# A Generalized Sidelobe Canceller Formulation for Multi-Rank Capon Beamforming*

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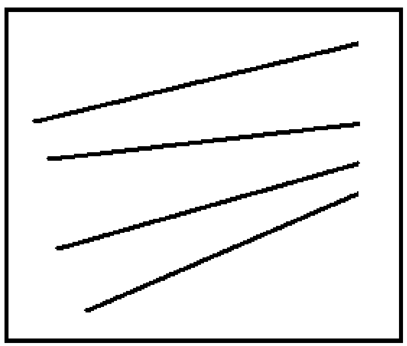
# Outline

1. Wavefront Models & Corresponding Subspace Signal Models
2. A Quick Review of Matched Subspace & Matched Direction Beamformers
3. The GSC for Linearly- & Quadratically-Constrained Capon Beamformers: The Multi-Rank Case
4. Preliminary Findings

# 1. Wavefront Models & Corresponding Subspace Signal Models

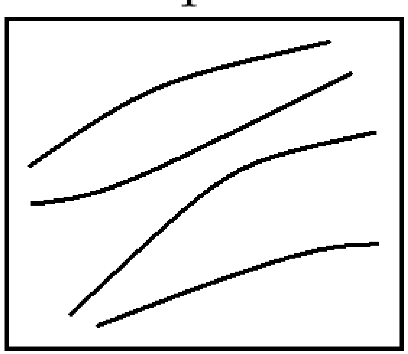
## 1.1 The ergodic case

Plane



$\underline{\psi} = [1, e^{j\theta}, \ldots, e^{j(L-1)\theta}]^T$

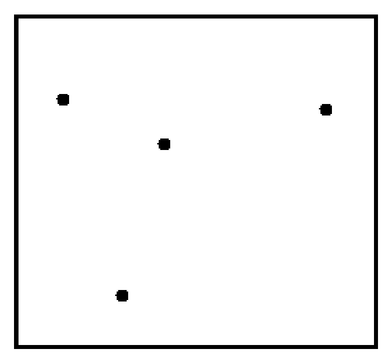
Slepian



$H\Lambda\underline{b} = \sum_1^r \underline{h}_i \lambda_i b_i$

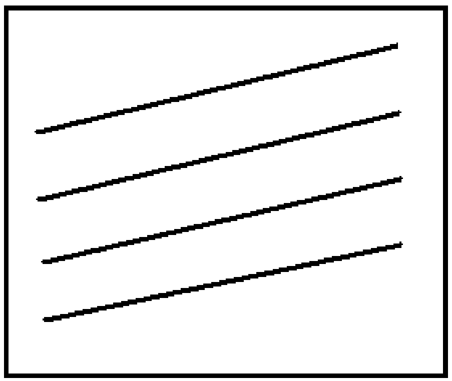
Slepian: $$R_{ss} = \int_{-\pi\beta}^{\pi\beta} (\underline{\psi}\,\underline{\psi}^\star)(e^{j\theta}) \frac{d\theta}{2\pi\beta} \cong H\Lambda^2 H^\star$$

(Randomly generated planewaves from a manifold are indistinguishable from randomly generated Slepians from a subspace . . . from second-order statistics.)

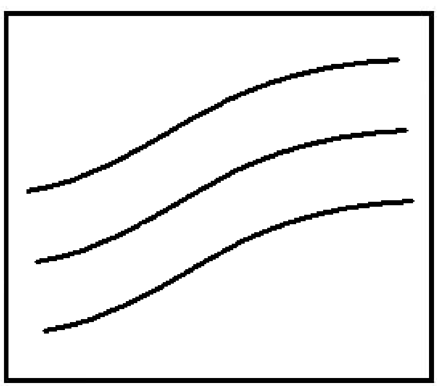


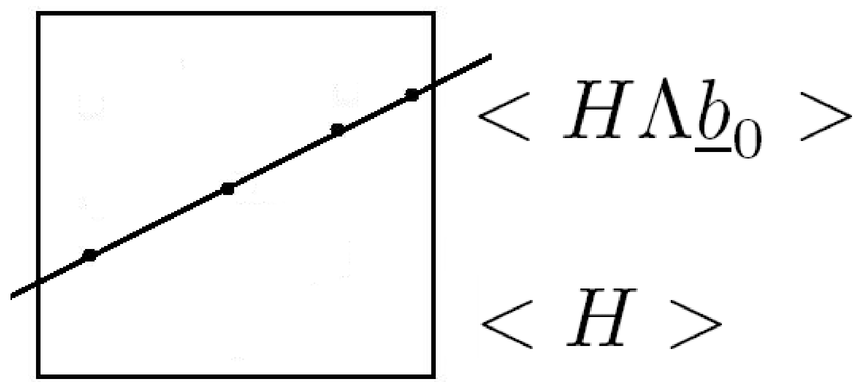
$< H >$

## 1.2 The non-ergodic Case



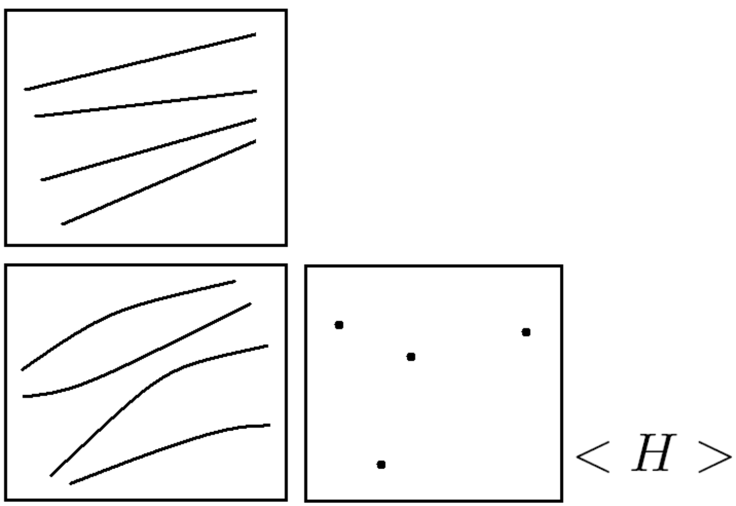
$R_{ss} = \underline{\psi}\,\underline{\psi}^\star(e^{j\theta})$ vs $H\Lambda\underline{b}_0\,\underline{b}_0^\star\Lambda H^\star$ distinguishable, but the Slepian model is not identifiable from second-order statistics.

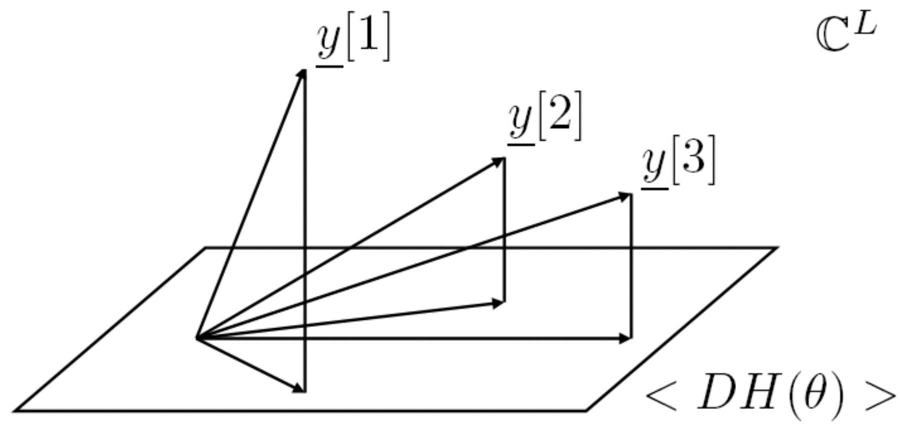




# 2. A Quick Review of Matched Subspace & Matched Direction Beamformers

## 2.1 Ergodic Case (Matched Subspace)

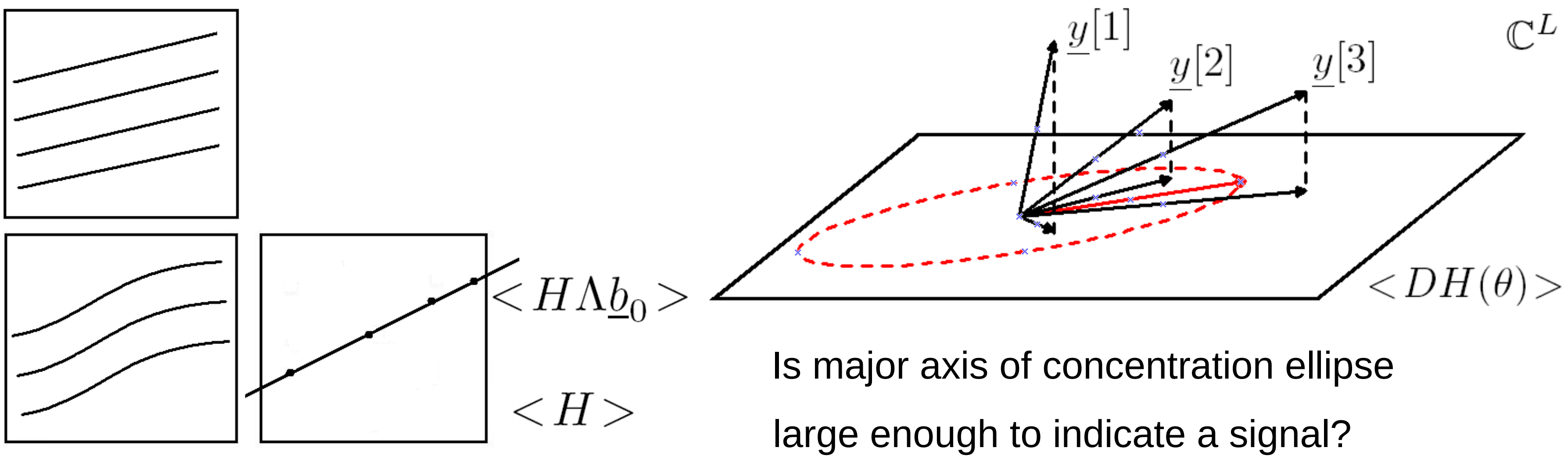



Are summed energies large enough to indicate a signal?

$$
\begin{aligned}
B(\theta) &= \mathrm{tr}\, Y^\star D(e^{j\theta}) P_H D^\star(e^{j\theta}) Y && : \quad \text{unsteer data/steer projection} \\
&= \mathrm{tr}\, Y^\star P_{D(e^{j\theta})H} Y && : \quad \text{steer subspace} \\
&= \textstyle\sum_1^M \sum_1^r |(D(e^{j\theta})\underline{h}_i)^\star \underline{y}[m]|^2 && : \quad \text{BF \& Div}
\end{aligned}
$$

## 2.2 Nonergodic Case (Matched Direction-Besson)




Is major axis of concentration ellipse large enough to indicate a signal?

$$B(\theta) = \text{max ev}(Y^\star D(e^{j\theta}) P_H D^\star(e^{j\theta}) Y)$$

: learns the line $H\Lambda \underline{b}_0$ from snapshots and uses it to match

# 3. The GSC for Linearly- & Quadratically-Constrained Capon Beamformers– The Multi-Rank Case, $W \in \mathbb{C}^{L \times r}$

3.1 Linear (Cox 1972/73)

$$\min_{W} \text{ tr } W^\star R W \quad \text{u.c.} \quad W^\star \Psi \Lambda = L^\star Q^\star$$

3.2 Quadratic (ASAP 2005)

$$\min_{W} \text{ tr } W^\star R W \quad \text{u.c.} \quad W^\star \Psi \Lambda^2 \Psi^\star W = D$$

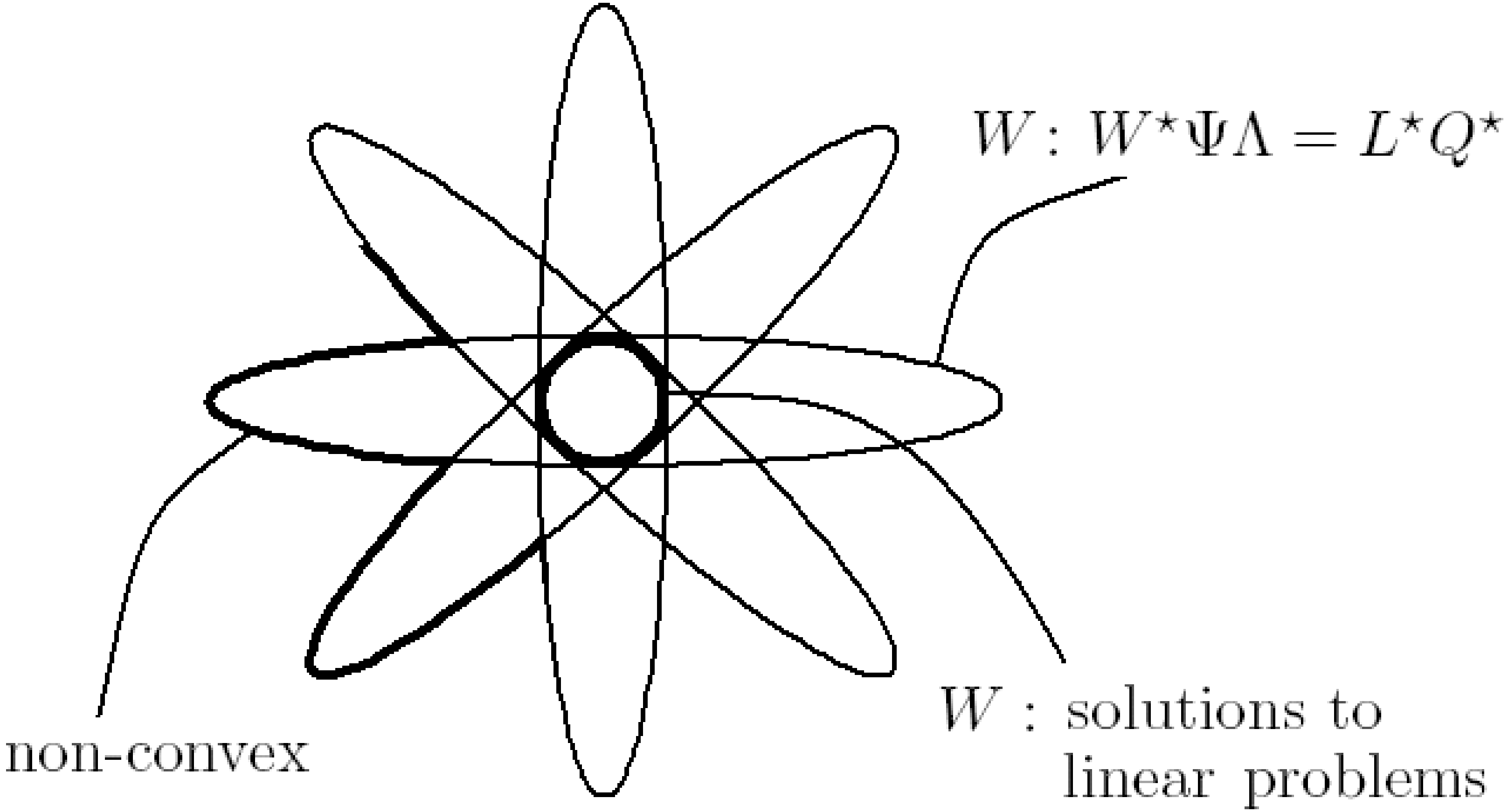
3.3 Connections & Comments

(a) If $L^\star L = D$, then solution to linear problem meets constraints of quadratic problem for every choice of unitary $Q$ ($Q^\star Q = I$).

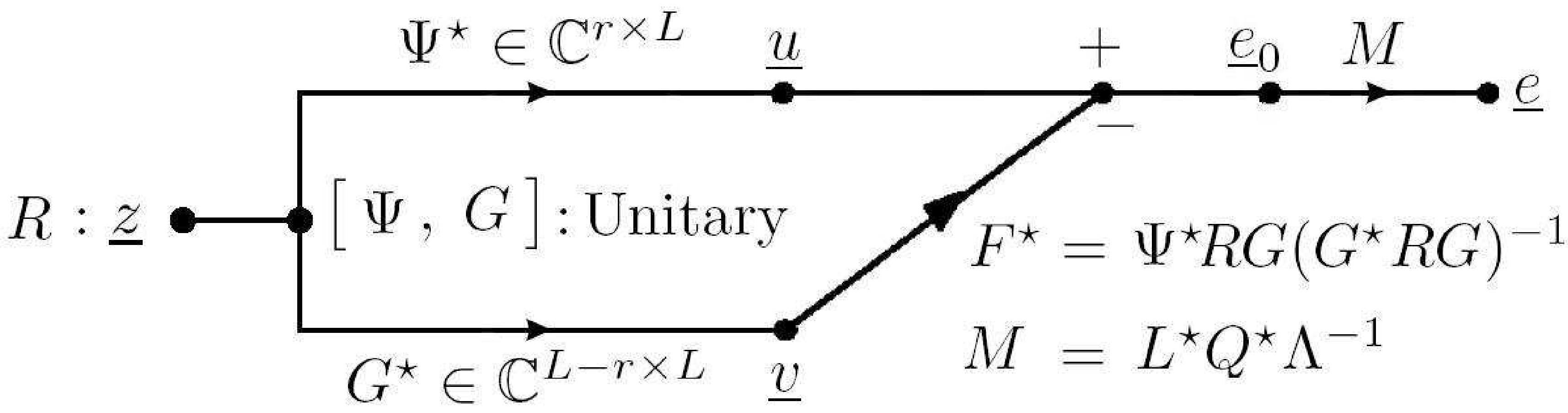
(b) Linear problem is convex (Lagrange). Quadratic problem is non-convex (majorization)

(c) The set of solutions to linear problem is a convex set of candidates for the quadratic problem.

## 3.4 Geometry


$W : W^\star \Psi \Lambda = L^\star Q^\star$
$W$ : solutions to linear problems
non-convex

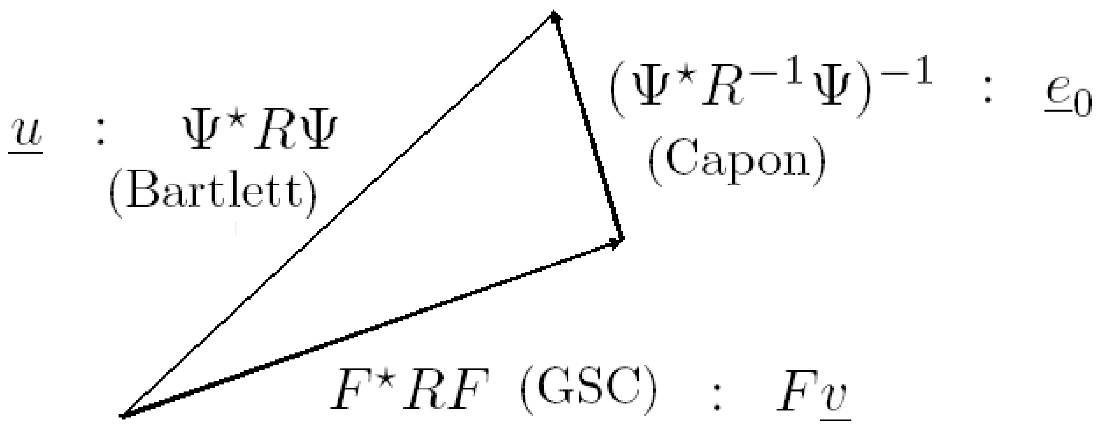
## 3.5 Solutions & GSC & Geometry




$$W_0^\star = \Psi^\star(I - RG(G^\star RG)^{-1}G^\star) = \Psi^\star(I-K): \text{ oblique}$$

$$P_0 = E\underline{e}_0\,\underline{e}_0^\star = (\Psi^\star R^{-1}\Psi)^{-1} = \Psi^\star R\Psi - F^\star RF$$

# 3.5 Solutions & GSC & Geometry (Continued)




This is Pythagorean decomposition of multi-rank Bartlett into multi-rank Capon plus multi-rank GSC.

$$(\Psi^\star R^{-1} \Psi)^{-1} \leq \Psi^\star R \Psi \quad \text{(Schwartz)}$$

Consideration of the term $M = L^\star Q^\star \Lambda^{-1}$ pre- and post-multiplies each term by $M$ and $M^\star$.

3.6 Question: $Q$ is a free unitary matrix. Is there a choice that minimizes output power

$$P = \text{tr}\, MP_0M^\star; \quad M = L^\star Q^\star \Lambda^{-1}$$

The answer is yes- it amounts to searching the circle in the atomic picture and the answer is

$$Q^\star \Lambda(\Psi^\star R^{-1}\Psi)\Lambda Q = \text{diag}(\mu_i)$$

$P = \text{tr}\, MP_0M^\star = \sum_1^r \frac{d_i}{\mu_i}$ (Multi-rank Capon)

$D = \text{diag}(d_i)$ & $\mu_i = ev_i(\Lambda(\Psi^\star R^{-1}\Psi)\Lambda) = ev_i(R_{ss}^{H/2} R^{-1} R_{ss}^{1/2})$

$\Lambda$ and $\Psi$ come from signal subspace *model* $R_{ss} = \Psi\Lambda^2\Psi^\star$!

## 4. Preliminary Findings

Design parameters are $R_{ss}\ \ (\Psi, \Lambda)$ & $D$. For example, if $d_i = \frac{\mu_i}{\sum_1^r \mu_i}$, then

$$P(\theta) = \frac{1}{\frac{1}{r}\sum_1^r ev_i(R_{ss}^{H/2}(\theta)R^{-1}R_{ss}^{1/2}(\theta))}$$

A design model for $R_{ss}$ is

$$\begin{aligned} R_{ss} \; &= \epsilon \underline{1}\,\underline{1}^\star + (1-\epsilon) \int_{-\pi\beta}^{\pi\beta} (\underline{\psi}\ \underline{\psi}^\star)(e^{j\theta}) \frac{d\theta}{2\pi\beta} \\ &\cong \epsilon \underline{1}\,\underline{1}^\star + (1-\epsilon)\Psi\Lambda^2\Psi^\star \end{aligned}$$

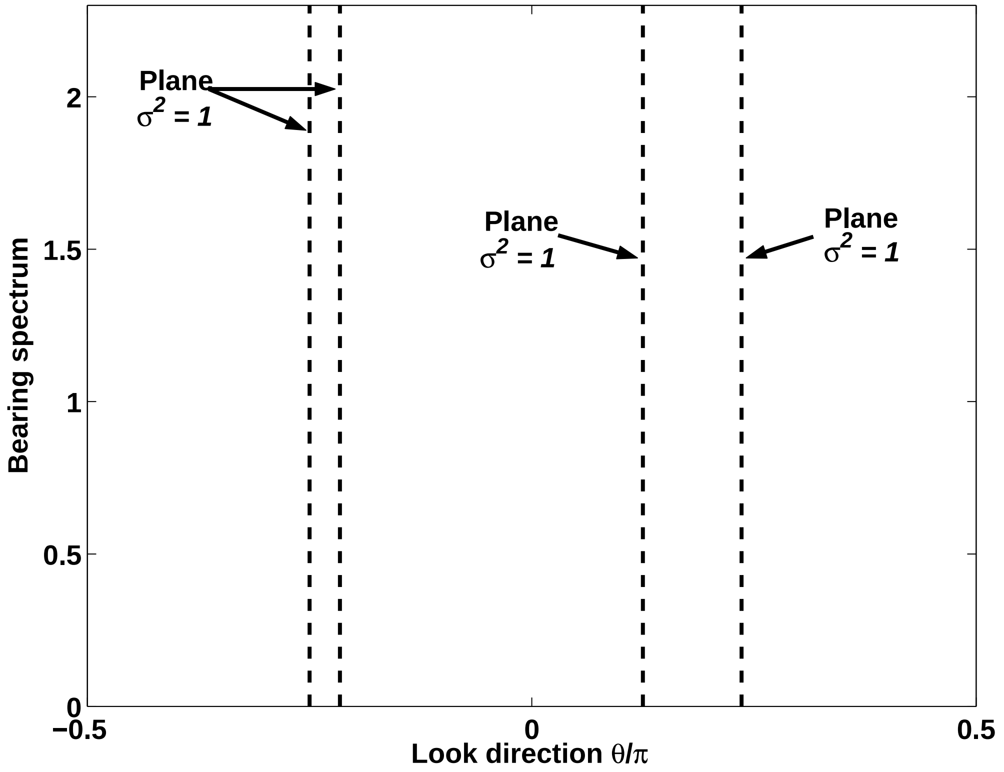
where $0 \leq \epsilon \leq 1$ determines our confidence in the planewave and Slepian models.

Pre- and post-multiplication of $R_{ss}$ by $D(e^{j\theta})$ and $D^\star(e^{j\theta})$ steers $R_{ss}$ to angle $\theta$, to get $R_{ss}(\theta)$.

# Example 1: Uncorrelated Planewaves

20-element ULA. Four uncorrelated planewaves with equal powers ($\sigma^2 = 1$). Two of them (LHS) within Rayleigh limit. Input SNR is $0$ dB.

$R_{ss}$ is rank-4 (rank-1 plane + rank-3 Slepian); $\epsilon = 0.75$; $\beta = 0.1$.



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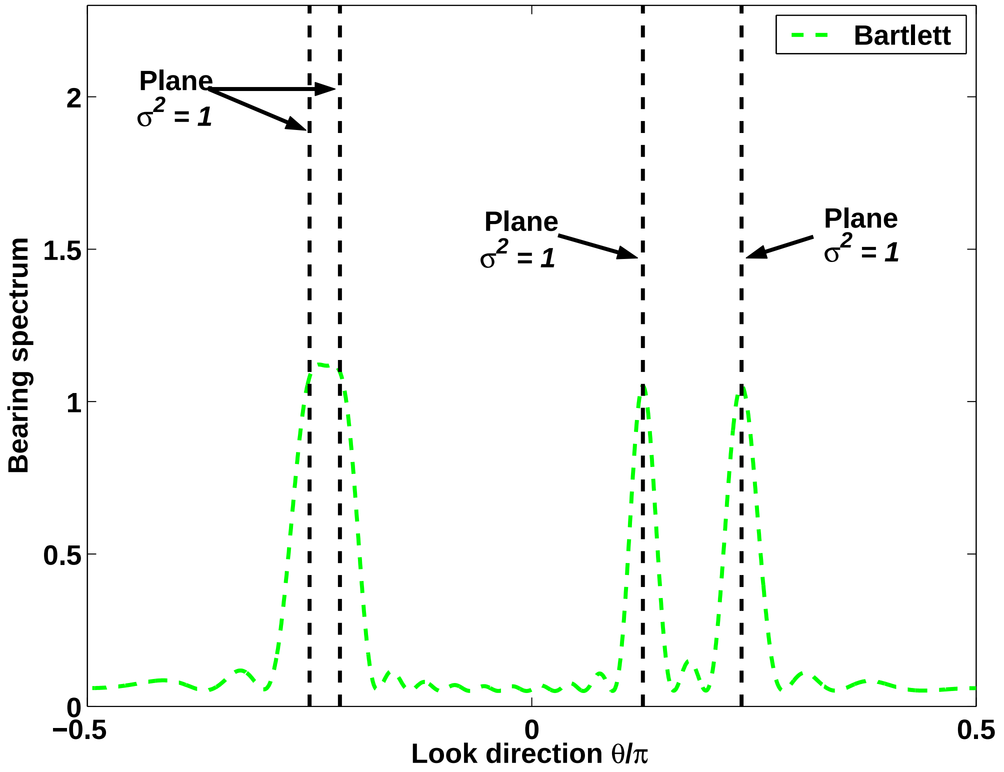

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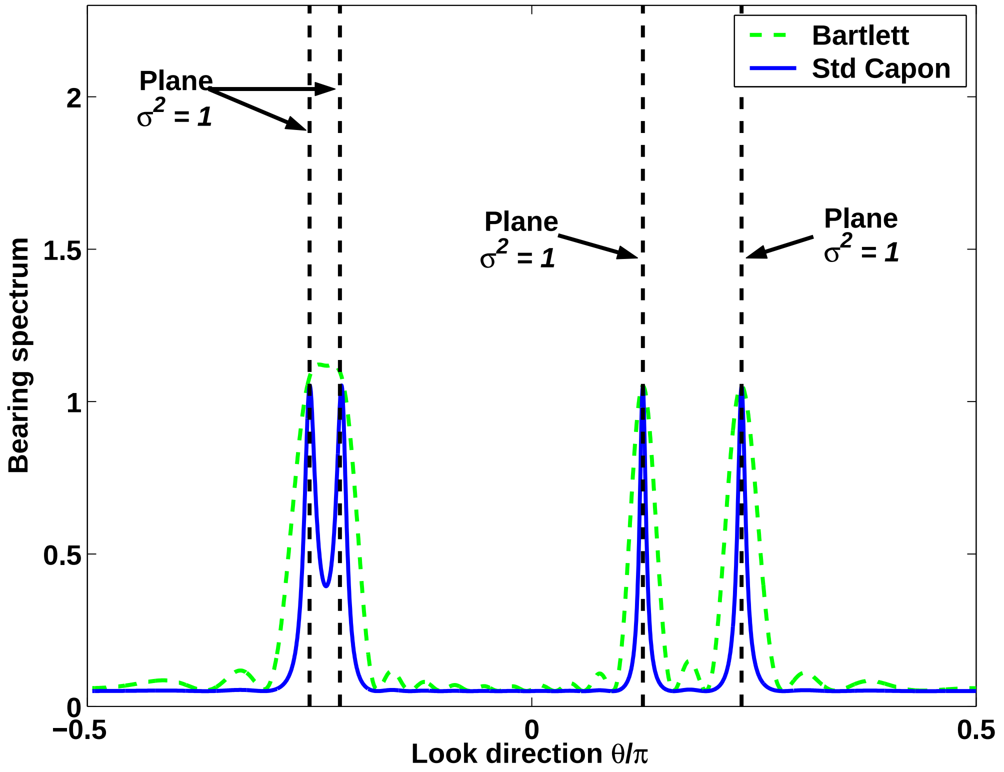

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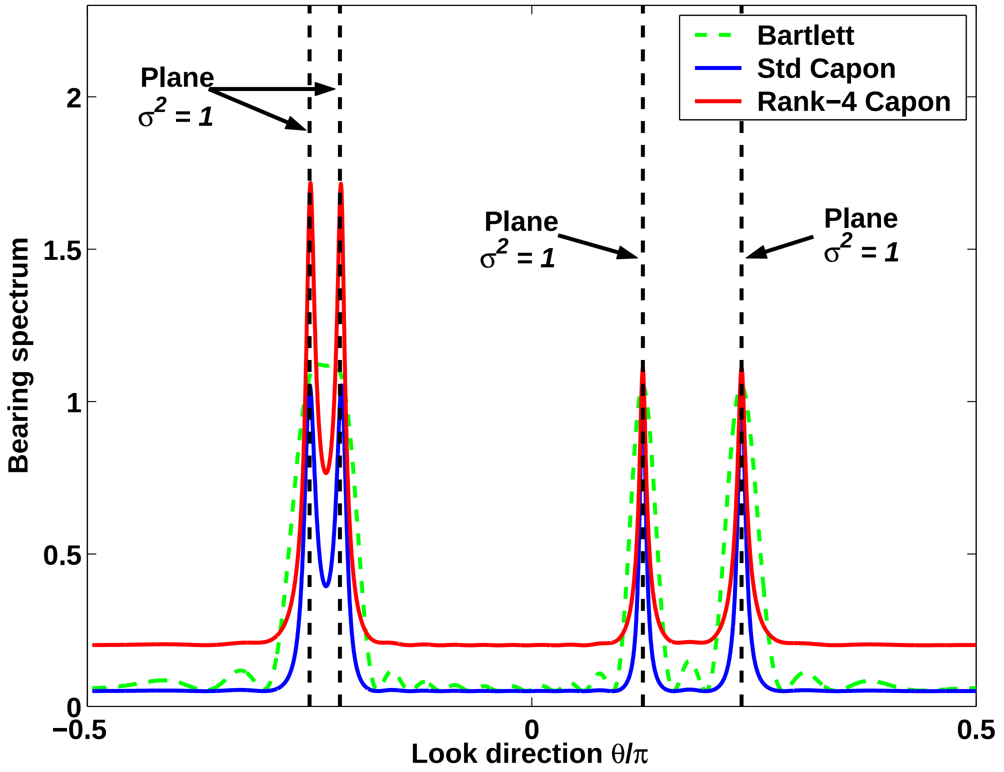
Bartlett, standard Capon, and rank-4 Capon all detect.

Standard Capon and rank-4 Capon resolve the closely-spaced planewaves.



# Example 2: Correlated Planewaves; Uncorrelated Planewaves

Two planewaves on the LHS are now correlated ($\rho = 0.95$), but have equal powers ($\sigma^2 = 1$). The RHS planewaves are uncorrelated, but one is much stronger than the other ($\sigma^2 = 20$ vs $\sigma^2 = 0.2$). Input SNR is $0$ dB for sources with unit power.

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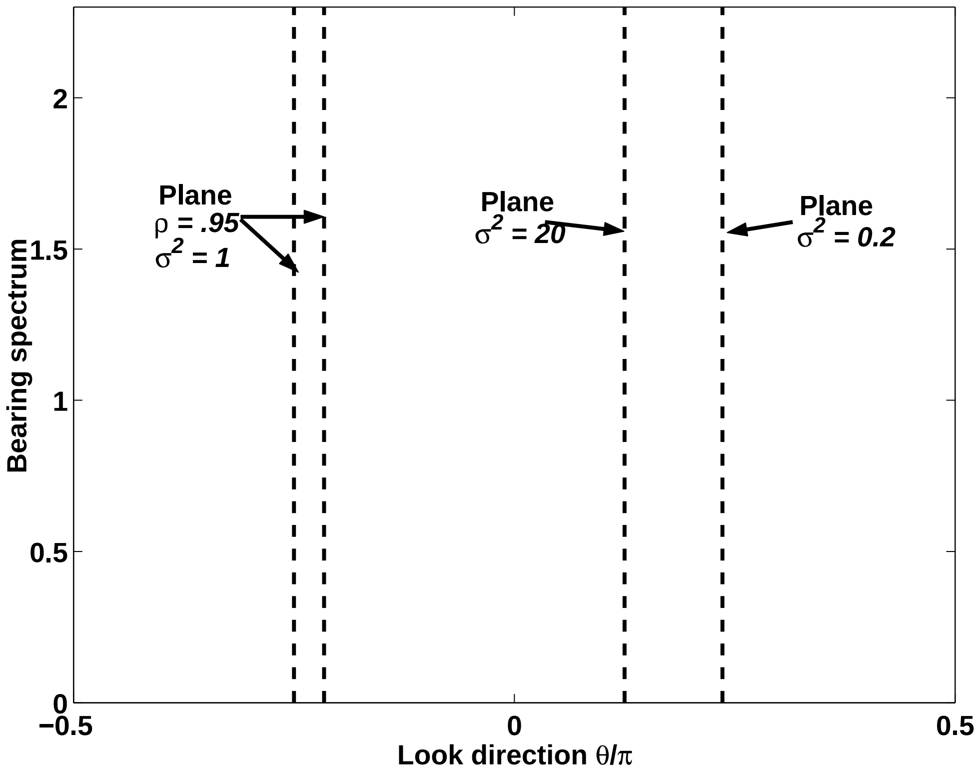

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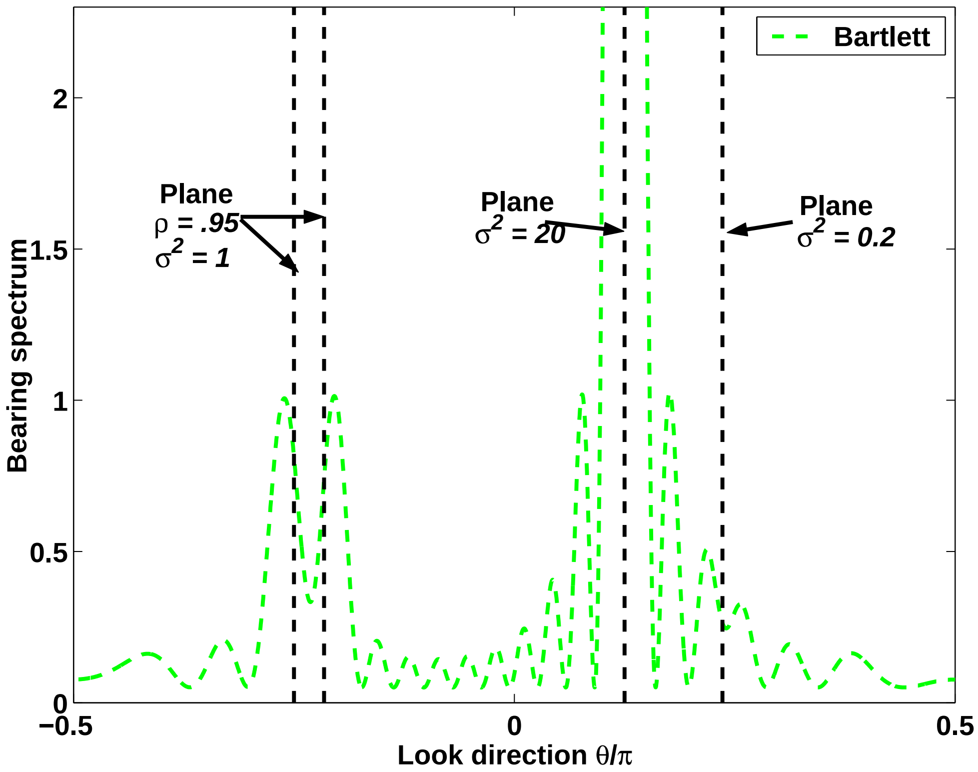
Bartlett does not detect the weak source.



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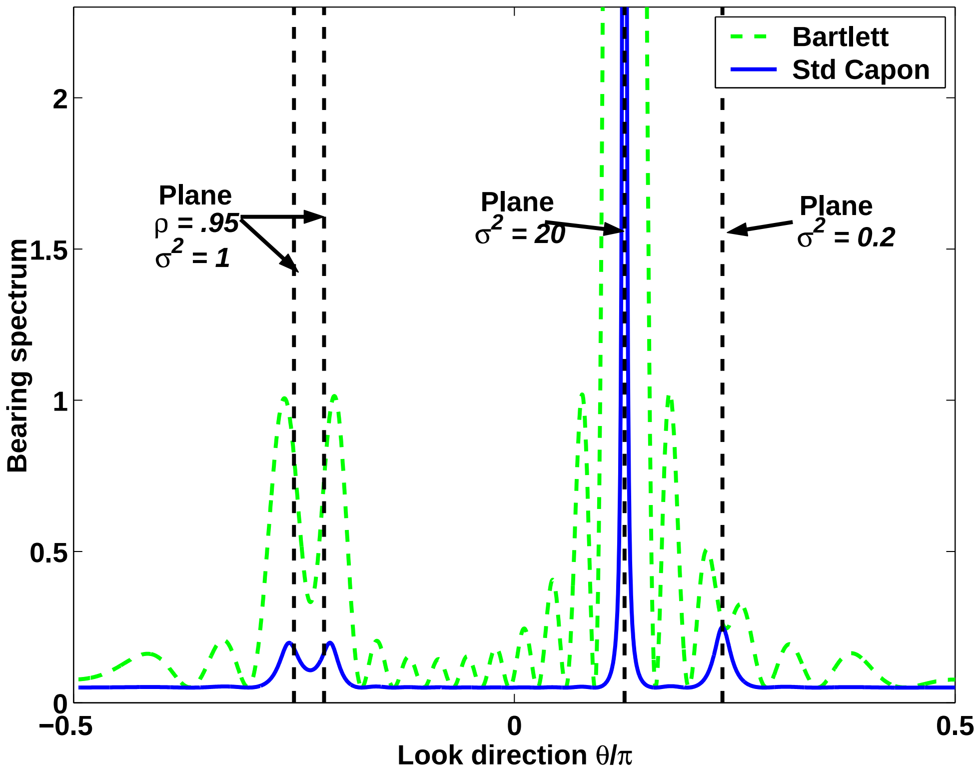
Standard Capon has low detectibility for correlated sources, and peaks slightly off the correct locations; Detects the weak planewave.



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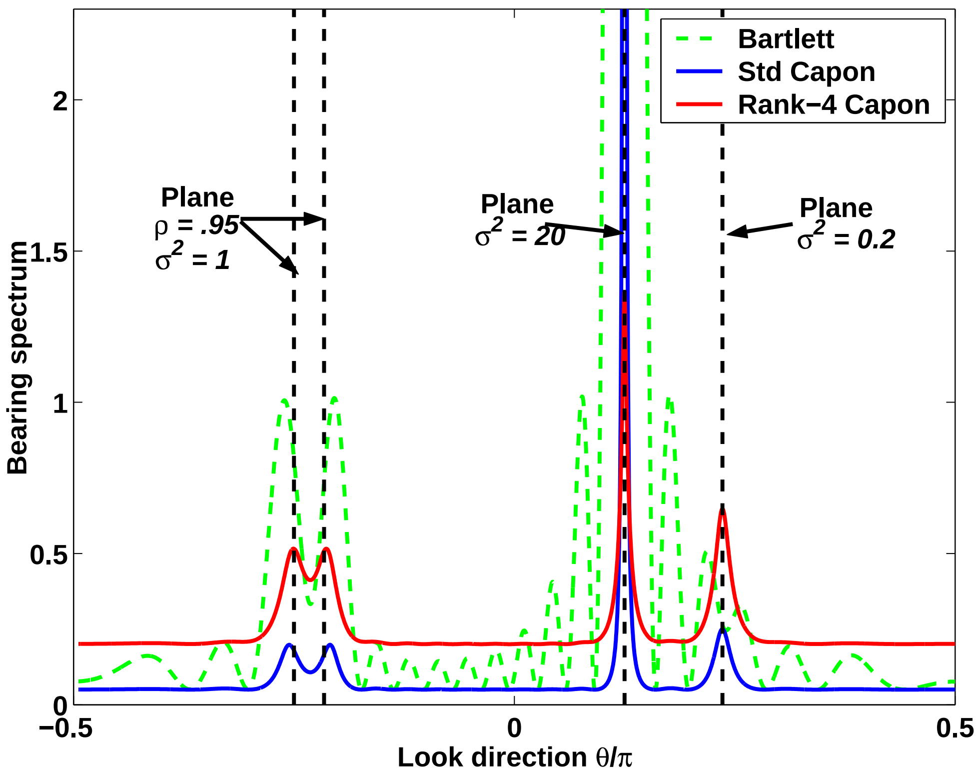
Standard Capon has low detectibility for correlated sources, and peaks slightly off the correct locations; Detects the weak planewave.

Rank-4 Capon has low detectibility for correlated sources, but resolves them at correct locations; Detects the weak planewave.



# Example 3: Slepian Waves

Two Slepian waves ($\beta = 0.1$, approx rank-3), with equal powers ($\sigma^2 = 1$), within Rayleigh limit (LHS sources). One strong planewave ($\sigma^2 = 10$) followed by one weak Slepian wave ($\sigma^2 = 0.1$) (RHS sources). Input SNR is $0$ dB for sources with unit power.

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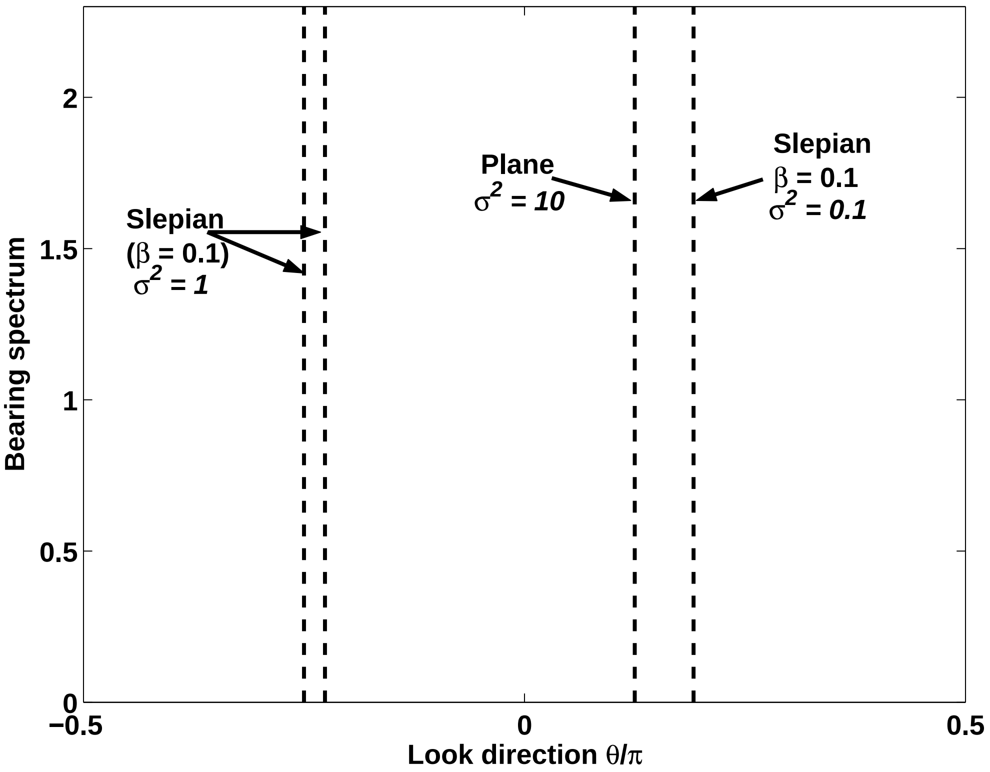

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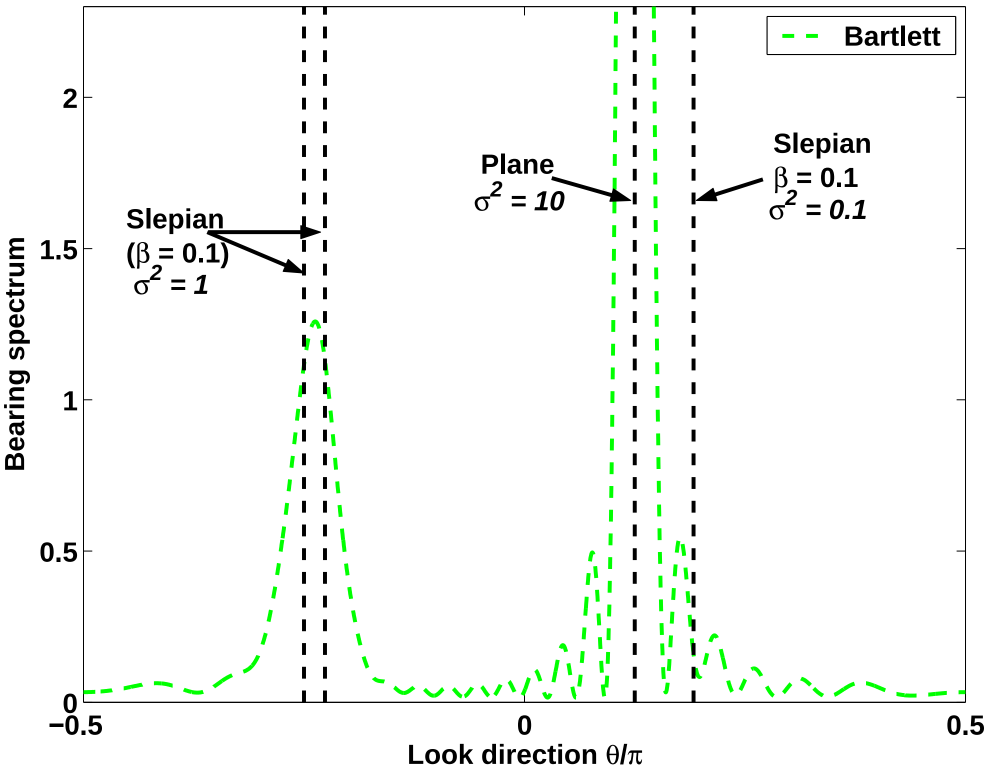

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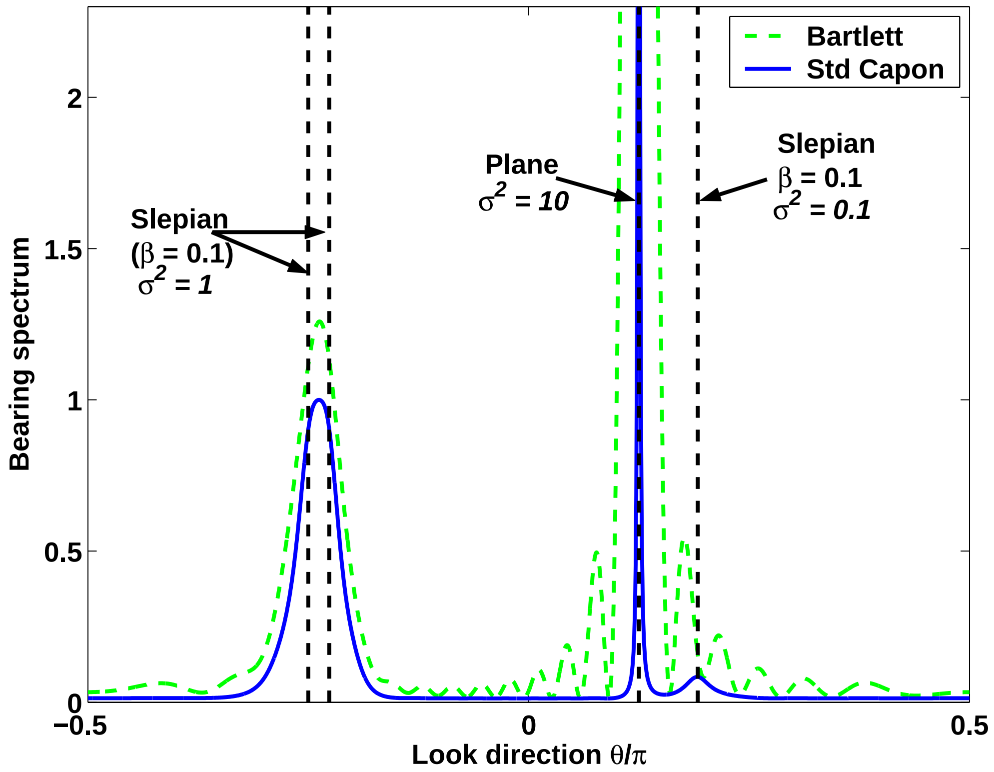
Standard Capon does not resolve the closely-spaced Slepians.



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Rank-4 Capon resolves the two Slepians, and the strong planewave from the weak Slepian.