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Interpolation of Some Concrete Symmetric Spaces

by

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A natural problem in interpolation theory is to find necessary and sufficient conditions for interpolation properties to hold among spaces of the same type. A necessary condition in terms of the fundamental functions of the spaces involved was studied by e.g. G. G. Lorentz and T. Shimogaki [7], E. I. Pustyl'nik [13], R. Sharpley [16]. In this paper we discuss examples of symmetric spaces for which this condition is also sufficient for the interpolation property.

1. Notations and Basic Lemmas

A pair $\bar{A} = (A_0, A_1)$ of Banach spaces is called a *Banach couple* if both A_0 and A_1 are continuously embedded into some Hausdorff topological vector space. We denote by $\Delta(\bar{A}) := A_0 \cap A_1$ and $\sum(\bar{A}) := A_0 + A_1$ with the natural norms. A Banach space A is called *intermediate* with respect to $\bar{A}$ if $\Delta(\bar{A}) \subset A \subset \sum(\bar{A})$ with continuous inclusions. If $\bar{A} = (A_0,A_1)$ and $\bar{B} = (B_0,B_1)$ are two Banach couples, then let $L(\bar{A},\bar{B})$ denote the Banach space of all linear operators $T: \sum(\bar{A}) \to \sum(\bar{B})$ such that the restrictions of T to the spaces A_i are bounded operators from A_i into B_i (i=0,1) with the norm

$$||T||_{L(\bar{A},\bar{B})} := \max \{ ||T||_{[A_0,B_0]} , ||T||_{[A_1,B_1]} \}.$$

If A and B are Banach spaces intermediate with respect to $\bar{A}$ and $\bar{B}$, respectively, then we say that (A,B) is an *interpolation couple* with respect to $\bar{A}$ and $\bar{B}$

if every operator $T \in L(\bar{A},\bar{B})$ is a bounded operator from A into B.

Remark 1. We may define the interpolation couple (A,B) in an analogous way for any subspace $A \neq \{0\}$ of $\sum(\bar{A})$, even if $\Delta(\bar{A}) \not\subset A$, and for any space $B \supset \Delta(\bar{B})$, even if $B \not\subset \sum(\bar{B})$. In this case we say that (A,B) is a *g-interpolation couple* (generalized) with respect to $\bar{A}$ and $\bar{B}$. Finally, let P denote the set of all functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi(s) \leq \max\{1,s/t\}\,\varphi(t)$ for all $s,t \in \mathbb{R}_+$, and $\tilde{P} := \{\varphi \in P;\ \varphi \text{ is concave}\}$. On P we define an involution by $\bar{\varphi}(t) := 1/\varphi(1/t)$. The following Lemma is easily verified:

Lemma 1. *Let* $\varphi_0, \varphi_1, \varphi \in P$ *and* $\psi(t) := \varphi_0(t)\,\varphi(\varphi_1(t)/\varphi_0(t))$ *for* $t \in \mathbb{R}_+$. *Then* $\psi \in P$; *moreover*, $\psi \in \tilde{P}$, *if* $\varphi_0, \varphi_1, \varphi \in \tilde{P}$.

In the sequel let A_0, A_1, A and B_0, B_1, B be symmetric Banach lattices (Banach function spaces) over the interval $I=(0,\ell)$, $0 < \ell \leq \infty$, with Lebesgue measure in the sense of [6], and with the fundamental functions φ_0, φ_1, φ and ψ_0, ψ_1, ψ, respectively. Recall that the fundamental function of a symmetric Banach lattice A is defined by $\varphi_A(t) = \|1_{(0,t)}\|_A$ where $1_{(0,t)}$ is the characteristic function of the interval $(0,t)$. Let $\varphi_{01} := \varphi_0/\varphi_1$ and ψ_0/ψ_1.

The smallest (largest) symmetric space contained in (containing) A with the same fundamental function is defined by

$$\Lambda_\varphi := \{x;\ \|x\|_{\Lambda_\varphi} := \varphi_A(0+)\,\|x\|_\infty + \int_0^\ell x^*(s)\,\varphi_A'(s)\,ds < \infty\}$$

$$(\, M_\varphi := \{x;\ \|x\|_{M_\varphi} := \sup_{t\in I} \varphi_A(t)\, x^{**}(t) < \infty\}\,).$$

Here x^* means the non-increasing rearrangement of the function x and $x^{**} := t^{-1}\int_0^t x^*(s)\,ds$.

A necessary condition for the interpolation of symmetric spaces is well known:

<u>Lemma 2.</u> *If* (A,B) *is an g-interpolation couple with respect to* $\bar{A}$ *and* $\bar{B}$, *then there exists a constant* $c > 0$ *such that for all* $s,t \in I$

$$(1) \qquad \frac{\psi(t)}{\varphi(s)} \leq c \max\left\{ \frac{\psi_0(t)}{\varphi_0(s)}, \frac{\psi_1(t)}{\varphi_1(s)} \right\}.$$

Moreover, (1) *is equivalent to each of the following conditions:*

$$(1a) \qquad \min\left\{ \frac{\varphi_0(s)}{\psi_0(t)}, \frac{\varphi_1(s)}{\psi_1(t)} \right\} \leq c \, \frac{\varphi(t)}{\psi(s)}$$

for some $c > 0$ *and all* $t,s \in I$;

$$(1b) \qquad \psi(t) \leq \psi_0(t)\, f\left(\frac{\psi_1(t)}{\psi_0(t)}\right), \quad \varphi_0(t)\, f\left(\frac{\varphi_1(t)}{\varphi_0(t)}\right) \leq c\, \varphi(t)$$

for some $f \in \tilde{P}$, $c>0$ *and all* $t \in I$.

For the proof Lemma 2 see [17] and [7], respectively.
In general, condition (1) is not sufficient for (A,B) to be an g-interpolation couple. Indeed, the following "weak converse" of Lemma 2 holds:

<u>Lemma 3.</u> *If* (1) *holds (or equivalently* (1a) *or* (1b) *), then*

$$L(\bar{A},\bar{B}) \subset [\Lambda_\varphi(A), M_\psi(B)].$$

with

$$\|T\|_{L(\bar{A},\bar{B})} \leq c \max_{i=0,1} \|T\|_{[\Lambda_{\varphi_i}, M_{\psi_i}]} \qquad (T \in L(\bar{A},\bar{B})).$$

For the proof of Lemma 3 recall that the Calderôn operator of the interpolation segment $\sigma := (\bar{A},\bar{B})$ is given by

$$S(\sigma)[x](t) := \int_0^\ell x(s)\, d\left(\min\left\{\frac{\varphi_0(s)}{\psi_0(t)}, \frac{\varphi_1(s)}{\psi_1(t)}\right\}\right) \qquad (x \in \Lambda_{\varphi_0} + \Lambda_{\varphi_1}).$$

Hence, the minimum occuring in (1a) can be expressed by

$$\min\left\{\frac{\varphi_0(s)}{\psi_0(t)}, \frac{\varphi_1(s)}{\psi_1(t)}\right\} = S(\sigma)[1_{(0,t)}](s),$$

and (1) is equivalent to

$$(1^*) \qquad S(\sigma)\,[\,1_{(0,t)}\,]\,(s)\cdot\psi(s) \leq c\ \varphi(t) \qquad (t,s\in I).$$

On the other hand, it can be shown (comp.[15,p. 497/498]) that for every operator $T\in L\,(\bar{A},\bar{B})$ one has

$$(Tx)^{**}\,(s) \leq (\max_{i=0,1} \|T\|_{[\Lambda_{\varphi_i}\,,\,M_{\psi_i}]})\cdot S(\sigma)\,[\,x^*\,](s)$$

for all $x\in\Lambda_{\varphi_0}+\Lambda_{\varphi_1}$. From (1*) it therefore follows that

$$(\,T\,1_{(0,t)})^{**}\,(s)\ \psi\,(s) \leq M\,c\,\varphi\,(t)$$

with $M := \max_{i=0,1} \|T\|_{[\Lambda_{\varphi_i}, M_{\psi_i}]}$ and c= constant of (1a).

Finally, by passing from $1_{(0,t)}$ to simple functions and then to limits of simple functions one obtains Lemma 3.

Remark 2. If the space B belongs to Sharpley's class $\mathcal{U}$ (see also [5]), then the assertion of Lemma 3 means that every operator $T\in L\,(\bar{A},\bar{B})$ is of weak type (A,B), provided (1) holds. This way of looking at the lemma explains, why condition (1) is not sufficient for the g-interpolation property, in general. In the sequel, we collect some concrete examples of Banach lattices for which - nevertheless - (1) is sufficient.

2. Interpolation of Lorentz and Marcinkiewicz spaces

In [8] it is proved that a couple $(\Lambda_\varphi\,,\,\Lambda_\psi)$ is an g-interpolation couple with respect to $(\Lambda_{\varphi_0}, \Lambda_{\varphi_1})$ and $(\Lambda_{\psi_0}, \Lambda_{\psi_1})$, if and only if (1) holds. In [16] a shorter proof of this statement is given.

We now discuss the interpolation property for couples of the type (M,M) and (Λ,M), respectively, giving new proofs of the results of [17] , [13] .

Theorem 1 (R. Sharpley). *A couple* $(M_\varphi\,,\,M_\psi)$ *is an g-interpolation couple with respect to* $(M_{\varphi_0}\,,\,M_{\varphi_1})$ *and* $(M_{\psi_0}\,,\,M_{\psi_1})$, *if and only if* (1) *holds.*

Theorem 2 (E.I. Pustyl`nik). *Let* $\min\{\varphi_0(0+), \varphi_1(0+)\} = 0$.
A couple $(\Lambda_\varphi, M_\psi)$ *is an g-interpolation couple with respect to* $(\Lambda_{\varphi_0}, \Lambda_{\varphi_1})$ *and* (M_{ψ_0}, M_{ψ_1}), *if and only if* (1) *holds.*

Our proof of the above two theorems is based upon the following lemma [4].

Lemma 4. *Let* $\bar{A} = (A_0, A_1)$ *be any Banach couple,* $K(t,a;\bar{A})$ *the Peetre K-functional with respect to* $\bar{A}$ *(with* $t > 0$, $a \in \sum(\bar{A})$ *), and* $\bar{M}_\psi := (M_{\psi_0}, M_{\psi_1})$.
Then $T \in L(\bar{A}, \bar{M}_\psi)$, *if and only if*

$$(2) \qquad (Ta)^{**}(t) \leq \frac{c}{\psi_0(t)} K\left(\frac{\psi_0(t)}{\psi_1(t)}, a; \bar{A}\right)$$

with some $c > 0$ *and for all* $t > 0$, $a \in \sum(\bar{A})$.

Proof. If $a = a_0 + a_1 \in \sum(\bar{A})$ with $a_i \in A_i$ $(i=0,1)$ and $t>0$, then

$$(Ta)^{**}(t) \leq (Ta_0)^{**}(t) + (Ta_1)^{**}(t) \leq c_0 \frac{||a_0||_{A_0}}{\psi_0(t)} + c_1 \frac{||a_1||_{A_1}}{\psi_1(t)}$$

$$= \frac{c}{\psi_0(t)} \left(||a_0||_{A_0} + \frac{\psi_0(t)}{\psi_1(t)} ||a_1||_{A_1} \right).$$

Passing to the infimum over all representations, we obtain (2).
Conversely, if (2) holds, then, since $K(t,a;\bar{A}) \leq t^i ||a_i||_{A_i}$ for $a \in A_i$ $(i=0,1)$, we have

$$(Ta)^{**}(t) \leq \begin{cases} \dfrac{c}{\psi_0(t)} ||a||_{A_0} & \text{if } a \in A_0 \\[2ex] \dfrac{c}{\psi_0(t)} \dfrac{\psi_0(t)}{\psi_1(t)} ||a||_{A_1} = \dfrac{c}{\psi_1(t)} ||a||_{A_1} & \text{if } a \in A_1, \end{cases}$$

yielding $T \in L(\bar{A}, \bar{M}_\psi)$.

<u>Proof of Theorem 1</u>. Since $K(t,x; M_{\varphi_0}, M_{\varphi_1}) \approx ||x^{**} \min\{\varphi_0, \varphi_1\}||_\infty$ (see [18]), for any operator $T \in L(\bar{M}_\varphi, \bar{M}_\psi)$ and for all $x \in M_\varphi \subset M_{\min\{\varphi_0, \varphi_1\}} = M_{\varphi_0} + M_{\varphi_1}$ it follows from Lemma 4 that

$$(Tx)^{**}(t) \leq 2c \, ||x^{**} \min\{\frac{\varphi_0}{\psi_0(t)}, \frac{\varphi_1}{\psi_1(t)}\}||_\infty$$

$$\leq \frac{2c'}{\psi(t)} ||x^{**} \varphi||_\infty = \frac{2c'}{\psi(t)} ||x||_{M_\varphi}$$

with some $c' > 0$. Hence T is a bounded operator from M_φ into M_ψ.

<u>Proof of Theorem 2.</u> Since

$$K(t,x; \Lambda_{\varphi_0}, \Lambda_{\varphi_1}) = \int_0^\ell x^*(s)\, d\,(\min\{\varphi_0(s), t\varphi_1(s)\})$$

(see [15]), for any

$T \in L(\bar{\Lambda}_\varphi, \bar{M}_\psi)$ and all $x \in \Lambda_\varphi \subset \Lambda_{\min\{\varphi_0, \varphi_1\}} = \Lambda_{\varphi_0} + \Lambda_{\varphi_1}$ it follows from Lemma 4 that

$$(Tx)^{**}(t) \leq c \int_0^\ell x^*(s)\, d\,(\min\{\frac{\varphi_0(s)}{\psi_0(t)}, \frac{\varphi_1(s)}{\psi_1(t)}\})$$

$$\leq \frac{c'}{\psi(t)} \int_0^\ell x^*(s)\, d\,\varphi(s) = \frac{c'}{\psi(t)} ||x||_{\Lambda_\varphi}$$

with some $c' > 0$; i.e. $T \in [\Lambda_\varphi, M_\psi]$.

3. Calderón-Lozanovskiǐ Spaces and Interpolation

Let $\bar{A} = (A_0, A_1)$ be a couple of Banach lattices on (Ω, μ) and let $\varphi \in P$. Sometimes we regard φ as a function on $\mathbb{R}_+ \times \mathbb{R}_+$ by defining $\varphi(s,t) := s\,\varphi(t/s)$. We denote by $\varphi(\bar{A})$ the *Calderón-Lozanovskiǐ space* of all classes of measurable functions x on Ω such that $|x| \leq \lambda\, \varphi(|x_0|, |x_1|)$ μ-a.e. for some $x_i \in A_i$ with $||x_i||_{A_i} \leq 1$ $(i=0,1)$ and some $\lambda < \infty$. If $\varphi \in \tilde{P}$, then $\varphi(\bar{A})$ with the norm $||x||_\varphi := \inf \lambda$ is a Banach lattice intermediate to $\bar{A}$.
If, in particular, $\varphi(t) = t^\theta$, $0 < \theta < 1$, then $\varphi(\bar{A}) = A_0^{1-\theta} A_1^\theta$. This space was

introduced by A.P. Calderon [3]. Moreover, he considered the space $\varphi(\bar{A})$ with $A_1 = L_\infty^{\sim}$ as a generalization of an Orlicz space. The properties of the spaces $\varphi(\bar{A})$ were studied in detail by G. Ja. Lozanovskiǐ [9] .
In [8] Lozanovskiǐ proved that $\varphi(\bar{A})$ is not an interpolation functor even if $\varphi(t) = t^\Theta$. A. P. Calderón, P. P. Zabreiko, V. A. Šestakov , and G. Ja. Lozanovskiǐ gave some conditions such that $A_0^{1-\Theta} A_1^\Theta$ becomes an interpolation functor. These results were extended to $\varphi(\bar{A})$ by V. I. Ovchinnikov [12]. In particular, he proved

Theorem (Ovchinnikov [12, Thm.3]). *Let* $\bar{A} = (A_0,A_1)$ *and* $\bar{B} = (B_0,B_1)$ *be two couples of Banach lattices on* (Ω, μ) *and* (Ω_1, μ_1)*, respectively, and let* $\varphi \in P$. *If* $\varphi(\bar{B})$ *has the Fatou property, then* $(\varphi(\bar{A}), \varphi(\bar{B}))$ *is an interpolation couple with respect to* $\bar{A}$ *and* $\bar{B}$.

From this theorem the following can be derived:

Theorem 3. *Assume that* $\bar{\psi}_0(\bar{B})$ *and* $\bar{\psi}_1 = (\bar{B})$ *have Fatou norms, and let* $I = \mathbb{R}_+$. *If* (1) *holds, then* $(\bar{\varphi}(\bar{A}), \bar{\psi}(\bar{B}))$ *is an g-interpolation couple with respect to* $(\bar{\varphi}_0(\bar{A}), \bar{\varphi}_1(\bar{A}))$ *and* $(\bar{\psi}_0(\bar{B}), \bar{\psi}_1(\bar{B}))$.

For the proof of this theorem we use a lemma on reiteration, namely:

Lemma 5. *Let* $\varphi_0, \varphi_1, \varphi, f \in P$, $c>0$, *and* $g(t) := \varphi_0(t)\, f(\varphi_1(t) / \varphi_0(t))$. *If* $g < c\,\varphi$, *then* $f(\varphi_0(\bar{A}), \varphi_1(\bar{A})) \subset \varphi(\bar{A})$; *moreover, if* $\varphi < c\, g$, *then* $\varphi(\bar{A}) \subset f(\varphi_0(\bar{A}), \varphi_1(\bar{A}))$.

The proof of Lemma 5 is analogous to a Lemma of [3,p.166] where the case $f(t) = t^\Theta$ is considered.

Proof of Theorem 3. Let $T \in L((\bar{\varphi}_0(\bar{A}), \bar{\varphi}_1(\bar{A})), (\bar{\psi}_0(\bar{B}), \bar{\psi}_1(\bar{B})))$.
Since $(\bar{f}(\bar{\psi}_0(\bar{B}), \bar{\psi}_1(\bar{B})))'' = \bar{f}(\bar{\psi}_0(\bar{B})'', \bar{\psi}_1(\bar{B})'')$, see [9], it follows from

Ovchinnikov's theorem that T is a bounded operator from $\bar{f}(\varphi_0(\bar{A}), \bar{\varphi}_1(\bar{A}))$ into $\bar{f}(\bar{\psi}_0(\bar{B}), \bar{\psi}_1(\bar{B}))$. If (1) holds for $I = \mathbb{R}_+$, then by Lemma 2 one has

$$\bar{\varphi} < c\ \bar{\varphi}_0\ \bar{f}(\bar{\varphi}_1 / \bar{\varphi}_0) \quad \text{and} \quad \bar{\psi}_0\ \bar{f}(\bar{\psi}_1 / \bar{\psi}_0) < \bar{\psi}$$

for some $f \in \mathcal{P}$ and some $c > 0$. On account of Lemma 5 this yields that $T \in [\bar{\varphi}(\bar{A}), \bar{\psi}(\bar{B})]$.

If $\varphi \in \mathcal{P}$ and $\bar{A} = (A_0, A_1)$ is a couple of symmetric spaces on I, then the space $\varphi(\bar{A})$ is also symmetric and has a fundamental function equivalent to $||1_{(0,t)}|| \approx \varphi_0(t) / \varphi_{01}(t)$. From Lemma 2 and Theorem 3 we therefore obtain a characterization of the interpolation property for *symmetric* Calderón-Lozanovskiǐ spaces.

Theorem 4. *Let* $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ *be two couples of symmetric spaces on* $\mathbb{R}_+$ *such that* B_0 *and* B_1 *have the Fatou property, and* $\varphi_{01}(\mathbb{R}_+) = \psi_{01}(\mathbb{R}_+) = \mathbb{R}_+$. *The couple* $(\bar{\varphi}(\bar{A}), \bar{\psi}(\bar{B}))$ *is an g-interpolation couple with respect to* $(\bar{\varphi}_0(\bar{A}), \bar{\varphi}_1(\bar{A}))$ *and* $(\bar{\psi}_0(\bar{B}), \bar{\psi}_1(\bar{B}))$ *if and only if* (1) *holds.*

As an application of this theorem we finally give interpolation theorems for Orlicz spaces and for Musielak-Orlicz spaces. Indeed, if M is a convex Orlicz function and L_M the corresponding Orlicz space, then L_M can be written as a Calderón-Lozanovskiǐ space, namely $L_M = \varphi(L_1, L_\infty)$ with $\varphi(t) := t\,M^{-1}(1/t)$. Applying Theorem 4 to this case one therefore immediately has

Theorem 5. *A couple of Orlicz spaces* (L_M, L_N) *is an g-interpolation couple with respect to* (L_{M_0}, L_{M_1}) *and* (L_{N_0}, L_{N_1}) *if and only if condition* (1) *holds, or, equivalently, iff*

$$\frac{M^{-1}(u)}{N^{-1}(v)} < c \max\left\{ \frac{M_0^{-1}(u)}{N_0^{-1}(v)}, \frac{M_1^{-1}(u)}{N_1^{-1}(v)} \right\} \qquad (u, v > 0).$$

This theorem was stated in [14] without proof.

If (Ω,μ) is a complete σ-finite measure space, and M a convex Musielak-Orlicz function with respect to Ω, then $L_{\{M\}}$ denotes the corresponding Musielak-Orlicz space with Luxemburg norm (for details see [11]). This space is symmetric if $M(u,t)$ does not depend upon t, and we have

Theorem 6. *Let* $L_{\{M_0\}}, L_{\{M_1\}}, L_{\{M\}}$ *and* $L_{\{N_0\}}, L_{\{N_1\}}, L_{\{N\}}$ *be Musielak-Orlicz spaces on* (Ω,μ) *and* (Ω_1, μ_1), *respectively. We assume that for all* $u > 0$

(3) $$M^{-1}(u,t) = \varphi(M_0^{-1}(u,t), M_1^{-1}(u,t))$$

$$N^{-1}(u,s) = \varphi(N_0^{-1}(u,s), N_1^{-1}(u,s))$$

for some $\varphi \in P$, *where the inverse functions are taken with respect to* u *for every fixed* $t \in \Omega \setminus E$, $\mu(E) = 0$, *and* $s \in \Omega_1 \setminus E_1$, $\mu_1(E_1) = 0$. *Then* $(L_{\{M\}}, L_{\{N\}})$ *is an interpolation couple with respect to* $(L_{\{M_0\}}, L_{\{M_1\}})$ *and* $(L_{\{N_0\}}, L_{\{N_1\}})$.

The proof follows from Theorem 5, the equality $L_{\{N\}}'' = L_{\{N\}}$ and the fact that (3) implies $L_{\{M\}} = \varphi(L_{\{M_0\}}, L_{\{M_1\}})$ and $L_{\{N\}} = \varphi(L_{\{N_0\}}, L_{\{N_1\}})$.

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