$. Indeed,

$$\frac{\partial}{\partial x}\left(x \ln \frac{x}{x-s}\right) = \ln \frac{x}{x-s} - \frac{s}{x-s} < 0,$$

and

$$\frac{\partial}{\partial s}\left(\frac{1}{s} \ln \frac{x}{x-s}\right) = \frac{1}{s^2}\left(\frac{s}{x-s} - \ln \frac{x}{x-s}\right) > 0$$

for $x > s > 0$.

For the operator K_γ the following theorem holds [2]:

Theorem A. *Let $1 < p \le q < \infty$, $\gamma > \frac{1}{p}$, and $u(x)$ - be a non-increasing function. Then the operator K_γ defined by (6) is bounded from L_p to L_q if and only if $A = \sup\limits_{x>0} A(x) < \infty$ and, moreover, $\|K_\gamma\|_{L_p \to L_q} \approx A$, where*

$$A(x) = \left(\int_0^x u^{p'}(s) s^{\gamma p'} ds\right)^{\frac{1}{p'}} \left(\int_x^\infty \frac{v^q(t)}{t^q} dt\right)^{\frac{1}{q}}.$$

The corresponding result for the case $q < p$ reads (see [2]):

Theorem B. *Let $p > 1$, $0 < q < p < \infty$ and $\gamma > \frac{1}{p}$. Let u be a non-increasing function on R_+. Then the operator K_γ defined by (6) is bounded from L_p to L_q if and only if*

$$B = \left(\int_0^\infty \left(\int_x^\infty \frac{v^q(t)}{t^q} dt \right)^{\frac{p}{p-q}} \left(\int_0^x u^{p'}(s) s^{p'\gamma} ds \right)^{\frac{p(q-1)}{p-q}} u^{p'}(x) x^{p'\gamma} dx \right)^{\frac{p-q}{pq}} < \infty$$

and, moreover, $\|K_\gamma\|_{L_p \to L_q} \approx B$.

Remark 2.1. *In the case $1 < q < p < \infty$, the constant B is equivalent to the constant*

$$\widetilde{B} = \left(\int_0^\infty \left(\int_x^\infty \frac{v^q(t)}{t^q} dt \right)^{\frac{q}{p-q}} \left(\int_0^x u^{p'}(s) s^{p'\gamma} ds \right)^{\frac{q(p-1)}{p-q}} \frac{v^q(x)}{x^q} dx \right)^{\frac{p-q}{pq}}.$$

3 The main results

Our first main result reads:

Theorem 3.1. *Let $1 < p \leq q < \infty$, $\gamma > \frac{1}{p}$, and $u(x)$ - be a non-increasing function. Then the operator K_γ defined by (6) is compact from L_p to L_q if and only if $A < \infty$ and $\lim_{x \to 0^+} A(x) = \lim_{x \to \infty} A(x) = 0$.*

The corresponding result for the case $q < p$ reads:

Theorem 3.2. *Let $1 < q < p < \infty$ and $\gamma > \frac{1}{p}$. Let u be a non-increasing function on R_+. Then the operator K_γ defined by (6) is compact from L_p to L_q if and only if $B < \infty$.*

We define

$$A^*(x) = \left(\int_x^\infty \frac{v^{p'}(t)}{t^{p'}} \right)^{\frac{1}{p'}} \left(\int_o^x s^{q\gamma} u^q(s) ds \right)^{\frac{1}{q}}, \quad A^* = \sup_{x>0} A^*(x),$$

and

$$B^* = \left(\int_0^\infty \left(\int_x^\infty \frac{v^q(t)}{t^{p'}} \right)^{\frac{q(p-1)}{p-q}} \left(\int_0^x s^{q\gamma} u^q(s) \right)^{\frac{q}{p-q}} x^{q\gamma} u^q(x) dx \right)^{\frac{p-q}{pq}}.$$

We consider the operator K_γ^* (defined by (7)) and its action from L_p to L_q. If $1 < p, q < \infty$, then the operator K_γ^* is bounded (compact) from L_p to L_q if and only if the operator K_γ is bounded (compact) from $L_{q'}$ to $L_{p'}$. In this case the conditions $1 < p \leq q < \infty$ and $1 < q < p < \infty$ are equivalent to the conditions $1 < q' \leq p' < \infty$ and $1 < p' < q' < \infty$, respectively. Therefore from Theorems 3.1 and 3.2, we have the following:

Theorem 3.3. *Let $1 < p \leq q < \infty$ and $\gamma > \frac{1}{p}$. Then the operator K_γ^* defined by (7) is compact from L_p to L_q if only if $A^* < \infty$ and*
$\lim_{x\to 0^+} A^*(x) = \lim_{x\to\infty} A^*(x) = 0.$

Theorem 3.4. *Let $1 < q < p < \infty$ and $\gamma > \frac{1}{p}$. Then the operator K_γ^* defined by (7) is compact from L_p to L_q if only if $B^* < \infty$.*

4 Proofs of the main results

Proof of Theorem 3.1. Necessity. Let the operator K_γ be compact from L_p to L_q. Then the operator is bounded and therefore, by Theorem A, $A < \infty$. First, we prove that $\lim_{z\to 0^+} A(z) = 0$.

Consider the family of functions $\{f_t\}_{t\in I}$, where

$$f_t(x) = \chi_{(0,t)}(x)u^{p'-1}(x)x^{(p'-1)\gamma}(x)\left(\int_0^t u^{p'}(s)s^{p'\gamma}(s)ds\right)^{-\frac{1}{p}}. \tag{1}$$

Then

$$\int_0^\infty |f_t(x)|^p dx = \left(\int_0^t u^{p'}(s)s^{p'\gamma}(s)ds\right)^{-1}\int_0^t u^{p'}(x)x^{p'\gamma}(x)dx \equiv 1. \tag{2}$$

Next we show that the family of functions $\{f_t\}$ converges weakly to zero in L_p. Let $g \in L_{p'} = (L_p)^*$.

Applying the Hölder inequality and using (2) we have that

$$\int_0^\infty f_t(x)g(x)dx \leq \left(\int_0^t |f_t(x)|^p dx\right)^{\frac{1}{p}}\left(\int_0^t |g(x)|^{p'}dx\right)^{\frac{1}{p'}}$$

$$= \left(\int_0^t |g(x)|^{p'}dx\right)^{\frac{1}{p'}}.$$

Since $g \in L_{p'}$, then the last integral converges to zero as $t \to 0^+$, which means the weak convergence to zero for the family of functions $\{f_t\}$. Then, by the compactness of the operator K_γ from L_p to L_q

$$\lim_{z\to 0^+} ||K_\gamma f_t||_q = 0. \tag{3}$$

Since $\ln\frac{x}{x-s} \geq \frac{s}{x}$ for $x > s > 0$ we find that

$$||K_\gamma f_t||_q^q = \int_0^\infty v^q(x)\left(\int_0^x u(s)s^{\gamma-1}(s)\ln\frac{x}{x-s}f_t(s)ds\right)^q dx$$

$$\geq \int_t^\infty \frac{v^q(x)}{x^q}\left(\int_0^t u(s)s^\beta(s)f_t(s)w(s)ds\right)^q dx$$

$$= \left(\int_0^t u^{p'}(s)s^{p'\gamma}(s)ds\right)^{-\frac{q}{p}}\left(\int_0^t u^{p'}(s)s^{p'\gamma)}(s)ds\right)^q \int_t^\infty \frac{v^q(x)}{x^q}dx$$

$$= (A(t))^q \,. \tag{4}$$

By combining (3) and (4) we obtain that $\lim\limits_{t\to 0^+} A(t) = 0$.

Now we prove that $\lim\limits_{t\to\infty} A(t) = 0$.

The compactness of the operator $K_\gamma : L_p \to L_q$ implies the compactness of the dual operator (7) from $L_{q'}$ to $L_{p'}$.

We introduce the family of functions $\{g_t\}_{t\in I}$, where

$$g_t(x) = \chi_{(t,\infty)}(x)\left(\int_t^\infty \frac{v^q(x)}{x^q}dx\right)^{-\frac{1}{q'}} \frac{v^{q-1}(x)}{x^{q-1}}.$$

Since $A < \infty$, then the function g_t is well defined.

In view of the equality

$$\int_0^\infty |g_t(x)|^{q'}dx = \left(\int_t^\infty \frac{v^q(x)}{x^q}dx\right)^{-1}\left(\int_t^\infty \frac{v^q(x)}{x^q}dx\right) = 1$$

for $f \in L_q = (L_{q'})^*$ we see that

$$\int_0^\infty f(x)g_t(x)dx \leq \left(\int_t^\infty |f(x)|^q dx\right)^{\frac{1}{q}}\left(\int_t^\infty |g_t(x)|^{q'}dx\right)^{\frac{1}{q'}}$$

$$= \left(\int_t^\infty |f(x)|^q dx\right)^{\frac{1}{q}}.$$

Consequently, $\lim\limits_{t\to\infty}\int_0^\infty f(x)g_t(x)dx = 0$ for any $f \in L_q$, which means the weak convergence to zero of the family of functions $\{g_t\}$. Then, by the compactness of the operator K_γ^* from $L_{q'}$ to $L_{p'}$, it follows that

$$\lim_{t\to\infty} \|K_\gamma^* g_t\|_{p'} = 0. \tag{5}$$

Again using that $\ln\frac{x}{x-s} \geq \frac{s}{x}$ for $x > s > 0$, we obtain that

$$\|K_\gamma^* g_t\|_{p'}^{p'} \geq \int_0^t |u(s)s^{\gamma-1}(s)|^{p'}\left(\int_t^\infty v(x)\ln\frac{x}{x-s}g_t(x)dx\right)^{p'} w^{1-p'}(s)ds$$

$$\geq \int_0^t u^{p'}(s)s^{p'\gamma}(s)ds\left(\int_t^\infty \frac{v^q(x)}{x^q}dx\right)^{-\frac{p'}{q'}}\left(\int_a^t \frac{v^q(x)}{x^q}dx\right)^{p'} = A^{p'}(t). \qquad (6)$$

By combining (5) and (6) it follows that $\lim_{t\to\infty} A(t) = 0$. Thus, the necessity is proved.

Sufficiency. Let $A < \infty$ and $\lim_{z\to 0^+} A(z) = \lim_{z\to\infty} A(z) = 0$.

For $0 < c < d < \infty$ we define

$$P_c f = \chi_{(0,c]} f, \;\; P_{cd} f = \chi_{(c,d]} f, \;\; Q_d f = \chi_{(d,\infty)} f.$$

Then

$$f = P_c f + P_{cd} f + Q_d f$$

and since $P_c K_\gamma P_{cd} \equiv 0$, $P_c K_\gamma Q_d \equiv 0$, $P_{cd} K_\gamma Q_d \equiv 0$, we have that

$$K_\gamma f = P_{cd} K_\gamma P_{cd} f + P_c K_\gamma P_c f + P_{cd} K_\gamma P_c f + Q_d K_\gamma f. \qquad (7)$$

We show that the operator $P_{cd} K_\gamma P_{cd}$ is compact from L_p to L_q. Since $P_{cd} K_\gamma P_{cd} f(x) = 0$ for $x \in I \setminus (c,d)$, then it is enough to show that the operator $P_{cd} K_\gamma P_{cd}$ is compact from $L_p(c,d)$ to $L_q(c,d)$. This, in turn, is equivalent to compactness of the operator

$$Tf(x) = \int_c^d K(x,s) f(s) ds$$

from $L_p(c,d)$ to $L_q(c,d)$ with the kernel

$$K(x,s) = u(s)s^{\gamma-1}v(x)\chi_{(c,d)}(x-s)\ln\frac{x}{x-s}.$$

Next we note that there are the points $2^i, 2^n$, $n > i$ such that $2^i \leq c < 2^{i+1}$, $2^{n-1} < d \leq 2^n$. We assume that the numbers c and d are chosen so that $2^{i+1} < 2^{n-1}$. Then arguing as in the estimates of I_1 and I_2 in Theorem 3.1 in [2] (see Theorem A), we find that

$$\int_c^d\left(\int_c^d |K(x,s)|^{p'}ds\right)^{\frac{q}{p'}}dx = \int_c^d v^q(x)\left(\int_c^x u^{p'}(s)s^{p'(\gamma-1)}\left(\ln\frac{x}{x-s}\right)^{p'}ds\right)^{\frac{q}{p'}}dx$$

$$\ll \sum_{k=i}^{n-1}\int_{2^k}^{2^{k+1}} v^q(x)\left(\int_0^{2^{k-1}} u^{p'}(s)s^{p'(\gamma-1)}\left(\ln\frac{x}{x-s}\right)^{p'}ds\right)^{\frac{q}{p'}}dx$$

$$+\sum_{k=i}^{n-1}\int_{2^k}^{2^{k+1}} v^q(x)\left(\int_{2^{k-1}}^{x} u^{p'}(s)s^{p'(\gamma-1)}\left(\ln\frac{x}{x-s}\right)^{p'}ds\right)^{\frac{q}{p'}}dx$$

$$\leq \mu(n-i+1)A < \infty,$$

where the constant μ does not depend on i and n. Therefore, on the basis of Kantarovich condition [2] (page 420), the operator T is compact from $L_p(c,d)$ to $L_q(c,d)$, which is equivalent to the compactness of the operator $P_{cd}K_\gamma P_{cd}$ from L_p to L_q .

From (7) it follows that

$$\|K_\gamma - P_{cd}K_\gamma P_{cd}\| \le \|P_cK_\gamma P_c\| + \|P_{cd}K_\gamma P_c\| + \|Q_dK_\gamma\|. \tag{8}$$

We show that the right side of (8) tends to zero at $c \to 0^+$ and $d \to \infty$. Then it follows that the operator K_γ as the uniform limit of compact operators is compact from L_p to L_q.

By Theorem A we have that

$$\begin{aligned}\|P_cK_\gamma P_c f\|_q &= \left(\int_0^c v^q(x)\left|\int_0^x u(s)s^{\gamma-1}\ln\frac{x}{x-s}f(s)ds\right|^q dx\right)^{\frac{1}{q}}\\ &\ll \sup_{0<z<c}\left(\int_0^z u^{p'}(s)s^{p'\gamma}ds\right)^{\frac{1}{p'}}\left(\int_z^c v^q(x)x^{-q}dx\right)^{\frac{1}{q}}\|f\|_p\\ &\le \sup_{0<z<c} A(z)\|f\|_p.\end{aligned}$$

Consequently, $\|P_cK_\gamma P_c\| \ll \sup\limits_{0<z<c} A(z)$. Hence,

$$\lim_{c\to 0^+}\|P_cK_\gamma P_c\| \ll \lim_{c\to 0^+}\sup_{0<z<c} A(z) = \lim_{c\to 0^+} A(c) = 0. \tag{9}$$

Let $v_d = Q_dv$. Then, by Theorem A, we find that

$$\begin{aligned}\|Q_dK_\gamma f\|_q &= \left(\int_0^\infty v_d^q(x)\left|\int_0^x u(s)s^{\gamma-1}\ln\frac{x}{x-s}f(s)ds\right|^q dx\right)^{\frac{1}{q}}\\ &\ll \sup_{0<z}\left(\int_0^z u^{p'}(s)s^{p'\gamma}ds\right)^{\frac{1}{p'}}\left(\int_z^\infty v_d^q(x)x^{-q}dx\right)^{\frac{1}{q}}\|f\|_p\\ &\le \sup_{d<z} A(z)\|f\|_p.\end{aligned}$$

Therefore,

$$\lim_{d\to\infty}\|Q_dK_\gamma\| \ll \lim_{d\to\infty} A(d) = 0. \tag{10}$$

Now we will prove that

$$\lim_{c\to 0^+}\|P_{cd}K_\gamma P_c\| = 0. \tag{11}$$

We put $v_{cd} = P_{cd}v$ and $u_c = P_cu$. It is obvious that the function u_c is non-increasing. Therefore, according to Theorem A, we get that

$$\begin{aligned}\|P_{cd}K_\gamma P_c f\|_q &= \left(\int_0^\infty v_{cd}^q(x)\left|\int_0^x u_c(s)s^{\gamma-1}\ln\frac{x}{x-s}f(s)ds\right|^q dx\right)^{\frac{1}{q}}\\ &\ll \sup_{0<z}\left(\int_0^z u_c^{p'}(s)s^{p'\gamma}ds\right)^{\frac{1}{p'}}\left(\int_z^\infty v_{cd}^q(x)x^{-q}dx\right)^{\frac{1}{q}}\|f\|_p\end{aligned}$$

$$\le A(c)\|f\|_p.$$

and we conclude that equality (11) holds.

From (9), (10) and (11) it follows that the right side of (8) tends to zero at $c \to 0^+$ and $d \to \infty$. Hence, also the sufficiency is proved. The proof is complete. □

Proof of Theorem 3.2. Necessity. Let the operator K_γ be compact from L_p to L_q. Then the operator is bounded and therefore, by Theorem B, $B < \infty$.
Sufficiency. Let $A < \infty$. Here we have $K_\gamma f = P_d K_\gamma P_d f + P_d K_\gamma Q_d f + Q_d K_\gamma f$. Therefore

$$\|K_\gamma - P_d K_\gamma P_d\| \le \|P_d K_\gamma Q_d\| + \|Q_d K_\gamma\|. \tag{12}$$

Since $d < \infty$, then from the Ando theorem and its generalizations (see e.g. [10]) the operator $P_d K_\gamma P_d$ is compact from $L_p(0,d)$ to $L_q(0,d)$, which is equivalent to the compactness of it from L_p to L_q. We show that the right-hand side (12) tends to zero as $d \to \infty$. Then the operator K_γ is compact from L_p to L_q as the uniform limit of compact operators. Similarly as in the proof of Theorem 3.1 we find that

$$\|Q_d K_\gamma f\|_q = \left(\int_0^\infty v_d^q(x) \left| \int_0^x u(s) s^{\gamma-1} \ln \frac{x}{x-s} f(s) ds \right|^q dx \right)^{\frac{1}{q}}.$$

Then, by Theorem 3.1,

$$\|Q_d K_\gamma\| \ll \left(\int_d^\infty \left(\int_z^\infty u^{p'}(s) s^{p'\gamma} ds \right)^{\frac{q(p-1)}{p-q}} \right.$$

$$\left. \times \left(\int_d^z v^q(x) x^{-q} dx \right)^{\frac{q}{p-q}} v^q(z) z^{-q} dz \right)^{\frac{(p-q)}{pq}}.$$

From this estimate and the fact that $B < \infty$ it follows that

$$\lim_{d\to\infty} \|Q_d K_\gamma\| = 0. \tag{13}$$

Let $v_{dd} = P_d v$ and $u_d = Q_d u$. Then, using again Theorem A, we obtain that

$$\|P_d K_\gamma Q_d f\|_q = \left(\int_0^\infty v_{dd}^q(x) \left| \int_0^x u_d(s) s^{\gamma-1} \ln \frac{x}{x-s} f(s) ds \right|^q dx \right)^{\frac{1}{q}}$$

$$\ll \left(\int_d^\infty u^{p'}(s) s^{p'\gamma} ds \right)^{\frac{1}{p'}} \left(\int_0^d v^q(x) x^{-q} dx \right)^{\frac{1}{q}} \|f\|_p = A(d)\|f\|_p. \tag{14}$$

We also note that, by Remark 2.1, $B \approx \widetilde{B}$. Since

$$A(d) \ll \widetilde{B}(d,\infty)$$

$$= \left(\int_d^\infty \left(\int_x^\infty \frac{v^q(t)}{t^q} dt \right)^{\frac{q}{p-q}} \left(\int_0^x u^{p'}(s) s^{p'\gamma} ds \right)^{\frac{q(p-1)}{p-q}} \frac{v^q(x)}{x^q} dx \right)^{\frac{p-q}{pq}},$$

then from (14) we have that $\lim_{d\to\infty} \|P_d K_\gamma Q_d\| = 0$. From this and from (13) it follows that the right-hand side of (12) tends to zero at $d \to \infty$. Therefore also the sufficiency part is proved. The proof is complete. □

Finally, we remark that as mentioned before the proofs of Theorems 3.3 and 3.4 follow by using Theorems 3.1 and 3.2, respectively, and a standard duality argument.

We also include the following final remark:

Remark 4.1. *This paper is an essentially improved and enlarged version of the paper [1] (in Russian).*

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Paper 7

Additive weighted L_p estimates of some classes of integral operators involving generalized Oinarov kernels

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Additive weighted L_p estimates of some classes of integral operators involving generalized Oinarov kernels

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Keywords: weighted Hardy inequalities, weighted functions, fractional order operator
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Abstract: Inequalities of the form

$$\|u\mathcal{K}f\|_q \le C\left(\|\rho f\|_p + \|vHf\|_p\right), \quad f \ge 0,$$

are considered, where $\mathcal{K}$ is an integral operator of Volterra type and H is the Hardy operator. Under some assumptions on the kernel $\mathcal{K}$ we give necessary and sufficient conditions for such an inequality to hold.

1 Introduction

Let $I = (0, +\infty)$, $1 \le p, q < \infty$. Let $u(\cdot)$, $v(\cdot)$ and $\rho(\cdot)$ be weighted functions, i.e. positive measurable functions on I. Let $\mathcal{K}^+$, $\mathcal{K}^-$, H^+ and H^- be integral operators of the form

$$\mathcal{K}^+ f(x) = \int_0^x K(x,s)f(s)ds, \quad \mathcal{K}^- f(x) = \int_x^\infty K(t,x)f(t)dt,$$

$$H^+ f(x) = \int_0^x f(s)ds, \quad H^- f(x) = \int_x^\infty f(s)ds, \quad x > 0,$$

where $K(x,s) \ge 0$ as $x \ge s \ge 0$.

Denote by L_p the set of all measurable functions f such that

$$\|f\|_p := \left(\int_0^\infty |f(x)|^p \, dx\right)^{\frac{1}{p}} < \infty.$$

Inequalities of the form

$$\left\|uHf\right\|_q \leq C\left\|vf\right\|_p, \tag{1}$$

where H is some of the operators H^+, H^-, K^+ and K^- are called Hardy type inequalities in the literature. For the standard Hardy operators H^+ and H^- almost everything is nowadays known, see e.g. the books [4], [5], [12] and [3] and the references given there. However, for the case with a general positive kernel $k(x, y)$ a characterization of the weights so that (1) holds for K^+ or K^- is a long standing open question. However, for some kernels and parameters the answer of this open question is known. The most typical such example is when $k(x, y)$ is a so called Oinarov kernel (in particular satisfying (4) below) and when $1 < p \leq q < \infty$ or $0 < q < p < \infty, p \geq 1$. See especially Chapter 2 in [4] and the references therein. Later on R.Oinarov [9] generalized such results to cover also the case with so called generalized Oinarov conditions, for definitions and some of these results see Section 2.

In this paper we consider the following more general additive weighted inequalities

$$||u\mathcal{K}^+f||_q \leq C\left(||\rho f||_p + ||vH^+f||_p\right), \quad f \geq 0, \tag{2}$$

and

$$||u\mathcal{K}^-f||_q \leq C\left(||\rho f||_p + ||vH^-f||_p\right), \quad f \geq 0. \tag{3}$$

In particular, our results give new information related to the open question mentioned above.

Inequalities of the form (2)-(3) were considered in [6, 7, 10, 11, 8]. In [8] the inequalities (2)-(3) have been studied assuming that the kernels $K(\cdot,\cdot)$ of the operators $\mathcal{K}^+, \mathcal{K}^-$ satisfy "Oinarov's condition", i.e., that there exist a number $d \geq 1$ such that the relation

$$d^{-1}\left(K(x,t) + K(t,s)\right) \leq K(x,s) \leq d\left(K(x,t) + K(t,s)\right) \tag{4}$$

holds for $x \geq t \geq s > 0$.

In this paper we study the inequalities (2)-(3) when the kernels of the operators $\mathcal{K}^+$ and $\mathcal{K}^-$ satisfy weaker conditions than the conditions (4), namely, we assume that the kernels of the operators $\mathcal{K}^+$ and $\mathcal{K}^-$ belong to the classes $\mathcal{O}_n^+$, $\mathcal{O}_n^-$, $n \geq 0$, respectively, which was first introduced in [9]. (for definitions see Section 2)

This paper is organized as follows: In Section 3 we present our main results with proofs. In order not to disturb our presentations we present some Preliminaries of independent interest in Section 2.

Conventions: If A and B are functionals, then $A \ll B$ means that there exist a constant $C > 0$ independent of the arguments of the functionals A and B and the inequality $A \leq CB$ holds. In the case $A \ll B \ll A$ we write $A \approx B$.

2 Preliminaries

In [9] the classes $\mathcal{O}_n^+$ and $\mathcal{O}_n^-$ of the kernels of the form $\mathcal{K}^+$, $\mathcal{K}^-$ are defined for each $n \geq 0$. We agree to write $K(\cdot,\cdot) \equiv K_n^{\pm}(\cdot,\cdot)$, if $K(\cdot,\cdot) \in \mathcal{O}_n^{\pm}$.

Let $K^+(\cdot,\cdot)$ and $K^-(\cdot,\cdot)$ be nonnegative measurable functions in $\Omega = \{(x,s) : x \geq s \geq 0\}$ and besides the function $K^+(\cdot,\cdot)$ is non-decreasing in the first argument and $K^-(\cdot,\cdot)$ is non-increasing in the second argument.

We say that the function $K(\cdot,\cdot) \equiv K_0^{\pm}(\cdot,\cdot)$ belongs to the class $\mathcal{O}_0^{\pm}(\Omega)$ if only if $K_0^+(x,s) = v(s) \geq 0$, $K_0^-(x,s) = u(x) \geq 0$ for all $(x,s) \in \Omega$.

The classes $\mathcal{O}_n^{\pm}, n = 1,2,...$ are defined recursively as follows: Let the classes $\mathcal{O}_i^{\pm}(\Omega)$, $i = 0,1,\ldots,n-1$, $n \geq 1$ be defined. Then $K(\cdot,\cdot) \equiv K_n(\cdot,\cdot) \in \mathcal{O}_n^{\pm}(\Omega)$ if and only if there exist functions

$K_i^{\pm}(\cdot,\cdot) \in \mathcal{O}_i^{\pm}(\Omega)$, $i = 0, 1, \ldots, n-1$ such that

$$K_n^+(x,s) \approx \sum_{i=0}^{n} K_{n,i}^+(x,t) K_i^+(t,s), \tag{1}$$

$$K_n^-(x,s) \approx \sum_{i=0}^{n} K_i^-(x,t) K_{i,n}^-(t,s), \tag{2}$$

when $0 < s \leq t \leq x < \infty$ and $K_{n,n}^{\pm}(\cdot,\cdot) \equiv 1$, where the functions $K_{n,i}^+(\cdot,\cdot)$, $K_{i,n}^-(\cdot,\cdot)$, $i = 0, 1, \ldots, n-1$, generally speaking, are arbitrary nonnegative measurable functions defined on Ω, satisfying the conditions (1) or (2), respectively. In fact, these functions can be defined in the following form (see [9]):

$$K_{n,i}^+(x,t) = \inf_{0<s\leq t} \frac{K_n^+(x,s)}{K_i^+(t,s)},$$

$$K_{i,n}^-(t,s) = \inf_{t<x} \frac{K_n^-(x,s)}{K_i^-(x,t)}, \quad i = 0, 1, \ldots, n-1.$$

From (1) and (2) we have for $n = 1$ that the functions $K_1^+(\cdot,\cdot)$, $K_1^-(\cdot,\cdot)$ belong to the classes $\mathcal{O}_1^+$, $\mathcal{O}_1^-$, respectively, if there exist functions $v_1 \geq 0$ and $u_1 \geq 0$ such that

$$K_1^+(x,s) \approx K_{1,0}^+(x,t) v_1(s) + K_1^+(t,s),$$

$$K_1^-(x,s) \approx K_1^-(x,t) + K_{0,1}^-(t,s) u_1(x),$$

respectively, for all $x \geq t \geq s > 0$.

In particular, we note that each function, satisfying the condition (4), belong to $\mathcal{O}_1^+$ and $\mathcal{O}_1^-$. However, functions from $\mathcal{O}_1^+$ and $\mathcal{O}_1^-$ need not to satisfy the condition (4). For example, the functions $K_1^+(x,s) = x^\beta - (x-s)^\beta$ and $K_1^+(x,s) = \ln^\gamma \frac{(x+1)^\beta}{s}$, $x \geq s > 0$, $\gamma > 0$, $\beta > 1$, do not satisfy the condition (4). However, they belong to the class $\mathcal{O}_1^+(\Omega)$ since

$$x^\beta - (x-s)^\beta \approx (x-t)^{\beta-1} s + t^\beta - (t-s)^\beta, \quad x \geq t \geq s > 0,$$

and

$$\ln^{\gamma}\frac{(x+1)^{\beta}}{s} \approx \ln^{\gamma}\frac{x+1}{t+1} + \ln^{\gamma}\frac{(t+1)^{\beta}}{s}, \quad x \geq t \geq s > 0.$$

Consider the inequality (1) with $H = K^{+}$ or $H = K^{-}$, i.e.

$$\|u\mathcal{K}f\|_{q} \leq C\|vf\|_{p}, \tag{3}$$

where $\mathcal{K}$ is one of the operators $\mathcal{K}^{+}$ or $\mathcal{K}^{-}$. The following Theorems were proved in [9]:

Theorem A^{+}**.** *Let* $1 < p \leq q < \infty$ *and the kernel of the operator* $\mathcal{K}^{+}$ *belong to the class* $\mathcal{O}_{n}^{+}(\Omega)$, $n \geq 0$. *Then the inequality* (3) *holds for the operator* $\mathcal{K}^{+}$ *if and only if one of the conditions*

$$A_{1}^{+} = \sup_{z>0}\left(\int_{z}^{\infty} u^{q}(x)\left(\int_{0}^{z}\left|K^{+}(x,s)v^{-1}(s)\right|^{p'} ds\right)^{\frac{q}{p'}} dx\right)^{\frac{1}{q}} < \infty,$$

$$A_{2}^{+} = \sup_{z>0}\left(\int_{0}^{z} v^{-p'}(s)\left(\int_{z}^{\infty}\left|K^{+}(x,s)u(x)\right|^{q} dx\right)^{\frac{p'}{q}} ds\right)^{\frac{1}{p'}} < \infty$$

holds and for the best constant $C > 0$ *in* (3) *holds the relation* $A_{1}^{+} \approx C \approx A_{2}^{+}$.

Theorem A^{-}**.** *Let* $1 < p \leq q < \infty$ *and the kernel of the operator* $\mathcal{K}^{-}$ *belongs to the class* $\mathcal{O}_{n}^{-}(\Omega)$, $n \geq 0$. *Then the inequality* (3) *holds for the operator* $\mathcal{K}^{-}$ *if and only if one of the conditions:*

$$A_{1}^{-} = \sup_{z>0}\left(\int_{0}^{z} u^{q}(x)\left(\int_{z}^{\infty}\left|K^{-}(x,s)v^{-1}(s)\right|^{p'} ds\right)^{\frac{q}{p'}} dx\right)^{\frac{1}{q}} < \infty,$$

$$A_{2}^{-} = \sup_{z>0}\left(\int_{z}^{\infty} v^{-p'}(s)\left(\int_{0}^{z}\left|K^{-}(x,s)u(x)\right|^{q} dx\right)^{\frac{p'}{q}} ds\right)^{\frac{1}{p'}} < \infty$$

holds and $A_1^- \approx C \approx A_2^-$, *where* $C > 0$ *is the best constant from* (3).

Let $1 < p < \infty$. We introduce the functions

$$\varphi(x) = \left\{ \inf_{0<t<x} \left[\left(\int_t^x \rho^{-p'}(s)ds \right)^{-\frac{1}{p'}} + \left(\int_t^\infty v^p(s)ds \right)^{\frac{1}{p}} \right] \right\}^{-1},$$

and

$$\psi(x) = \left\{ \inf_{x<t} \left[\left(\int_x^t \rho^{-p'}(s)ds \right)^{-\frac{1}{p'}} + \left(\int_0^t v^p(s)ds \right)^{\frac{1}{p}} \right] \right\}^{-1}.$$

The following result was proved in [8]:

Theorem B^+**.** *Let* $1 < p < \infty$, g *is a nonnegative non-increasing function and the functions* ρ, v *satisfy the conditions* $\rho^{-1} \in L_{p'}^{loc}(I)$, $v \in L_p(t, \infty)$, $t > 0$, *and* $\varphi(0) = 0$. *Then*

$$\sup_{f\geq 0} \frac{\int_0^\infty f(s)g(s)ds}{\|\rho f\|_p + \|vH^+ f\|_p} \approx \left(\int_0^\infty g^{p'}(s)d\varphi^{p'}(s) \right)^{\frac{1}{p'}}, \tag{4}$$

where $\varphi(0) = \lim_{x\to 0} \varphi(x)$.

Also the next result was formulated in [8]:

Theorem B^-**.** *Let* $1 < p < \infty$, g *is a nonnegative non-decreasing function and the functions* ρ, v *satisfy the conditions* $\rho^{-1} \in L_{p'}^{loc}(I)$, $v \in L_p(t, \infty)$, $\forall t > 0$, *and* $\psi(\infty) = 0$. *Then*

$$\sup_{f\geq 0} \frac{\int_0^\infty f(s)g(s)ds}{\|\rho f\|_p + \|vH^- f\|_p} \approx \left(\int_0^\infty g^{p'}(s)d(-\psi^{p'}(s)) \right)^{\frac{1}{p'}}, \tag{5}$$

where $\psi(\infty) = \lim_{x\to\infty} \psi(x)$.

Remark: The assertion in Theorem B^- was given without proof in [8]. However, this result is crucial for the proof of one of our main result so for completeness we present a proof also of Theorem B^- as a part of our main results given in the next Section.

3 The main results

Our first main result reads:

Theorem 3.1. *Let* $1 < p \leq q < \infty$, $\varphi(0) = 0$, $\rho^{-1} \in L_{p'}^{loc}(I)$, $v \in L_p(0,t)$, $t > 0$, *and the kernel of the operator* $\mathcal{K}^+$ *belongs to the class* $\mathcal{O}_n^-(\Omega)$, $n \geq 0$. *Then the inequality* (2) *holds if and only if one of the conditions*

$$E_1^+ = \sup_{z>0} \left(\int_z^\infty \left(\int_0^z K^{p'}(x,s) d\varphi^{p'}(s) \right)^{\frac{q}{p'}} u^q(x) dx \right)^{\frac{1}{q}} < \infty,$$

$$E_2^+ = \sup_{z>0} \left(\int_0^z \left(\int_z^\infty K^q(x,s) u^q(x) dx \right)^{\frac{p'}{q}} d\varphi^{p'}(s) \right)^{\frac{1}{p'}} < \infty$$

holds. Moreover, for the sharp constant $C > 0$ *in* (2) *it holds that* $E_1^+ \approx E_2^+ \approx C$.

The corresponding main result for the operator $\mathcal{K}^-$ reads:

Theorem 3.2. *Let* $1 < p \leq q < \infty$, $\psi(\infty) = 0$, $\rho^{-1} \in L_{p'}^{loc}(I)$, $v \in L_p(t,\infty)$, $t > 0$, *and the kernel of the operator* $\mathcal{K}^-$ *belongs to the class* $\mathcal{O}_n^+(\Omega)$, $n \geq 0$. *Then the inequality* (3) *holds if and only if one* *of the conditions*

$$E_1^- = \sup_{z>0}\left(\int_z^\infty \left(\int_0^z K^q(x,s)u^q(s)ds\right)^{\frac{p'}{q}} d\left(-\psi^{p'}(x)\right)\right)^{\frac{1}{p'}} < \infty,$$

$$E_2^- = \sup_{z>0}\left(\int_0^z \left(\int_z^\infty K^{p'}(x,s)d\left(-\psi^{p'}(x)\right)\right)^{\frac{q}{p'}} u^q(s)ds\right)^{\frac{1}{q}} < \infty$$

holds. In this case $E_1^- \approx E_2^- \approx C$*, where* $C > 0$ *is the sharp constant in* (3).

We will begin by proving Theorem 3.2. However, since this proof heavily depends on the (unproved) Theorem B^- we first prove this Theorem.

Proof of Theorem B^-**:** First we assume that the inequalities

$$\left(\int_0^\infty \left(\int_t^\infty f ds\right)^{p-1} f(t)\psi^{-p}(t)dt\right)^{\frac{1}{p}} \ll \left(\|\rho f\|_p + \|vH^- f\|_p\right), \quad f \geq 0 \tag{1}$$

and

$$\left(\|\rho f\|_p + \|vH^- f\|_p\right) \ll \left(\int_0^\infty |f(t)|^p(\psi)^{-1}\left|\frac{d\psi}{dt}\right|^{1-p} dt\right)^{\frac{1}{p}} \tag{2}$$

hold.

By virtue of (2) and the principle of duality in L_p spaces we have

$$\sup_{f\geq 0} \frac{\int_0^\infty f(s)g(s)ds}{\|\rho f\|_p + \|vH^- f\|_p} \gg \sup_{f\geq 0} \frac{\int_0^\infty f(s)g(s)ds}{\left(\int_0^\infty |f|^p\psi^{-1}|\frac{d\psi}{dt}|^{1-p}\right)^{\frac{1}{p}}}$$

$$= \left(\int_0^\infty g^{p'} \left(\psi^{-1} \left| \frac{d\psi}{dt} \right|^{1-p} \right)^{1-p'} dt \right)^{\frac{1}{p'}} = \left(\int_0^\infty g^{p'} \psi^{p'-1} \frac{d\psi}{dt} dt \right)^{\frac{1}{p'}}$$

$$= \left(\frac{1}{p'} \right)^{\frac{1}{p}} \left(\int_0^\infty g^{p'}(t) d\psi^{p'}(t) \right)^{\frac{1}{p'}}. \tag{3}$$

Moreover, from the results of [1] the inequality

$$\int_0^\infty f g ds \le \left(\int_0^\infty \left(\int_t^\infty f(s) ds \right)^{p-1} f(t) \psi^{-p}(t) dt \right)^{\frac{1}{p}}$$

$$\times \left(\int_0^\infty g^{p'}(s) d\psi^{p'}(s) \right)^{\frac{1}{p'}}, \ f \ge 0, \tag{4}$$

holds for all functions g, which are non-negative and non-decreasing.

Therefore, according to (1) and (4), we have

$$\sup_{f \ge 0} \frac{\int_0^\infty f(s) g(s) ds}{\|\rho f\|_p + \|vH^- f\|_p} \ll \sup_{f \ge 0} \frac{\int_0^\infty f(s) g(s) ds}{\left(\int_0^\infty \left(\int_t^\infty f ds \right)^{p-1} f(t) \psi^{-p}(t) dt \right)^{\frac{1}{p}}}$$

$$\le \left(\int_0^\infty g^{p'}(s) d\psi^{p'}(s) \right)^{\frac{1}{p'}}.$$

This estimate combined with (3) implies (5). And now we prove (1). First, we note that by definition ψ is a non-increasing function. Let $f \ge 0$ and $k \in Z$. Assume that $T_k = \{x \in I : \int_x^\infty f(s) ds \le 2^{-k}\}$, $x_k = \inf T_k$, if $T_k \ne 0$ and $x_k = \infty$, if

$T_k = \varnothing$. Let $Z_0 = \{k \in Z : x_k < \infty\}$. From the definition x_k it follows that $2^{-(k+1)} \le \int_x^\infty f(s)ds \le 2^{-k}$ for $x_k \le x \le x_{k+1}$, $k \in Z_0$, $\int_{x_k}^{x_{k+1}} f(s)ds = 2^{-(k+1)}$, $I = \bigcup_{k\in Z_0} [x_k, x_{k+1})$.

Thus

$$\left(\int_0^\infty \left(\int_t^\infty f(s)ds\right)^{p-1} f(t)\psi^{-p}(t)dt\right)^{\frac{1}{p}}$$

$$= \left(\sum_{k\in Z_0} \int_{x_k}^{x_{k+1}} \left(\int_t^\infty f(s)ds\right)^{p-1} f(t)\psi^{-p}(t)dt\right)^{\frac{1}{p}}$$

$$\le \left(\sum_k \psi^{-p}(x_{k+1}) \int_{x_k}^{x_{k+1}} \left(\int_t^\infty f(s)ds\right)^{p-1} f(t)dt\right)^{\frac{1}{p}}$$

$$\le \left(\sum_k \left[\left(\int_{x_{k+1}}^{x_{k+2}} \rho^{-p'}ds\right)^{-\frac{1}{p'}} + \left(\int_0^{x_{k+2}} v^p ds\right)^{\frac{1}{p}}\right]^p 2^{-k(p-1)} \cdot 2^{-(k+1)}\right)^{\frac{1}{p}}$$

$$\ll \left(\sum_k \left(\int_{x_{k+1}}^{x_{k+2}} \rho^{-p'}ds\right)^{-\frac{p}{p'}} 2^{-kp}\right)^{\frac{1}{p}} + \left(\sum_k 2^{-kp} \int_0^{x_{k+2}} v^p ds\right)^{\frac{1}{p}} := I_1 + I_2. \quad (5)$$

We estimate I_1 and I_2 separately .

By the Hölder inequality we have

$$I_1 = \left(\sum_k 2^{2p} \left(\int_{x_{k+1}}^{x_{k+2}} \rho^{-p'}ds\right)^{1-p} \left(\int_{x_{k+1}}^{x_{k+2}} f(t)dt\right)^p\right)^{\frac{1}{p}}$$

$$\ll \left(\sum_k \int_{x_{k+1}}^{x_{k+2}} |\rho f|^p dt \right)^{\frac{1}{p}} \le \|\rho f\|_p \tag{6}$$

and

$$I_2 = \left(\sum_{k \in Z_0} 2^{-kp} \sum_{i \le k} \int_{x_{i+1}}^{x_{i+2}} v^p ds \right)^{\frac{1}{p}} \le \left(\sum_i \int_{x_{i+1}}^{x_{i+2}} v^p ds \sum_{k \ge i} 2^{-kp} \right)^{\frac{1}{p}}$$

$$\ll \left(\sum_i \int_{x_{i+1}}^{x_{i+2}} v^p ds 2^{-(i+2)p} \right)^{\frac{1}{p}} \le \left(\sum_i \int_{x_{i+1}}^{x_{i+2}} v^p ds \left(\int_s^{\infty} f(t) dt \right)^p \right)^{\frac{1}{p}}$$

$$\le \|vH^- f\|_p.$$

This inequality together with (5) and (6) implies (1).

Finally, we prove (2). Let $0 < x < z$. From the definition of ψ we find

$$\psi^{p'}(x) \le \sup_{x<t<z} \frac{\int_x^t \rho^{-p'} ds}{\left[1 + \left(\int_x^t \rho^{-p'}(s) ds \right)^{\frac{1}{p'}} \left(\int_0^z v^p ds \right)^{\frac{1}{p}} \right]^p}$$

$$+ \sup_{z<t} \frac{\int_x^z \rho^{-p'} ds + \int_z^t \rho^{-p'} ds}{\left[1 + \left(\int_x^z \rho^{-p'}(s) ds + \int_z^t \rho^{-p'} ds \right)^{\frac{1}{p'}} \left(\int_0^t v^p ds \right)^{\frac{1}{p}} \right]^p}$$

$$\le 2 \int_x^z \rho^{-p'} ds + \psi^{p'}(z).$$

We note that $0 < \psi^{p'}(x) - \psi^{p'}(z) \leq 2\int\limits_x^z \rho^{-p'} ds$. Hence, the function ψ is locally absolutely continuous and

$$p'\psi^{p'-1}(z)\left(-\frac{d\psi}{dz}\right) = \lim_{x\to z} \frac{\psi^{p'}(x) - \psi^{p'}(z)}{z-x}$$

$$\leq 2\lim_{x\to z} \frac{1}{z-x}\int\limits_x^z \rho^{-p'} ds = 2\rho^{-p'}(z).$$

for almost all $z \in I$. Therefore,

$$\rho^p(z)\psi(z)\left|\frac{d\psi}{dz}\right|^{p-1} \ll 1 \quad \text{or}$$

$$\rho^p(z) \ll \psi^{-1}(z)\left|\frac{d\psi}{dz}\right|^{1-p} \quad \text{a.e. } z \in I. \tag{7}$$

According to (7) we have

$$\|f\rho\|_p \ll \left(\int\limits_0^\infty |f|^p \psi^{-1}(z)\left|\frac{d\psi}{dz}\right|^{1-p} dz\right)^{\frac{1}{p}}. \tag{8}$$

By the Hardy inequality (see e.g. [4]) we obtain

$$\|vH^- f\|_p \ll \left(\int\limits_0^\infty |f|^p \psi^{-1}(z)\left|\frac{d\psi}{dz}\right|^{1-p} dz\right)^{\frac{1}{p}} \tag{9}$$

since

$$\sup_{z>0}\left(\int\limits_0^z v^p ds\right)^{\frac{1}{p}}\left(\int\limits_z^\infty \psi^{p'-1}(t)(-\psi'(t))dt\right)^{\frac{1}{p'}}$$

$$= \left(\frac{1}{p'}\right)^{\frac{1}{p'}} \sup_{z>0}\left(\int\limits_0^z v^p ds\right)^{\frac{1}{p}} \psi(z) \ll 1.$$

By combining (8) and (9) we get (2). Theorem B^- is proved.

Proof of Theorem 3.2: Let $C > 0$ be the sharp constant in (3). Then, by using the duality principle in L_q, $1 < q < \infty$, we have

$$C = \sup_{f\geq 0} \frac{\|u\mathcal{K}^- f\|_q}{\|\rho f\|_p + \|vH^- f\|_p} = \sup_{f\geq 0} \sup_{0\leq g\in L_{q'}} \frac{\int_0^\infty gu\mathcal{K}^- f ds}{\left(\|\rho f\|_p + \|vH^- f\|_p\right)\|g\|_{q'}}$$

$$= \sup_{g\geq 0} \frac{1}{\|g\|_{q'}} \sup_{f\geq 0} \frac{\int_0^\infty f(x)(\mathcal{K}^+ gu)dx}{\|\rho f\|_p + \|vH^- f\|_p}.$$

Hence, by using the fact that the function $(\mathcal{K}^+ gu)(x)$ is non-decreasing we can apply Theorem B^- to obtain that

$$C \approx \sup_{0\leq g\in L_{q'}} \frac{\left(\int_0^\infty (\mathcal{K}^+ gu)^{p'}(x) d(-\psi^{p'}(x))\right)^{\frac{1}{p'}}}{\|g\|_{q'}} = \widetilde{C}.$$

Therefore, the inequality (3) is equivalent to the inequality

$$\left(\int_0^\infty (\mathcal{K}^+ gu)^{p'}(x) d(-\psi^{p'}(x))\right)^{\frac{1}{p'}} \leq \widetilde{C}\left(\int_0^\infty |g(t)|^{q'} dt\right)^{\frac{1}{q'}}, \quad g \geq 0,$$

or the inequality

$$\left(\int_0^\infty (\mathcal{K}^+ g)^{p'}(x) d(-\psi^{p'}(x))\right)^{\frac{1}{p'}} \leq \widetilde{C}\left(\int_0^\infty \left|u^{-1} g\right|^{q'} dt\right)^{\frac{1}{q'}}, \quad g \geq 0, \quad (10)$$

and $C \approx \widetilde{C}$.

The inequality (10) is the inequality of the form (3). Since $1 < p \leq q < \infty$ implies that $1 < q' \leq p' < \infty$, then applying

Theorem A^+ to the inequality (10), we get that the inequality (10) holds if and only if one of the conditions

$$A_1^* = \sup_{z>0}\left(\int_z^\infty\left(\int_0^z \left|K^+(x,s)u(s)\right|^q ds\right)^{\frac{p'}{q}} d\left(-\psi^{p'}(x)\right)\right)^{\frac{1}{p'}} = E_1^- < \infty,$$

$$A_2^* = \sup_{z>0}\left(\int_0^z u^q(s)\left(\int_z^\infty \left|K^+(x,s)\right|^{p'} d\left(-\psi^{p'}(x)\right)\right)^{\frac{q}{p'}}\right)^{\frac{1}{q}} = E_2^- < \infty$$

holds and, moreover, $\widetilde{C} \approx E_1^- \approx E_2^-$. But $C \approx \widetilde{C}$ and, thus, also $C \approx E_1^- \approx E_2^-$. The proof is complete.

Proof of Theorem 3.1: The proof is similar to that of Theorem 3.2 so we omit the details. We only remark that in this case we use Theorem B^+ and Theorem A^- instead of Theorem B^- and Theorem A^+, respectively.

Finally, we will consider the case $p = 1$. In this case for $f \geq 0$ we have

$$\|\rho f\|_1 + \|vH^+ f\|_1 = \int_0^\infty \rho(t)f(t)dt + \int_0^\infty v(t)\int_0^t f(s)dsdt$$

$$= \int_0^\infty \rho(t)f(t)dt + \int_0^\infty f(s)\int_s^\infty v(t)dtds = \int_0^\infty f(s)\left(\rho(s) + \int_s^\infty v(t)dt\right)ds$$

$$= \int_0^\infty w^+(s)f(s)ds; \quad \text{where } w^+(s) \equiv \rho(s) + \int_s^\infty v(t)dt,$$

and

$$\|\rho f\|_1 + \|vH^- f\|_1 = \int_0^\infty \rho(t)f(t)dt + \int_0^\infty v(t)\int_t^\infty f(s)dsdt$$

$$= \int_0^\infty f(s)\left(\rho(s) + \int_0^s v(t)dt\right)ds = \int_0^\infty w^-(s)f(s)ds,$$

$$\text{where } w^-(s) \equiv \rho(s) + \int_0^s v(t)dt.$$

Therefore, in the case $p = 1$ the inequalities (2) and (3) have the forms

$$\|u\mathcal{K}^+ f\|_q \leq C^+\|w^+ f\|_1, \quad f \geq 0, \tag{11}$$

$$\|u\mathcal{K}^- f\|_q \leq C^-\|w^- f\|_1, \quad f \geq 0, \tag{12}$$

respectively, i.e. the problem in this case reduces to the problem boundedness of the operators $\mathcal{K}^+, \mathcal{K}^-$ from $L_{1,w^\pm}$ to $L_{q,u}$.

Thus, on the basis of Theorem 4 of Chapter XI from [2], we have the following:

Proposition 3.3. *Let $p = 1$ and $1 \leq q < \infty$. Then the inequalities* (2) *and* (3) *hold if and only if*

$$C^+ = \operatorname{ess\,sup}_{s>0}\left\{\left(\int_s^\infty \left|u(x)K^+(x,s)\right|^q dx\right)^{\frac{1}{q}}\left(\rho(s) + \int_s^\infty v(t)dt\right)^{-1}\right\} < \infty,$$

and

$$C^- = \operatorname{ess\,sup}_{x>0}\left\{\left(\int_0^x \left|u(s)K^-(x,s)\right|^q ds\right)^{\frac{1}{q}}\left(\rho(s) + \int_0^x v(t)dt\right)^{-1}\right\} < \infty$$

hold, respectively. Moreover, for the best constant C in (2) *and* (3), *it yields that $C^+ \approx C$ and $C^- \approx C$, respectively.*

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