REITERATION FOR AND EXACT RELATIONS BETWEEN SOME REAL INTERPOLATION SPACES

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ABSTRACT. For the real interpolation spaces $(A_0,A_1)_{f,q}$ with a parameter function f we prove a general reiteration result, where we need not assume some separation condition between the corresponding parameter functions. As one application we obtain a sharp embedding result between the spaces $(A_0,A_1)_{(\theta,b),q}$ obtained by using the function parameter $f(t) = t^{\theta}(1+|\log t|)^{b}$. This result may be regarded as a generalization of some well-known embeddings between Lorentz-Zygmund spaces.

INTRODUCTION. Let $\theta,\theta_1,\theta_0,q,q_0,q_1$ and q_θ denote real numbers satisfying $0 < \theta,\theta_0,\theta_1 < 1$, $0 < q,q_0,q_1 \le \infty$ and $1/q_\theta = (1-\theta)/q_0+\theta/q_1$, where $1/\infty = 0$. Let (A_0,A_1) be a compatible pair of quasi-Banach spaces. The Lions-Peetre real interpolation spaces $(A_0,A_1)_{\theta,q}$ are well understood and widely used in different kinds of applications, see e.g. the books [2],[3],[5],[7],[17] (and compare with our Section 1). In particular, we note that *the reiteration formula*

$$((A_0,A_1)_{\theta_0,q_0},(A_0,A_1)_{\theta_1,q_1})_{\theta,q} = (A_0,A_1)_{\eta,q},$$

where $\eta = (1-\theta)\theta_0+\theta\theta_1$ holds if $\theta_0 \neq \theta_1$ (see e.g. [3,p.50]) or if $q = q_\theta$ (see e.g. [5,p.186] or our Theorem 1.1). Unfortunately, if none of these conditions is satisfied, then the situation is much more complicated. In this paper we present exact descriptions of the more general (parameter function) spaces

$$((A_0,A_1)_{f_0,q_0},(A_0,A_1)_{f_1,q_1})_{f,q} \tag{0.1}$$

in cases which do not include the complicated cases described above. (The real interpolation spaces $(A_0,A_1)_{f,q}$ with a parameter function f are described in Section 1). Moreover we introduce the spaces $(A_0,A_1)_{(\theta,b),q}$ which generalizes the usual Lorentz-Zygmund spaces

$L^{p,q}(\log L)^b$ (see [1]) in a natural way. We prove a sharp embedding theorem between these spaces which in particular generalizes the well-known embedding between the $L^{p,q}(\log L)^b$-spaces (see [1,p.31]).

The paper is organized in the following way: In Section 1 we give some preliminaries including some necessary theory about real interpolation. In Section 2 we present the announced descriptions of the spaces (0.1). We remark that the proofs are easy consequences of a fundamental estimate of Brudnyi-Krugljak (see (1.1)) and some fairly new descriptions of real interpolation spaces between weighted L^p and Lorentz-spaces (see [9] and [15]). In Section 3 we introduce the spaces $(A_0,A_1)_{(\theta,b),q}$ and prove the announced embedding theorem between these spaces. Moreover we prove that this embedding is in a sense the sharpest possible.

Conventions: C denotes any positive constant (not the same in different appearances). The equivalence symbol $f(t) \approx g(t)$ means that $af(t) \le g(t) \le bf(t)$ for some constants $a,b > 0$ and all $t>0$. Two quasi-normed spaces A and B are considered as equal, and we write $A = B$, whenever their quasi-norms are equivalent.

1. Preliminaries.

We consider the Lebesgue space $L^q_* = L^q(0,\infty,dt/t)$. The class Q_ε, $\varepsilon>0$, consists of the functions $\psi:(0,\infty) \to (0,\infty)$ satisfying

$$\|\psi\|_{L^1_*} = 1, \quad t^{-\varepsilon}\psi(t) \text{ is decreasing and } t^{\varepsilon}\psi(t) \text{ is increasing.}$$

Let (Ω,Σ,μ) be a σ-finite measure space and let $\omega = \omega(x)$ be a weight function on Ω, i.e. let ω be a measurable and positive function on Ω. Let $E = E(\mu)$ be an ideal quasi-normed subspace of the space $S(\mu)$ of all μ-measurable functions, which are finite almost everywhere. The weighted ideal quasi-normed space $E(\omega) = E(\omega,\mu)$ consists of all $a \in S(\mu)$ satisfying

$$\|a\|_{E(\omega)} = \|a\omega\|_E < \infty.$$

Let a^* denote the nonincreasing rearrangement of a. The weighted Lorentz space $L^{\varphi q}(\omega,\mu)$ consists of all $a \in S(\mu)$ for which $(a\omega)^*$ belongs to the space $L^q_*(\varphi)$ endowed with quasi-norm

$$\|a\|_{L^{\varphi q}(\omega,\mu)} = \|(a\omega)^*\|_{L^q_*(\varphi)},$$

see e.g. [9]. (The function $a\omega$ is rearranged with respect to the measure μ.) If $\omega=1$, then $L^{\varphi q}(\omega,\mu) = L^{\varphi q}(\mu) = L^{\varphi q}$. In particular, if $\varphi(t) = t^{1/p}(1+|\log t|)^b$, then $L^{\varphi q}$ coincides with the usual

Lorentz-Zygmund spaces $L^{p,q}(\log L)^b$ investigated by Bennett and Rudnick in [1].

Let $\bar{A} = (A_0,A_1)$ denote a quasi-Banach pair. The K-functional $K(t,a)$ is defined for every $a \in A_0+A_1$ and $t>0$ as

$$K(t,a) = K(t,a,\bar{A}) = \inf_{a=a_0+a_1} (\|a_0\|_{A_0}+t\|a_1\|_{A_1}).$$

Let E be an ideal quasi-normed subspace of $S(\mu)$, where $\mu = dt/t$ and $\Omega = (0,\infty)$. The real interpolation space $\bar{A}_E$ is defined as the set of all $a \in A_0+A_1$ satisfying

$$\|a\|_{\bar{A}_E} = \|K(t,a,\bar{A})\|_E < \infty.$$

In particular if $E = L^q_*(\frac{1}{f})$, where f is a <u>parameter function</u>, we obtain <u>the real interpolation spaces</u> $\bar{A}_{f,q}$ <u>with a parameter function</u> f. The necessary hypothesis on the parameter function f can be given in several essentially equivalent ways. In this paper we use the Matuszewska-Orlicz indices α_f and β_f defined for every $f\in B$, where B denotes the class of all continuous functions $f:(0,\infty) \to (0,\infty)$ such that, for every $t>0$,

$$\bar{f}(t) = \sup_{s>0} (f(st)/f(s)) < \infty.$$

The definition of α_f and β_f are as follows

$$\alpha_f = \sup_{0<t<1} \frac{\log \bar{f}(t)}{\log t} = \lim_{t\to 0+} \frac{\log \bar{f}(t)}{\log t},$$

$$\beta_f = \inf_{t>1} \frac{\log \bar{f}(t)}{\log t} = \lim_{t\to\infty} \frac{\log \bar{f}(t)}{\log t}.$$

It is well-known that $-\infty < \alpha_f \le \beta_f < \infty$. See [8] for more information about these and other indices. In this paper we assume that the parameter functions f belong to the class $B_0= \{f\in B:\ 0 < \alpha_f \le \beta_f < 1\}$.

Example 1.1. Let $f(t) = t^{\theta}(1+|\log t|)^b$, $0 < \theta < 1$, $b\in\mathbb{R}$. Then $\bar{f}(t) = t^{\theta}(1+|\log t|)^{|b|}$ and $\alpha_f = \beta_f = \theta$ and, thus, $f\in B_0$. In this case we denote the spaces $A_{f,q}$ by $A_{(\theta,b),q}$. For the case $b=0$ we have the usual parameter spaces $A_{\theta,q}$.

More information concerning real interpolation with a parameter function between quasi-Banach pairs can be found in [14] and the references given there. For the Banach case these spaces are in fact special cases of the spaces $(A_0,A_1)_\Phi{}^K$ already studied by Peetre in [11].

Later on we need the following fundamental estimate by Brudnyi-Krugljak (see [4],[5] and also Nilsson [10]):

Let (A_0, A_1) be a quasi-Banach pair and (E_0, E_1) any pair of interpolation spaces between ℓ^∞ and $\ell^\infty((2^{-i})_i)$. Then, for any $a \in A_0 + A_1$ and $t > 0$,

$$K(t, a, \bar{A}_{E_0}, \bar{A}_{E_1}) \approx K(t, (K(2^i, a, \bar{A}))_i, E_0, E_1). \tag{1.1}$$

2. Reiteration.

First of all we note that using the crucial estimate (1.1) with $E_j = \ell^{q_j}(\omega_j)$, where $\omega_j = (1/f_j(2^i))_i$, $i \in Z$, $j = 0, 1$, we obtain that

$$(\bar{A}_{f_0, q_0}, \bar{A}_{f_1, q_1}) = \bar{A}_E, \text{ where } E = \left[L_*^{q_0}(\tfrac{1}{f_0}), L_*^{q_1}(\tfrac{1}{f_1})\right]_{f,q} \tag{2.1}$$

In the general case it is difficult to give simple descriptions of the spaces E (compare e.g. with our Lemma 2.2) but in some special cases such simple descriptions are known. For example, in the diagonal cases $q_0 = q_1 = q$, $0 < q \le \infty$ and $0 < q_0, q_1 < \infty$, $q = q_\theta$, $f(t) = t^\theta$ we have

$$E = L_*^q(1/f_2) \quad \text{with} \quad f_2 = f_0 f(f_1/f_0). \tag{2.2}$$

See [9, Theorem 2] or [14, Lemma 3.1] and [3, p.115]. Moreover, if the functions f_1 and f_2 are separated from each other by some suitable index condition (corresponding to the condition $\theta_0 \ne \theta_1$ in the parameter case) we can use the following lemma:

Lemma 2.1. *Let ω_0 and ω_1 be positive weight functions on Ω, $\omega_{01} = \omega_0/\omega_1$ and $\omega = \omega_0/f(\omega_{01})$. Let $0 < p_0, p_1, q \le \infty$, $f \in B_0$, $\omega_0, \omega_1, \omega_{01} \in B$ with $\alpha_{\omega_i} > 0$, $i = 0, 1$ and $\alpha_{\omega_{01}} > 0$ or $\beta_{\omega_{01}} < 0$. If $b = b(t)$ is a positive and continuous function on $(0, \infty)$ such that $b(t)\, t^c$ is increasing or decreasing for some constant c, then*

$$\|b\|_{(L_*^{p_0}(\omega_0), L_*^{p_1}(\omega_1))_{f,q}} \approx \|b\|_{L_*^q(\omega)}.$$

Lemma 2.1 is proved in [9, p.771]. See also [15]. The following theorem generalizes the usual parameter versions of the reiteration theorem to the function parameter case.

Theorem 2.1. *Let $0 < q, q_0, q_1 \le \infty$, $f, f_0, f_1 \in B_0$, $f_{10} = f_1/f_0$ and $f_2 = f_0 f(f_1/f_0)$. Then*

$$(\bar{A}_{f_0, q_0}, \bar{A}_{f_1, q_1})_{f,q} = \bar{A}_{f_2, q} \tag{2.3}$$

if one of the following conditions holds:

(a) $q_0 = q_1 = q$.

(b) $f(t) = t^{\theta}$, $q = q_{\theta}$, $0 < q_0, q_1 < \infty$.

(c) $\alpha_{f_{10}} > 0$ or $\beta_{f_{10}} < 0$.

Proof. Let (a) or (b) be satisfied. Then (2.2) holds and, thus, according to (2.1), (2.3) is satisfied. Moreover, using Lemma 2.1 with $\omega_i = 1/f_i$, $i=1,2$ and $b(t) = K(t,a,A_0,A_1)$ we find that also the assumption (c) together with (2.1) imply (2.3) and the proof is complete.

Remark. Concerning Theorem 2.1 (a) and (b) see also [5,p.186] and [14,Examples 4.1–4.2] or [9,Example 1]. Other proofs of Theorem 2.1 (c) can be found in [14,p.211] and (at least for the case $q < \infty$) in [6,Theorem 2.1].

In order to be able to treat the general off-diagonal case $q_0 = q_1 \neq q$ we need the following lemma:

Lemma 2.2. *Let* $f \in B_0$, $0 < p,q \le \infty$, $\gamma = 1/p-1/q$ *and* $\varepsilon_0 = \min(\alpha_f, 1-\beta_f)/|\gamma|$, $\gamma \neq 0$. *If* ω_0 *and* ω_1 *are weight functions on* Ω, $\omega_{01} = \omega_0/\omega_1$ *and* $\omega = \omega_0/f(\omega_{01})$, *then, for any* $\varepsilon \in (0,\varepsilon_0)$,

$$(L^P(\omega_0),L^P(\omega_1))_{f,q} = \begin{cases} \bigcap_{\psi\in Q_\varepsilon} L^P(\omega(\psi\circ\omega_{01})^{\gamma}) & \text{if } q > p, \\ \bigcup_{\psi\in Q_\varepsilon} L^P(\omega(\psi\circ\omega_{01})^{\gamma}) & \text{if } q < p. \end{cases}$$

A proof of Lemma 2.2 can be found in [9,p.769]. See also [15].

Theorem 2.2. *Let* $f, f_0, f_1 \in B_0$, $f_{10} = f_1/f_0$, $f_2 = f_1 f(f_{10})$, $0 < p,q \le \infty$, $\gamma = 1/q-1/p$, $\gamma \neq 0$ *and* $\varepsilon_0 = \min(\alpha_f, 1-\beta_f)/|\gamma|$. *Then, for any* ε, $0 < \varepsilon < \varepsilon_0$,

$$(\overline{A}_{f_0,p},\overline{A}_{f_1,p})_{f,q} = \begin{cases} \bigcap_{\psi\in Q_\varepsilon} \overline{A}_{f_{2,\psi},p} & \text{if } q > p, \\ \bigcup_{\psi\in Q_\varepsilon} \overline{A}_{f_{2,\psi},p} & \text{if } q < p. \end{cases}$$

where $f_{2,\psi} = f_2(\psi\circ f_{10})^{\gamma}$.

Proof. We use (2.1) together with Lemma 2.2 where $\Omega = (0,\infty)$, $d\mu = dt/t$, $\omega_i = 1/f_i$, $i = 0,1$, and the proof follows.

Remark. If $A_0 = L(\Omega)$ and $A_1 = L^{\infty}(\Omega)$, then

$$K(t,a,A_0,A_1) = \int_c^t a^*(u)du, \quad 0 < t < \infty, \tag{2.4}$$

see [11] or [3,p.109]. We conclude that in this case the spaces $(A_0,A_1)_{f,q}$ can be identified with generalized Lorentz spaces of the type $L^{\varphi q}$. Therefore, by combining Theorem 2.1(c) with Theorem 2.2 we obtain a new proof of Theorem 3.1 in [13].

We close this section by stating a description of the spaces (0.1) also for the most complicated off-diagonal case $q_0 \neq q_1$, $q_\theta \neq q$.

Theorem 2.3. *Let $f, f_0, f_1 \in B_0$, $0 < q, q_0, q_1 \le \infty$, $q_\theta \neq q$, $q_0 \neq q_1$,*

$$f_2 = f_0^{1/(q_1/q_0-1)} f_1^{1/(q_0/q_1-1)}, \quad f_3 = (f_1/f_0)^{1/(1/q_0-1/q_1)}$$

and $\varphi(t) = t^{1/q_0}/f(t^{1/q_0-1/q_1})$. Then

$$(\bar{A}_{f_0,q_0}, \bar{A}_{f_1,q_1})_{f,q} = \bar{A}_E,$$

where $E = L^{\varphi q}(f_2, f_3 dt/t)$.

Proof. The proof is similar to those of Theorems 2.1 and 2.2. In this case we use Theorem 4 in [9] (with $\omega_i = 1/f_i$, $i = 0,1$, and $d\mu = dt/t$) instead of (2.1)- Lemma 2.1 and Lemma 2.2, respectively.

3. A sharp embedding between the spaces $\bar{A}_{(\theta,b),q}$.

First of all we point out the following obvious embeddings:

$$\bar{A}_{(\theta,b),q_0} \subset \bar{A}_{(\theta,b),q_1} \quad \text{if } q_0 \le q_1 \tag{3.1}$$

and

$$\bar{A}_{(\theta,b_0),q} \subset \bar{A}_{(\theta,b_1),q}, \quad \text{if } b_0 \le b_1.$$

The next theorem may be regarded as a complement of (3.1)-(3.2) which gives us precise information about the importance of the quantity $1/q-b$ in the definition of the spaces $A_{(\theta,b),q}$.

Theorem 3.1. *Let $0 < q_1 < q_0 \le \infty$ and $-\infty < b_0, b_1 < \infty$. If $1/q_0-b_0 > 1/q_1-b_1$, then*

$$\bar{A}_{(\theta,b_0),q_0} \subset \bar{A}_{(\theta,b_1),q_1}. \tag{3.3}$$

Proof. We choose a and θ_0 such that $1-\theta < a < 1$ and $\theta-1+a < \theta_0 a < \min(\theta,a)$. Let $f(t) = t^{\theta_0}$, $f_0(t) = t^{\theta-\theta_0 a}(1+|\log t|)^{b_0}$ and $f_1(t) = t^{\theta+(1-\theta_0 a)}(1+|\log t|)^{b_0}$ Then $f_{10}(t) = f_1(t)/f_0(t) = t^a$ and $f_2(t) = f_0(t)f(f_{10}(t)) = t^{\theta}(1+|\log t|)^{b_0}$. Therefore, according to Theorem 2.1(c) and Theorem 2.2, we find that the assumption $q_0 > q_1$ implies that, for any fixed ε, $0 < \varepsilon < \varepsilon_0$,

(3.4)
$$\overline{A}_{(\theta,b_o),q_o} = \bigcap_{\psi\in Q_\varepsilon} \overline{A}_{f_{z,\psi},q_1},$$

where $f_{z,\psi} = \psi^{\gamma}(t^{a})t^{\theta}(1+|\log t|)^{b_o}$; $\gamma = 1/q_o-1/q_1$. We assume that $a \in \overline{A}_{(\theta,b_o),q_o}$ and choose ε such that $0 < \varepsilon < \varepsilon_o$ and $\psi \in Q_\varepsilon$ such that $\psi(t) \approx (1+|\log t|)^{-1-\varepsilon}$. Then, by (3.4), we find that $a \in A_{(\theta,b),q_1}$ for

$$b = b_o-(1+\varepsilon)(1/q_o-1/q_1) > b_o+1/q_1-1/q_o.$$

Thus (3.3) holds for every b_1 satisfying

$$b_o+1/q_1-1/q_o < b_1 \le b_o+(1+\varepsilon_o)(1/q_1-1/q_o).$$

Hence, in view of (3.2) we conclude that (3.3) holds for every $b_1 > b_o+1/q_1-1/q_o$ and the proof is complete.

Next we prove a statement showing that Theorem 3.1 is a way best possible.

PROPOSITION 3.2. *Let $0 < q_1 < q_o \le \infty$ and $-\infty < b_o, b_1 < \infty$. If $1/q_o-b_o = 1/q_1-b_1$, then (3.3) does not hold in general.*

Proof. Let $A_o = L^r(\Omega)$, $0 < r < \infty$, and $A_1 = L^{\infty}(\Omega)$. Then

$$K(t,a,A_o,A_1) \approx \left(\int_0^{t^r} (a^*(u))^r du\right)^{1/r},$$

see [3,p.109]. Therefore, according to a suitable variant of Hardy's inequality (see e.g. Lemma 3.2 in [14] applied with $\psi(t) = t^{1/r}$ and $f(t) = 1/t^{\theta}(1+|\log t|)^{b}$, we obtain that

$$\|a\|_{\overline{A}_{(\theta,b),q}} \le C \left(\int_0^{\infty} (t^{-\theta}(1+|\log t|)^{-b})^q \left(\int_0^{t^r} (a^*(u))^r du\right)^{q/r} \frac{dt}{t}\right)^{1/q}$$

$$\le C \left(\int_0^{\infty} (t^{-\theta/r}(1+|\log t|)^{-b})^q \left(\int_0^{t} (a^*(u))^r du\right)^{q/r} \frac{dt}{t}\right)^{1/q}$$

$$\le C \left(\int_0^{\infty} (t^{(1-\theta)/r}(1+|\log t|)^{-b})^q (a^*(t))^q \frac{dt}{t}\right)^{1/q}$$

Since this inequality trivially holds in the opposite direction (with another constant) we have that

(3.5)
$$\overline{A}_{(\theta,b),q} = L^{p,q}(\log L)^{-b}, \quad p = r/(1-\theta).$$

Let $q_o > 0$ and consider the function

$$f(t) = \begin{cases} t^{-1/p}\left(1+\log \frac{1}{t}\right)^{b_1-\frac{1}{q_1}}\left(1+\log \log \frac{e}{t}\right)^{-a}, & 0 < t \le 1, \\ 0, & t > 1. \end{cases}$$

where a is a real number to be chosen later on. According to (3.5) and the assumption $b_1 = b_o+1/q_1-1/q_o$ we find that

(3.6)
$$\|a\|_{\bar{A}_{(\theta,b_1),q_1}} = \|a\|_{L^{p,q_1}(\log L)^{-b_1}} = \int_o^1 \frac{1}{t(1+\log 1/t)(1+\log\log 1/t)^{aq_1}}\, dt$$

and

(3.7)
$$\|a\|_{\bar{A}_{(\theta,b_o),q_o}} = \|a\|_{L^{p,q_o}(\log L)^{-b_o}} = \int_o^1 \frac{1}{t(1+\log 1/t)(1+\log\log 1/t)^{aq_o}}\, dt.$$

We choose a satisfying $1/q_o < a \le 1/q_1$ and use (3.6)–(3.7) to obtain that

(3.8)
$$a \in \bar{A}_{(\theta,b_o),q_o} \quad \text{but} \quad a \notin \bar{A}_{(\theta,b_1),q_1}.$$

We only need to modify the arguments used above to see that (3.8) holds for the case $q_o = \infty$ too. The proof is complete.

Corollary 3.3. *Let $0 < p < \infty$, $0 < q_1 < q_o \le \infty$ and $-\infty < b_o, b_1 < \infty$.*
a) If $1/q_o+b_o > 1/q_1+b_1$, then

(3.9)
$$L^{p,q_1}(\log L)^{b_1} \supset L^{p,q_o}(\log L)^{b_o}$$

b) If $1/q_o+b_o = 1/q_1+b_1$, then the inclusion (3.9) does not hold in general.

Proof. We choose r, $0 < r < \infty$, and θ, $0 < \theta < 1$, such that $p = r/(1-\theta)$. Let $A_o = L^r(\Omega)$, $0 < r < \infty$, and $A_1 = L^{\infty}(\Omega)$. Then, in view of (3.5), we find that (3.3) implies (3.9). The statement in b) was proved in Proposition 3.2.

Remark. Another proof of (3.9) can be found in [1,p.31]. See also [13].

Remark. As seen in this paper it is difficult to describe the spaces

(3.10)
$$(\bar{A}_{\theta_o,q_o}, \bar{A}_{\theta_o,q_1})_{\theta,q}, \quad q_o \ne q_1,$$

in off-diagonal cases $q \ne q_\theta$. These difficulties appear usually as well in special cases e.g. when we are concerned with scales of Besov, Lorentz, weighted L^p or operator ideal spaces, etc. (see e.g. [9],[13],[15],[16] and the references given in these papers). One possibility to avoid this type of troubles can be to replace the scale (3.10) by the scale

$$(3.11) \qquad (\bar{A}_{\theta_\alpha,q_0}, \bar{A}_{\theta_\alpha,q_1})_{(\theta,b),q_\theta}, \quad b = \frac{1}{q} - \frac{1}{q_\theta}.$$

On the other hand, according to Theorem 3.1 (and Proposition 3.2), we see that the scales in (3.10) and (3.11) are very closed related and on the other hand we note that our Theorem 2.1(b) gives a fairly uncomplicated description of the reiteration spaces (3.11).

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The general reiteration result for the K-method of interpolation has the form $K_\Phi(K_{\Phi_0}, K_{\Phi_1}) = K_\Psi$, where Φ, Φ_0, Φ_1 and Ψ are some Banach function lattices and Ψ is constructed from Φ_0, Φ_1 and Φ. The authors calculate Ψ under the condition that Φ, Φ_0 and Φ_1 have the form L_p with a weight and under some additional constraints. They also establish sharp embeddings $K_{\Phi_0} \hookrightarrow K_{\Phi_1}$ in the case when Φ_0 and Φ_1 are L_p spaces, with a weight of the form $t^\theta(1+|\log t|)^b$. The latter results generalize the well-known embeddings for Lorentz-Zygmund spaces.

{For the entire collection see MR 92m:46004.}

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Function spaces, Proc. 2nd Int. Conf., Poznań/Pol. 1989, Teubner-Texte Math. 120, 238-247 (1991).

[For the entire collection see Zbl. 731.00012.]

For the real interpolation spaces $(A_0, A_1)_{f,q}$ with a parameter function f we prove a general reiteration result, where we need not assume some separation condition between the corresponding parameter functions. As one application we obtain a sharp embedding result between the spaces $(A_0, A_1)_{(\theta,b),q}$ obtained by using the function parameter $f(t) = t^\theta(1+|\log t|)^b$. This result may be regarded as a generalization of some well-known embeddings between Lorentz-Zygmund spaces. *Summary.*