Model Reduction using LMIs

A. Helmersson

Department of Electrical Engineering

Linköping University

S-581 83 Linköping, Sweden

tel: +46 13 281622

fax: +46 13 282622

email: andersh@isy.liu.se

May 25, 1994

33rd CDC

Abstract

This paper presents an iterative two-step LMI method for improving the H_∞ model error compared to Hankel norm reduction. The improvement of the Hankel norm reduced model is usually not significant, typically a few percents only. For practical use the LMI reduction scheme is usually not worth-while, but it can be interesting from the theoretical and principal point of view.

Keywords: LMIs, model reduction, H-infinity.

1 Introduction

Truncated balanced realization and Hankel norm reduction [5] are well-behaved algorithms for finding reduced order models. In this paper the selection of model is performed using the H_∞ norm as criterion. When reducing

a model of order n to order k , the Hankel norm reduced model is optimal if $k = n - 1$. In other cases the H_∞ model error is bounded by the sum of the $n - k$ smallest Hankel singular values. If these values are close to each other, one can expect that the Hankel norm reduction can be improved in some cases.

During recent years linear matrix inequalities (LMIs) have attracted a lot of interest for solving H_∞ problems. Since the model reduction problem can be considered as a special case of H_∞ design with confined degree of the controller, it seems natural to adopt LMIs for model reduction. The LMI formulation does not yield a convex problem when the degree of the controller is confined, it is not guaranteed that the solution will converge to a (global) minimum. However, we can expect improvement compared to the Hankel model reduction in most cases (if $k < n - 1$).

The model reduction algorithm proposed is based on an iterative two-step LMI scheme. The convergence of the algorithm is not analyzed but examples show that both linear (geometric) and sub-linear convergence can occur.

The paper is outlined as follows. In section 2 the results on the optimal Hankel norm reduction is summarized and in section 3 the LMI formulation for H_∞ problems is given. The algorithm is presented in section 4 and it is applied to a number of examples in section 5.

1.1 Notations

X^T denotes the transpose of X ; $X > (\geq) 0$ a symmetric, positive definite (semidefinite) matrix; $\text{diag}[X_1, X_2]$ a block-diagonal matrix composed of X_1 and X_2 ; $\|x\|_2$ the L_2 -norm of the vector x ; $\bar{\sigma}(X)$ the maximal singular value of X and $\|\cdot\|_\infty$ the H_∞ norm.

2 Optimal Hankel Norm Reduction

One well-behaved method for model reduction is the Optimal Hankel-norm approximation (see [5] for a comprehensive treatment). It is based on balanced realizations and Hankel singular values. The Hankel-norm model reduction starts by finding the balanced realization of the system.

According to the theory for Hankel-norm model reduction [5], the H_∞

model error, here denoted γ , is bounded by

$$\gamma \leq \sum_{i=k+1}^n \sigma_i$$

where we have assumed that the Hankel singular values σ_i are ordered

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_k \geq \sigma_{k+1} \geq \dots \geq \sigma_n \geq 0.$$

For any reduced model of order k we have that

$$\sigma_{k+1} \leq \gamma$$

Thus, the Hankel-norm model reduction is optimal in the H_∞ sense if $k = n - 1$, but not necessarily so in the general case.

3 Linear Matrix Inequalities (LMIs)

3.1 Bounded Real Lemma

In this paper linear matrix inequalities (LMIs) are used for analyzing the stability of systems with uncertainties. We first consider the problem of finding the H_∞ -norm of a stable system G defined by $G(s) = D + C(sI - A)^{-1}B$. Then $\|G\|_\infty < \gamma$ is equivalent to the existence of a symmetric, positive definite matrix $P > 0$ satisfying

$$\begin{bmatrix} PA + A^T P & PB & C^T \\ B^T P & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0. \quad (1)$$

This inequality is called the Bounded Real Lemma, see e.g. [6], [3]. This inequality will be used in different shapes in this paper.

3.2 Solving LMIs

During the last years methods for solving LMIs have been developed and made available [2], [4]. Methods are still being elaborated for higher efficiency and user flexibility.

4 Algorithm

We assume that the model to be reduced is defined by the state-space representation $G(s) = D + C(sI - A)^{-1}B$ and the reduced model by $\hat{G}(s) = \hat{D} + \hat{C}(sI - \hat{A})^{-1}\hat{B}$. The model reduction problem can be stated as an optimization problem. Minimize $\|\tilde{G}\|_\infty = \|G - \hat{G}\|_\infty$ with respect to $\hat{G}$.

Thus, the model reduction problem can be considered as an optimization problem subject to a matrix inequality constraint. Find the smallest possible γ with respect to $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ and $P > 0$ such that

$$\begin{bmatrix} \tilde{A}^T P + P \tilde{A} & P \tilde{B} & \tilde{C}^T \\ \tilde{B}^T P & -\gamma I & \tilde{D}^T \\ \tilde{C} & \tilde{D} & -\gamma I \end{bmatrix} < 0 \quad (2)$$

with

$$\left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array} \right] = \left[\begin{array}{cc|c} A & 0 & B \\ 0 & \hat{A} & \hat{B} \\ \hline C & -\hat{C} & D - \hat{D} \end{array} \right]. \quad (3)$$

By partitioning the matrix P into

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} > 0 \quad (4)$$

we can rewrite the inequality to

$$\left[\begin{array}{cccc} A^T P_{11} + P_{11} A & A^T P_{12} + P_{12} \hat{A} & P_{11} B - P_{12} \hat{B} & C^T \\ \hat{A}^T P_{12}^T + P_{12}^T A & \hat{A}^T P_{22} + P_{22} \hat{A} & P_{12}^T B - P_{22} \hat{B} & \hat{C}^T \\ B^T P_{11} - \hat{B}^T P_{12}^T & B^T P_{12} - \hat{B}^T P_{22} & -\gamma I & D^T - \hat{D}^T \\ C & \hat{C} & D - \hat{D} & -\gamma I \end{array} \right] < 0 \quad (5)$$

If $(\hat{A}, \hat{B})$ are kept constant the matrix inequality is linear in P and $(\hat{C}, \hat{D})$. Also, if P_{12} and P_{22} are kept constant it is linear in P_{11} and $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$.

The following iterative two-step algorithm can be used even if it does not guarantee (global) convergence.

- (i) Start with a $\hat{G}$ obtained from, for example, Hankel model reduction or truncated balanced realization.

- (ii) Keeping $(\hat{A}, \hat{B})$ constant, minimize γ subject to (5) with respect to $(P, \hat{C}, \hat{D})$;
- (iii) Keeping (P_{12}, P_{22}) constant, minimize γ subject to (5) with respect to $(P_{11}, \hat{A}, \hat{B}, \hat{C}, \hat{D})$;
- (iv) Repeat (ii) and (iii) until the solution converges given some criterion.

In step (iii), keeping (P_{12}, P_{22}) constant is not as restrictive as it might seem. Since, a similarity transformation can be performed on (A, B, C, D) without changing $\hat{G}$ only the subspace defined by P_{12} is significant. Thus, step (ii) finds the proper subspace and step (iii) minimizes γ with respect to this.

4.1 Input-Output Duality

There exists a symmetry between input and output. By replacing (A, B, C, D) with (A^T, C^T, B^T, D^T) and P with P^{-1} we get equivalent problems. This observation may be useful in certain applications.

5 Examples

5.1 Numerical Algorithms

In these examples a direct Matlab implementation of a method of centers for minimizing generalized eigenvalues [2] has been used. The implementation gives an unfair comparison since more efficient and flexible LMI solver are now available, e.g. [4].

The LMI reduction has been compared to Hankel norm reduction. The following commands in the μ -Analysis and Synthesis Toolbox [1] has been used for obtaining the Hankel norm reduced model

```
>> [bal, sv] = sysbal (sys);
```

where **bal** is the balanced realization of the system **sys** and **sv** contains the Hankel singular values. Then the Hankel-norm reduced model is obtained using the command

```
>> hank = hankmr (bal, sv, k, 'd');
```

where **k** is the order of the reduced system **hank**.

5.2 A Non-Iterative Example

We start by giving a very simple example of model reduction. We want to replace a dynamical system with a constant (D -matrix). This can be obtained without iterations since the matrix inequality is linear in P and $\hat{D}$; A and B do not exist. For example consider the system

$$G(s) = \frac{1}{(s+1)(s+5)}$$

with Hankel singular values 0.111236 and 0.012361. Hankel-norm model reduction yields $\hat{G}_H(s) = 0.1$, while the LMI method gives $\hat{G}_0(s) = 0.08513$. The model errors are 0.124722 and 0.115684 respectively.

5.3 Example 1

We first consider the third-order system

$$G(s) = \frac{1}{s+1} + \frac{1}{s^2+s+4}$$

We will now try to reduce this to a first order system. The Hankel singular values are 0.71443, 0.19114, 0.10171. Thus, by reducing the model order to one the model error can not be lower than $\sigma_2 = 0.19114$. The Hankel-norm model reduction guarantees a maximum H_∞ error of $\sum \sigma_i$, where the sum is taken over the removed states. In this case we get 0.2928. The Hankel-norm reduced model is

$$\hat{G} = 1.43514 \frac{1 - 0.106451s}{1 + 0.825176s}$$

The LMI reduced model is

$$\hat{G}_1(s) = 1.41322 \frac{1 - 0.110249s}{1 + 0.817918s}$$

The model errors are 0.197142 and 0.192139 respectively. The LMI reduced model is only marginally better than the Hankel-norm reduced one (about 2.6%). The Hankel-norm model reduced model is in this case quite close to its theoretical bound and much improvements using other techniques cannot be obtained.

In Figure 1 the error convergence versus iteration is shown. The plot gives $\gamma - \gamma^*$ where γ^* is the minimum value obtained during the iterative search.

Figure 1: γ -convergence of Example 1

Usually it is also the value obtained in the last iteration but in this example we have run the algorithm to the extremes of the numerical precision of the computer. The convergence is linear (or geometric) in this case, that is the remaining error is scaled by an approximately constant factor less than one in each iteration. In this case the factor is about 0.3. The convergence rate depends on how well the LMIs are performed in each iterative step. If the LMIs, which are also iterative, are run into higher accuracy by modifying the stopping criterion, then the convergence rate of the model reduction iterations are faster. However, it is not known which trade-off offers the best overall computational performance.

In Figure 2 the model errors are shown for the Hankel-norm reduced (dashed line) and the LMI reduced (solid line) models.

Figure 2: Model errors of Example 1

5.4 Example 2

The next example is somewhat more difficult and can introduce numerical problems. The system to be reduced is

$$G(s) = \frac{1}{(s+1)(s^2+s+10)}$$

The Hankel-norm singular values are 0.07608, 0.04925 and 0.02317. The Hankel model reduction yields

$$\hat{G}_H(s) = 0.14017 \frac{1 - 0.63286s}{1 + 1.801126s}$$

with a model error of 0.057330. The LMI method reduces the error close to 0.05, which apparently is the optimal value. The LMI model converges to

$$\hat{G}_1(s) = \frac{1}{20} \frac{1-s}{1+s}$$

The convergence rate is in this case very slow, probably due to the fact that the original system and the optimally reduced system have a common pole.

Figure 3: Model error convergence versus iteration of Example 2

The convergence rate is not linear (geometric) as in the previous examples (See Figure 3). The difficulty may be due to the fact that the model error converges towards an all-pass second order system

$$G(s) - \hat{G}_1(s) = \frac{1}{20} \frac{s^2 - s + 10}{s^2 + s + 10}$$

which is of order 2 only compared to 5 of the augmented system $\tilde{G}$.

5.5 Example 3

Next we will look at a fifth order system

$$G(s) = \frac{1}{s^2 + s + 4} + \frac{2}{s^2 + 2s + 2} + \frac{1}{2s + 1}$$

The Hankel singular values are 1.286225, 0.279441, 0.11875, 0.027257 and 0.026797. We will try to approximate this system to first and third order using the two model reduction schemes: Hankel and LMI.

Using Hankel-norm model reduction we obtain model errors of 0.05299 and 0.3230 for the third and first order models respectively.

Reducing the model to first order using the LMI method consumed 125 CPU seconds (on a Sparc-station SLC) for the first iteration (giving an error of 0.3003) and 195 CPU seconds for the next nine iterations. The model error after ten iterations was 0.29879 (7.5% reduction compared to the Hankel-norm model reduction). The geometric convergence rate after five iterations is 0.4-0.5. Only two or three iterations are needed for obtaining a good approximation to the (locally) optimal solution.

During the search for a third order model using the LMI approach numerical difficulties emerged and the LMI algorithm had to be modified slightly with respect to step length selection. After the modification the LMI reduction performed well without any further problems. The LMI model error obtained was 0.027280, which is very close to the theoretical limit imposed by the Hankel singular value, $\sigma_4 = 0.027257$. The model error is shown in Figure 4. In the figure the Hankel-norm reduced (dashed line) and the LMI reduced (solid line) models are compared. Also, the theoretical limit (σ_4 : dash-dotted line) is shown. Note that the LMI model error is very close to an all-pass transfer function, even though it does not seem to converge to an exact all-pass transfer function in this case.

Most of the improvement by using the LMI method relative to the Hankel-norm reduction comes already in the first iteration. Subsequent steps improve the error but are not that significant.

5.6 Summary

The model errors obtained for the different examples are summarized in the table below. As can be seen the improvement in error norm by using LMI reduction is usually not significant. However, the LMI model comes very close to the theoretical bound set by the Hankel singular values.

Example	$n \rightarrow k$	Hankel	LMI	σ_{k+1}
0	$2 \rightarrow 0$	0.1247	0.1157	0.1112
1	$3 \rightarrow 1$	0.1971	0.1921	0.1911
2	$3 \rightarrow 1$	0.05733	0.05000	0.04925
3	$5 \rightarrow 3$	0.05299	0.02728	0.02726
3	$5 \rightarrow 1$	0.3230	0.2988	0.2794

Figure 4: Model error versus frequency using third-order approximations of Example 3

6 Conclusions

A model reduction algorithm using linear matrix inequalities (LMIs) have been implemented and studied on a number of examples. The algorithm starts by using the Hankel-norm reduced model and performs improvements using an iterative two-step scheme.

In all examples studied a better approximation in the H_∞ sense have been found provided that the reduced model is of order $n - 2$ or less, where n is the order of the original model. Often, the improvement is not significant (few percents) but one case with better result has been found. The computational load with the LMI-solver implemented is high and it is usually not worth-while to employ the LMI algorithm unless optimal performance is a necessity. In the examples studied the LMI reduced model has been very close to the theoretical limit defined by the largest Hankel singular value that was removed (σ_{k+1}).

The convergence of the algorithm has not been analyzed. In the examples studied both linear (geometric) and sub-linear convergence occur. Already in the first step most of the improvement compared to the Hankel norm

reduction is achieved.

References

- [1] G. Balas, J. Doyle, K. Glover, A. Packard, and R. Smith. *μ -Analysis and Synthesis Toolbox for Use with Matlab, User's Guide*. The MathWorks, Inc., 1993.
- [2] S. Boyd and L. El Ghaoui. Method of centers for minimizing generalized eigenvalues. *Linear Algebra and Applications*, 1993.
- [3] P. Gahinet and P. Apkarian. An LMI-based parametrization of all H_∞ controllers with applications. In *IEEE Proceedings of the 32nd Conference on Decision and Control*, volume 1, pages 656–661, San Antonio, Texas, December 1993.
- [4] P. Gahinet and A. Nemirovskii. General-purpose LMI solvers with benchmarks. In *IEEE Proceedings of the 31st Conference on Decision and Control*, volume 3, pages 3162–3165, San Antonio, Texas, December 1993.
- [5] K. Glover. All optimal Hankel-norm approximations of linear multivariable systems and their L^∞ -error bounds. *International Journal of Control*, 39(6):1115–1193, 1984.
- [6] C. Scherer. *The Riccati Inequality and State-Space H_∞ -Optimal Control*. Ph. D. Dissertation, Universitat Wurtzburg, Germany, 1990.