Combined Electric, Electromagnetic and Thermal Modeling based on a PEEC Approach

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Abstract

The thermal design of printed circuit boards (PCB:s) has become a critical process in the early development stages of electronic packages. This increasingly important design has been brought about by rapid advancements in chip technology and packaging techniques. This master’s thesis combines the partial element equivalent circuits (PEEC) formulation to give mathematical models for development of heat and temperature in electric circuits and networks. The numerical stability of solving integral and differential equations for PEEC currents and the electrothermal solution, respectively, has also been discussed in this work. Computer implementation and simulation of the electrothermal modeling of a lightning protection system (LPS) show that the finite difference methods are applicable to configure the integral equation-based method of PEEC with the differential equation-based method for solving the conduction of heat in such electrical networks. In this model combination, however, there will be restrictions on numerical solution of the thermal modeling in which carefulness in choosing the number of time step-sizes is of absolute importance.
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Preface

This master’s thesis is conducted to Dept. of Computer Science and Electrical Engineering at Luleå University of Technology. I wish to express my special gratitude to Dr. Jonas Ekman for his careful, thorough and patient revision of this work and his valuable comments from which this work has profited considerably in both contents and readability. I would like to thank my supervisor Prof. Jerker Delsing who gave me the opportunity to work within this interesting field. I am grateful to Åke Wisten and Bo Kjellmert for helping me, among others, to understand different concepts within electromagnetism. I would also like to thank people on the department of Mathematics specially Rheinhold Näslund for helping me with the concepts of boundary value problems and the method of Green’s functions and Inge Söderquist for helping me with the numerical parts of this work. I am also grateful to my friend Andreas Almquist on Dept. of Machine Elements for giving me valuable advice about how to treat differential equations in technical processes.
This section presents the relevance of the topic of the thesis together with a short survey of current techniques within electrothermal- and electromagnetic- modeling.

1.1 Introduction

The thermal design of printed circuit boards (PCB’s) has become a critical process in the early development stages of electronic packages. This increasingly important design consideration has been brought about by rapid advancements in chip technology and packaging techniques. Chip packages provide the computer industry with power densities in excess of 100W/cm² [2]. These developments have prompted thermal engineers to put a strong emphasis on optimal chip location early in the design process. In determining the optimal chip configuration, it is usually to consider several chip/board model range from analytical and computer-aided numerical models to experimental procedures. Experimental testing is the most accurate method of obtaining correlations and temperature distributions for any particular PCB. For this reason, experiments are usually performed on fabricated models of proposed PCB’s before mass-producing them. However, the advantage of obtaining data from experimentation must be weighted against the large amount of time and expense required to conduct such experiments. If the initial experimental results suggest a relocation of chips, the PCB model must be re-fabricated and the experimental procedure repeated. As chip packages are made denser, the number of such design alterations will continue to increase as thermal engineers strive to produce optimal chip/board configurations. This reality limits the cost effectiveness of experimentation as a practical means of determining initial temperature distributions across PCB surfaces. Hence, if a numerical or analytical solution exists for a given PCB model, experimental testing is usually postponed until the final design stage, leaving the initial design stages to more cost effective techniques. There are analytical models [3, 4, 5] that are frequently used to provide PCB temperature distributions. The applicability of these models is often limited to simple geometries and boundary conditions. Even with these limitations, analytical modeling offers many advantages over experimental modeling. With the help of computers, analytical solutions can be obtained much quicker than
introduction. While these features make analytical models more cost effective than experimentation, it must be noted that many real-life problems involving complex geometries do not have readily available analytical solutions.

One of the primary causes of failure in electronic components is interconnection failure in the leads and solder balls that connect an individual package (first level) to a printed circuit board (second level) [6]. The primary cause of failure in this inputs/outputs is due to cyclic thermomechanical deformation. This is a form of fatigue failure, which is a strain driven failure as opposed to stress driven failures due to vibration and shock [7]. The cyclic thermomechanical deformation is caused by expansion mismatches between the package and printed circuit board (PCB) upon which the package is mounted. The expansion mismatch is caused by differences in thermal expansion coefficients between the PCB and package and by differences in thermal loading of the components. The package/PCB assembly undergoes cyclic thermal loading resulting from environmental temperature changes and from power dissipation. Power dissipation places nonuniform thermal loads on packages. Environmental temperatures place uniform thermal loads on the components. Although the effects of uniform thermal loading on electronic packaging has been studied extensively, few works have attempted to correlate the thermomechanical behavior of packages under power dissipation and uniform thermal loading [8].

Due to the large coefficient of thermal expansion (CTE) mismatch between the silicon die and the organic substrate, die fracture becomes one of the major failure modes during reliability testing, including thermal cycling and thermal shock. This failure mode will be more pronounced for larger die sizes arising from the demand for higher input/output counts. For plastic ball grid array (PBGA) packages, Wilson [9] and Nakayama [10] have suggested the need for using local heat transfer coefficients on the package surfaces in order to solve the coupled temperatures and stress distributions. Several studies have proposed an integrated flow-thermomechanical method [11] – [14]. Their approach has integrated CFD (computational fluid dynamics), transient heat transfer, and thermal stress analyses to study solder joint fatigue. A similar integrated CFD-thermomechanical analysis was performed to evaluate the transient effect of thermal shock on die cracking. The computational fluid dynamics (CFD) method was used to obtain the thermal boundary conditions surrounding the package [15].

This thesis explores the possibility to combine a partial element equivalent circuit (PEEC) electric and electromagnetic modeling method with a suitable thermal modeling method. The PEEC method is an integral based method in which an electromagnetic problem is transformed into a discrete circuit equivalent. This enables a combined electric and electromagnetic solution using circuit techniques. The possibility to combine a PEEC solution with a thermal modeling method would result in a fully integrated tool for combined electric, electromagnetic, and thermal characterization making use of already developed, circuit based, thermal models for electric components.

The following section gives a short introduction to current electromagnetic and electrothermal modeling techniques. Chapter 2 gives the theoretical background to electrothermal modeling. Chapter 3 details a finite difference method suitable for the thermal modeling required in this thesis. Chapter 4 gives a review of the PEEC method while Chapter 5 combines the thermal and electric/electromagnetic modeling methods in a flow graph and using numerical examples. Finally conclusions are presented with a
1.2 Electromagnetic Modeling Techniques

Numerical modeling of electromagnetic properties are used by, for example, the electronics industry to:

1. *Ensure functionality of electric systems.* System performance can be degraded due to unwanted electromagnetic interference coupling into sensitive parts.

2. *Ensure compliance with electromagnetic compatibility (EMC) regulations and directives.* To prevent re-designs of products and ensure compliance with directives post-production.

The techniques for solving field problems, Maxwell’s equations, can be classified as experimental, analytical (exact), or numerical (approximate). The experimental techniques are expensive and time consuming but are still widely used. The analytical solution of Maxwell’s equations involves, among others, Separation of variables and Series expansions, but are not applicable in a general case. The numerical solution of field problems became possible with the availability of high performance computers. The most popular numerical techniques are (1) Finite difference methods, (2) Finite element methods, (3) The method of moments, and (4) The partial element equivalent circuit (PEEC) method. The differences in the numerical techniques have their origin in the basic mathematical approach and therefore make one technique more suitable for a specific class of problem compared to the others. Typical classes of problems in the area of EM modeling are:

- Printed circuit board (PCB) simulations (mixed circuit and EM problem).
- Electromagnetic field strength and pattern characterization.
- Antenna design.

Further, the problems presented above require different kinds of analysis in terms of:

- Requested solution domain (time and/or frequency).
- Requested solution variables (currents and/or voltages or electric and/or magnetic fields).

This categorization of EM problems into classes and requested solutions in combination with the complexity of Maxwell’s equations emphasizes the importance of using the right numerical technique for the right problem to enable a solution in terms of accuracy and computational effort.

In the following sections, four different types of EM computational techniques are briefly presented. The first three, FEM, Mom, and FDM are the most common techniques used today for simulating EM problems. The fourth technique, the PEEC method, widely used within signal integrity (SI) as indicated above, is presented briefly in this section and more careful in Chapter 4.
\section*{1.2.1 Finite Element Method}

The finite element method \cite{16} is a powerful numerical technique for handling problems involving complex geometries and heterogeneous media. The method is more complicated than the previously mentioned FDTD method but also applicable to a wider range of problems. FEM is based on the differential formulation of Maxwell’s equations in which the complete field space is discretized. The method is applicable in both the time and frequency domain. In the method, partial differential equations (PDEs) are solved by a transformation to matrix equations \cite{17}. This is done by minimizing the energy for a PDE using the mathematical concept functional, $F$ \cite{18}, where the energy can be obtained by integrating the (unknown) fields over the structure volume. The procedure \cite{19, 20} is commonly explained by considering a PDE described by the function $u$ with corresponding driving, excitation, function $f$ as:

$$Lu = f \quad (1.1)$$

where $L$ is a PDE operator. For example, Laplace’s equation is given by $L = \nabla^2$, $u = V$, and $f = 0$. The next step is to discretize the solution region into finite elements for which the functional can be written. The functional for each FEM element, $F_e$, is then calculated by expanding the unknown fields as a sum of known basis functions, $u_{e_i}$, with unknown coefficients, $\alpha_i$. The total functional is solely dependent on the unknown coefficients $\alpha_i$ and can be written as

$$F = \sum_{\forall e} F_e \quad (1.2)$$

where $e$ is the number of finite elements in the discretized structure and

$$F_e = \sum_{\forall i} \alpha_i u_{e_i} \quad (1.3)$$

where $i$ depend on what kind of finite elements are used in the discretization. The last step is to minimize the functional for the entire region and solve for the unknown coefficients $\alpha_i$. This requires the partial derivatives of $F$ with respect to each unknown node coefficient, $\alpha$, to be zero, i.e.

$$\frac{\partial F}{\partial \alpha_i} = 0, \quad \forall i \quad (1.4)$$

The method offers great flexibility to model complicated geometries with the use of nonuniform elements.

As for the FDTD method the FEM delivers the result in field variables, $\vec{E}$ and $\vec{H}$, for general EM problems at all locations in the discretized domain and at every time or frequency point. To obtain structured currents and voltages post-processing is needed for the conversion.

\section*{1.2.2 Finite Difference Methods}

In this section a finite difference time domain method, (FDTD), is briefly explained. The method is widely used within EM modeling mainly due to its simplicity. The FDTD
method can be used to model arbitrarily heterogeneous structures, for instance, PCBs and the human body.

In the FDTD method finite difference equations are used to solve Maxwell’s equations for a restricted computational domain. The method requires the whole computational domain to be divided, or discretized, into volume elements (cells) for which Maxwell’s equations have to be solved. The volume element sizes are determined by considering two main factors [19]:

1. **Frequency.** The cell size should not exceed $\frac{\lambda}{10}$, where $\lambda$ is the wavelength corresponding to the highest frequency in the excitation.

2. **Structure.** The cell sizes must allow the discretization of thin structures.

The volume elements are not restricted to cubical cells, parallelepiped cells can also be used with a side to side ratio not exceeding 1:3 [19], mainly to avoid numerical problems. After discretizing the structure, the electromagnetic field components, $E_X$, $E_Y$, $E_Z$, $H_X$, $H_Y$, and $H_Z$, are defined for the cells, for example as shown in Fig. 1.1.

![Figure 1.1: FDTD cell with indicated field components.](image)

If the field components are defined as in Fig. 1.1 the resulting FDTD method is based according to the well known Yee formulation [21]. There are other FDTD methods that are not based on the Yee cell and thus have another definition of the field components. To be able to apply Maxwell’s equations in differential form to the Yee cell the time and spatial derivatives are written as partial derivatives. Finally, by substituting time and spatial partial derivatives using finite difference expressions, results in the FDTD equations [22]. The equations are then solved by:

1. Calculating the electric field components for the complete structure.

2. Advance time by $\frac{\Delta t}{2}$.

3. Calculate the magnetic field components for the complete structure based on the electric field components calculated in 1.
4. Advance time by $\Delta t$ and continue to 1.

The FDTD method delivers the result in field variables, $\vec{E}$ and $\vec{H}$, at all locations in the discretized domain and at every time point. To obtain structured currents and voltages post-processing is needed for the conversion.

### 1.2.3 Method of Moments

Method of moments (MoM) [23] is based on the integral formulation of Maxwell’s equations. This basic feature makes it possible to exclude the air around the objects in the discretization. The method is usually employed in the frequency domain but can also be applied to time domain problems.

In the MoM, integral based equations, describing as an example the current distribution on a wire or a surface, are transformed into matrix equations easily solved using matrix inversion. When using the MoM for surfaces a wire-grid approximation of the surface can be utilized as described in [19]. The wire formulation of the problem simplifies the calculations and is often used for far field calculations.

The starting point for the theoretical derivation [19, 23], is a linear (integral) operator, $L$, involving the appropriate Green’s function $G(\vec{r}, \vec{r})$ applied to an unknown function, $I$, where $f$ is the known excitation function for the system as

$$L I = f \quad (1.5)$$

For example, (1.5) can be the Pocklington Integral Equation, describing the current distribution $I(z')$ on a cylindrical antenna, written as

$$\int_{-l/2}^{l/2} I(z') \left( \frac{\partial^2}{\partial z^2} + k^2 \right) G(z, z') = j\omega\epsilon E_z \quad (1.6)$$

Then the wanted function, $I$, can be expanded into a series of known functions, $u_i$, with unknown amplitudes, $I_i$, resulting in

$$I = \sum_{i=1}^{n} I_i u_i \quad (1.7)$$

where $u_i$ are called basis (or expansion) functions. To solve for the unknown amplitudes, $n$ equations are derived from the combination of (1.5) and (1.7) by the multiplication of $n$ weighting (or testing) functions, integrating over the wire length, and the formulation of a suitable inner product [24]. This results in the transformation of the problem into a set of linear equations which can be written in matrix form as

$$[Z][I] = [V] \quad (1.8)$$

where the matrices $[Z]$, $[I]$, and $[V]$ are referred to as generalized impedance, current, and voltage matrices and the desired solution for the current $I$ is obtained by matrix inversion. The Method of Moments describes the basis of all the electromagnetic analysis techniques in this chapter. The unknown solution is expressed as a sum of known basis functions.
where the weighting coefficients corresponding to the basis functions are determined for best fit. The same process applied to differential equations is known as a "weighted-residual" method [25] or the finite element method. The MoM delivers the result in system current densities $\vec{J}$ and/or voltages at all locations in the discretized structure and at every frequency point (depending on the integral equation in (1.5)). To obtain the results in terms of field variables post-processing is needed for the conversion.

The well-known computer program Numerical Electromagnetics Code, often referred to as NEC [26], utilizes the MoM for the calculation of the electromagnetic response for antennas and other metal structures.

### 1.2.4 Partial Element Equivalent Circuit Method

The basis of the PEEC method [27, 28, 29] originates from VLSI inductance calculations performed by Dr. Albert E. Ruehli at IBM T.J. Watson Research Center, during the first part of 1970. Dr. Ruehli was working with electrical interconnect problems and understood the benefits of breaking a complicated problem into basic partitions, for which inductances could be calculated, to model the inductive behavior of the complete structure [27, 30]. By doing so, return current paths need not to be known a priori as required for regular (loop) inductance calculations.

The concept of partial inductance was first introduced by Rosa [31] in 1908, further developed by Grover [32] in 1946 and Hoer & Love [33] in 1965. However, Dr. Ruehli included the theory of partial coefficients of potential and introduced the partial element equivalent circuit (PEEC) theory in 1972 [34]. Significant contributions in the development of the PEEC method includes:

- The inclusion of dielectrics [35].
- The equivalent circuit representation with coefficients of potential [36].
- The retarded partial element equivalent circuit representation [37].
- PEEC models to include incident fields, scattering formulation [38].
- Nonorthogonal PEECs [39].

The interest and research effort for the PEEC method have increased during the last five year period. The reasons can be an increased need for combined circuit and EM simulations and the increased performance of personal computers enabling large EM system simulations. This development reflects on the areas of current PEEC research, for instance, model order reduction (MOR), model complexity reduction, and general speed up.

In this work, a combination of PEEC and equation of conduction, as a coupled analysis, is represented. This combination develops mathematical models to determine temperature increasing in different electrical circuits and networks whose electrical currents is determined by PEEC method. A finite-difference method is employed to determine temperature rise in a lightning protection system (LPS). The electric current in the LPS will be solved by the PEEC method and this electric current has a role as the source function in the equation for heat conduction in the system.
1.3 Electrothermal Modeling Techniques

There are several methods that study thermal development of circuit boards and electric networks; boundary element methods (BEM), compact thermal models (CTM), finite element methods (FEM), and finite difference methods (FD) are among the most commonly used methods in this area. Some progress has also been made for solving partial differential equations (PDE:s) by the recent development of finite difference techniques. The method of moments (MOM), volume integral method (VIM), boundary element method (BEM) and boundary integral method (BIM) are the integral forms of PDE:s that are based on the Green’s function method\(^1\) of boundary value problems.

Following, some of the well-known methods in this area are represented.

1.3.1 Compact Thermal Methods (CTM)

A well-known method in electrothermal modeling is compact thermal models. A compact model, by definition, is a simplification of a detailed model of a device [40], or ”A compact model is a simple network comprising a limited number of thermal resistors, connecting the critical part of the device (usually the junction) to the outer parts of the device, and is dependent of the boundary conditions applied” [41]. A compact model is represented by a series of equations that are solved iteratively for each set of boundary conditions, thereby optimizing the values of the resistances in the compact model. Two available software packages for the generation of compact models are Pstar and Optimize.

Compact thermal models are suited for embedding in the design environments that are employed by the electronics industries, and the compact models may be incorporated into the component libraries linked to PCB thermal analysis software packages. These models are characterized by the fact that in a compact model, the thermal conductivity of a package is included as a parameter [42]. Some compact models are consisted of a cost function which is to be minimized in the optimization procedure. In some study, the CTMs are optimized to predict junction temperature to within some percent and heat flows to within some another percent while minimizing a special designed cost function [43].

1.3.2 Transmission Line Modeling (TLM)

The method of Transmission Line Modeling (TLM), can be applied to coupled electromagnetic fields and thermal problems. TLM is a time-domain differential modeling method and may be simply understood as a technique whereby a three-dimensional circuit consisting of transmission line segments is constructed so that voltages and currents in this circuit are exact copies of the electromagnetic field occurred in that part of space [44]. A combined electromagnetic and thermal model based on a circuit description can be constructed by TLM. The electric current is, indeed, the heat source in this combined modeling, by which quantities such as thermal conductivity, losses and the dielectric constant in a material may be updated, as the temperature varies by time [45, 46, 47].

\(^1\)More about the Green’s functions method in the next chapters.
1.3.3 Finite Element Methods (FEM)

The finite element method for partial differential equations was originally developed for use in civil engineering, but it is now used for approximating the solutions to partial differential equations (PDEs) that arise in all areas of applied mathematics. One of the advantages of the FEM over the other numerical methods for solving PDEs is the relative ease with which the boundary conditions of the problem are handled. Irregularly shaped boundaries and boundaries involving derivatives are difficult to handle using, for example, finite difference techniques. In FD, each boundary condition involving a derivative must be approximated by a difference quotient at the grid points, and irregular shaping of the boundary makes placing the grid points difficult. Instead, in FEM the construction procedure is independent of the particular boundary conditions of the problem since the finite-element method includes the boundary condition in a functional that is being minimized [48].

ANSYS is a general purpose finite-element program which has been under continuous development since its first commercial release in 1970. The package includes complete pre- and post-processing together with a wide variety of solution types, including stress, thermal, electromagnetics, fluid flow and combinations of these as coupled analysis [49, 50].

1.3.4 Finite Difference Methods (FD)

The finite-difference methods employ the Taylor’s series expansion. By these methods, one can approximate boundary value problems directly by systems of algebraic equations. In such methods, derivatives in boundary value problems are replaced by finite-difference approximations which in the next stage will be solved as a linear equation system.

For a generalized cable system, consisting of a finite number of insulated electrical conductors carrying heat sources, a two-dimensional finite-difference model for heat transfer can be developed. Then, some ultimate simultaneous equations are to be solved by FD for each unknown nodal temperature in the cable system [51]. The effect of boundary condition and the introduced errors on temperature response at the heat source can be studied by an analytic thermal modeling of power electronic systems [52]. In this modeling, an analysis about the choice of thermal model circuit networks, equivalent to discretization of the heat equation by FD, is given.
The French mathematician Jean Fourier (1768-1890) first observed and presented the general principles of heat conduction. He was determined that it was in the first place necessary to distinguish and define with precision the elementary properties which determine the action of heat. According to him, all the phenomena which depended on the action resolved themselves into a very small number of general and simple facts. Fourier could bring back every physical problem of this kind to an investigation of mathematical analysis. From these general facts, it was sufficient for him to submit each substance to three fundamental observations, he concluded: Different bodies in fact do not possess in the same degree the same power to contain heat, to receive it or transmit it across their surfaces, nor to conduct it through the interior of their masses. These are the three specific quantities which Fourier’s theory distinguished and showed how to measure.

Heat can be transferred through the conduction, convection and radiation. In electroheat applications all this mechanisms are important and can act independently or in combinations, often in the presence of volumetric heat sources. Heat transfer plays a central role in electroheat applications.

### 2.1 Heat Conduction and Electric Current

The rate of heat transfer from a body at a higher to one at a lower temperature is given by Fourier’s law[53]:

\[
q = -Ak \frac{d\theta}{dx}
\]

(2.1)

where \( k \) is the thermal conductivity, \( d\theta/dx \) is the temperature gradient in the direction of heat flow \( x \) and \( A \) is the area through which heat flows. The thermal conductivity, as it is seen in the equation above, depends on the temperature. The higher the thermal
conductivity of an isotropic material\textsuperscript{1} the higher is the transfer of heat. Generally, we are interested in functions which, apart from the governing differential equation, fulfill other conditions. Depending on the character of the problem, it may be different types of differential equations. Two important conditions that usually occur when solving differential equations are initial and boundary conditions\textsuperscript{2}. The mathematical interpretation of initial and boundary conditions are not regarded as conditions which the temperature $\theta$ must satisfy on the surface itself or at the instant $t = 0$. They are taken as limiting conditions. The boundary conditions are supposed to be that, for fixed $t > 0$, the given combination of the temperature and its derivatives is to tend to the prescribed value as we approach a point of the surface. The initial conditions are understood as, for a fixed point within the region, the temperature is to tend to the prescribed value as $t \to 0$.

An important issue when solving boundary and initial value problems is the concept of well-posed problems. The governing equations and auxiliary conditions are well-posed mathematically if the following three conditions are met:

1. the solution exists,
2. the solution is unique,
3. the solution depends continuously on the auxiliary data.

The question of existence does not usually create any difficulty. However, the usual cause of non-uniqueness is a failure to properly match the auxiliary conditions to the type of partial differential equations. For some types of differential equations governing, for example, irrotational flows, the potential equation and boundary layer equations, the appropriate initial and boundary value conditions are well established. In general an under-prescription of boundary conditions leads to non-uniqueness and an over-prescription to nonphysical solutions adjacent to the boundary in question.

Now, we consider a rod of small cross-section. We assume that the rod is so thin that the temperature at all points of the section is the same. In addition, we suppose that the rod has a constant area of cross-section $A$, perimeter $p$, conductivity $K$, density $\rho$, specific heat $c$, diffusivity $\kappa$, temperature $\theta$ and surface conductance $H$. In one-dimensional case, consider the rod to lie along the $x-$axis. The rate at which heat flows over the face $x$ into the volume element bounded between sections $x$ and $x + dx$ is\textsuperscript{1}

\[-K \frac{\partial \theta}{\partial x} A.\]

\textsuperscript{1}An isotropic media is a media whose structure and properties in neighborhood of any point are the same relative to all directions through the point. Because of this symmetry, the flux vector at a point must be along the normal to the isothermal surface through the point, and in the direction of falling temperature.

\textsuperscript{2}These conditions are called auxiliary conditions in mathematical literature and they are specified in three ways:

- Dirichlet condition where $u = f$ on $\partial R$,
- Neumann condition where $\partial u/\partial n = f$ or $\partial u/\partial s = g$ on $\partial R$,
- mixed or Robin condition where $\partial u/\partial n + ku = f, k > 0$ on $\partial R$.

where $\partial/\partial n$ denotes the outward normal derivative and $\partial R$ denotes the boundary region.
2.1. Heat Conduction and Electric Current

The rate of heat that flows across the surface element \( x + dx \) is

\[
\left( -K \frac{\partial \theta}{\partial x} - K \frac{\partial^2 \theta}{\partial x^2} dx - ... \right) A.
\]

The difference between the two above quantities is the rate at which the element \( dx \) gains heat, that is

\[
AK \frac{\partial^2 \theta}{\partial x^2} dx.
\] (2.2)

When radiation occurs in the rod at temperature \( \theta_0 \), the rate of lost heat at the surface is

\[
H(\theta - \theta_0) p dx
\] (2.3)

Thus, the total rate of gain \( G_{\text{total}} \) is the difference between (2.2) and (2.3), that is

\[
G_{\text{total}} = AK \frac{\partial^2 \theta}{\partial x^2} dx - H(\theta - \theta_0) p dx
\] (2.4)

Ultimately, the total rate of gain is defined as

\[
Ac\rho \frac{\partial \theta}{\partial t} dx
\] (2.5)

Setting equations (2.3) and (2.4) equal gives

\[
Ac\rho \frac{\partial \theta}{\partial t} dx = AK \frac{\partial^2 \theta}{\partial x^2} dx - H(\theta - \theta_0) p dx
\]

Thus we have

\[
\frac{\partial \theta}{\partial t} = \frac{AK}{c\rho} \frac{\partial^2 \theta}{\partial x^2} - \frac{Hp}{Ac\rho} (\theta - \theta_0)
\]

Setting

\[
\frac{Hp}{Ac\rho} = \nu \quad \text{and} \quad \frac{AK}{c\rho} = \kappa
\]

gives

\[
\frac{\partial \theta}{\partial t} = \kappa \frac{\partial^2 \theta}{\partial x^2} - \nu(\theta - \theta_0)
\] (2.6)

If no radiation occurs on the surface of the rod, that is \( \nu = 0 \), equation (2.6) becomes

\[
\frac{\partial \theta}{\partial t} = \kappa \frac{\partial^2 \theta}{\partial x^2}
\] (2.7)

which is the temperature distribution along the rod. This equation is a prototypical parabolic partial differential equation\(^3\).

\(^3\)Linear, homogeneous partial differential equations of the second order in two independent variables have the form

\[
A(x, y)u_{xx} + B(x, y)u_{xy} + C(x, y)u_{yy} + D(x, y)u_x + E(x, y)u_y + H(x, y)u = 0
\]

where \( A, B, C, D, E, \) and \( H \) are the coefficients of the equation. In developing the general theory of such equations, it is helpful to classify them as hyperbolic, parabolic, or elliptic according to the scheme used in studying conic sections: \( B^2 - 4AC > 0 \) (hyperbolic), \( B^2 - 4AC = 0 \) (parabolic), and \( B^2 - 4AC < 0 \) (elliptic). This classification scheme is important in determining numerical method to find a solution.
2.1.1 Equation of Conduction for a Rod Heated by an Electric Current

The rate of generation of heat along x-axis and in the length $dx$ in the above situation is

$$\frac{j}{A\sigma}I^2dx$$

(2.8)

where $I$ is the D.C. current in the rod measured in Ampere. $\sigma$ and $A$ are the electrical conductivity and the cross-section area of the rod respectively. $j$, is the number of calories in a joule, that is $j \approx 0.239$. To construct a model in this situation, we have to add the above term to (2.7), that is the term corresponding to the rate of gain of heat in the rod because the electrical current $I$. This can be written as

$$\frac{\partial \theta}{\partial t} = \kappa \frac{\partial^2 \theta}{\partial x^2} + \frac{j}{\rho c A^2 \sigma}I^2$$

(2.9)

where $c$ and $\rho$ are specific heat and density, as before. We may further suppose situations where the electric current $I$ is time-dependent. In the case of time-varying end conditions, we can formulate the general problem as following:

$$\theta_{xx} = \frac{1}{\kappa} \theta_t - q, \quad 0 < x < l, t > 0$$

(2.10)

with boundary conditions as

$$\theta(0, t) = \alpha(t), \quad \theta(l, t) = \beta(t), \quad t > 0$$

(2.11)

and initial conditions as

$$\theta(x, 0) = f(x), \quad 0 < x < l$$

(2.12)

The last three equations above represent a non-homogeneous problem. Problems are classified as non-homogeneous if either the partial differential equation or boundary conditions are nonhomogeneous. The term $q$ in (2.10), is proportional to the heat source, as it is seen in (2.9). Our first step to solve the non-homogeneous system above is to convert the problem to a homogeneous problem where the boundary conditions are homogeneous. Things are beginning to get clearer by taking an example for a case when the heat source does not change in time\(^4\).

We are to solve the non-homogeneous problem where

$$\theta_{xx} = \theta_t - ae^{-bx}, \quad 0 < x < 1, t > 0$$

(2.13)

Boundary cond. : $\theta(0, t) = 0, \theta(1, t) = 0, \quad t > 0$

Initial cond. : $\theta(x, 0) = \frac{a}{b^2}(1 - e^{-bx}), \quad 0 < x < 1$

We begin to solve the problem by writing

$$\theta(x, t) = P(x) + v(x, t)$$

(2.14)

\(^4\)These cases are called steady cases.
2.2 Equation of Conduction with Time-varying End Conditions

which gives

\[ \theta_{xx} = \theta_t + v_{xx} \]  
\[ \theta_t = 0 + v_t \]  

Comparing (2.13) and (2.15) gives

\[ P''(x) + v_{xx} \equiv v_t - ae^{-bx} \]  

which thereby gives

\[ P''(x) = -ae^{-bx} \]  

with \( P(0) = 0, P(1) = 0 \) and \( v_{xx} = v_t \). By twice integrating:

\[ P(x) = \frac{a}{b^2}(1 - e^{-bx}) - \frac{a}{b^2}(1 - e^{-b})x \]

Now, we have the originally nonhomogeneous problem converted to a homogeneous one as

\[ v_{xx} = v_t, 0 < x < 1, t > 0 \]  

Boundary cond. : \( v(0,t) = 0, v(1,t) = 0 \)

Initial cond. : \( v(x,0) = \frac{a}{b^2}(1 - e^{-b})x \)

The originally nonhomogeneous problem is now reduced to the homogeneous problem (2.18). After solving this problem, we can insert \( P(x) \) and \( v(x,t) \) in (2.14) to determine our main problem, that is, determining \( \theta(x,t) \).

There are a variety of methods for solving the nonhomogeneous problems above. Laplace and Fourier transforming and the method of Green’s functions are methods to be mentioned in this context. Numerically, one can solve the above differential equation by, for example, finite difference methods (FD) or finite element method (FEM). Solving the problem by eigenfunction expansion method (Fourier method) is the other possibility.

2.2 Equation of Conduction with Time-varying End Conditions

We set

\[ \frac{j}{\rho c A^2 \sigma}I^2 = q(x,t) \]  

where the electric current \( I \) is supposed to be a constant D.C. current. Now, equations

\[ \theta_{xx} = \frac{1}{\kappa} \theta_t - q, \quad 0 < x < L, t > 0 \]  

B.C. : \( \theta(0,t) = \alpha(t), \quad \theta(L,t) = \beta(t), \quad t > 0 \)

I.C. : \( \theta(x,0) = f(x), \quad 0 < x < L \)
Electrothermal Modeling

describe a wire whose temperature at the ends are changing by time in a general case. B.C. and I.C. stand for boundary conditions and initial condition respectively. To be solved, this system will be converted into two simpler problems, as we did in the previous example

\[ \theta(x, t) = K(x, t) + V(x, t) \] (2.21)

which gives

\[ \theta_{xx} = K_{xx} + V_{xx} \] (2.22)

\[ \theta_t = K_t + V_t \]

with boundary conditions as

\[ K(0, t) + V(0, t) = \alpha(t) \] (2.23)

\[ K(L, t) + V(L, t) = \beta(t) \]

and initial condition as

\[ K(x, 0) + V(x, 0) = f(x) \] (2.24)

Our intention now is to convert the above equations into one problem where the boundary conditions are homogeneous. By selecting any differentiable function \( K(x, t) \) satisfying

\[ K(0, t) = \alpha(t), K(L, t) = \beta(t) \] (2.25)

then \( V(x, t) \) will feature homogeneous boundary conditions. An appropriate choice of \( K \) can be

\[ K(x, t) = \alpha(t) + \left[ \beta(t) - \alpha(t) \right] \frac{x}{L} \] (2.26)

which implies that \( K_{xx}(x, t) = 0 \). Now, solving process of \( V(x, t) \) is reduced to

\[ V_{xx} = \frac{1}{\kappa} V_t + \frac{1}{\kappa} K_t(x, t) - q(x, t), \quad 0 < x < L, t > 0 \] (2.27)

**B.C.** : \( V(0, t) = 0, V(L, t) = 0, \quad t > 0 \)

**I.C.** : \( V(x, 0) = f(x) - K(x, 0), \quad 0 < x < L \)

Indeed, equations above are reduced to a problem with homogeneous boundary conditions which can be solved by, for example, eigenfunction expansion method.

### 2.3 Eigenfunction Expansion Method

Equations 2.27, as previously mentioned, feature a problem with homogeneous boundary conditions. For convenience, we rewrite the equation as

\[ V_{xx} = \frac{1}{\kappa} V_t + Q(x, t), \quad 0 < x < L, t > 0 \] (2.28)

**B.C.** : \( V(0, t) = 0, V(L, t) = 0, \quad t > 0 \)

**I.C.** : \( V(x, 0) = F(x), \quad 0 < x < L \)
with
\[ Q(x, t) = \frac{1}{\kappa} K_t(x, t) - q(x, t) \] \hspace{1cm} (2.29)
\[ F(x) = f(x) - K(x, 0) \]

We assume that the solution of (2.28) can be written in the form\[5 3, 59, 60, 61\].
\[ V(x, t) = \sum_{n=1}^{\infty} E_n(t)\Psi_n(x) \] \hspace{1cm} (2.30)

where \( \Psi_n(x) \) are eigenfunctions belonging to the associated eigenvalue problem\[5\]
\[ X'' + \lambda X = 0 \] \hspace{1cm} (2.31)

and boundary conditions as above. \( E_n(t) \) are time-dependent coefficients to be determined. We assume that termwise differentiation\[6\] is permitted. In this case
\[ V_t(x, t) = \sum_{n=1}^{\infty} E'_n(t)\Psi_n(x) \] \hspace{1cm} (2.32)
\[ V_{xx}(x, t) = \sum_{n=1}^{\infty} E_n(t)\Psi''_n(x) \] \hspace{1cm} (2.33)

which together with (2.31) gives
\[ V_{xx}(x, t) = -\sum_{n=1}^{\infty} \lambda_n E_n(t)\Psi_n(x) \] \hspace{1cm} (2.34)

This is a result of applying the superposition principle which can be deduced as \( \Psi''_n(x) = -\lambda_n \Psi_n(x) \) from (2.31). Next, rewriting the partial differential equation in (2.28) as
\[ \kappa V_{xx} = V_t + \kappa Q(x, t) \] \hspace{1cm} (2.35)

and inserting the expressions (2.32) and (2.33) into the right-hand side of (2.34), we obtain
\[ \kappa V_{xx} = \sum_{n=1}^{\infty} [E'_n(t) + \kappa \lambda_n E_n(t)]\Psi_n(x) \] \hspace{1cm} (2.36)

\[ ^5 \text{Clearly } V(x, t), \text{ satisfies the prescribed homogeneous boundary conditions, since each eigenfunction } \Psi_n(x) \text{ does.} \]
\[ ^6 \text{The operation of termwise differentiation of an infinite series is valid according to: Corollary If } f_k(x) \text{ has a continuous derivative on } [a, b] \text{ for each } k = 1, 2, 3, \ldots \text{ and if } \sum_{k=1}^{\infty} f_k(x) \text{ converges to } S(x) \text{ on } [a, b] \text{ and if the series } \sum_{k=1}^{\infty} f'_k(x) \text{ converges uniformly to } g(x) \text{ on } [a, b] \text{ then } S'(x) = g(x) \text{ for every } x \in [a, b]; \text{ equivalently } \frac{d}{dx} \sum_{k=1}^{\infty} f_k(x) = \sum_{k=1}^{\infty} \frac{d}{dx} f_k(x) \ldots \text{” Introduction to Mathematical Analysis page 206-William Parynski, Philip W. Zipse.} \]
The right-hand side of equation above is interpreted as a generalized Fourier series\(^7\) of the function \(\kappa V_{xx}\) for a fixed value of \(t\). Thus, the Fourier coefficients are defined by

\[
E_n'(t) + \kappa \lambda_n E_n(t) = \frac{1}{\|\Psi_n(x)\|^2} \int_0^L Q(x, t) \Psi_n(x) dx \tag{2.37}
\]

for \(n = 1, 2, \ldots\)

where \(\|\Psi_n(x)\|\) is defined as the norm of \(\Psi_n(x)\) with the relation

\[
\|\Psi_n(x)\|^2 = \int_0^L [\Psi_n(x)]^2 dx, \text{ for } n = 1, 2, \ldots \tag{2.38}
\]

Equation 2.36 as a first-order linear differential equation, has the general solution

\[
E_n(t) = \left( c_n + \frac{1}{\kappa} \int_0^t \exp\left( \frac{1}{\kappa} \lambda_n \right) P_n(\tau) d\tau \right) \exp\left( -\frac{1}{\kappa} \lambda_n t \right) \tag{2.39}
\]

for \(n = 1, 2, 3, \ldots\) and the assumption that \(\lambda_n \neq 0\) for all \(n\). It has to be added that \(c_n\) are arbitrary constants. In the equation above, \(P_n(t)\) is defined as

\[
P_n(t) = \frac{1}{\|\Psi_n(x)\|^2} \int_0^L Q(x, t) \Psi_n(x) dx, \text{ for } n = 1, 2, 3, \ldots \tag{2.40}
\]

Now, by substituting (2.39) into (2.30) we finally get

\[
V(x, t) = \sum_{n=1}^\infty \left( c_n + \frac{1}{\kappa} \int_0^t \exp\left( \frac{1}{\kappa} \lambda_n \right) P_n(\tau) d\tau \right) \exp\left( -\frac{1}{\kappa} \lambda_n t \right) \Psi_n(x) \tag{2.41}
\]

For determining the arbitrary coefficients \(c_n, n = 1, 2, 3, \ldots\), we shall force equation 2.40 to satisfy the prescribed initial condition in (2.28). By setting \(t = 0\) in (2.40) and inserting the result into the initial condition, we obtain

\[
V(x, 0) = F(x) = \sum_{n=1}^\infty c_n \Psi_n(x) \tag{2.42}
\]

which gives

\[
c_n = \frac{1}{\|\Psi_n(x)\|^2} \int_0^L F(x) \Psi_n(x) dx, \text{ for } n = 1, 2, 3, \ldots \tag{2.43}
\]

\(^7\) These series can be used in developing infinite series like Fourier series and have the general form \(f(x) = \sum_{n=1}^\infty c_n \phi_n(x)\) for \(x_1 < x < x_2\), where the set of functions \(\{\phi_n(x)\}\) is orthogonal on the specified interval with respect to a given weighting function \(w(x) > 0\), that is \(\int_{x_1}^{x_2} \phi_n(x) \phi_k(x) w(x) dx = 0\), for all \(k \neq n\).
The original problem was
\[
\theta(x, t) = V(x, t) + K(x, t)
\]
where
\[
K(x, t) = \alpha(t) + [\beta(t) - \alpha(t)] \frac{x}{L}
\]

Hence
\[
\theta(x, t) = \sum_{n=1}^{\infty} \left( c_n + \frac{1}{\kappa} \int_0^t \exp\left( \frac{1}{\kappa} \lambda_n \tau \right) P_n(\tau) d\tau \right) \exp\left( -\frac{1}{\kappa} \lambda_n t \right) \Psi_n(x) + \alpha(t) + [\beta(t) - \alpha(t)] \frac{x}{L} + (2.44)
\]

with
\[
P_n(t) = \frac{1}{\|\Psi_n(x)\|^2} \int_0^L Q(x, t) \Psi_n(x) dx, \text{ for } n = 1, 2, 3, ...
\]
\[
Q(x, t) = \frac{1}{\kappa} K_1(x, t) + \frac{j}{\rho c A^2 \sigma} I^2
\]

As previously mentioned, the electric current \( I \) in the above equation is a constant D.C. current.
Integral equations and the method of Green’s functions are frequently used to solve initial value problems within both heat conduction and electromagnetism. In this chapter, we will attempt to familiarize with these concepts. It must be mentioned that this is a short and concise approach of these subjects and the reader is referred to another sources for a deeper understanding [53, 54, 55].

### 3.1 Integral Equations

Integral equations arise in physics, chemistry, biology and engineering applications modelled by initial value problems for a finite interval \([a, b]\). They arise naturally in observational sciences such as astronomy, seismology and spectrometry. Integral equations can also result from Green’s function methods, including the PEEC method presented in Ch. 4, or boundary element methods for solving differential equations. An integral equation, by definition, is an equation in which the unknown function \(u(x)\) to be determined appears under the integral sign. A typical form of an integral equation in \(u(x)\) is of the form[62, 63].

\[
u(x) = f(x) + \int_{\alpha(x)}^{\beta(x)} K(x, xt)u(xt)dxt
\]

where \(K(x, t)\) is called the kernel of the integral equation, and \(\alpha(x)\) and \(\beta(x)\) are the limits of the integration in the equation above. It is easily observed that the unknown function \(u(x)\) appears under the integral sign in most other cases. It is important to point out that the kernel \(K(x, t)\) and the function \(f(x)\) in the above equation are given in advance. Our goal is to determine \(u(x)\) that will satisfy (3.1), and this may be achieved by using different techniques.

Integral equations arise indeed as representation forms of differential equations and one important issue concerning integral equations is how to convert boundary and initial value problems to the form of an integral equation. Depending on which type of
differential equation will be handled, there will be different types of integral equations to be solved. For example, Fredholm and Volterra integral equations arise from different origins and applications; The former origins from boundary value problems and the latter from initial value problems. Based on this fact that integral equations arise from distinct origins, different techniques and approaches will be used to determine the solution of each type of integral equations. In Fredholm integral equation, the integral is taken over a finite interval with fixed limits of integration. However, in Volterra integral equations, at least one limit of the range of integration is a variable, and the upper limit is the most commonly used with a variable limit. For other cases there are other approaches like Integro-Differential equations and Singular integral equations but studying these approaches is beyond the scope of this work. We will next try to familiarize us with relatively simple and frequently used integral equations.

3.1.1 Fredholm and Volterra Linear Integral Equations

In the standard form of Fredholm linear integral equations, the limits of integration $a$ and $b$ are constants. This type of integral equations is given by

$$\phi(x)u(x) = f(x) + \lambda \int_a^b K(x, xt)u(xt)dx, \quad a \leq x, xt \leq b$$  \hspace{1cm} (3.2)

where $K(x, t)$ is the kernel of the integral equation and the function $f(x)$ are given in advance. $\lambda$ is a parameter and $u(x)$ under the integral sign is a linear function whose power is one. The value of $\phi(x)$ gives different kinds of Fredholm linear integral equations. We have

$$f(x) + \lambda \int_a^b K(x, xt)u(xt)dx = 0$$  \hspace{1cm} (3.3)

if $\phi(x) = 0$. Equation 3.3 is called Fredholm integral equation of the first kind. If $\phi(x) = 1$, we have

$$u(x) = f(x) + \lambda \int_a^b K(x, xt)u(xt)dx$$  \hspace{1cm} (3.4)

Equation 3.4 is called Fredholm integral equation of the second kind. In Volterra linear integral equations, the limits of integration are functions of $x$ rather than constants. The standard form of Volterra integral equations is given by

$$\phi(x)u(x) = f(x) + \lambda \int_a^x K(x, xt)u(xt)dx$$  \hspace{1cm} (3.5)

where the unknown function $u(x)$ under the integral sign is a linear function of $x$. We have

$$f(x) + \lambda \int_a^x K(x, xt)u(xt)dx = 0$$  \hspace{1cm} (3.6)

if $\phi(x) = 0$. Equation 3.6 is called Volterra integral equation of the first kind. When $\phi(x) = 1$, equation 3.5 becomes

$$u(x) = f(x) + \lambda \int_a^x K(x, xt)u(xt)dx$$  \hspace{1cm} (3.7)

which is Volterra integral equation of the second kind.
3.1. Integral Equations

3.1.2 Analytic and Numerical Solution of Integral Equations

A solution of an integral equation on the interval of integration is a function \( u(x) \) which satisfies the given integral equation. A crucial issue when solving integral equations is the concept of existence of a solution. Moreover, when a solution does exist, it must be unique. Next important issue is that if a solution exists for an integral equation, it can possibly be given in form of elementary functions like polynomial or exponential functions. However, it is not always possible to give the solution in a closed form like, for example, an exponential function. In such cases, one may obtain a solution in series form. This kind of solution will be applied for numerical approximations. A better accuracy in such cases occurs by taking more terms in the series. An integral equation can be thought of a system of algebraic equations when one intends to solve it numerically. For example, a Fredholm integral equation of the first kind which has the form

\[
\lambda \int_{a}^{b} K(x, x') u(x') \, dx' = f(x)
\]

with known kernel \( K \) and known function \( f \) is to be solved for \( u \). For numerical solution of equation 3.8, the intention is to convert this equation to a linear system like \( Ax = y \).

In many fields of science and engineering, the kernel \( K \) represents the response function of an instrument, \( f \) represents measured data and \( u \) is the underlying signal that is sought. This solution may be extremely sensitive to perturbations in such systems. These perturbations occur in form of varying the input data \( f \). The mathematical reason to this sensitivity is that the integration operation is a smoothing operation and determining the integrand from the integral is an opposite operation. This sensitivity occurs even when we have converted the integral equation to a system of linear equations of the form \( Ax = y \). In such cases, the coefficient matrix \( A \) will be ill-conditioned and the solution vector \( x \) will get elements that are smaller than the characteristic machine epsilon for the software. One has to perform the conditioning of the resulting linear equation systems with different known techniques. If we, for example, try to solve the ill-conditioned linear equation system iteratively, we may use conditioned conjugated gradient algorithm.

A standard technique for solving integral equations numerically is to use a quadrature formula to replace the integral by an approximating finite sum

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} K(x_i, t_j) w_j u(t_j) = f(x_i)
\]

where \( w_j \) and \( t_j \) are nodes respectively weights of the quadrature rule. Quadrature formulas are often used in the PEEC method when calculating partial elements for nonorthogonal geometries [39]. The equation above represents a system of a linear algebraic equations \( Ax = y \), where the coefficient matrix elements \( a_{ij} = K(x_i, t_j) w_j \), the input data \( y_i = f(x_i) \) and \( x_j = u(t_j) \). An approximated discrete sample of values are now obtained for the underlying function \( u \).

3.1.3 The Method of Green’s Functions

When a physical system is subject to some external disturbance, a non-homogeneity arises in the mathematical formulation of the problem, either in the differential equation or in
the auxiliary conditions or both. When the differential equation is nonhomogeneous, a particular solution of the equation can be found by applying either the method of undetermined coefficients or the variation of parameter technique. In general, however, such techniques lead to a particular solution that has no special physical significance. Green’s functions are specific functions that develop general solution formulas for solving nonhomogeneous differential equations. Importantly, this type of formulation gives an increased, physical, knowledge since all Green’s function has a physical significance. This function measures the response of a system due to a point source somewhere on the fundamental domain, and all other solutions due to different source terms are found to be superpositions of the Green’s function. There are, however, cases where Green’s functions fail to exist. We cannot always construct Green’s functions associated with boundary value problems but for problems associated with initial value problems, Green’s functions can always be constructed.

Green’s mathematics is nearly all devised to solve very general physical problems. Green’s first interest was in electrostatics. The inverse-square law had recently been established experimentally, and he wanted to calculate how this determined the distribution of charge on the surfaces of conductors. He made great use of the electrical potential, and gave it that name, and one of the theorems that he proved in this work became famous and is known as Green’s theorem. It relates the properties of mathematical functions at the surfaces of a closed volume to other properties inside. The powerful method of Green’s functions involves what are now called Green’s functions, $G(x, x')$. If we have a differential equation $Ly = F(x)$, where $L$ is a linear differential operator, then the solution can be written as

$$y(x) = \int_0^x G(x, x')F(x')dx'$$

(3.10)

To see this, consider the equation

$$\frac{dy}{dx} + ky = F(x)$$

which can be solved by the standard integrating factor technique to give

$$y = e^{-kx} \int_0^x e^{kx'}dx' = \int_0^x e^{k(x-x')}F(x')dx'$$

so that $G(x, x') = e^{-k(x-x')}$. This technique may be applied to other more complicated systems. In an electrical circuit the Green’s function is the current due to an applied voltage pulse. In electrostatics the Green’s function is the potential due to a change applied at a particular point in space. In general the Green’s function is, as mentioned

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1George Green, 1793-1841, was one of the most remarkable of nineteenth century physicists, a self-taught mathematician whose work has contributed greatly to modern physics.

2Consider a set of functions $\phi_n$ for $n = 1, 2, ..., N$. If each number of the functions $\phi_n$ is a solution to the partial differential equation $L\phi = 0$, with $L$ as a linear operator and with some prescribed boundary conditions, then the linear combination $\phi_N = \phi_0 + \sum_{n=1}^N a_n\phi_n$ also satisfies $L\phi = g$. Here, $g$ is a known excitation or source. This fundamental concept is verified in different mathematical literature.
earlier, the response of a system to a stimulus applied at a particular point in space or time. This concept has been readily adapted to quantum physics where the applied stimulus is the injection of a quantum of energy. It is in the quantum domain that the application of Green’s functions to physical problems has grown most spectacularly in the past few decades.

3.1.4 Green’s Functions and Eigenfunctions

If we can solve the eigenvalue problem associated with the operator \( L \), then we can find the Green’s function. We know that the eigenvalue problem\([53, 59, 60, 61]\)

\[
Lu = \lambda u, \quad a < x < b
\]  

(3.11)

where \( L \) includes the boundary conditions, has infinite many eigenvalues and corresponding orthonormal eigenfunctions \( \lambda_n \) and \( \phi_n \), where \( n = 1, 2, 3, \ldots \), respectively. Moreover, the eigenfunctions form a basis for the square integrable functions on \((a, b)\). Therefore we assume that the solution \( u \) is given in terms of the eigenfunctions as

\[
u(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)
\]  

(3.12)

where the coefficients \( c_n \) are to be determined. Further, the given function \( f \) forms our source term in the nonhomogeneous differential equation

\[
Lu = f \quad \text{or} \quad u = L^{-1}f
\]  

(3.13)

where \( L^{-1} \) is the opposite operator to the operator \( L \). Now, we write the given function \( f \) in terms of the eigenfunctions as

\[
f(x) = \sum_{n=1}^{\infty} f_n \phi_n(x),
\]  

(3.14)

with

\[
f_n = \int_{a}^{b} f(\xi) \phi_n(\xi) d\xi
\]  

(3.15)

Combining (3.12), (3.13), and (3.14) gives

\[
L \left( \sum_{n=1}^{\infty} c_n \phi_n(x) \right) = \sum_{n=1}^{\infty} f_n \phi_n(x)
\]  

(3.16)

The linearity associated with superposition principle concerning the eigenvalue problem can be shown as

\[
L \left( \sum_{n=1}^{\infty} c_n \phi_n(x) \right) = \sum_{n=1}^{\infty} c_n L(\phi_n(x))
\]  

(3.17)

But

\[
\sum_{n=1}^{\infty} c_n L(\phi_n(x)) = \sum_{n=1}^{\infty} c_n \lambda_n \phi_n(x) = \sum_{n=1}^{\infty} f_n \phi_n(x)
\]  

(3.18)
which finally yields
\[ L \left( \sum_{n=1}^{\infty} c_n \phi_n(x) \right) = \sum_{n=1}^{\infty} f_n \phi_n(x) \] (3.19)

By comparing (3.18) and (3.19), we get
\[ c_n = \frac{1}{\lambda_n} \text{ and } f_n = \frac{1}{\lambda_n} \int_{a}^{b} f(\xi)\phi_n(\xi) d\xi \text{ for } n = 1, 2, 3, \ldots \] (3.20)

Further
\[ u(x) = \sum_{n=1}^{\infty} c_n \phi_n(x) \] (3.21)
\[ = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \left( \int_{a}^{b} f(\xi)\phi_n(\xi) d\xi \right) \phi_n(x) \]

Now, we suppose that an interchange of summation and integral is allowed. In this case (3.21) can be written as
\[ u(x) = \int_{a}^{b} \left( \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(\xi)}{\lambda_n} \right) f(\xi) d\xi \] (3.22)

But by definition of Green’s function
\[ u(x) = L^{-1} f = \int_{a}^{b} g(x, \xi) f(\xi) d\xi \] (3.23)

Now, \( u(x) \) can be expressed in terms of Green’s functions if we compare the last two equations, i.e.
\[ g(x, \xi) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(\xi)}{\lambda_n} \] (3.24)

which is the Green’s function associated with the eigenvalue problem (3.11) with the differential operator \( L \).

### 3.2 Differential Equations

The physical situations behind technical and scientific questions are modelled by expressing the laws of mechanics and physics in terms of questions that relate derivatives and integrals. Common variables in these models are time, position, velocity, acceleration, mass, density, momentum etc. Information about the physical process being modelled is gained by solving for some of these variables, i.e. by computing the solution of the differential equation in terms of the other variables, which are assumed to be known data. Calculus is the basic study of differential equations and their solutions.
3.2. Differential Equations

Sometimes it is possible to find an exact solution of a differential equation. This solution may be expressed as a combination of elementary functions or as a trigonometric or power series. This type of solutions are now partially automated in mathematical softwares for symbolic computation. Within the modeling of the modern technical and scientific problems, however, solutions of differential equations cannot be found explicitly in terms of known mathematical functions. The alternative is to determine an approximated solution for a given data through numerical computations. The basic idea is to discretize a given differential equation to obtain a system of equations with a finite number of unknowns in form of the linear equation system $Ax = b$, where $x$ is the unknown vector, $A$ is a coefficient matrix, and $b$ is a given (source) vector. The error in a good numerical method decreases with the increasing of the number of unknowns and computational work in a computer. The rapid increase of computer power has opened new possibilities for this approach.

One way to solve differential equations with boundary values numerically is to use finite difference methods. These methods use finite difference approximation to convert differential equations into algebraic equations.

3.2.1 The Finite Difference Method (FDM)

Considering the Taylor series expansions

$$f(x + a) = f(x) + \frac{f'(x)}{1!}a + \frac{f''(x)}{2!}a^2 + \frac{f'''(x)}{3!}a^3 + \frac{f^{(4)}(x)}{4!} + ... \tag{3.25}$$

and

$$f(x - a) = f(x) - \frac{f'(x)}{1!}a + \frac{f''(x)}{2!}a^2 - \frac{f'''(x)}{3!}a^3 + \frac{f^{(4)}(x)}{4!} - ... \tag{3.26}$$

when $f(x)$ and its first derivatives $f'(x)$, $f''(x)$, $f'''(x)$, $f^{(4)}(x)$, ... are continuous in an interval about $x=a$. Then one can obtain

\[f'(x) = \frac{f(x + a) - f(x)}{a} - \frac{f''(x)}{2}a + ...\]

\[\approx \frac{f(x + a) - f(x)}{a}\]  

(3.27)

\[f'(x) = \frac{f(x) - f(x - a)}{a} + \frac{f''(x)}{2}a + ...\]

\[\approx \frac{f(x) - f(x - a)}{a}\]  

(3.28)

Relation 3.27 is called the forward difference formula and the latter one, (3.28), backward difference formula. As it is seen, the dominant term in the remainder of series in both of the approximations is $o(n)$. The first derivative can also be written as

\[f'(x) = \frac{f(x + a) - f(x - a)}{2a} - \frac{f''(x)}{3}a^2 + ...\]

\[\approx \frac{f(x + a) - f(x - a)}{2a}\]  

(3.29)
which is called the centered difference formula. The approximation in (3.29) is second order accurate because the dominant term in the remainder of the series is $o(a^2)$. One can express the second derivative of $f(x)$ analogously by using Taylor series expansions. Adding (3.25) and (3.26) gives

$$f(x + a) - f(x - a) = f(x) + \frac{f'(x)}{1!}a + \frac{f''(x)}{2!}a^2 + \frac{f'''(x)}{3!}a^3 + \frac{f^{(4)}(x)}{4!}a^4 +$$

$$\ldots + f(x) - \frac{f'(x)}{1!}a + \frac{f''(x)}{2!}a^2 - \frac{f'''(x)}{3!}a^3 +$$

$$\frac{f^{(4)}(x)}{4!}a^4 + \ldots$$

$$= a^2 f''(x) + 2f(x) + \frac{f^{(4)}(x)}{12}a^4$$

Hence

$$a^2 f''(x) = f(x + a) + f(x - a) - 2f(x) - \frac{f^{(4)}(x)}{12}a^4 + \ldots$$

Refereed to the equation above, $f''(x)$ can be expressed as

$$f''(x) = \frac{f(x + a) - 2f(x) + f(x - a)}{a^2} - \frac{f^{(4)}(x)}{12}a^2 + \ldots$$

(3.30)

The approximation of the second derivative of $f(x)$ as

$$f''(x) = \frac{f(x + a) - 2f(x) + f(x - a)}{a^2}$$

(3.31)

is the second order accurate because the vanishing $a^2$-term in (3.30) and (3.31).

### 3.2.2 Solving the Equation of Conduction by FDM

Now, we try to solve our original problem for heat conduction due to an electric current by FD. By this method, one can solve nonhomogeneous partial differential equations numerically and obtain it in matrix form to be solved as a linear equation system. Referred to (2.28) in Ch 2.3

$$\frac{\partial^2 V(x, t)}{\partial x^2} = \frac{1}{\kappa} \frac{\partial V(x, t)}{\partial t} + Q(x, t), \quad 0 < x < L, t > 0$$

_B.C._ : $V(0, t) = 0, V(L, t) = 0$, $t > 0$

_I.C._ : $V(x, 0) = F(x)$, $0 < x < L$

with

$$Q(x, t) = \frac{1}{\kappa} \frac{\partial K(x, t)}{\partial t} - q(x, t), \quad 0 < x < L, t > 0$$

$$F(x) = f(x) - K(x, 0)$$

$$q(x, t) = -\frac{j}{\rho c A^2 \sigma} I^2$$
Our intention from the beginning was to obtain

\[
\theta(x, t) = K(x, t) + V(x, t)
\]

with

\[
\theta(x, 0) = f(x) = K(x, 0) + V(x, 0)
\]

For finite difference discretizing of the boundary value problem above, we first select an integer \( m > 0 \) and define \( h = L/m \) with selecting a time step size \( k \). Assuming \((x_i, t_j)\) as the grid points where \( x_i = ih \) and \( t_j = jk \) for \( i = 0, 1, 2, \ldots \) and \( j = 0, 1, 2, \ldots \) respectively. By using Taylor expansion, we form the difference formula for \( t \theta \) with the grid points where

\[
\frac{\partial V}{\partial t}(x, t_j) \simeq \frac{V(x_i, t_j + k) - V(x_i, t_j)}{k} - \frac{k \partial^2 V}{2 \partial t^2}(x_i, \tau_j)
\]

for some \( \tau_j \) in \((t_{j-1}, t_{j+1})\). We also form the difference formula in \( x \) as

\[
\frac{\partial^2 V}{\partial x^2}(x, t_j) \simeq \frac{V(x_{i+h}, t_j) - 2V(x_i, t_j) + V(x_{i-h}, t_j)}{h^2} - \frac{h^2 \partial^4 V}{12 \partial x^4}(\xi_i, t_j)
\]

for some \( \xi_i \) in \((x_{i-1}, x_{i+1})\). By comparing the last equations and the boundary value problem, we obtain

\[
\frac{V(x_{i+h}, t_j) - 2V(x_i, t_j) + V(x_{i-h}, t_j)}{h^2} - \frac{h^2 \partial^4 V}{12 \partial x^4}(\xi_i, t_j) + \frac{1}{\kappa} \left( \frac{V(x_i, t_j + k) - V(x_i, t_j)}{k} - \frac{k \partial^2 V}{2 \partial t^2}(x_i, \tau_j) \right) = Q(x_i, t_j)
\]

The error in this approximation will be

\[
\varepsilon_{ij} = \frac{k \partial^2 V}{2 \partial t^2}(x_i, \tau_j) - \frac{1}{\kappa} \frac{h^2 \partial^4 V}{12 \partial x^4}(\xi_i, t_j)
\]

if we approximate \( V(x_i, t_j) \) by \( W(x_i, t_j) \) so that

\[
\frac{W_{i,j+1} - W_{i,j}}{k} - \frac{1}{\kappa} \frac{W_{i,j+1} - 2W_{i,j} + W_{i-1,j}}{h^2} - Q_{i,j} = 0
\]

which can be solved for \( W_{i,j+1} \) as

\[
W_{i,j+1} = \left( 1 - \frac{2k}{\kappa h^2} \right) W_{i,j} + \frac{k}{\kappa h^2} (W_{i+1,j} + W_{i-1,j}) + kQ_{i,j}
\]

By setting \( \frac{k}{\kappa h^2} = r \), we can rewrite the equation above as

\[
W_{i,j+1} = (1 - 2r) W_{i,j} + r(W_{i+1,j} + W_{i-1,j}) + kQ_{i,j}
\]

for \( i = 1, 2, \ldots, m-1 \) and \( j = 1, 2, \ldots \). The initial condition \( V(x, 0) = F(x) \) or in discretized form as \( W_0 = F(x) \), for each \( i = 0, 1, 2, \ldots, m-1 \). These values may be used in the
difference equation above to find value of $W_{i1}$ for each $i = 1, 2, ..., m - 1$. The boundary conditions $V(0, t) = V(L, t) = 0$ give values for $W_0 = W_{11} = W_{21} = ... = W_{m1} = 0$ so that once the procedure is applied again, all the approximations $W_{i1}, W_{i2}, W_{i3}, ..., W_{i,m-1}$ may be obtained in a similar manner. Finally, by rewriting the known elements in $Q$ as $Q_{i1}, Q_{i2}, ..., Q_{i,m-1}$, we can write the approximate difference solution in matrix form as

$$
\vec{W}^{(j)} = A\vec{W}^{(j-1)} + k\vec{Q}^{(j)}
$$

(3.37)

with

$$
\vec{W}^{(0)} = [F(x_1), F(x_2), ..., F(x_{m-1})]^T
$$

and

$$
\vec{W}^{(j)} = [W_{1j}, W_{2j}, ..., W_{m-1,j}]^T
$$

where $A$ is the tri-diagonal coefficient matrix associated with this explicit difference method\(^3\) and has a size of $(m - 1) \times (m - 1)$ and it may be written as

$$
A = \begin{bmatrix}
(1 - 2r) & r & 0 \\
r & (1 - 2r) & r \\
0 & r & (1 - 2r) \\
. & . & . \\
. & . & . \\
. & . & . \\
0 & r & (1 - 2r)
\end{bmatrix}
$$

(3.38)

3.2.3 Derivation of Difference Equations by Reduction to a System of ODEs

The equation

$$
\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2}, \quad 0 < x < L, t > 0
$$

(3.39)

B.C. : $\theta(0, t) = \theta(L, t) = 0, \quad t > 0$

I.C. : $\theta(x, 0) = g(x), \quad 0 \leq x \leq L$

Difference approximation of the right-hand side of (3.39) gives

$$
\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{h^2} \left\{ \theta(x - h, t) - 2\theta(x, t) + \theta(x + h, t) \right\} + o(h^2)
$$

where $h$ is the step size in $x$-axis. The equations above, together give

$$
\frac{\partial \theta}{\partial t} = \frac{1}{h^2} \left\{ \theta(x - h, t) - 2\theta(x, t) + \theta(x + h, t) \right\} + o(h^2)
$$

(3.40)

\(^3\)Explicit finite difference formulas consider solution values only at previous levels. Both for ODEs and PDEs, implicit methods are stable for a much greater range of step-sizes. Implicit methods are, so called, unconditionally stable which means that there is no restriction on the relative sizes of $\Delta t$ and $\Delta x$, i.e. time and room step-sizes respectively. Explicit methods are, by this definition of stability, conditional stable which thereby means that relative sizes of $\Delta t$ and $\Delta x$ are restricted.
By subdividing the interval $0 \leq x \leq L$ into $N$ equal subintervals by the gridlines $x_i = ih$, for $i = 0, 1, 2, ..., N - 1$ along $t$-axis. Further, We suppose that $V_i(t)$ approximates the exact solution, which in this case is $\theta_i(t)$ and forms a system of $(N-1)$ ordinary differential equations as

$$
\frac{d\theta_1}{dt} = \frac{1}{h^2}(\theta_0 - 2\theta_1 + \theta_2)
$$

$$
\frac{d\theta_2}{dt} = \frac{1}{h^2}(\theta_1 - 2\theta_2 + \theta_3)
$$

$$
\vdots
$$

$$
\frac{d\theta_{N-1}}{dt} = \frac{1}{h^2}(\theta_{N-2} - 2\theta_{N-1} + \theta_N)
$$

with known boundaries $\theta_0$ and $\theta_N$. In matrix form, the above system of ODEs can be written as

$$
\frac{d}{dt} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_{N-2} \\ \theta_{N-1} \end{pmatrix} = \frac{1}{h^2} \begin{pmatrix} -2 & 1 & & & 1 \\ 1 & -2 & 1 & & \\ & 1 & -2 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_{N-2} \\ \theta_{N-1} \end{pmatrix} + \frac{1}{h^2} \begin{pmatrix} \theta_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \vec{b}
$$

or

$$
\frac{d\vec{\theta}(t)}{dt} = \vec{A}\vec{\theta}(t) + \vec{b}
$$

(3.41)

where $\vec{\theta}(t) = (\theta_1, \theta_2, ..., \theta_{N-1})^T$, $\vec{b}$ is a column vector consists of both zeros and known boundary values, and $\vec{A}$ is

$$
\vec{A} = \frac{1}{h^2} \begin{pmatrix} -2 & 1 & & & 1 \\ 1 & -2 & 1 & & \\ & 1 & -2 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{pmatrix}
$$

(3.42)

The size of the matrix $\vec{A}$ is $(N - 1) \times (N - 1)$. The scalar differential equation

$$
\frac{d\theta(t)}{dt} = A\theta(t) + b
$$

has the solution

$$
\theta(t) = -\frac{b}{A} + (g + \frac{b}{A})exp(At)
$$

(3.43)
where \( A \) and \( b \) are independent of \( t \) and \( \theta(t) \) satisfies the initial condition \( \theta(0) = g \). It can be shown that the solution of (18) satisfying the initial condition \( \hat{\theta}(0) = (g_1, g_2, ..., g_{N-1})^T = \bar{g} \) is

\[
\hat{\theta}(t) = -(\bar{A})^{-1}\bar{b} + \{exp(t\bar{A})\}(\bar{g} + (\bar{A})^{-1}\bar{b})
\]

(3.44)

where

\[
exp(t\bar{A}) = \frac{1}{h^2} \begin{pmatrix}
exp(-2t) & exp(t) \\
exp(-2t) & exp(t) \\
& & \ddots & \ddots \\
& & & & exp(t) \\
1 & & & & exp(-2t)
\end{pmatrix}
\]

We define \( k \) as a small perturbation in \( t \)-domain so that

\[
\hat{\theta}(t + k) = -(\bar{A})^{-1}\bar{b} + \{exp(t + k)\bar{A}\}(\bar{g} + (\bar{A})^{-1}\bar{b})
\]

\[
= -(\bar{A})^{-1}\bar{b} + \{exp(k\bar{A})\}\{exp(t\bar{A})\}(\bar{g} + (\bar{A})^{-1}\bar{b})
\]

If all boundary values are zero, then

\[
\hat{\theta}(t + k) = \{exp(k\bar{A})\}\hat{\theta}(t)
\]

The perturbed solution, \( \tilde{\theta}(t) \), is now

\[
\tilde{\theta}(t) = -(\bar{A})^{-1}\bar{b} + \{exp(t\bar{A})\}(\bar{g} + (\bar{A})^{-1}\bar{b})
\]

We form now the perturbation vector as

\[
\tilde{\theta}(t) - \hat{\theta}(t) = \{exp(t\bar{A})\}(\tilde{g} - \bar{g})
\]

As it is seen, the perturbation vector \( \tilde{e}(t) = \tilde{\theta}(t) - \hat{\theta}(t) \) at time \( t \) is related to the initial perturbation vector as

\[
\tilde{\theta}(0) - \hat{\theta}(0) = \{1\}(\tilde{g} - \bar{g}) = \tilde{e}(0)
\]

which yields

\[
\tilde{e}(t) = \{exp(t\bar{A})\}\tilde{e}(0)
\]

(3.45)

This last equation may be important when we intend to find the error in our finite difference approximation. Convergence and stability in this difference method are depending on our choice of the number of time step-sizes which forms one of the error sources in finite difference methods. If we add a source term like, for example, \( f(t)\sin(\omega t) \), one can determine \( \hat{\theta}(t) \) as following

\[
\hat{\theta}(t) = -(\bar{A})^{-1}\bar{b} + \{exp(t\bar{A})\}g - (A^2 + \omega^2\bar{I})^{-1}\left[\omega \cos(\omega t)\bar{I} + \sin(\omega t)\bar{A}\right]\tilde{f}(t)
\]

(3.46)
Equation 3.46 may be numerically too expensive because the inverse and square matrix operations.

This chapter has presented both integral and differential equations in a simplified manner. In the following chapters, the integral equation based PEEC method will be presented and combined with a differential equation based thermal modeling method, Equation 3.35. This results in the combination of two numerical solutions in parallel, one considering the integral equation based system and one considering the differential equation based system. Further, information between the two systems will be transferred and utilized in the combined solution, from integral equation- to differential equation based method.
This section describes the time domain PEEC formulation for orthogonal structures discretized using orthogonal cells. The PEEC theory is based upon Maxwell’s integral equations in electromagnetism but before studying PEEC theory, there is a need to introduce some basic concepts within electromagnetism.

4.1 Basic Concepts in Electromagnetism

In constructing the electrostatic model we define an electric field intensity vector $\vec{E}$ and an electric flux density vector, $\vec{D}$. The fundamental governing differential equations are

$$\vec{\nabla} \times \vec{E} = 0$$  \hspace{1cm} (4.1)
$$\vec{\nabla} \cdot \vec{D} = \rho_v$$

where $\rho_v$ is volume charge density. For linear and isotropic media, $\vec{E}$ and $\vec{D}$ are related by relation

$$\vec{D} = \epsilon \vec{E}$$  \hspace{1cm} (4.2)

The fundamental governing equations for magnetostatic model are

$$\vec{\nabla} \cdot \vec{B} = 0$$  \hspace{1cm} (4.3)
$$\vec{\nabla} \times \vec{H} = \vec{J}$$

where $\vec{B}$ and $\vec{H}$ are defined as magnetic flux density vector and magnetic field intensity vector respectively. $\vec{B}$ and $\vec{H}$ are related as

$$\vec{H} = \frac{1}{\mu} \vec{B}$$  \hspace{1cm} (4.4)
where $\mu$ is defined as relative permeability of the medium which is measured in $H/m$. The media is assumed to be linear and isotropic. Equations 4.1 and 4.3 are known as Maxwell’s equations and form the foundation of electromagnetic theory. As it is seen in the above relations, $\vec{E}$ and $\vec{D}$ in the electrostatic model are not related to $\vec{B}$ and $\vec{H}$ in the magnetostatic model. The coexistence of static electric fields and magnetic electric fields in a conducting medium causes an electromagnetostatic field and a time-varying magnetic field gives rise to an electric field. The latter phenomenon is verified by numerous experiments. Static models are not suitable for explaining time-varying electromagnetic phenomena. Under time-varying conditions it is necessary to construct an electromagnetic model in which the electric field vectors $\vec{E}$ and $\vec{D}$ are related to the magnetic field vectors $\vec{B}$ and $\vec{H}$ and in such situations, the equivalent equations are constructed as

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$  \hspace{1cm} (4.5)$$
$$\nabla \times \vec{H} = \vec{J}$$  \hspace{1cm} (4.6)$$
$$\nabla \cdot \vec{D} = \rho_v$$  \hspace{1cm} (4.7)$$
$$\nabla \cdot \vec{B} = 0$$  \hspace{1cm} (4.8)

where $\vec{J}$ is current density. As it is seen, the Maxwell’s equations above are in differential form. However, we can not apply these differential equations when explaining electromagnetic phenomena in a physical environment. It is convenient to convert the differential forms into their integral-form equivalents. There are several techniques to convert differential equations into integral equations but in the above cases, one may apply Stoke’s theorem to obtain integral form of Maxwell’s equations after taking the surface integral of both sides of the equations over an open surface $S$ with contour $C$. The result will be constructed as following table:

Maxwell’s equations

<table>
<thead>
<tr>
<th>Differential form</th>
<th>Integral form</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$</td>
<td>$\oint_L \vec{H} \cdot d\vec{l} = I + \int_S \frac{\partial \vec{D}}{\partial t} \cdot d\vec{S}$</td>
</tr>
<tr>
<td>$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$</td>
<td>$\oint_C \vec{E} \cdot d\vec{l} = -\frac{\partial \Phi}{\partial t}$</td>
</tr>
<tr>
<td>$\nabla \cdot \vec{D} = \rho_v$</td>
<td>$\int_S \vec{D} \cdot d\vec{S} = Q$</td>
</tr>
<tr>
<td>$\nabla \cdot \vec{B} = 0$</td>
<td>$\int_S \vec{B} \cdot d\vec{S} = 0$</td>
</tr>
</tbody>
</table>

where $Q$ is electric charge in Coulomb.
4.2 Derivation of the PEEC Method

This presented derivation of the PEEC method is based, to a large extent, on the work presented in [56].

4.2.1 Derivation of Electric Field Integral Equation

The theoretical derivation of the PEEC method starts from the expression of the total electric field in free space, $\vec{E}^T(\vec{r}, t)$, by using the magnetic vector and electric scalar potentials, $\vec{A}$ and $\phi$ respectively.

$$\vec{E}^T(\vec{r}, t) = \vec{E}^i(\vec{r}, t) - \frac{\partial \vec{A}(\vec{r}, t)}{\partial t} - \nabla \phi(\vec{r}, t)$$ (4.13)

where $\vec{E}^i$ is a potential applied external electric field. If the observation point, $\vec{r}$, is on the surface of a conductor, the total electric field can be written as

$$\vec{E}^T(\vec{r}, t) = \frac{\vec{J}(\vec{r}, t)}{\sigma}$$ (4.14)

in which $\vec{J}(\vec{r}, t)$ is the current density in a conductor and $\sigma$ is the conductivity of the conductor. Combining the above equations results in

$$\vec{E}^i = \frac{\vec{J}(\vec{r}, t)}{\sigma} + \frac{\partial \vec{A}(\vec{r}, t)}{\partial t} + \nabla \phi(\vec{r}, t)$$ (4.15)

To transform (4.15) into the electric field integral equation (EFIE) the definitions of the electromagnetic potentials, $\vec{A}$ and $\phi$ are used. The magnetic vector potential, $\vec{A}$, at the observation point $\vec{r}$ is given by

$$\vec{A}(\vec{r}, t) = \sum_{k=1}^{K} \mu \int_{v_k} G(\vec{r}, \vec{r}^\prime) \vec{J}(\vec{r}^\prime, t_d) dv_k$$ (4.16)

in which the summation is over $K$ conductors and $\mu$ is the permeability of the medium. Since no magnetic material medium are considered in this thesis $\mu = \mu_0$. In (4.16) the free space Green’s function is used and is defined as

$$G(\vec{r}, \vec{r}^\prime) = \frac{1}{4\pi} \frac{1}{|\vec{r} - \vec{r}^\prime|}$$ (4.17)

$\vec{J}$ is the current density at a source point $\vec{r}^\prime$ and $t_d$ is the retardation time between the observation point, $\vec{r}$, and the source point given by

$$t_d = t - \frac{|\vec{r} - \vec{r}^\prime|}{c}$$ (4.18)

where $c = 3 \cdot 10^8 m/s$. The electric scalar potential, $\phi$, at the observation point $\vec{r}$ is given by

$$\phi(\vec{r}, t) = \sum_{k=1}^{K} \frac{1}{\epsilon_0} \int_{v_k} G(\vec{r}, \vec{r}^\prime) q(\vec{r}^\prime, t_d) dv_k$$ (4.19)
in which $\epsilon_0$ is the permittivity of free space and $q$ is the charge density at the source point. Combining (4.15), (4.16) and (4.19) results in the well known electric field integral equation (EFIE) or mixed potential integral equation (MPIE) that is to be solved according to

$$
\hat{n} \times \vec{E}'(\vec{r}, t) = \hat{n} \times \left[ \frac{\vec{J}(\vec{r}, t)}{\sigma} \right] + \hat{n} \times \left[ \sum_{k=1}^{K} \mu \int_{v_k} G(\vec{r}, \vec{r}') \frac{\partial \vec{J}(\vec{r}, t_d)}{\partial t} \, dv_k \right] + \hat{n} \times \left[ \sum_{k=1}^{K} \nabla \frac{1}{\epsilon_0 J_{vk}} \int_{v_k} G(\vec{r}, \vec{r}') q(\vec{r'}, t_d) \, dv_k \right]
$$

(4.20)

where $\hat{n}$ is the surface normal to the body surfaces.

The transformation of the EFIE in (4.20) into the PEEC formulation starts by expanding the current- and charge-densities according to this section. This results in a general form of the EFIE for the PEEC formulation from which the equivalent circuit can be derived.

### 4.2.2 PEEC Current Density Expansion

The total current density, $\vec{J}$, in (4.20) is expanded in the PEEC formulation to include the conduction current density, $\vec{J}^C$, due to the losses in the material and a polarization current density, $\vec{J}^P$, due to the dielectric material properties resulting in

$$
\vec{J} = \vec{J}^C + \vec{J}^P
$$

(4.21)

where

$$
\vec{J}^C = \sigma \vec{E}
$$

(4.22)

$$
\vec{J}^P = \epsilon_0 (\epsilon_r - 1) \frac{\partial \vec{E}}{\partial t}
$$

(4.23)

For perfect conductors, the total current density $\vec{J}$ reduces to $\vec{J}^C$. While for perfect dielectrics the total current density reduces to $\vec{J}^P$. The polarization current density is added in the differential form of the generalized Ampere’s circuital law according to

$$
\nabla \times \vec{H} = \vec{J}^C + \epsilon_0 (\epsilon_r - 1) \frac{\partial \vec{E}}{\partial t} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}
$$

(4.24)

which is reduced to the original form

$$
\nabla \times \vec{H} = \vec{J}^C + \epsilon_0 \frac{\partial \vec{E}}{\partial t}
$$

(4.25)

for $\epsilon_r = 1$. In this way the displacement current due to the bound charges for the dielectrics with $\epsilon_r > 1$ are treated separately from the conduction currents due to the free charges.
4.2. Derivation of the PEEC Method

4.2.3 PEEC Charge Density Expansion

The charge density $q^T$, indicating the combination of the free, $q^F$, and bound, $q^B$, charge density is given as

$$q^T = q^F + q^B$$

This allows the modeling of the displacement current due to the bound charges for dielectrics with $\epsilon_r > 1$ separately from the conducting currents due to the free charges. For perfect conductors, the total charge density $q^T$ reduces to $q^F$. While for perfect dielectrics the total charge density reduces to $q^B$.

The resulting EFIE for the PEEC formulation can then be written as

$$\hat{n} \times \vec{E}'(\vec{r}, t) = \hat{n} \times \left[ \frac{\vec{J}^C(\vec{r}, t)}{\sigma} \right]$$

$$+ \hat{n} \times \left[ \sum_{k=1}^{K} \mu \int_{v_k} G(\vec{r}, \vec{r}') \frac{\partial \vec{J}^C(\vec{r}', t_d)}{\partial t} dv_k \right]$$

$$+ \hat{n} \times \left[ \sum_{k=1}^{K} \epsilon_0(\epsilon_r - 1) \mu \int_{v_k} G(\vec{r}, \vec{r}') \frac{\partial^2 \vec{E}(\vec{r}', t_d)}{\partial t^2} dv_k \right]$$

$$+ \hat{n} \times \left[ \sum_{k=1}^{K} \nabla \epsilon_0 \int_{v_k} G(\vec{r}, \vec{r}') q^T(\vec{r}', t_d) dv_k \right]$$

4.2.4 Interpretation as equivalent circuit

The conversion from integral equation, (4.27), to equivalent circuit formulation is detailed in this section. The PEEC formulation for a strict conductor environment is detailed, for the dielectric formulation review reference [35]. The exclusion of dielectric bodies and external fields reduces (4.27) to

$$0 = \hat{n} \times \left[ \frac{\vec{J}^C(\vec{r}, t)}{\sigma} \right]$$

$$+ \hat{n} \times \left[ \sum_{k=1}^{K} \mu \int_{v_k} G(\vec{r}, \vec{r}') \frac{\partial \vec{J}^C(\vec{r}', t_d)}{\partial t} dv_k \right]$$

$$+ \hat{n} \times \left[ \sum_{k=1}^{K} \epsilon_0 \sum_{v_k} G(\vec{r}, \vec{r}') q^F(\vec{r}', t_d) dv_k \right]$$

Note that the system of equations in (4.28) have two unknowns, the conduction current density, $\vec{J}^C$, and the charge density, $q^F$. To solve the system of equations the following procedure is employed:
1. The current densities are discretized into volume cells that gives a 3D representation of the current flow. This is done by defining rectangular pulse functions

\[ P_{\gamma nk} = \begin{cases} 1, & \text{inside the } nk\text{-th volume cell} \\ 0, & \text{elsewhere} \end{cases} \quad (4.29) \]

where \( \gamma = x, y, z \) indicates the current component of the \( n\):th volume cell in the \( k\):th conductor.

![Thin conducting plate diagram](image)

Figure 4.1: 2-D Discretization of current density and surface charge distribution.

2. The charge densities are discretized into surface cells that gives a 2D representation of the charge over the corresponding volume cell, Fig. 4.1 This is done by defining the rectangular pulse functions

\[ p_{mk} = \begin{cases} 1, & \text{inside the } mk\text{-th surface cell} \\ 0, & \text{elsewhere} \end{cases} \quad (4.30) \]
for the charge density on the \(m\):th volume cell of the \(k\):th conductor. Using the definitions in (4.29) and (4.30) the current and charge densities can be written as

\[
\vec{J}^C_{\gamma k}(\vec{r}, t_d) = \sum_{n=1}^{N_{\gamma k}} P_{\gamma nk} J_{\gamma nk}^C(\vec{r}_{\gamma nk}', t_{\gamma nk}) \tag{4.31}
\]

\[
q^T_k(\vec{r}, t_d) = \sum_{m=1}^{M_k} p_{mk} q_{mk}(\vec{r}_{mk}', t_{mk}) \tag{4.32}
\]

where

\[
t_{\gamma nk} = t - \frac{|\vec{r} - \vec{r}_{\gamma nk}'|}{v} \tag{4.33}
\]

\[
t_{mk} = t - \frac{|\vec{r} - \vec{r}_{mk}'|}{v}
\]

The vector \(\vec{r}_{\gamma nk}'\) is the source position vector indicating the center of the \(n\):th volume cell of the \(k\):th conductor in the \(\gamma\) discretization and \(\vec{r}_{mk}'\) is the source position vector indicating the center of the \(m\):th surface cell of the \(k\):th conductor. In (4.31), the summation is over all the volume cells in conductor \(k\) with \(\gamma\) directed current while in (4.32), the summation is over all the surface cells in conductor \(k\).

Pulse functions are also used for the testing functions resulting in a Galerkin solution. The inner product is defined as a weighted volume integral over a cell as

\[
< f, g >= \frac{1}{a} \int_v f(\vec{r})g(\vec{r}) \, dv \tag{4.34}
\]

Combining (4.28), (4.31), (4.32), and (4.34) while using the inner product defined in (4.31) results in a systems of equations given by

\[
0 = \hat{n} \times \left[ \frac{\vec{J}^C(\vec{r}, t)}{\sigma} \right] \\
+ \hat{n} \times \left[ \sum_{k=1}^{K} \sum_{n=1}^{N_{\gamma k}} \mu \int_{v_{\gamma nk}} G(\vec{r}, \vec{r}_{\gamma nk}') \frac{\partial P_{\gamma nk} J_{\gamma nk}^C(\vec{r}_{\gamma nk}', t_{\gamma nk})}{\partial t} \, dv_{\gamma nk} \, dt \right] \tag{4.35}
\]

\[
+ \hat{n} \times \left[ \sum_{k=1}^{K} \sum_{m=1}^{M_k} \epsilon_0 \int_{v_{mk}} G(\vec{r}, \vec{r}_{mk}') p_{mk} q_{mk}(\vec{r}_{mk}', t_{mk}) \, dv_{mk} \right]
\]

Equation 4.35 is the basic discretized version of the electric field integral equation for the PEEC method from which the partial elements can be identified as will be shown in the following paragraphs.
4.2.4.1 Partial Inductances

The basic expression for partial inductances can be derived from the second term in (4.35) by using:

- The free space Green’s function.
- The expression \( I_{\gamma m} = J_{\gamma m} a_m \) for the total current, \( I_{\gamma m} \), through a cross sectional area, \( a_m \).

This results in

\[
\sum_{k=1}^{K} \sum_{n=1}^{N_{\gamma k}} \frac{\mu}{4\pi} \frac{1}{a_{\gamma nk} a_{\gamma nk}} \int_{v_{\gamma nk}} \int_{v_{\gamma nk}} \frac{\partial}{\partial t} I_{\gamma nk}(\vec{r}_{\gamma nk}(t), t_{\gamma nk}) \left| \vec{r} - \vec{r}' \right| dv_{\gamma nk} dv' \tag{4.36}
\]

and can be interpreted as the inductive voltage drop, \( v_L \), over the corresponding volume cell. By defining the partial inductance as

\[
L_{p\alpha\beta} = \frac{\mu}{4\pi} \frac{1}{a_{\alpha} a_{\beta}} \int_{v_{\alpha}} \int_{v_{\beta}} \frac{1}{\left| \vec{r}_{\alpha} - \vec{r}_{\beta} \right|} dv_{\alpha} dv_{\beta} \tag{4.37}
\]

can be rewritten as

\[
v_L = \sum_{k=1}^{K} \sum_{n=1}^{N_{\gamma k}} L_{p\alpha\beta} \frac{\partial}{\partial t} I_{\gamma nk}(t - \tau_{\gamma nk}) \tag{4.38}
\]

where \( \tau_{\gamma nk} \) is the center to center delay between the volume cells \( v_{\gamma nk} \). Equation 4.37 is the basic definition for the partial self and mutual inductance using the volume formulation. It is from this definition that simplified and analytical formulas for the partial inductances for special geometries have been developed.

The interpretation of the second term in (4.35) as the inductive voltage drop (using the partial inductance concept) results in:

- The connection of nearby nodes using the partial self inductance \( L_{p\alpha\alpha} \) of the corresponding volume cell \( \alpha \).
- The mutual inductive coupling of all volume cells using the concept of partial mutual inductance.

A voltage source has been used to sum all the inductive (magnetic field) couplings from all other volume cells, corresponding to the summation in (4.39). This voltage source is defined as

\[
V_m^L(t) = \sum_{\forall n, n \neq m} L_{p_{mn}} \frac{\partial i_n(t - \tau_{mn})}{\partial t} \tag{4.39}
\]

Where \( i_n(t - \tau_{mn}) \) is the current through volume cell \( n \) at an earlier instance in time, \( (t - \tau_{mn}) \). A PEEC model only consisting of partial inductances is entitled a \( (L_p)\)PEEC model, Fig. 4.2.
4.2. Derivation of the PEEC Method

Figure 4.2: PEEC model for volume cell \( m \) connecting node \( i \) and \( j \) where \( L_{p_{mm}} \) is the partial self inductance for the volume cell and \( V^L_m \) accounts for the mutual inductance (magnetic field) coupling from other volume cells, 4.39.

4.2.4.2 Coefficients of Potential

The basic definition for partial coefficients of potential can be derived from the third term in (4.35) by using the following approximations:

- The charges only resides on the surface of the volumes, i.e. converting the volume integral to a surface integral.
- The integral in the \( \gamma \) coordinate can be calculated using a finite difference (FD) approximation according to

\[
\int_v \frac{\partial}{\partial \gamma} F(\gamma) dv \approx a \left[ F\left( \gamma + \frac{l_m}{2} \right) - F\left( \gamma - \frac{l_m}{2} \right) \right]
\]

This results in

\[
\sum_{k=1}^{K} \sum_{m=1}^{M_k} q_{mk}(t_{mk}) \int_{s_{mk}} \int_{s_{mk}} \frac{1}{\left| \vec{r}^+ - \vec{r}^- \right|} ds t - q_{mk}(t_{mk}) \int_{s_{mk}} \int_{s_{mk}} \frac{1}{\left| \vec{r}^+ - \vec{r}^- \right|} ds t
\]

which can be interpreted as the capacitive voltage drop, \( v_C \), over the actual cell and the vectors \( \vec{r}^+ \) and \( \vec{r}^- \) are associated with the positive and negative end of the cell respectively. By defining the partial coefficient of potential as

\[
p_{ij} = \frac{1}{S_i S_j 4\pi \varepsilon_0} \int_{S_i} \int_{S_j} \frac{1}{\left| \vec{r}_i - \vec{r}_j \right|} dS_j dS_i \]

the capacitive voltage drop can be written as

\[
v_C = \sum_{k=1}^{K} \sum_{m=1}^{M_k} Q_{mk}(t - t_{mk})[pp^+_i(m_k) - pp^-_i(m_k)]
\]

using the total charge, \( Q_{mk} \), of the \( mk \)th cell.
Figure 4.3: (P)PEEC model for one surface cell/node $i$ where $P_{ii}$ is the partial self coefficient of potential for the surface cell and $V_i^C$ accounts for the mutual capacitance (electric field) coupling from other surface cells in Eq. (4.42).

From the basic definition in (4.42) a number of simplified and analytical formulas for partial coefficients of potential can be derived for special geometries configurations.

The interpretation of the third term in (4.35) as self and mutual (partial) coefficient of potential (capacitive) coupling results in:

- The connection of each surface cell (node) to infinity through self partial (pseudo-) capacitances.
- Mutual capacitive couplings of all surface cells (nodes).

This is illustrated in Fig. 4.3 where a voltage source, $V_i^C$ has been used to sum all the capacitive (electric field) couplings from all other surface cells.

The voltage source is defined as

$$V_i^C(t) = \sum_{j,j \neq i} \frac{P_{ij}}{P_{jj}} V_j(t - \tau_{ij})$$  \hspace{1cm} (4.44)

where $V_j(t - \tau_{ij})$ is the voltage over the pseudo-capacitance $\frac{1}{P_{jj}}$, of the $j$:th node, at an earlier instance in time, $(t - \tau_{ij})$. A PEEC model only consisting of partial coefficients of potential is entitled a (P)PEEC model.

4.2.4.3 Resistances

The first term in (4.35) can be shown to equal the resistive voltage drop over the volume cell. By assuming a constant current density over the volume cell the term is rewritten
4.2. Derivation of the PEEC Method

Figure 4.4: (R)PEEC model for volume cell \( m \) connecting node \( i \) and \( j \),

as

\[
\frac{\bar{J}_\gamma^C}{\sigma_\gamma} = \frac{I_\gamma}{a_\gamma \sigma_\gamma} \tag{4.45}\]

where \( a_\gamma \) is the cross section of the volume cell normal to the \( \gamma \) direction. The resistance is then calculated as

\[
R_\gamma = \frac{l_\gamma}{a_\gamma \sigma_\gamma} \tag{4.46}\]

where \( l_\gamma \) is the volume cell length in the \( \gamma \) direction.

The interpretation of the first term in (4.35) as the voltage drop in a conductor results in a lumped resistance connection between the nodes in the PEEC model. A PEEC model only consisting of volume cell resistances is entitled a (R)PEEC model, Fig. 4.4.

4.2.4.4 Combined \((L_p)\)PEEC, \((P)\)PEEC, and \((R)\)PEEC Models.

When partial inductances are used in the \((R)\)PEEC model a series connection of the resistance and partial inductance is made. This results in a \((L_p, R)\)PEEC model, Fig. 4.5.

Figure 4.5: \((L_p, R)\)PEEC model for volume cell \( m \) connecting node \( i \) and \( j \).
The inclusion of partial coefficients of potential results in a \((L_p, R, P)\)PEEC model, Fig. 4.6. In the figure one surface cell at each node is used to account for the capacitive coupling to corresponding node.

![Cell diagram](image)

**Figure 4.6**: \((L_p, R, P)\)PEEC model for volume cell \(m\) connecting node \(i\) and \(j\).

### 4.2.5 Solution of Time- and Frequency Domain PEEC Models

For the solution of PEECs in the time and frequency domain an *Admittance Method* or a *Modified Nodal Analysis* (MNA) [57] method can be used. The Admittance Method produces a minimal but dense system matrix to obtain the voltages in the structure. The MNA solves for both voltages and currents in a structure and therefore produces a larger, and sparse, system matrix. The MNA method is widely used in modern circuit analysis software due to its full-spectrum properties and flexibility to include additional circuit elements. The choice between the two methods depend on the specific problem at hand and the computational resources available.

### 4.3 Practical PEEC Modeling

The basic procedure for creating PEEC models is illustrated in Fig. 4.7 illustrating all the essential blocks required in a PEEC based electromagnetic solver. Shortly, a graphical tool is needed to draw and edit a structure, a routine then performs the discretization of the structure, the PEEC engine then calculates the partial elements and creates and solves the linear system (for both time and frequency domain simulations). Finally, if the
system is stable, the solution variables (currents and voltages) are exported to a graph viewer for further inspection. The workflow in Fig. 4.7 will be used in later chapters to explain the inclusion of the thermal modeling with an existing PEEC solver.

Figure 4.7: Work flow when creating PEEC models.
In this chapter, we are combining the equation of conduction with a PEEC based electric and electromagnetic simulation. In the first section, we begin with a simple example, electrothermal modeling of a current carrying wire solved by an analytic solution. A more complicated problem, a lightning protection system (LPS), is studied in last section. This model employs the finite difference method for the numerical modeling of the heat equation while a \((L_p, C, R)\)PEEC model accounts for the electric and electromagnetic behavior.

5.1 Mathematical Modeling of Temperature Rise in a Wire

Our modeling of temperature was based upon an experiment from which we obtained data for, among others, PEEC volume cell currents. In all of our cases, the PEEC currents are proportional to the source term everywhere there is a differential equation. For simplicity, we begin with a theoretical model to determine the temperature distribution in a metallic rod heated by an electric current. In our study, we use the model of conduction of heat in a wire in one dimension and the constant electric current \(I\) is supposed to have been continuing for a long time. Thus, the generation of heat in length \(dx\) of the wire is

\[
\frac{jI^2}{\sigma A^2} \, dx
\]

where \(A\) and \(\sigma\) are the cross section area and conductivity of the wire respectively and \(j = 0.239\ldots\) is the number of calories in a joule. The mathematical model to determine the steady temperature distribution in this situation is[1]

\[
K \frac{d^2\theta}{dx^2} + \frac{jI^2}{\sigma A^2} = 0
\]  

(5.1)
which can be solved both by analytic and numerical methods. The analytic solution of this equation is

$$\theta(x) = -\frac{1}{2} j I^2 x^2 + C_1 x + C_2$$

by given constants as

$$K = 400 \text{ W m} K, \sigma = 5.8 \times 10^8 \Omega^{-1} m^{-1}, A = 5 \times 5 \text{ mm}^2$$

for copper. The constants $C_1$ and $C_2$ will be determined for different boundary conditions in the equation above. We can determine these constants by, for example, the Dirichlet type boundary condition as

$$\theta(x = 0) = \theta(x = L) = 273^o K, \text{ for } 0 < x < L.$$

By applying this boundary condition and the above numerical values, one can obtain the values for the constants as $C_1 = 0.8241379312 \times 10^{-5} \times I^2 \times L$ and $C_2 = 273$ in Eq. 5.2, which is the model for the steady case of the temperature rise in a wire due to an electric current $I$. Fig. 5.1 displays the steady temperature rise in a wire made of copper due to different steady electric currents. In this model situation, the surface of the wire is supposed to be impervious to heat and the current $I$ is supposed to have been flowing long enough for reaching the steady rate of temperature. The ends of the wire are kept in $273^o K(0^o C)$.
5.2 Electrothermal modeling of an LPS

The consequences of electromagnetic interference in sensitive equipments may be limited or even eliminated by reasonable selection of lightning protection systems (LPS) and its ground system, Fig. 5.3, or, when this is not possible, by the placing of that equipment elsewhere with less interference. Using more sensitive equipment, protection projects against electromagnetic interference, such as lightning protection structures, have become more and more rigorous. Since it is impossible to avoid lightning on structures, the
knowledge of this behavior along the grounding cables and its electromagnetic fields in
the interior of a building become very important to predict some possible problems with
the equipment.

A protection system is made of a set of conductors that are intended to conduct the
lightning induced currents to earth, avoiding in this way any damage to the building.
The terminations of the conductors which are supposed to be transmission lines, may be
another conductors, the earth, the grounding system or the lightning channel. The pur-
pose of the ground system is to dissipate the current to earth to secure human protection,
and to supply with a reference of potential to the electrical and electronic systems.

5.2.1 Thermal Modeling Technique

Our PEEC data concerning the electric current in the LPS were both room- and time-
dependent. Hence, we ought to model the temperature as a quantity that was depending
on both room and time. An important issue in our simulation was so that data in form
of boundary conditions for temperature were missing. A numerical solution in the next
sections is given for the case where the endpoints of the LPS are hold in 0°C.

In mathematical terms, such condition could be interpreted as boundary conditions
in form of Dirichlet conditions. However, there were premises that let us to interpret
the problem as a partial differential equation with Neumann boundary conditions. The
reason was the very short time of a lightning which was of order $10^{-4}s$, during which the
system wouldn’t ”have time” to exchange heat with surrounding. This factor would allow
the system to be interpreted as a well-insulated one and that is the case where Neumann
boundary conditions can be applied. In mathematical terms, this can be formulated as
the following partial differential equation

$$\frac{\partial^2 \theta(x, t)}{\partial x^2} = \frac{1}{\kappa} \frac{\partial \theta(x, t)}{\partial t} - q(x, t), \quad 0 < x < L, t \geq 0 \quad (5.5)$$
5.2. Electrothermal modeling of an LPS

\[ B.C. : \quad \frac{\partial \theta(0, t)}{\partial x} = 0, \quad \frac{\partial \theta(L, t)}{\partial x} = 0, \quad t \geq 0 \]

\[ I.C. : \quad \theta(x, 0) = f(x), \quad 0 < x < L \]

where \( q(x, t) \) is squared proportional to the electric current in our case and \( f(x) \) is the temperature distribution in the LPS at the beginning. The above PDE can be solved in terms of eigenvalue functions as

\[ \theta(x, t) = \sum_{n=1}^{\infty} W_n(t) \phi_n(x) \quad (5.6) \]

To solve the above eigenvalue problem by the method of separation of variables, we begin with finding eigenfunctions and eigenvalues of equations

\[ X''(x) + \lambda X(x) = 0, \quad \text{with} \quad X'(0) = X'(L) = 0 \quad (5.7) \]

\[ W'(t) + \frac{1}{\kappa} \lambda W(t) = 0 \]

Now, we can solve the first equation with the prescribed boundary conditions. For \( \lambda = 0 \)

\[ X''(x) = 0 \implies X(x) = C_1 + C_2x \quad (5.8) \]

which with the prescribed boundary conditions, we obtain

\[ X(x) = C_1 \]

where \( C_1 \) is an arbitrary constant. We set this arbitrary constant equal to 1 and write the first eigenvalue and the associated eigenfunction as

\[ \phi_0(x) = 1 \quad \text{with} \quad \lambda_0 = 0 \quad (5.9) \]

For the cases when \( \lambda > 0 \) or \( \lambda = k^2 > 0 \), (5.7) has a general solution as

\[ X(x) = C_1 \cos kx + C_2 \sin kx \quad (5.10) \]

This equation and values for \( k \) can also be found by the above prescribed boundary conditions. Finally, for the case of non-trivial solution of the eigenvalue problem, we obtain

\[ \sin k\pi = 0 \implies kL = n\pi, \quad \text{for} \quad n = 1, 2, 3, ... \quad (5.11) \]

Hence

\[ \lambda_n = \frac{n^2\pi^2}{L^2}, \quad \text{for} \quad n = 1, 2, 3, ... \quad (5.12) \]

The result summarized as

\[ \lambda_0 = 0, \phi_0(x) = 1, n = 0 \quad (5.13) \]

\[ \lambda_n = \frac{n^2\pi^2}{L^2}, \phi_n(x) = \cos \frac{n\pi x}{L}, \quad \text{for} \quad n = 1, 2, 3, ... \]
It can be shown that there are no solutions for the above eigenvalue problem in the cases where \( \lambda < 0 \). The second eigenvalue problem in (5.7) may be solved in the same way and the result can be summarized as

\[
W_0(t) = 1, \lambda_0 = 0, n = 0 \quad (5.14)
\]

\[
W_n(x) = c_n \exp\left(-\frac{n^2 \pi^2}{\kappa L^2} t\right), \lambda_n = \frac{n^2 \pi^2}{L^2}, \text{ for } n = 1, 2, 3, \ldots
\]

We suppose that termwise differentiation is allowed. Thus, it can be concluded that

\[
\frac{\partial \theta(x, t)}{\partial t} = \sum_{n=1}^{\infty} W_n'(t) \phi_n(x) and \quad \frac{\partial^2 \theta(x, t)}{\partial x^2} = \sum_{n=1}^{\infty} W_n(t) \phi_n''(x)
\]

\[
= - \sum_{n=1}^{\infty} \lambda_n W_n(t) \phi_n(x)
\]

The PDE in (5.5) may be written as

\[
\kappa q(x, t) = \frac{\partial \theta(x, t)}{\partial t} - \kappa \frac{\partial^2 \theta(x, t)}{\partial x^2}
\]

\[
= \sum_{n=1}^{\infty} [W_n'(t) + \lambda_n W_n(t)] \phi_n(x)
\]

The term \( W_n'(t) + \lambda_n W_n(t) \) is room-independent and it contains, in addition, terms of exponential functions for a fixed value \( t \). Hence, the right-hand side of the equation above can be interpreted as a generalized Fourier series of the function on the left-hand side, i.e. \( \kappa q(x, t) \). Thus

\[
W_n'(t) + \lambda_n W_n(t) = \kappa \frac{1}{\|\phi_n(x)\|^2} \int_0^L q(x, t) \phi_n(x) dx, \text{ for } n = 1, 2, 3, \ldots \quad (5.17)
\]

where

\[
\|\phi_n(x)\|^2 = \int_0^L [\phi_n(x)]^2 dx, \text{ for } n = 1, 2, 3, \ldots \quad (5.18)
\]

The ordinary differential equation above can be solved by method of integrating factor. The integrating factor for the equation is \( \exp(a^2 \lambda_n t) \). By multiplying this factor in both sides of the differential equation above, we obtain

\[
\exp(a^2 \lambda_n t) W_n'(t) + \exp(a^2 \lambda_n t) \lambda_n W_n(t) = \frac{\exp(a^2 \lambda_n t) \kappa}{\|\phi_n(x)\|^2} \int_0^L q(x, t) \phi_n(x) dx,
\]

for \( n = 1, 2, 3, \ldots \)
which gives
\[
\exp(a^2\lambda_n t)W_n(t) = c_n + \int_0^t \left( \frac{a^2 \exp(a^2\lambda_n t)}{\|\phi_n(x)\|^2} \int_0^L q(x, t)\phi_n(x)dx \right) dt \tag{5.19}
\]
Solving for \(W_n(t)\) yields
\[
W_n(t) = \left( c_n + a^2 \int_0^t \exp(a^2\lambda_n t)P_n(\tau)d\tau \right) \exp(-a^2\lambda_n t), \text{ for } n = 1, 2, 3, ... \tag{5.20}
\]
where
\[
P_n(\tau) = \frac{1}{\|\phi_n(x)\|^2} \int_0^L q(x, t)\phi_n(x)dx \tag{5.21}
\]
Equation (5.6) together with the above relations give
\[
\theta(x, t) = \sum_{n=1}^{\infty} \left( c_n + a^2 \int_0^t \exp(a^2\lambda_n t)P_n(\tau)d\tau \right) \exp(-a^2\lambda_n t)\phi_n(x) \tag{5.22}
\]
for \(n = 1, 2, 3, ...\)

and
\[
\theta(x, t) = W_0(t)\phi_0(x), \lambda_0 = 0 \text{ and } n = 0
\]
The only unknown variable in the right-hand side of the equation above is now the constants \(c_n\) which can be determined by utilizing the prescribed initial condition in (5.5) that is by setting \(t\) equal zero in (5.22). Thus
\[
\theta(x, 0) = f(x) = \sum_{n=1}^{\infty} c_n\phi_n(x)
\]
which thereby yields
\[
c_n = \frac{1}{\|\phi_n(x)\|^2} \int_{x=0}^{L} f(x)\phi_n(x)dx, \text{ for } n = 1, 2, 3, ... \tag{5.23}
\]
(5.22) is an analytic solution to the electrothermal problem formulated as the PDE with the concerned boundary conditions in (5.5). The major limitation in our study case is that the source term which is included in \(P_n(\tau)\) is a vector of scalars that cannot be presented by a mathematical function. This means the exclusion of possibility to integrate this source term. The reason is that every usual numerical method fails for approximation or interpolation of scalar vectors such as \(q(x, t)\) with too high number of the vector entries. The number of entries in \(q(x, t)\) was 400 i.e. the same number for time step-sizes on the discretization of the LPS according to the PEEC formulation of volume currents. In this case, a pressing issue was to solve the problem by a numerical method. We chose the finite difference method.
5.2.2 Numerical Example

This section employs the combined model for thermal, electric, and electromagnetic analysis developed in previous sections. The model is used to characterize a lightning protection system (LPS) often occurring in buildings. More specifically, the model employs the finite difference method for the numerical modeling of the heat equation while a $(L_p, C, R)$PEEC model accounts for the electric and electromagnetic behavior. In our first implementation, we discretized the LPS in a total of 20 PEEC volume cells, with 5 PEEC cells in every side of the LPS. The LPS was excited using a realistic current source, Fig. 5.5, similar to a lightning strike [58]. The lightning strike generate well over 3.5 kA in each LPS arm close to the strike position. For the two other arms, the effect is not severe due to the effective shortening resistors show in Fig. 5.3.

By using PEEC model calculations described in Chapter 4, volume cell currents was obtained from a computer program. Now, the temperature development on the LPS may be calculated by, for example, the finite difference methods. The problem in this study case is that there are restrictions about discretizing the sides of the LPS. This choice of discretization should be according to the number of PEEC cells on this side of the LPS. With these circumstances, the best result is obtained for a fine, 15 PEEC cells, discretization.

As previously mentioned, the real boundary and initial conditions concerning the thermal equation for the LPS were missing. In this case, we choose these prescribed conditions as idealized ones. By way of introduction, we were now to solve the governing PDE for thermal modeling of one of the LPS sides. We supposed that this LPS side was well-insulated with the endpoints hold on $0^\circ$C. This kind of problem is already solved in (3.36) by FDM. The governing PDE for the electrothermal modeling of the one of LPS sides is

\[
\frac{\partial \theta}{\partial t} = \frac{K}{\rho c} \frac{\partial^2 \theta}{\partial x^2} + \frac{1}{\rho c A} Q
\]  

(5.24)

We supposed further that the LPS was made by copper with the numerical data as following:

\[
K = 400 \frac{W}{mK}, \quad \rho = 8954 \frac{kg}{m^3}, \quad c = 383.1 \frac{J}{kgK}, \quad A = 5 \times 5 mm^2
\]  

(5.25)

and

\[
Q = RI^2
\]  

(5.26)

where $K$ is thermal conductivity, $A$ cross section area of the LPS, $\rho$ density, and $c$ specific heat for copper. $Q$ in the PDE represents power per length unity as a function of $R$, the electrical resistance, and $I$, the electric current. $\theta$ and $t$ represent the temperature and time respectively. In all of the calculations, the length $x$ is equal to 1m (not a realistic LPS). The only source term in the above PDE is supposed to be caused by ohm heating (PEEC volume currents, $I$) which are based on different discretizations of the LPS, 5, 10, or 15 cells for each arm, Fig. 5.3. The PDE is solved for one of the LPS sides by FDM with 15 room step-sizes, Fig. 5.8.

The work performed in the computer implementation is summarized in the flow graph in Fig. 5.4. The figure is based on the previously presented flowgraph in Fig. 4.7 but has now been completed for thermal modeling possibilities. As it is shown, stability is an
5.2. Electrothermal modeling of an LPS

Important part of the flow chart and studied in two different moments; The first moment of stability has to do with the integral equation-based method of PEEC on the left hand side of the flow chart. These integral equations are resulted from the employed Green’s functions included in the PEEC method for obtaining, for example, $I$, the PEEC currents and $R$, the partial element resistances. These already designed Green’s functions may be sensitive to perturbations in the input data (=source terms). This kind of sensitivity is generally a result of an inverse mathematical operator to integration which is a smoothing operator. One may solve integral equations by a quadrature rule to replace the integral by an approximating finite sum. A linear equation system such as $Ax = y$ will be the result of the quadrature rule, where $A$, the coefficient matrix may be subject to ill-conditioning. The sensitivity and ill-conditioning are inherent in the problem and is not due to the method of solving it. There are, however, methods that set about the problem of ill-conditioning in such cases. These methods and techniques require additional information to obtain physically meaningful solution. The second moment of stability should be studied when solving the thermal differential equation, depicted in the right hand side of the flow chart. This PDE which consists of source terms may be solved by FDM. The source term in the case of the LPS was a function of the power which in its turn was a mathematical function of PEEC volume currents $I$ and resistances $R$ in the left hand side of the flow chart. Other source terms in the PDE were excluded in these simulation. Once we intend to solve the PDE by FDM, we have to set about the choice of a suitable finite difference technique. Different FD techniques have different stability issues. The important limitations in this stage is that restrictions occur when choosing room step-sizes in the different FD regimes. Once we formulate the problem in the left hand side of the flow chart by PEEC formulation, there will be too costly to change the number of PEEC cells. Therefore, the FD approximation should be according to this choice of PEEC discretization. Our strategy will be to choose an explicit FD method such as forward difference method which is relatively cheap, assuming that the required condition is accomplished. Explicit methods within FDM require $r < 0.5$ in (3.36). This possibility may be limited with the restriction about choosing room step-sizes in the FDM with respect to number of PEEC cells.

Once solving of the electrothermal equation in the right hand side of the flow chart by an explicit method fails, one can employ an implicit method instead. Implicit methods within FDM are so called unconditionally stable methods. Applying such methods excludes the instability of the FD regime even if $r > 0.5$ in (3.36). However, one should be careful when choosing the time step-sizes. A possible strategy is to solve the electrothermal equation by starting with no source term in the equation. In the case of solving the equation without any source terms, one has the possibility to solve it by analytic solutions. Any choice of the time step-sizes for solving the PDE without source terms will be tested and compared with the analytic solution. The best choice is when the numerical solution and the analytic one coincide as exact as possible. This choice of the number of time step-sizes will even give a reliable solution once we intend to solve the PDE with the source term. By employing this strategy, we will succeed to reduce the numerical error in the case of applying implicit methods for the LPS.
Figure 5.4: Flowgraph describing the combined tool for electric, electromagnetic, and thermal modeling.

As seen in Fig. 5.8, the temperature rise in one of the most affected LPS arms is approx. 7 °C. This temperature rise is for the 5 × 5 mm cross section of the LPS arms. By reducing the cross section, the temperature would increase. The performed numerical example could be ‘scaled’ to a conductor geometry delivering power (current) to components on a PCB. Typical conductor cross sections at PCB level is 50 µm × 100 µm (T × W) and current levels in the order of Ampere.

5.2.3 Discussion

Stability in the thermal modeling based on the PEEC method is of importance and it has to be studied and discussed, at least, in two different phases; When we are solving PEEC currents and potentials in the first phase, we should set about instability that occurs when solving integral equations numerically. Solving integral equations leads generally
5.2. Electrothermal modeling of an LPS

Figure 5.5: The current source exciting the LPS, lightning strike.

Figure 5.6: PEEC currents in the LPS in time step-size 11.
Figure 5.7: Lightning strike position in the LPS.

Figure 5.8: Temperature distribution on one of the LPS sides.
to ill-conditioning in the ultimate system of linear equations. Solving the differential equation describing the thermal process of the problem in the second phase may require a numerical method. Depending on which numerical method we use, one should think about the stability of the method. In our case, we applied the finite difference method for solving the thermal modeling in different situations. In the case of solving the differential equation for the temperature development in the LPS, backward difference method was used because the unconditional property of this numerical method. We were restricted to discretize the sides of the square-shaped LPS in 15 mesh points as a maximum.
Chapter 6

Conclusions

6.1 Results

This thesis shows that PEEC based electric and electromagnetic modeling can be combined alt. completed with a thermal modeling technique. The presented implementation utilizes a finite difference method for solving the heat conduction. The finite difference method is implemented in parallel with the PEEC solution and utilizes:

1. PEEC model discretizations (volume cell discretization),

2. PEEC model partial elements (partial resistances),

3. PEEC model results (volume cell currents).

to solve for the heat conduction (temperature).

In this work, the electromagnetic behavior of circuit is presented with partial element equivalent circuits (PEEC) which is based upon an electrical field integral equation solution to Maxwell’s equation. The electrothermal modeling of the LPS (our study case) is based on the equation of conduction in one dimension. It is a partial differential equation with the source term as a mathematical function of electric power which, in its turn is a function of both PEEC model for volume cell currents and partial element resistances. The numerical stability when solving integral and differential equations for PEEC volume cell currents and the electrothermal solution, respectively, should be treated and discussed, at least, in two different moments. At the first moment, one has to set about the occurrence of instability for numerical solving of integral based PEEC volume cell currents. Generally, numerical solving of integral equations might, in many cases, lead to numerical instability when solving the ultimate linear equations in the numerical method. The next moment of instability may occur when the equation of conduction will be solved by a numerical method such as finite difference method. At this moment, carefulness in choosing the number of time step-sizes is a crucial issue to avoid instability and to predict the error in the numerical solution of the differential equation.
6.2 Limitations

During the process of determination of the electrothermal modeling of the LPS and solving the concerned partial differential equation, we should set about following issues:

- The only possible solution for the electrothermal modeling in this study case was a numerical method. This limitation occurred since to solve the governed differential equation analytically, one should approximate or interpolate vectors of scalars with some polynomial function. These vectors had a role as the source term in the partial differential equation describing the thermal modeling. The problem was the large dimension of this vector that could not be approximated or interpolated into analytic functions such as polynomials with the usual interpolation- or approximation methods. Every method fails when designing such interpolation polynomials for these scalar vectors with such high number of entries.

- To combine the PEEC integral method and the differential thermal model the real boundary condition was not available. The boundary and initial conditions in the study of the LPS should be chosen in an idealized form. In this case we chose a Dirichlet type of boundary condition on which the LPS was well-insulated with the endpoints at 0°C.

- For numerical solving of the electrothermal modeling of the LPS, one should choose time- and spatial step-sizes according to the number of PEEC cells because it would be too costly to change the number of PEEC cells. This limitation might cause instability in the numerical method when solving the differential equation describing the thermal model. Conditional stable methods such as forward difference methods could be unstable if choosing this regularly low number of spatial step-sizes in PEEC discretization. In such cases, unconditional stable methods like backward difference method would be utilized which contains another complicated methods to estimate the numerical error.

6.3 Further Work

The thesis have introduced several important areas for further investigation as defined in these subsections.

6.3.1 Electric and Electromagnetic Modeling with Temperature Feedback

One very interesting continuation would be the possibility to use the developed electrothermal modeling technique with a temperature feedback in the PEEC simulations. At this stage, the PEEC solution routine is providing the thermal solution routine with information (volume cell currents, partial element values, and discretization). Suppose, that the thermal solution could provide the PEEC solution with temperature data and
that the PEEC solution routine in the next time step could update the electric and electromagnetic model under this new, new, conditions. This is illustrated in Fig. 6.1, based in the original flow graph in Fig. 5.4, where the communication is from the PEEC solution (R[Ω] and I[A]) to the thermal solution and back (T[°C]). This would enable a temperature dependent electromagnetic modeling of components, packages, interconnects and complete systems.

6.3.2 Thermo-mechanical-deformation

When the temperature of a bar rises from an initial temperature $T_o$ to a final temperature $T_f$ this temperature rise produces an elongation of the bar that affects the original length and diameter of the bar. When a beam is subjected to thermal loads, it will be deformed. The deformation will consist of axial elongation and of bending. The phenomenon of de-
formation caused by thermal loads is also valid for electronics and electronic components; With the development of modern electronic packages, many electronic components experience extreme thermal environments because power dissipation which, in its turn, is caused by electric currents through an impedance. Leads and solder balls that connect an individual package to a circuit board are subjected to failure because such kind of thermal loading. This is a form of fatigue failure caused by a cyclic-thermo-mechanical deformation. The thermal system of package/PCB undergoes a cyclic thermal loading resulting from environmental changes and from power dissipation. The PEEC formulation has a capability that enables one to prescribe several moments of the concerned electromagnetic modeling analytically. By applying the PEEC formulation, these analytical solutions will decrease drastically the solving process of a thermal modelling. In a further work one can utilize these easily earned PEEC currents to determine the power dissipation in the thermal system of package/PCB and which thereby its thermo-mechanical deformation.

6.3.3 Integral Equation Formulated Thermal Modeling Approach

The developed solver is using an integral equation based PEEC electric and electromagnetic modeling technique in parallel with a differential equation based finite difference thermal modeling method. In a second stage it would be possible to explore integral equation based methods for solving the thermal problem, in parallel or as a post-processor to the PEEC solution.

6.3.4 Radiation losses

In most practical cases the flux of heat from the surface is not a linear function of the temperature difference between the surface and its surroundings. However, in some cases, the flux of heat from the surface obeys \( H(\theta - \theta_0) \), where \( H \) is called the 'Outer' or 'Surface' Conductivity and \( \theta_0 \) is the temperature of surroundings[1]. Let \( I \) be the current, measured in amperes, in one of the LPS sides (a wire made of copper), and let \( \sigma \) be the electrical conductivity of this wire. Then the rate of generation of heat in length \( dx \) of the wire is

\[
\frac{jI^2}{\omega\sigma} dx
\]

where \( j = 0.239... \), as mentioned in Chapter 2, is the number of calories in a joule. This term is the new contribution to the rate of gain of heat which was missing in (2.6). We suppose that the wire is of area \( A \), perimeter \( p \) and surface conductance \( H \). Then we have

\[
\frac{1}{\rho c} \frac{\partial \theta}{\partial t} = K \frac{\partial^2 \theta}{\partial x^2} - \frac{p}{A} H(\theta - \theta_0) + \frac{jI^2}{A^2\sigma}
\]

(6.2)

In this model situation, the term \( H(\theta - \theta_0) \) can be approximated by[1]

\[
5.48 \times 10^{-12} E\theta_0^3(\theta - \theta_0)
\]

(6.3)

where \( E = 0.03 \) is the emissivity of copper at \( 20^\circ C \). If the flux across the surface of the wire is proportional to the temperature difference between the surface and the
surrounding medium, the boundary condition will be

\[ K \frac{\partial \theta}{\partial t} + H(\theta - \theta_0) = 0 \]  

(6.4)

Fig. 6.2 is a plot of the model situation above. The boundary equation is referred to the classical works as radiation boundary condition. The assumption that the wire loses heat by radiation to its surroundings at a rate proportional to the temperature difference is only satisfactory if this temperature difference is small: if not, it must be replaced by the statement that the loss of heat should be modelled by equations for 'black-body radiation'[1].
References


Conclusions


Following, some important definitions in mathematical physics are represented. It should be mentioned that this short appendix cannot be enough to a complete understanding of these rather deep concepts. For a more detailed description, the reader is referred to another sources in this area of mathematics[53, 59, 60, 61].

**Definition 1 Linear Operators**

Let $u_1$ and $u_2$ be functions and $c_1$ and $c_2$ constants. Then, the function $c_1u_1+c_2u_2$ is called the linear combination of $u_1$ and $u_2$. Further, one can define a class of functions with a common domain of definition such that each linear combination of any two functions in that class remains in it. This class of functions is called a function space. A linear operator on this space is an operator $L$ that transforms each function $u$ of the space into a function $Lu$ with following properties

$$L(c_1u_1 + c_2u_2) = c_1Lu_1 + c_2Lu_2 \quad (1.1)$$

$$L(u_1 + u_2) = Lu_1 + Lu_2, \quad L(c_1u_1) = c_1Lu_1 \quad (1.2)$$

A generalization of (1.1) gives

$$L \left( \sum_{n=1}^{N} c_n u_n \right) = \sum_{n=1}^{N} c_n Lu_n \quad (1.3)$$

This last equation can be proved, for example, by mathematical induction.

**Definition 2 Principle of Superposition**

Let $L$ be a linear differential operator. Thus, every linear homogeneous differential equation has the form

$$Lu = 0 \quad (1.4)$$

As an example, if

$$L = A \frac{\partial^2}{\partial x^2} + B \frac{\partial^2}{\partial y \partial x} + C \frac{\partial^2}{\partial y^2} + D \frac{\partial}{\partial x} + E \frac{\partial}{\partial y} + F \quad (1.5)$$
then the linear homogeneous partial differential equation
\[ Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + u_y + Fu = 0 \] (1.6)
has the form (1). Coefficients \( A \) through \( F \) may be functions of \( x \) and \( y \) and \( u = u(x, y) \). Apparently, the variables \( x \) and \( y \) are restricted to be on the boundary of a domain. Now, suppose that \( u_n \) with \( n = 1, 2, 3, ..., N \) denotes functions that satisfy Eq.(4). Hence \( Lu_n = 0 \) for each \( n \). By property (3), it is evident that each linear combination of those functions also satisfies (4). That is what we call principle of superposition of solutions.

**Definition 3** Regular Sturm-Liouville Problems

Let \( L \) be a one-dimensional differential operator in the interval \( I = [a, b] \), where \( a \) and \( b \) are real constants. Then
\[ Lu = \frac{1}{\omega(x)}[-\frac{d}{dx}(p(x)\frac{du}{dx}) + q(x)u] = \frac{1}{\omega}[-(pw)\gamma + qu] \] (1.7)
with the domain
\[ D_L = \{ u \in C^2(I) \mid A_0u(x_0) + B_0u(x_0) = 0, A_1u(x_1) + B_1u(x_1) = 0 \} \] (1.8)
with
\[
\begin{align*}
p & \in C^1(I), \ p(x) > 0 \\
q & \in C^0(I), \ q(x) \geq 0 \\
\omega & \in C^0(I), \ \omega(x) \geq 0
\end{align*}
\]
where \( A_0 \) and \( B_0 \geq 0 \) and \( A_1 \) and \( B_1 \geq 0 \). However, \( A_0 \) and \( B_0 \) cannot be equal zero together. The same is true for \( A_1 \) and \( B_1 \). A function which is continuous but which has a discontinuous first derivative is termed a \( C^0 \)-continuous in a majority of mathematical literature. Generally, \( C^k \)-continuity means that the \( k \)-derivative is continuous. The boundary condition in above relations is of the type \( Au + b\frac{\partial u}{\partial n} \), where \( n \) is the outward normal. The operator \( L \) in this case is called a Sturm-Liouville operator of the regular type.